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Nonlinear elliptic system with a singular Hardy potential

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List of symbols

Ω	Domain in $\mathbb{R}^N$, usually bounded with $0 \in \Omega$
$\partial\Omega$	Boundary of the domain Ω
N	Space dimension ($N \geq 3$)
∇u	Gradient of u
Δ	Laplacian operator
$(-\Delta)^s$	Fractional Laplacian operator
$\operatorname{div}(\cdot)$	Divergence operator
$\lambda, \lambda_1, \lambda_2$	Positive parameters, related to Hardy potential strength
γ	Weight exponent in the Hardy-type potential
Ω_T	Parabolic cylinder $\Omega \times (0, T)$
u_t, v_t	Partial derivatives with respect to time
Λ_N	Optimal Hardy constant: $\Lambda_N = \left(\frac{N-2}{2}\right)^2$
$\Lambda_{N,\gamma,a}$	Best constant in the Caffarelli–Kohn–Nirenberg inequality
$\mu(\lambda), \mu^+(\lambda)$	Characteristic exponents, roots of $s^2 - (N-2)s + \lambda = 0$
$\mu(\lambda, \gamma, a)$	Generalized exponent for the quasilinear case
$\mu_s(\lambda)$	Characteristic exponents related to the fractional Hardy operator $(-\Delta)^s - \lambda x ^{-2s}$
$P_\lambda(s)$	Polynomial $s^2 - (N-2)s + \lambda$
$D_\gamma^{1,a}(\Omega)$	Weighted Sobolev space with norm involving $ x ^{-a\gamma}$
$C_0^\infty(\Omega)$	Space of smooth functions compactly supported in Ω
$B_r(0)$	Ball of radius r centered at the origin
w_λ	Radial weak solution or supersolution associated with parameter λ
$u_0(x)$	Initial condition for parabolic problems
ϕ, ψ, φ	Test functions (usually in $C_0^\infty(\Omega)$)
$L^p(\Omega)$	Lebesgue space with integrability exponent p

Introduction

The study of elliptic and parabolic partial differential equations (PDEs) with nonlinearities has attracted considerable attention in recent years due to their broad applications in physical and geometric models. In particular, weighted systems involving Hardy-type potentials naturally arise in problems with singularities at the origin or in contexts where symmetry breaking and degeneracy occur. These systems are not only mathematically rich but also present significant analytical challenges, especially when combined with critical growth nonlinearities.

In this thesis, we focus on nonlinear systems where the interplay between singular weights, cooperative structures, and critical nonlinearities produces rich and complex dynamics. Our objective is to establish rigorous mathematical results that reveal new qualitative behaviors in both elliptic and parabolic settings.

More specifically, we analyze weighted, cooperative elliptic and parabolic systems whose leading operators are, respectively, the Laplacian and the Heat operator. The elliptic systems describe stationary states of diffusion processes, while the parabolic systems account for their time evolution. In both cases, the introduction of singular weights such as Hardy-type potentials combined with cooperative interactions between components, creates a framework that goes far beyond classical variational methods. This makes it necessary to carefully define the notion of solution and to develop new tools for their analysis.

The study of weighted and cooperative systems has attracted considerable attention in recent decades, with numerous results concerning existence, nonexistence, regularity, and blow-up phenomena. Nevertheless, the presence of critical nonlinearities and singular potentials continues to pose challenging open questions. The purpose of this thesis is: first, to situate our work within the existing literature and highlight the most relevant contributions; and second, to present our own results, which extend the theory to new weighted and cooperative frameworks.

A common feature of the systems studied here is their critical character. The delicate balance between diffusion, singular potentials, and nonlinear interactions makes the analysis particularly subtle, but also highly meaningful. Indeed, our findings demonstrate that classical models, once perturbed by weighted terms and cooperative effects, may display unexpected and sometimes surprising behaviors.

This thesis is organized into two main parts:

Part I: Elliptic Case. We study weighted elliptic systems involving the Laplacian with lower-order terms. Particular attention is given to the cooperative structure of the system where the singularity depends indirectly with the operator, which plays a decisive role in the existence and nonexistence of positive solutions.

Part II: Parabolic Case. In the first part, we study a class of parabolic systems involving a Hardy-type potential. We analyze the interplay between the coupling effects and the singular nature of the potential, and we establish results concerning the global existence of solutions as well as possible blow-up phenomena. In the second part, we study the parabolic system in the fractional setting. We investigate local existence and nonexistence results for three main types of nonlinearities. In a particular case, we also analyze the Fujita critical exponent, as well as the phenomena of global existence and finite-time blow-up.

In both parts, our main goal is to identify optimal conditions for the existence of solutions in the appropriate sense, and to describe in detail the influence of weights and cooperative effects. By combining techniques from elliptic and parabolic theory, critical point analysis, and functional inequalities, we obtain results that enrich the current understanding of weighted and cooperative PDE systems.

Motivation of the Problems

A central theme of this thesis is the role of singular weights, in particular the Hardy potential, together with nonlinear terms depending either on the solution itself or on its gradient. Much of the analysis is devoted to understanding how these ingredients compete or cooperate within elliptic and parabolic systems. The importance of Hardy-type singularities in the theory of partial differential equations can be traced back to the seminal contribution of Jean Leray [48], who encountered Hardy-type structures in his study of the Navier Stokes system in three dimensions. In that context, singular terms appeared naturally in connection with energy estimates, and the Hardy inequality provided a fundamental tool to control them.

Since then, Hardy-type inequalities have become a cornerstone of modern PDE theory, not only because they capture the delicate balance between diffusion and singularity, but also because they mark the threshold separating well-posedness from ill-posedness.

A fundamental result is the classical Hardy inequality: for all $u \in C_c^\infty(\mathbb{R}^N)$ with $N \geq 3$, one has

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \frac{1}{\Lambda_N} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \quad \Lambda_N = \left(\frac{N-2}{2} \right)^2. \quad (1)$$

The constant Λ_N is sharp but never attained, and extremal functions behave like $u(x) = |x|^{-\frac{N-2}{2}}$, which lie outside the natural Sobolev space $H^1(\mathbb{R}^N)$. This lack of compactness is at the heart of many analytical difficulties.

One of the most striking illustrations of the impact of Hardy-type terms was given by Baras and Goldstein [11], who studied the heat equation with an inverse-square potential:

$$u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N. \quad (2)$$

They proved that a critical threshold is dictated by the Hardy constant Λ_N : if $\lambda \leq \Lambda_N$, then nontrivial solutions exist, while if $\lambda > \Lambda_N$, no nontrivial nonnegative solution exists, not even in a weak sense. This dichotomy illustrates the decisive role of the Hardy potential in the dynamics of parabolic equations.

In parallel, Peral and Vázquez [51] investigated nonlinear parabolic models arising in combustion, such as

$$u_t - \Delta u = \lambda e^u, \quad B_r(0) \quad (3)$$

where the singular Hardy term interacts with a nonlinear source $f(u)$. Their analysis revealed delicate phenomena of competition between singular diffusion and nonlinear growth, including global existence, blow-up, and threshold behavior depending on the balance between the parameters.

The role of the Hardy potential also extends into quantum mechanics, where it arises in the spectral study of Schrödinger operators of the form

$$i\partial_t \psi = -\Delta \psi + V(x) \psi, \quad (4)$$

with ψ denoting the wave function and $V(x) = -\frac{\lambda}{|x|}$ a regular potential. In this setting, Fefferman [39] emphasized the borderline nature of the operator

$$L = -\Delta - \frac{\lambda}{|x|^2}, \quad (5)$$

whose boundedness from below is guaranteed precisely when $\lambda < \Lambda_N$. This observation plays a central role in the mathematical analysis of the stability of matter, since crossing the Hardy threshold may lead to collapse of the system.

Altogether, these contributions highlight that the Hardy potential is not merely a technical artifact but a universal structure influencing the qualitative behavior of solutions across elliptic, parabolic, and quantum models. It embodies a critical balance between diffusion, singularity, and nonlinearity, which continues to shape contemporary research in PDE theory. Moreover, when combined with cooperative interactions in systems or with parabolic dynamics, the Hardy potential can give rise to new phenomena, including the competition between global existence and finite-time blow-up. This delicate balance between diffusion, singular potentials, and nonlinear effects constitutes one of the driving motivations of the present thesis.

In summary, this work is motivated by the aim of understanding how Hardy-type singularities, cooperative structures, and critical nonlinearities interact in both elliptic and parabolic systems. Addressing these questions requires extending classical techniques and developing new analytical approaches. The results obtained here highlight the richness of the interplay between singular potentials and nonlinear diffusion, and they illustrate the extent to which Hardy-type terms shape the behavior of solutions.

Description

Part I: Elliptic Systems

The chapter 2 is devoted to the study of weighted nonlinear elliptic systems involving Hardy-type singular potentials and superlinear nonlinearities. The presence of weights and singular terms makes the analysis delicate. More precisely, we consider

$$\left\{ \begin{array}{ll} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} = v^p + f_1(x) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} = u^q + f_2(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \right. \quad (6)$$

where $f_1, f_2 \in L^\infty$.

We first establish non-existence results for positive supersolutions under explicit

conditions on the nonlinear exponents and the Hardy parameters. These thresholds are shown to be sharp by proving that, outside the critical region, supersolutions actually exist. In the critical case, where the balance between the singular potential and the nonlinear terms is particularly subtle, we refine the analysis by constructing singular test functions and deriving asymptotic estimates.

In this chapter, we investigate the existence and non-existence of positive weak supersolutions to weighted elliptic systems involving Hardy-type singular potentials. The main difficulty arises from the competition between the singular weights and the nonlinear source terms, which creates a delicate balance between solvability and blow-up near the origin.

We establish a sharp non-existence result: if the nonlinear exponents (p, q) lie above the critical curve

$$\max \left\{ \mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1} \right\} = 2(\gamma+1).$$

then the system admits no nontrivial nonnegative supersolution.

Conversely, when the parameters (p, q) fall below this curve, we construct explicit radial functions and verify that they satisfy the weak formulation, thereby proving the existence of positive supersolutions. This shows that the critical curve is optimal and precisely separates the region of non-existence from the region of existence.

The study is extended to a quasilinear version of the system, where the operator takes the form of a weighted a -Laplacian. In this framework, we introduce new critical exponents, determine the corresponding non-existence region, and construct explicit supersolutions to demonstrate the sharpness of the thresholds.

For the quasilinear system

$$\begin{cases} -\operatorname{div}(|x|^{-a\gamma}|\nabla u|^{a-2}\nabla u) - \lambda_1 \frac{u^{a-1}}{|x|^{a(\gamma+1)}} = v^p + f_1(x) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-a\gamma}|\nabla v|^{a-2}\nabla v) - \lambda_2 \frac{v^{a-1}}{|x|^{a(\gamma+1)}} = u^q + f_2(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (7)$$

we establish both existence and non-existence results for positive weak supersolutions. The analysis is governed by a critical curve that separates the solvable

region from the non-existence region. Specifically, if the pair of exponents (p, q) lies above the curve

$$\max \left\{ \mu(\lambda_1, \gamma, a) \frac{pq - (a-1)^2}{p+1}, \mu(\lambda_2, \gamma, a) \frac{pq - (a-1)^2}{q+1} \right\} = a(\gamma+1),$$

then no nontrivial positive weak supersolution exists. Conversely, when (p, q) lie strictly below this curve, explicit supersolutions can be constructed, showing that the condition is sharp.

This description highlights the persistence of the existence and non existence in the quasilinear setting and emphasizes the fundamental role of the critical curve in determining the threshold for positive solutions.

The main tools employed include Hardy-type inequalities adapted to the weights, Picone's inequality, careful asymptotic analysis near the singularity, and the method of sub- and supersolutions. Altogether, the chapter provides a nearly complete description of the threshold phenomenon for the existence of positive solutions in such weighted systems, and highlights the role of singular potentials in shaping the existence–non-existence landscape.

This chapter is based on the results appearing in [13].

In Chapter 3, we turn our attention to the linear cooperative case, described by the system

$$\begin{cases} -\Delta u &= \lambda_1 \frac{v}{|x|^2} + f & \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{u}{|x|^2} + g & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega \end{cases} \quad (8)$$

where $0 < \lambda_1, \lambda_2$ and $f, g \in L^1(\Omega)$. In this framework, the main focus is on the existence and non-existence of positive supersolutions, and on the asymptotic behavior of solutions near singularities or in the presence of the cooperative structure.

The key result can be summarized as follows: if

$$\sqrt{\lambda_1 \lambda_2} > \Lambda_N,$$

then the system (8) admits no positive supersolution. This highlights the critical role of the product of the parameters λ_1 and λ_2 in determining solvability. Conversely, when the parameters satisfy

$$\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N,$$

one can establish the existence of positive supersolutions, possibly under additional conditions.

This chapter, therefore, illustrates how the cooperative nature of the system influences the threshold for existence, complementing the previous analysis of singular and weighted systems. The interplay between the cooperative structure and the spectral threshold Λ_N plays a central role in the existence–nonexistence dichotomy.

We will focus our analysis on the following two cases:

$$i) \quad f = v^p, \quad g = u^q, \quad ii) \quad f = |\nabla v|^p, \quad g = |\nabla u|^q. \quad (9)$$

As a first application, we consider the linear potential case, where the nonlinearities depend only on the functions themselves. The main result in this framework can be formulated in terms of the critical curves

$$h(p) = \frac{2}{\mu(\sqrt{\lambda_1 \lambda_2})} + \frac{\mu(\sqrt{\lambda_1 \lambda_2}) + 2}{p \mu(\sqrt{\lambda_1 \lambda_2})}, \quad \hat{h}(q) = \frac{2}{\mu(\sqrt{\lambda_1 \lambda_2})} + \frac{\mu(\sqrt{\lambda_1 \lambda_2}) + 2}{q \mu(\sqrt{\lambda_1 \lambda_2})}.$$

Then, if the pair of exponents (p, q) satisfies

$$q = h(p) \quad \text{or} \quad p = \hat{h}(q),$$

the system

$$\begin{cases} -\Delta u - \lambda_1 \frac{v}{|x|^2} = v^p, & \text{in } \Omega, \\ -\Delta v - \lambda_2 \frac{u}{|x|^2} = u^q, & \text{in } \Omega, \\ u, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \quad (10)$$

admits no positive weak supersolution. This demonstrates that the critical curves $h(p)$ and $\hat{h}(q)$ define a sharp threshold separating the existence and non-existence regimes for positive solutions.

This chapter is based on the results appearing in [8].

Part II: Parabolic Systems

In chapter 4, the corresponding parabolic systems are also treated, exhibiting a similar structure and critical thresholds.

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + f(v, \nabla v), & \text{in } \Omega_T, \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + g(u, \nabla u), & \text{in } \Omega_T, \end{cases} \quad (11)$$

where $\Omega \subseteq \mathbb{R}^N$ with $N \geq 3$, is an open bounded Lipschitz domain with $0 \in \Omega$ or $\mathbb{R}^N$, and $\Omega_T := \Omega \times (0, T)$ for some $T > 0$. We will focus our analysis in the cases:

$$i) f = v^p, g = u^q, \quad ii) f = |\nabla v|^p, g = |\nabla u|^q, \quad iii) f = v^p, g = |\nabla u|^q. \quad (12)$$

The fundamental local existence and nonexistence results are determined by a critical curve that separates the region of parameters leading to the existence of solutions from the region where solutions cannot exist. These results are summarized in the following theorem.

Theorem 0.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin.*

(1) *Suppose that $f = v^p$, and $g = u^q$. Then*

(i) *if*

$$\frac{p(q+1)}{p+1} \geq 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} \geq 1 + \frac{1}{1+\mu(\lambda_2)}, \quad (13)$$

then system (5.1) has no non-trivial positive very weak supersolutions.

(ii) *If*

$$\frac{p(q+1)}{p+1} < 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{1}{1+\mu(\lambda_2)}, \quad (14)$$

then system (5.1) admits positive supersolutions.

(2) *Suppose that $f = |\nabla v|^p$, and $g = |\nabla u|^q$. Then*

(i) *if*

$$\frac{p(q+1)}{p+1} > 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} > 1 + \frac{1}{1+\mu(\lambda_2)}, \quad (15)$$

then system (5.1) has no non-trivial positive very weak supersolutions.

(ii) If

$$\frac{p(q+1)}{p+1} < 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{1}{1+\mu(\lambda_2)}, \quad (16)$$

then system (5.1) has positive supersolutions.

(3) Suppose that $f = v^p$, and $g = |\nabla u|^q$. Then

(i) if

$$\frac{p(q+2)}{2p+1} > 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)+1}{q+2} > 1 + \frac{1}{\mu(\lambda_2)}, \quad (17)$$

then system (5.1) has no non-trivial positive very weak supersolutions.

(ii) If

$$\frac{p(q+2)}{2p+1} < 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)+1}{q+2} < 1 + \frac{1}{\mu(\lambda_2)}, \quad (18)$$

then system (5.1) admits positive supersolutions.

Following the study of the heat system in the local case, we proceed to analyze the Cauchy problem in order to establish global existence results.

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + v^p, & \text{in } \mathbb{R}^N \times (0, T), \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + u^q, & \text{in } \mathbb{R}^N \times (0, T), \end{cases} \quad (19)$$

Our main goal is to characterize the critical exponents that determine whether positive solutions blow up in finite time or exist globally. We introduce the notion of *very weak solutions* suitable for handling singular initial data and unbounded solutions. Using test function techniques, we establish a Fujita-type blow-up criterion for the system. More precisely, if the exponents p and q satisfy

$$\frac{p(q+1)}{p+1} \leq 1 + \frac{2}{N-\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} \leq 1 + \frac{2}{N-\mu(\lambda_2)},$$

then any positive very weak solution blows up in finite time. Conversely, for exponents above these critical values, we construct global supersolutions showing the existence of positive solutions for all times. This section thus extends the classical Fujita exponent to systems with singular potentials, providing a precise threshold

for blow-up versus global existence.

This chapter is based on the results appearing in [15].

In chapter 5, the corresponding fractional parabolic systems can also be analyzed, exhibiting a similar structure together with the appearance of critical thresholds governing existence and nonexistence phenomena. More precisely, we consider the system

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + f(v, \nabla v), & \text{in } \Omega_T, \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + g(u, \nabla u), & \text{in } \Omega_T, \end{cases} \quad (20)$$

where $0 < s < 1$, $\Omega \subseteq \mathbb{R}^N$ ($N > 2s$) is either a bounded Lipschitz domain containing the origin or the whole space $\mathbb{R}^N$, and $\Omega_T := \Omega \times (0, T)$ for some $T > 0$.

In this framework, we focus on the following typical nonlinearities:

$$(i) f = v^p, g = u^q, \quad (ii) f = |\nabla v|^p, g = |\nabla u|^q, \quad (iii) f = v^p, g = |\nabla u|^q.$$

As in the local case, the qualitative behavior of solutions is governed by the presence of a critical curve in the (p, q) -plane, which separates the region of parameters ensuring the existence of positive supersolutions from the region where no nontrivial solutions can occur. These threshold phenomena are described through suitable adaptations of Hardy-type constants associated with the fractional operator.

More precisely, one can establish existence and nonexistence results for very weak supersolutions depending on the interplay between the exponents (p, q) and the fractional Hardy potentials. The corresponding criteria involve modified critical quantities reflecting the nonlocal nature of $(-\Delta)^s$ and the singular weights $|x|^{-2s}$.

Following the analysis of the fractional heat system in bounded domains, we also investigate the associated Cauchy problem in the whole space:

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + v^p, & \text{in } \mathbb{R}^N \times (0, T), \\ v_t + (-\Delta)^s v = \lambda \frac{v}{|x|^{2s}} + u^p, & \text{in } \mathbb{R}^N \times (0, T). \end{cases} \quad (21)$$

Our main objective is to identify the critical exponents that determine whether solutions exist globally in time or blow up in finite time. To handle possible

singularities and lack of regularity, we work within the framework of *very weak solutions*. By employing suitable test function arguments adapted to the fractional setting, we derive a Fujita-type criterion for systems involving singular potentials.

More precisely, the competition between diffusion, potential terms, and nonlinear effects leads to critical thresholds that extend the classical Fujita exponent to the fractional and singular context. These results provide a sharp characterization of the transition between global existence and finite-time blow-up, highlighting the influence of both the fractional diffusion and the Hardy-type potentials.

This chapter is based on the results appearing in [\[14\]](#)

Chapter 1

Preliminaries and Functional Settings

This chapter gathers the essential mathematical concepts, definitions, and technical results that form the foundation of the analysis carried out in this thesis. Given the complexity and variety of the problems considered including singular elliptic and parabolic systems with Hardy-type potentials and both local and weighted nonlinearities it is crucial to establish a consistent framework and notation from the outset.

We begin by recalling fundamental function spaces and inequalities that will be instrumental in handling regularity, integrability, and singular behavior of solutions. Subsequently, we introduce the notions of weak and super-solutions, explicit radial solutions for homogeneous problems with Hardy potentials, and weighted Sobolev spaces. Finally, we present several key inequalities including Hardy, Picone, and Caffarelli–Kohn–Nirenberg inequalities which are systematically used throughout the thesis.

1 Functional Spaces and Hardy Inequalities

In this section, we recall some functional spaces and fundamental inequalities that will be used throughout this work. These tools are essential for handling singular potentials, particularly those involving inverse-square type terms.

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain containing the origin. We denote by $H_0^1(\Omega)$ the Sobolev space defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1(\Omega)} := \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

Hardy Inequality

We first recall the classical Hardy inequality (see [25]), which plays a fundamental role in the study of singular problems.

Theorem. For all $u \in H_0^1(\Omega)$, one has

$$\int_{\Omega} |\nabla u|^2 dx \geq \Lambda_N \int_{\Omega} \frac{|u|^2}{|x|^2} dx. \quad (1.1)$$

The constant $\Lambda_N = \frac{(N-2)^2}{4}$ is optimal and reflects the critical nature of the inverse-square potential. This inequality shows that the singular term $|x|^{-2}$ is controlled by the Dirichlet energy.

Weighted Sobolev Space

To deal with more general singular structures, we introduce the weighted Sobolev space (see [49])

$$H_\gamma^1(\Omega) := \overline{C_c^\infty(\Omega \setminus \{0\})}^{\|\cdot\|_{H_\gamma^1}},$$

where

$$\|u\|_{H_\gamma^1(\Omega)} := \left(\int_{\Omega} |\nabla u|^2 |x|^{-2\gamma} dx + \int_{\Omega} u^2 |x|^{-2\gamma} dx \right)^{1/2}, \quad \gamma \in \mathbb{R}.$$

The exclusion of the origin in the test functions is necessary due to the presence of the singular weight.

Weighted Hardy Inequality

We now state the weighted version of Hardy's inequality (see [29]), which extends the previous result to the weighted framework.

Theorem. Let $\gamma < \frac{N-2}{2}$. Then for all $u \in C_c^\infty(\Omega \setminus \{0\})$,

$$\int_{\Omega} |\nabla u|^2 |x|^{-2\gamma} dx \geq \Lambda_{N,\gamma} \int_{\Omega} \frac{|u|^2}{|x|^{2(\gamma+1)}} dx,$$

where $\Lambda_{N,\gamma}$ denotes the optimal constant. This inequality is a key tool for handling singular potentials in both linear and nonlinear problems, as it ensures that the weighted energy controls the singular term $|x|^{-2(\gamma+1)}$. The condition $\gamma < \frac{N-2}{2}$ guarantees the positivity of the constant.

2 Parabolic Functional Spaces

In the study of parabolic equations and systems, we shall frequently use the following functional spaces. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let

$$Q_T = \Omega \times (0, T).$$

For $1 \leq p \leq \infty$, we denote by

$$L^p(0, T; X)$$

For a Banach space X and $1 \leq p \leq \infty$, we define

$$L^p(0, T; X) = \left\{ u : (0, T) \rightarrow X \text{ measurable} : \|u\|_{L^p(0, T; X)} < \infty \right\},$$

where

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p},$$

with the usual modification when $p = \infty$.

The natural energy space associated with linear parabolic problems is

$$L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)),$$

whose elements have square-integrable spatial gradients and are continuous in time with values in $L^2(\Omega)$.

We also introduce the parabolic Sobolev space

$$W(0, T) = \{u \in L^2(0, T; H_0^1(\Omega)) : u_t \in L^2(0, T; H^{-1}(\Omega))\},$$

equipped with the norm

$$\|u\|_{W(0, T)} = \|u\|_{L^2(0, T; H_0^1(\Omega))} + \|u_t\|_{L^2(0, T; H^{-1}(\Omega))}.$$

By the Lions–Magenes theorem, one has the continuous embedding

$$W(0, T) \hookrightarrow C([0, T]; L^2(\Omega)).$$

3 Weak and Super-Solutions for Systems with Hardy Potentials

In the study of singular elliptic and parabolic systems involving Hardy-type potentials, the notions of weak solutions and super-solutions are fundamental. They provide a suitable framework to handle the singularity at the origin and to treat nonlinearities of critical or subcritical type.

Weak Solutions

Let us consider the elliptic system

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} = f(u, v) & \text{in } \Omega, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} = g(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $0 < \mu_1, \mu_2 \leq \frac{(N-2)^2}{4}$ and f, g are given nonlinearities.

A pair $(u, v) \in H_\mu^1(\Omega) \times H_\nu^1(\Omega)$ is called a weak solution if

$$\int_\Omega |f(u, v) \phi| dx < \infty, \quad \int_\Omega |g(u, v) \psi| dx < \infty,$$

for every $(\varphi, \psi) \in C_c^\infty(\Omega \setminus \{0\}) \times C_c^\infty(\Omega \setminus \{0\})$, and

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - \mu_1 \int_{\Omega} \frac{u \varphi}{|x|^2} \, dx &= \int_{\Omega} f(u, v) \varphi \, dx, \\ \int_{\Omega} \nabla v \cdot \nabla \psi \, dx - \mu_2 \int_{\Omega} \frac{v \psi}{|x|^2} \, dx &= \int_{\Omega} g(u, v) \psi \, dx. \end{aligned}$$

This formulation ensures that all terms are well-defined in the presence of the singular potentials, thanks to the structure of the weighted Sobolev spaces.

Supersolutions

A pair $(\bar{u}, \bar{v}) \in H_\mu^1(\Omega) \times H_\nu^1(\Omega)$ is called a super-solution if

$$\int_{\Omega} |f(\bar{u}, \bar{v}) \varphi| \, dx < \infty, \quad \int_{\Omega} |g(\bar{u}, \bar{v}) \psi| \, dx < \infty,$$

for every nonnegative test function $(\varphi, \psi) \in C_c^\infty(\Omega \setminus \{0\}) \times C_c^\infty(\Omega \setminus \{0\})$, and

$$\begin{aligned} \int_{\Omega} \nabla \bar{u} \cdot \nabla \varphi \, dx - \mu \int_{\Omega} \frac{\bar{u} \varphi}{|x|^2} \, dx &\geq \int_{\Omega} f(\bar{u}, \bar{v}) \varphi \, dx, \\ \int_{\Omega} \nabla \bar{v} \cdot \nabla \psi \, dx - \nu \int_{\Omega} \frac{\bar{v} \psi}{|x|^2} \, dx &\geq \int_{\Omega} g(\bar{u}, \bar{v}) \psi \, dx. \end{aligned}$$

Super-solutions play a central role in the construction of solutions, in particular through monotone iteration schemes. They also provide useful a priori estimates and allow one to control the behavior of solutions near the singularity.

The use of weighted Sobolev spaces guarantees that both weak super-solutions are well-defined despite the presence of singular terms. Similar notions can be introduced for parabolic systems by considering time-dependent test functions and integrals over space-time domains. The construction of super-solutions generally requires a careful balance between the singular potentials and the nonlinear terms, especially in the presence of critical growth.

Weak Super-Solutions

Consider the parabolic system

$$\begin{cases} u_t - \Delta u - \mu_1 \frac{u}{|x|^2} = f(u, v) & \text{in } \Omega \times (0, T), \\ v_t - \Delta v - \mu_2 \frac{v}{|x|^2} = g(u, v) & \text{in } \Omega \times (0, T), \\ u = v = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases} \quad (1.3)$$

A pair

$$(\bar{u}, \bar{v}) \in L^2(0, T; H_\mu^1(\Omega)) \times L^2(0, T; H_\nu^1(\Omega))$$

with

$$\partial_t \bar{u}, \partial_t \bar{v} \in L^2(0, T; H^{-1}(\Omega))$$

is called a weak super-solution if

$$\int_0^T \int_\Omega |f(\bar{u}, \bar{v}) \phi| dx dt < \infty, \quad \int_0^T \int_\Omega |g(\bar{u}, \bar{v}) \psi| dx dt < \infty,$$

for every nonnegative test functions $(\phi, \psi) \in C_c^\infty(\Omega \times (0, T)) \times C_c^\infty(\Omega \times (0, T))$, and

$$\begin{aligned} - \int_0^T \int_\Omega \bar{u} \partial_t \phi dx dt + \int_0^T \int_\Omega \nabla \bar{u} \cdot \nabla \phi dx dt - \mu_1 \int_0^T \int_\Omega \frac{\bar{u} \phi}{|x|^2} dx dt &\geq \int_0^T \int_\Omega f(\bar{u}, \bar{v}) \phi dx dt, \\ - \int_0^T \int_\Omega \bar{v} \partial_t \psi dx dt + \int_0^T \int_\Omega \nabla \bar{v} \cdot \nabla \psi dx dt - \mu_2 \int_0^T \int_\Omega \frac{\bar{v} \psi}{|x|^2} dx dt &\geq \int_0^T \int_\Omega g(\bar{u}, \bar{v}) \psi dx dt. \end{aligned}$$

Moreover,

$$\bar{u}(\cdot, 0) \geq u_0, \quad \bar{v}(\cdot, 0) \geq v_0 \quad \text{a.e. in } \Omega.$$

For the reader's convenience, we recall the classical Schauder Fixed Point Theorem, which will be used in the proof of our existence results (see, e.g., [64, Theorem 11.2]).

Theorem 3.1 (Schauder Fixed Point Theorem). *Let X be a Banach space and let K be a nonempty, closed, bounded, and convex subset of X . Assume that*

$$T : K \longrightarrow K$$

is a continuous and compact mapping (that is, $T(K)$ is relatively compact in X). Then T has at least one fixed point in K ; namely, there exists $u \in K$ such that

$$T(u) = u.$$

We recall the classical Kato inequality, which will be used repeatedly throughout this work (see [37]).

Theorem 3.2 (Kato's inequality). *Let $u \in L^1_{\text{loc}}(\Omega)$ be such that $\Delta u \in L^1_{\text{loc}}(\Omega)$ in the sense of distributions. Then*

$$\Delta|u| \geq \text{sgn}(u) \Delta u$$

in the sense of distributions.

4 Radial Solutions to the Homogeneous Equation

The purpose of this part is to analyze the behavior in a neighborhood of the origin of the solutions to the homogeneous problem

$$-\Delta w = \lambda w |x|^{-2} \text{in } (\mathbb{R}^N \setminus \{0\}), \quad (1.4)$$

where $0 < \lambda \leq \Lambda_N$, in order to use this a priori information as a tool for proving the existence and nonexistence results. Indeed, we start by constructing explicit radial solutions to the equation. Hence the equation admits two classical radial solution $|x|^{-\mu(\lambda)}$ and $|x|^{-\mu^+(\lambda)}$ where

$$\mu(\lambda) := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda} \quad \text{and} \quad \mu^+(\lambda) := \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}. \quad (1.5)$$

In particular $\mu(\lambda)$ and $\mu^+(\lambda)$ are the roots of the polynomial

$$P_\lambda(s) := s^2 - s(N-2) + \lambda. \quad (1.6)$$

Remark 4.1. *Let $\lambda_1, \lambda_2 \leq \Lambda_N$. Assume that u, v are a non-negative measurable functions such that $u, v \neq 0$ with $u, v \in L^1_{\text{loc}}(\Omega)$ and $u|x|^{-2}, v|x|^{-2} \in L^1_{\text{loc}}(\Omega)$. Suppose that u, v satisfy $-\Delta u - \lambda_1 u|x|^{-2} \geq 0$ and $-\Delta v - \lambda_2 v|x|^{-2} \geq 0$ in the weak sense. Then there exists a positive constant C and a small ball $B_r(0) \subset \Omega$ such that*

$$u(x) \geq C|x|^{-\mu(\lambda_i)}, \quad \text{a.e. in } B_r(0),$$

where $\mu(\lambda_i)$, $i = 1, 2$ are given above. (see [50])

Remark 4.2. *For any $\lambda_1, \lambda_2 \leq \Lambda_N$ if (u, v) is a positive supersolution, then for any $B_r(0) \subset \Omega$ we have that*

$$\int_{B_r(0)} |x|^{-\mu(\lambda_1)} f(v, \nabla v) dx < +\infty, \quad \int_{B_r(0)} |x|^{-\mu(\lambda_2)} g(u, \nabla u) dx < +\infty. \quad (1.7)$$

We establish the fundamental consequences required for the subsequent analysis.

This leads to the following theorem.

Theorem 4.3. *Let $f \in L^1(\Omega)$, then problem*

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) = f \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.8)$$

has an unique entropy solution u such that $u \in \mathcal{D}_{\gamma,0}^q(\Omega)$ for all $q < \frac{N-2\gamma}{N-2\gamma-1}$.

5 Picone Inequality

A fundamental tool for comparison principles is the Picone inequality.

Theorem. Let $v \in D_{\gamma,\text{loc}}^{1,1}(\Omega)$ be a positive function such that

$$-\operatorname{div}(|x|^{-2\gamma}\nabla v)$$

is a nonnegative and bounded Radon measure. Then, for all $u \in D_{\gamma,0}^{1,1}(\Omega)$, one has

$$\int_{\Omega} |\nabla u|^2 |x|^{-2\gamma} dx \geq \int_{\Omega} \frac{u^2}{v} (-\operatorname{div}(|x|^{-2\gamma}\nabla v)) dx. \quad (1.9)$$

In the particular case $\gamma = 0$, if $w \in W_{\text{loc}}^{1,1}(\Omega)$ is nonnegative and satisfies

$$-\Delta w = g \in L_{\text{loc}}^1(\Omega),$$

then for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} |\nabla \varphi|^2 dx \geq \int_{\Omega} \frac{-\Delta w}{w} \varphi^2 dx. \quad (1.10)$$

This inequality is a key ingredient in comparison arguments and in the construction of super-solutions.

5.1 Applications to Linear Problems with Hardy Potential

Consider the linear problem

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda \frac{u}{|x|^{2(\gamma+1)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.11)$$

with $f \in L^1(\Omega)$. Then we have the next results about the existence and the behavior of a weak solution near the origin.

Lemma 5.1.

• Assume that u is a non-negative weak super-solution to Problem (1.11) where $\lambda \leq \Lambda_{N,\gamma}$. Then $|x|^{-\mu(\lambda,\gamma)} f \in L^1_{loc}(\Omega)$, where $\mu(\lambda,\gamma)$ is defined in (1.5).

• Now, if $f \in L^1(\Omega)$ and $\int_{\Omega} f|x|^{-\mu(\lambda,\gamma)} dx < \infty$, then Problem (1.11) has a unique weak solution u such that $u \in W_0^{1,a}(\Omega)$ for all $a < \frac{N-2\gamma}{N-2\gamma-1}$ and $\int_{\Omega} \frac{u}{|x|^{2(\gamma+1)+\mu(\lambda)}} dx < \infty$.

• If $f \geq 0$ in Ω , then $u(x) \geq C|x|^{-\mu(\lambda,\gamma)}$ in $B_r(0) \subset\subset \Omega$.
In the case where $f(x) = \frac{1}{|x|^\beta}$ with $2 < \beta < N - \mu(\lambda,\gamma)$, the weak positive solution u to Problem (1.11) satisfies

$$u \geq |x|^{-\beta+2(\gamma+1)} \text{ in } B_r(0) \subset\subset \Omega.$$

To end this section we state the next existence result that is the key in order to show the non existence result on the critical curve (see [2]).

Lemma 5.2. Consider the problem:

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla w_\lambda) - \lambda \frac{w_\lambda}{|x|^{2(\gamma+1)}} = \frac{c}{|x|^{\mu(\lambda)+2(\gamma+1)}} & \text{in } B_\eta(0), \\ w_\lambda = 0 & \text{in } \partial B_\eta(0), \end{cases} \quad (1.12)$$

where $\lambda \leq \Lambda_{N,\gamma}$, then $w_\lambda(x) = C|x|^{-\mu(\lambda,\gamma)}(\ln(\frac{\eta}{|x|}))$. Moreover, if u is a supersolution to Problem (1.12), then $u \geq w$ in $B_\eta(0)$.

6 Weighted Sobolev Spaces

Since we are considering nonlinear elliptic systems with admissible weights in Chapter 2, it is necessary to introduce the appropriate weighted Sobolev spaces in which the solutions are defined.

Let $1 \leq a < N$ and $-\infty < \gamma < \frac{N-a}{a}$. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain such that $0 \in \Omega$.

Definition of the spaces

We define the weighted Sobolev space $\mathcal{D}_\gamma^{1,a}(\Omega)$ as the completion of $\mathcal{C}^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{\mathcal{D}_\gamma^{1,a}(\Omega)} := \left(\int_{\Omega} (|u|^a + |\nabla u|^a) |x|^{-a\gamma} dx \right)^{1/a}.$$

Similarly, we define $\mathcal{D}_{\gamma,0}^{1,a}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the same norm.

Since Ω is bounded, a weighted Poincaré-type inequality holds. As a consequence, the norm in $\mathcal{D}_{\gamma,0}^{1,a}(\Omega)$ is equivalent to

$$\|u\|_{\mathcal{D}_{\gamma,0}^{1,a}(\Omega)} := \left(\int_{\Omega} |\nabla u|^a |x|^{-a\gamma} dx \right)^{1/a}.$$

These spaces are Banach spaces and provide a natural framework for studying elliptic problems involving singular weights.

Basic properties

The restriction $\gamma < \frac{N-a}{a}$ ensures that the weight $|x|^{-a\gamma}$ is locally integrable near the origin. In particular,

$$|x|^{-a\gamma} \in L_{\text{loc}}^1(\Omega),$$

which guarantees that the above integrals are well-defined.

Moreover, the space $\mathcal{D}_{\gamma,0}^{1,a}(\Omega)$ is continuously embedded into suitable weighted Lebesgue spaces, which is essential for handling nonlinear terms.

Caffarelli–Kohn–Nirenberg Inequality

A fundamental tool in the analysis of weighted Sobolev spaces is the Caffarelli–Kohn–Nirenberg inequality (see [29]), which will be used repeatedly throughout this work.

As a consequence of this inequality, one obtains the following weighted Hardy–Sobolev inequality.

Lemma 6.1. *Let $N \geq 1$, $a > 1$ and $-\infty < \gamma < \frac{N-a}{a}$. Then for all $u \in \mathcal{D}_{\gamma,0}^{1,a}(\Omega)$, the following inequality holds:*

$$\Lambda_{N,\gamma,a} \int_{\Omega} \frac{|u|^a}{|x|^{a(\gamma+1)}} dx \leq \int_{\Omega} |x|^{-a\gamma} |\nabla u|^a dx, \quad (1.13)$$

where

$$\Lambda_{N,\gamma,a} = \left(\frac{N - a(\gamma + 1)}{a} \right)^a.$$

Moreover, the constant $\Lambda_{N,\gamma,a}$ is optimal and is not attained.

Particular case $a = 2$

In the case $a = 2$, for simplicity of notation, we write

$$\Lambda_{N,\gamma,2} = \Lambda_{N,\gamma}.$$

In this setting, the space $\mathcal{D}_{\gamma,0}^{1,2}(\Omega)$ becomes a Hilbert space endowed with the norm

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 |x|^{-2\gamma} dx \right)^{1/2}.$$

This framework is particularly suitable for studying elliptic and parabolic problems with Hardy-type singular potentials.

7 Fractional Laplacian

In this section, we recall some basic definitions and properties of the fractional Laplacian and the associated Hardy potential.

The fractional Laplacian. Let $s \in (0, 1)$ and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be a sufficiently smooth function. The fractional Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad (1.14)$$

where P.V. stands for the Cauchy principal value and $C_{N,s}$ is a normalization constant.

Alternatively, if u is sufficiently smooth and decays suitably at infinity, the fractional Laplacian can be characterized in Fourier variables by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi). \quad (1.15)$$

Functional framework. We denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi < \infty \right\}.$$

For a bounded domain $\Omega \subset \mathbb{R}^N$, we define

$$H_0^s(\Omega) = \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.$$

Hardy potential. We consider the singular potential of Hardy type

$$V(x) = \frac{\lambda}{|x|^{2s}}, \quad x \in \Omega \subset \mathbb{R}^N, \quad (1.16)$$

where $\lambda > 0$.

A fundamental tool is the fractional Hardy inequality: there exists a constant $\Lambda_{N,s} > 0$ such that

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \leq \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad (1.17)$$

for all $u \in C_c^\infty(\mathbb{R}^N)$.

The optimal constant is given by

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$

The inequality was established in [43]. For a complete direct proof, we refer the reader to Section 9.2 of [50].

Energy functional. Associated with the operator $(-\Delta)^s - \lambda|x|^{-2s}$, we define the energy

$$\mathcal{E}(u) = \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s}} dx. \quad (1.18)$$

If $\lambda < \Lambda_{N,s}$, then the energy is coercive on $H_0^s(\Omega)$.

7.1 Fractional Picone Inequality

A fundamental tool in the study of comparison principles for nonlocal problems is the fractional Picone inequality (see [47]).

Theorem 7.1 (Fractional Picone Inequality). *Let $0 < s < 1$ and let $v : \mathbb{R}^N \rightarrow \mathbb{R}$ be a positive function such that $v > 0$ in Ω , and $(-\Delta)^s v$ is well defined in the weak sense. Then, for every $u \in C_c^\infty(\Omega)$, one has*

$$\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \geq \int_{\Omega} \frac{u^2}{v} (-\Delta)^s v dx. \quad (1.19)$$

7.2 Radial Solutions to the Fractional Homogeneous Equation

The purpose of this section is to analyze the behavior in a neighborhood of the origin of the solutions to the homogeneous problem

$$(-\Delta)^s w = \lambda \frac{w}{|x|^{2s}} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad (1.20)$$

in order to use this a priori information as a tool for proving existence and nonexistence results. Indeed, we begin by constructing explicit radial solutions to the equation. Hence, the equation admits two classical radial solutions

$$|x|^{-\mu_s(\lambda)} \quad \text{and} \quad |x|^{-\mu_s^+(\lambda)},$$

where

$$\mu_s(\lambda) := \frac{N-2s}{2} - \alpha, \quad \mu_s^+(\lambda) := \frac{N-2s}{2} + \alpha, \quad (1.21)$$

and the parameter $\alpha \in (0, \frac{N-2s}{2})$ is uniquely determined by

$$\alpha = 2^{2s} \frac{\Gamma\left(\frac{N+2s+2\alpha}{4}\right) \Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s+2\alpha}{4}\right) \Gamma\left(\frac{N-2s-2\alpha}{4}\right)}. \quad (1.22)$$

Remark 7.2. Assume that u, v are nonnegative measurable functions such that

$$u, v \not\equiv 0, \quad u, v \in L_{\text{loc}}^1(\Omega), \quad u|x|^{-2s}, v|x|^{-2s} \in L_{\text{loc}}^1(\Omega).$$

Suppose that

$$(-\Delta)^s u - \lambda_1 \frac{u}{|x|^{2s}} \geq 0, \quad (-\Delta)^s v - \lambda_2 \frac{v}{|x|^{2s}} \geq 0$$

in the weak sense. Then there exists a positive constant C and a small ball $B_r(0) \subset \Omega$ such that

$$u(x) \geq C|x|^{-\mu_s(\lambda_1)}, \quad v(x) \geq C|x|^{-\mu_s(\lambda_2)} \quad \text{a.e. in } B_r(0),$$

where $\mu_s(\lambda_i)$, $i = 1, 2$, are defined above (see [38]).

Remark 7.3. For any $\lambda_1, \lambda_2 \leq \Lambda_{N,s}$, if (u, v) is a positive supersolution, $f, g \in L^1(\Omega)$ then for any $B_r(0) \subset \Omega$ we have

$$\int_{B_r(0)} |x|^{-\mu_s(\lambda_1)} f(v, \nabla v) dx < +\infty, \quad (1.23)$$

and

$$\int_{B_r(0)} |x|^{-\mu_s(\lambda_2)} g(u, \nabla u) dx < +\infty. \quad (1.24)$$

8 Fujita-Type Phenomena with Hardy Potentials

The study of blow-up and global existence for nonlinear parabolic equations originates from the seminal work of Fujita [42], who considered the classical semilinear heat equation

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^N \times (0, T). \quad (1.25)$$

He proved the existence of a critical exponent

$$p_F = 1 + \frac{2}{N},$$

called the *Fujita exponent*, which separates global existence and finite-time blow-up:

- if $1 < p \leq p_F$, every nontrivial nonnegative solution blows up in finite time,
- if $p > p_F$, there exist global solutions for sufficiently small initial data.

8.1 Local Case with Hardy Potential

When a Hardy-type singular potential is introduced, the model becomes

$$u_t - \Delta u = \lambda \frac{u}{|x|^2} + u^p. \quad (1.26)$$

The presence of the inverse-square potential significantly affects the behavior of solutions.

For $\lambda < \lambda_N$, the operator $-\Delta - \lambda|x|^{-2}$ remains coercive, and one can define the quantity $\mu(\lambda) = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$.

In this setting, the Fujita exponent takes the form

$$p_F(\lambda) = 1 + \frac{2}{N - \mu(\lambda)}. \quad (1.27)$$

The critical exponent $p_F(\lambda)$ separates the regimes of global existence and finite-time blow-up. In the Hardy setting, however, solutions may be singular at the origin, and the classical notion of L^∞ -blow-up is no longer appropriate. Following [4], finite-time blow-up is understood in the weighted sense given below.

Definition 8.1. *Let $u = u(x, t)$ be a positive solution. We say that u blows up in finite time if there exists $T^* < \infty$ such that*

$$\lim_{t \rightarrow T^*} \int_{B_r(0)} |x|^{-\mu(\lambda)} u(x, t) dx = \infty$$

for every ball $B_r(0) \subset \Omega$.

8.2 Fractional Laplacian with Hardy Potential

We now consider the fractional counterpart

$$u_t + (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p, \quad s \in (0, 1). \quad (1.28)$$

The corresponding Hardy inequality involves the sharp constant

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$

The Fujita-type critical exponent is then given by

$$p_F(\lambda, s) = 1 + \frac{2s}{N - \mu_s(\lambda)}. \quad (1.29)$$

This generalizes both:

- the classical Fujita exponent ($\lambda = 0$),
- and the Hardy-perturbed local case ($s = 1$).

8.3 Discussion and Interpretation

The presence of the Hardy potential modifies the diffusion–reaction balance by introducing a singular weight at the origin. The quantity $\mu(\lambda)$ measures the strength of this singularity and plays a crucial role in:

- the scaling properties of the equation,
- the decay of the associated heat kernel,
- and the blow-up/global existence dichotomy.

In particular, as $\lambda \rightarrow \lambda_N$ (or $\Lambda_{N,s}$), the effective dimension decreases, leading to a stronger tendency toward blow-up.

These results motivate the study of more complex systems involving: coupled nonlinearities.

Part I

Elliptic Case

Chapter 2

Weighted System

The present chapter focuses on a class of weighted nonlinear elliptic by the presence of a superlinear zero-order term and a Hardy-type potential . Specifically, we consider the system

$$\left\{ \begin{array}{ll} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} = v^p + f_1(x) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} = u^q + f_2(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \right. \quad (2.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain containing the origin, $\gamma \in (-\infty, \frac{N-2}{2})$, $\lambda_1, \lambda_2 \geq 0$ and f_1, f_2 are nonnegative measurable functions satisfying appropriate assumptions.

1 Nonexistence Results

Since we are considering systems with L^1 data, it is necessary to specify the sense in which solutions are defined. We provide the following definition

Definition 1.1. Assume that $p, q \geq 1$. We say that $(u, v) \in \mathcal{D}_{\gamma, loc}^{1,1}(\Omega) \times \mathcal{D}_{\gamma, loc}^{1,1}(\Omega)$, with $u, v \geq 0$, is a weak solution to System (2.1) if $\frac{u}{|x|^{2(\gamma+1)}}, \frac{v}{|x|^{2(\gamma+1)}} \in L_{loc}^1(\Omega)$,

$u^p, v^q \in L^1_{loc}(\Omega), f_1, f_2 \in L^1_{loc}(\Omega)$ and for all $\varphi, \psi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \nabla \varphi \nabla u |x|^{-2\gamma} dx &= \lambda_1 \int_{\Omega} \frac{u}{|x|^{2(\gamma+1)}} \varphi dx + \int_{\Omega} v^p \varphi dx + \int_{\Omega} f_1(x) \varphi dx, \\ &\text{and} \\ \int_{\Omega} \nabla \psi \nabla v |x|^{-2\gamma} dx &= \lambda_2 \int_{\Omega} \frac{v}{|x|^{2(\gamma+1)}} \psi dx + \int_{\Omega} u^q \psi dx + \int_{\Omega} f_2(x) \psi dx. \end{aligned} \quad (2.2)$$

Under the same regularity assumptions on u, v , we say that (u, v) is a supersolution to System (2.1) if for all $\varphi, \psi \in \mathcal{C}_0^\infty(\Omega)$ with $\varphi, \psi \geq 0$, we have

$$\begin{aligned} \int_{\Omega} \nabla \varphi \nabla u |x|^{-2\gamma} dx &\geq \lambda_1 \int_{\Omega} \frac{u}{|x|^{2(\gamma+1)}} \varphi dx + \int_{\Omega} v^p \varphi dx + \int_{\Omega} f_1(x) \varphi dx, \\ &\text{and} \\ \int_{\Omega} \nabla \psi \nabla v |x|^{-2\gamma} dx &\geq \lambda_2 \int_{\Omega} \frac{v}{|x|^{2(\gamma+1)}} \psi dx + \int_{\Omega} u^q \psi dx + \int_{\Omega} f_2(x) \psi dx. \end{aligned} \quad (2.3)$$

In this part, we provide an alternative demonstration of the nonexistence result established in [30]. Our method relies on a Picone-type inequality combined with a detailed study of the asymptotic behavior of positive solutions (u, v) to system (2.1) in the neighborhood of the singularity at the origin.

Theorem 1.2. *Let $0 < \lambda_1, \lambda_2 \leq \Lambda_{N, \gamma}$. Suppose that*

$$\max \left\{ \mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1} \right\} > 2(\gamma+1).$$

Then, system (2.1) does not admit any positive weak supersolutions.

Proof. We proceed by contradiction, assuming that (u, v) represents a weak supersolution of System (2.1). Then, we obtain that $u(x) \geq C|x|^{-\mu(\lambda_1, \gamma)}$ and $v(x) \geq C|x|^{-\mu(\lambda_2, \gamma)}$ in $B_r(0) \subset \subset \Omega$. In addition, we have

$$\int_{B_r(0)} |x|^{-\mu(\lambda_1, \gamma)} v^p dx + \int_{B_r(0)} |x|^{-\mu(\lambda_2, \gamma)} u^q dx < \infty.$$

Hence it holds that, for r small enough,

$$\int_{B_r(0)} |x|^{-\mu(\lambda_1, \gamma) - p\mu(\lambda_2, \gamma)} dx + \int_{B_r(0)} |x|^{-\mu(\lambda_2, \gamma) - q\mu(\lambda_1, \gamma)} dx < \infty.$$

Thus, if $p \geq \frac{N-\mu(\lambda_1, \gamma)}{\mu(\lambda_2, \gamma)}$ or $q \geq \frac{N-\mu(\lambda_2, \gamma)}{\mu(\lambda_1, \gamma)}$, this leads to a contradiction. Therefore we can assume that $p < \frac{N-\mu(\lambda_1, \gamma)}{\mu(\lambda_2, \gamma)}$ and $q < \frac{N-\mu(\lambda_2, \gamma)}{\mu(\lambda_1, \gamma)}$.

Hence, fixed $r > 0$ as above, then using Picone inequality given in Proposition (1.9), for both equation in $B_r(0)$, we find that for all $\varphi \in \mathcal{C}_0^\infty(B_r(0))$:

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx &\geq \int_{B_r(0)} \frac{-\operatorname{div}(|x|^{-2\gamma} \nabla u)}{u} \varphi^2 \geq \lambda_1 \int_{B_r(0)} \frac{\varphi^2}{|x|^{2(\gamma+1)}} + \int_{B_r(0)} \frac{v^p}{u} \varphi^2, \\ \int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx &\geq \int_{B_r(0)} \frac{-\operatorname{div}(|x|^{-2\gamma} \nabla v)}{v} \varphi^2 \geq \lambda_2 \int_{B_r(0)} \frac{\varphi^2}{|x|^{2(\gamma+1)}} + \int_{B_r(0)} \frac{u^q}{v} \varphi^2. \end{aligned}$$

Adding both inequalities,

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx &\geq \frac{1}{2} (\lambda_1 + \lambda_2) \int_{B_r(0)} \frac{\varphi^2}{|x|^{2(\gamma+1)}} + \frac{1}{2} \int_{B_r(0)} \varphi^2 \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \\ &\geq \frac{1}{2} \int_{B_r(0)} \varphi^2 K(u, v) \end{aligned} \quad (2.4)$$

where $K(u, v) = \frac{v^p}{u} + \frac{u^q}{v}$. Let $\alpha = \frac{v}{u^{\frac{1}{p}}}$ and consider the function $L(\sigma) = \sigma^p + \frac{B}{\sigma}$

where $B = u^{\frac{pq-1}{p}}$, then L has a minimum for $\hat{\sigma} = \left(\frac{B}{p}\right)^{\frac{1}{p+1}}$ and

$$L(\sigma) \geq L(\hat{\sigma}) = (p+1)\hat{\sigma}^p = \frac{p+1}{p^{\frac{1}{p+1}}} B^{\frac{p}{p+1}} = 2u^{\frac{pq-1}{p+1}}.$$

Going back to (2.4), we obtain that

$$\int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx \geq C \int_{B_r(0)} \varphi^2 u^{\frac{pq-1}{p+1}} dx.$$

Now if we consider $a = \frac{u}{v^{\frac{1}{q}}}$, setting $L(\sigma) = \sigma^q + \frac{\tilde{B}}{\sigma}$ where $\tilde{B} = v^{\frac{pq-1}{q}}$, then $K(u, v) \geq C v^{\frac{pq-1}{q+1}}$. Thus

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx &\geq C \int_{B_r(0)} \varphi^2 (u^{\frac{pq-1}{p+1}} + v^{\frac{pq-1}{q+1}}) dx \\ &\geq C \int_{B_r(0)} \varphi^2 \left(|x|^{-\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}} + |x|^{-\mu(\lambda_2, \gamma) \frac{pq-1}{q+1}} \right) dx. \end{aligned} \quad (2.5)$$

If $\max\{\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1}\} > 2(\gamma+1)$, this leads to a contradiction with the Hardy inequality. $\square$

Remark 1.3. Since $\mu(\lambda_i, \gamma) > 0$ for $i = 1, 2$, then if $\gamma + 1 < 0$, we deduce that System (2.1) admits only non-positive solutions for all $p, q \geq 1$. If $\gamma + 1 = 0$, then System (2.1) admits only non-positive solutions for all p, q such that $pq > 1$.

Remark 1.4. In the critical cases $\max\{\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1}\} = 2(\gamma + 1)$, then the above arguments can not give any conclusion related to the existence or non existence result. To treat the critical cases, it is necessary to refine how the solution behaves when approaching the origin.

Notice that if $\max\{\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1}\} < 2(\gamma + 1)$, then we can construct a suitable radial supersolution in balls. See the next section.

We are now prepared to establish the nonexistence result in the critical curve.

Theorem 1.5. Assume that $\gamma + 1 \geq 0$ and that $p < \frac{N-\mu(\lambda_1, \gamma)}{\mu(\lambda_2, \gamma)}, q < \frac{N-\mu(\lambda_2, \gamma)}{\mu(\lambda_1, \gamma)}$. Suppose that

$$\max\left\{\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1}\right\} = 2(\gamma + 1).$$

If either $\mu(\lambda_1, \gamma) \frac{pq-1}{p+1} = 2(\gamma + 1)$, or $\mu(\lambda_2, \gamma) \frac{pq-1}{q+1} = 2(\gamma + 1)$, then System (2.1) has non positive solution.

Proof. We will prove the first estimation under the first conditions. The other case follows in a similar way. We give the proof in several cases.

The first case : $\lambda_1, \lambda_2 < \Lambda_{N, \gamma}$ and $\mu(\lambda_1, \gamma) \frac{pq-1}{p+1} = 2(\gamma + 1)$.

Let (u, v) be is a non-negative, non-trivial weak supersolution of System (2.1), then from the second equation in System (2.1), we deduce that

$$-\operatorname{div}(|x|^{-2\gamma} \nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} \geq u^q \geq c|x|^{-q\mu(\lambda_1, \gamma)}.$$

Recall that $\mu(\lambda_1, \gamma)(pq-1) = 2(\gamma+1)(p+1)$, then

$q\mu(\lambda_1, \gamma) = 2(\gamma+1) + \frac{2(\gamma+1)+\mu(\lambda_1, \gamma)}{p} > 2(\gamma+1)$. Hence by Lemma 5.1, we deduce that $v(x) \geq \frac{c}{|x|^{q\mu(\lambda_1, \gamma)-2(\gamma+1)}}$. Going back to the first equation in System (2.1), it holds that

$$-\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} \geq v^p \geq \frac{1}{|x|^{p(q\mu(\lambda_1, \gamma)-2(\gamma+1))}}.$$

Observe that, under our assumption $p(q\mu(\lambda_1, \gamma) - 2(\gamma+1)) = \mu(\lambda_1, \gamma) + 2(\gamma+1)$. Hence

$$-\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} \geq \frac{1}{|x|^{\mu(\lambda_1, \gamma)+2(\gamma+1)}}.$$

Thus, using Lemma 5.2, it holds that $u \geq w(\lambda_1) = \frac{c}{|x|^{\mu(\lambda_1, \gamma)}} \ln\left(\frac{\eta}{|x|}\right)$ in $B_\eta(0)$. Following the same computational steps as in the proof of Theorem 2.2, for $\varphi \in \mathcal{C}_0^\infty(B_r(0))$ with $r \ll \eta$, we arrive to

$$\int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx \geq C \int_{B_r(0)} \varphi^2 u^{\frac{pq-1}{p+1}} \geq C \int_{B_r(0)} \varphi^2 |x|^{-\mu(\lambda_1, \gamma) \frac{pq-1}{p+1}} \left(\ln\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-1}{p+1}} dx. \quad (2.6)$$

Since $\mu(\lambda_1, \gamma) \frac{pq-1}{p+1} = 2(\gamma+1)$, then we get

$$\int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} \geq C \int_{B_r(0)} \frac{\varphi^2}{|x|^{2(\gamma+1)}} \left(\ln\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-1}{p+1}} dx,$$

a result that contradicts the Hardy inequality.

The second case : $\lambda_1, \lambda_2 < \Lambda_{N, \gamma}$ and $\mu(\lambda_2, \gamma) \frac{pq-1}{q+1} = 2(\gamma+1)$.

Using the first equation in System (2.1), we deduce that

$$-\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} \geq v^p \geq |x|^{-p\mu(\lambda_2, \gamma)}.$$

As in the first case, using the fact that $\mu(\lambda_2, \gamma)(pq-1) = 2(\gamma+1)(q+1)$, then $\frac{c}{p\mu(\lambda_2, \gamma)} > 2(\gamma+1)$, thus using Lemma 5.1, we deduce that $u(x) \geq \frac{c}{|x|^{p\mu(\lambda_2, \gamma) - 2(\gamma+1)}}$.

Hence we get

$$-\operatorname{div}(|x|^{-2\gamma} \nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} \geq u^q \geq \frac{1}{|x|^{q(p\mu(\lambda_2, \gamma) - 2(\gamma+1))}} \geq \frac{C}{|x|^{\mu(\lambda_1, \gamma) + 2(\gamma+1)}}.$$

Thus, using Lemma 5.2, it holds that $v \geq w(\lambda_1) = \frac{c}{|x|^{\mu(\lambda_1, \gamma)}} \ln\left(\frac{\eta}{|x|}\right)$ in $B_\eta(0)$. Therefore, following the argument used in the proof of Theorem 2.2, for $\varphi \in \mathcal{C}_0^\infty(B_r(0))$ with $r \ll \eta$, we reach that

$$\int_{B_r(0)} |\nabla \varphi|^2 |x|^{-2\gamma} dx \geq C \int_{B_r(0)} \varphi^2 v^{\frac{pq-1}{q+1}} \geq C \int_{B_r(0)} \frac{\varphi^2}{|x|^{2(\gamma+1)}} \left(\ln\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-1}{q+1}} dx,$$

a result that contradicts the Hardy inequality.

The case $\lambda_1 = \Lambda_{N, \gamma}$. In this case $\mu(\lambda_1, \gamma) = \frac{N-2(\gamma+1)}{2}$. As above we have $v(x) \geq \frac{C}{|x|^{q \frac{N-2(\gamma+1)}{2} - 2(\gamma+1)}}$ in $B_r(0)$. Hence, using the fact that $\int_{B_r(0)} v^p |x|^{-\mu(\Lambda_{N, \gamma}, \gamma)} dx < \infty$, it follows that

$$\int_{B_r(0)} \frac{C}{|x|^{p(q \frac{N-2(\gamma+1)}{2} - 2(\gamma+1))}} |x|^{-\mu(\Lambda_{N, \gamma}, \gamma)} dx < \infty. \quad (2.7)$$

Thus

$$p \left(q \frac{N-2(\gamma+1)}{2} - 2(\gamma+1) \right) + \mu(\Lambda_{N,\gamma}, \gamma) < N.$$

Since $\mu(\Lambda_{N,\gamma}, \gamma)(pq-1) = 2(\gamma+1)(p+1)$, then

$$p \left(q \frac{N-2(\gamma+1)}{2} - 2(\gamma+1) \right) + \mu(\Lambda_{N,\gamma}, \gamma) = N, \quad (2.8)$$

a contradiction with (2.7). Consequently, the claim follows. $\square$

As a variant of the system (2.1), we can consider the next one:

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} = |x|^{-2\gamma}v^p + f_1(x) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} = |x|^{-2\gamma}u^q + f_2(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.9)$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded regular domain and $-\infty < \gamma < \frac{N-2}{2}$. In this case, the nonexistence result may be expressed in the following form.

Theorem 1.6. Assume that $0 < \lambda_1, \lambda_2 \leq \Lambda_{N,\gamma}$ and

$$\max \left\{ \mu(\lambda_1, \gamma) \frac{pq-1}{p+1}, \mu(\lambda_2, \gamma) \frac{pq-1}{q+1} \right\} \geq 2.$$

Then System (2.9) has non positive weak supersolution.

Our argument can be used also to treat the quasi-linear case. More precisely, we turn our attention to the system:

$$\begin{cases} -\operatorname{div}(|x|^{-a\gamma}|\nabla u|^{a-2}\nabla u) - \lambda_1 \frac{u^{a-1}}{|x|^{a(\gamma+1)}} = v^p + f_1(x) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-a\gamma}|\nabla v|^{a-2}\nabla v) - \lambda_2 \frac{v^{a-1}}{|x|^{a(\gamma+1)}} = u^q + f_2(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.10)$$

In order to analyze the solution's behavior close to the origin, we consider radial solutions of the equation

$$-\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) = \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \text{ in } \mathbb{R}^N.$$

Notice that, if w is a radial positive solution to the previous equation, then w solves the nonlinear differential equation

$$(r^{N-a\gamma-1}|w'|^{a-2}w')' + r^{N-a\gamma-1}\lambda \frac{w^{a-1}}{r^a} = 0. \quad (2.11)$$

Setting $w(x) = A|x|^{-\sigma}$, then $\sigma = \mu(\lambda, \gamma, a)$ or $\sigma = \hat{\mu}(\lambda, \gamma, a)$ where $0 < \mu(\lambda, \gamma, a) < \hat{\mu}(\lambda, \gamma, a)$ solve the algebraic equation

$$(a-1)|\xi|^a - (N-a(\gamma+1))|\xi|^{a-1} + \lambda = 0. \quad (2.12)$$

Notice that

$$\mu(\lambda, \gamma, a) < \Lambda_{N,\gamma,a} = \left(\frac{N-a(\gamma+1)}{a} \right)^a < \hat{\mu}(\lambda, \gamma, a),$$

moveover, if $\lambda \rightarrow \Lambda_{N,\gamma,a}$, then $\mu(\lambda, \gamma, a) = \hat{\mu}(\lambda, \gamma, a) = \frac{N-a(\gamma+1)}{a}$.

Consider now the equation

$$\begin{cases} -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} = h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.13)$$

as in Lemma 5.1, we have the next result about the behavior of any positive supersolution near the origin.

Lemma 1.7. $u \in \mathcal{D}_{loc}^{1,p}(\Omega)$ is a non-negative weak super-solution to Problem (2.13) where $\lambda \leq \Lambda_{N,\gamma,a}$ and $f \not\equiv 0$ in Ω , then $u(x) \geq C|x|^{-\mu(\lambda,\gamma,a)}$ in $B_r(0) \subset\subset \Omega$.

Consequently, we can formulate the next nonexistence result

Theorem 1.8. Assume that $0 < \lambda_1, \lambda_2 \leq \Lambda_{N,\gamma,a}$ and $p, q > a-1$ are such that

$$\max \left\{ \mu(\lambda_1, \gamma, a) \frac{pq - (a-1)^2}{p+1}, \mu(\lambda_2, \gamma, a) \frac{pq - (a-1)^2}{q+1} \right\} > a(\gamma+1).$$

Then System (2.10) has non positive weak supersolution.

Proof. Suppose, for the sake of contradiction, that System (2.10) admits a positive solution (u, v) . Then using Lemma 1.7, we deduce that $u(x) \geq C|x|^{-\mu(\lambda_1,\gamma,a)}$ and $v(x) \geq C|x|^{-\mu(\lambda_2,\gamma,a)}$ in $B_r(0) \subset\subset \Omega$.

Fixed $r > 0$ as above and let $\varphi \in \mathcal{C}_0^\infty(B_r(0))$, then using Picone inequality given in Proposition 1.9 for both equation in $B_r(0)$, it holds that

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx \geq \int_{B_r(0)} \frac{-\operatorname{div}(|x|^{-a\gamma} |\nabla u|^{a-2} \nabla u)}{u^{a-1}} |\varphi|^a \geq \int_{B_r(0)} \frac{v^p}{u^{a-1}} |\varphi|^a$$

and

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx \geq \int_{B_r(0)} \frac{-\operatorname{div}(|x|^{-a\gamma} |\nabla v|^{a-2} \nabla v)}{v^{a-1}} |\varphi|^a \geq \int_{B_r(0)} \frac{u^q}{v^{a-1}} |\varphi|^a.$$

Thus

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} \geq \frac{1}{2} \int_{B_r(0)} |\varphi|^a \left[\frac{v^p}{u^{a-1}} + \frac{u^q}{v^{a-1}} \right]. \quad (2.14)$$

Define $K_a(u, v) = \frac{v^p}{u^{a-1}} + \frac{u^q}{v^{a-1}}$. We set $\sigma = \frac{v^{a-1}}{u^{\frac{(a-1)^2}{p}}}$, then we consider the function

$$L(\sigma) = \sigma^p + \frac{B}{\sigma} \text{ where } B = u^{\frac{pq-(a-1)^2}{p}}, \text{ then } L \text{ has a minimum for } \hat{\sigma} = \left(\frac{a-1}{p} B \right)^{\frac{a-1}{p+(a-1)}}$$

and

$$L(\sigma) \geq L(\hat{\sigma}) = \left(\left(\frac{a-1}{p} \right)^{\frac{p}{p+(a-1)}} + \left(\frac{p}{a-1} \right)^{\frac{a-1}{p+(a-1)}} B^{\frac{p}{p+(a-1)}} \right) = C(a, p) u^{\frac{pq-(a-1)^2}{p+(a-1)}}.$$

Hence we deduce that

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} \geq C(p, a) \int_{B_r(0)} |\varphi|^a u^{\frac{pq-(a-1)^2}{p+(a-1)}} dx.$$

In the same way, setting $\sigma = \frac{u^{a-1}}{v^{\frac{(a-1)^2}{q}}}$, then for $\tilde{L}(\sigma) = \sigma^{\frac{q}{a-1}} + \frac{\tilde{B}}{\sigma}$ with $\tilde{B} = v^{\frac{pq-(a-1)^2}{q}}$,

then $K(u, v) \geq C(a, q) v^{\frac{pq-(a-1)^2}{q+1}}$. Thus

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx &\geq C(p, q, a) \int_{B_r(0)} |\varphi|^a \left(u^{\frac{pq-(a-1)^2}{p+1}} + v^{\frac{pq-(a-1)^2}{q+1}} \right) dx \\ &\geq C \int_{B_r(0)} \varphi^2 \left(|x|^{-\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1}} + |x|^{-\mu(\lambda_2, \gamma, a) \frac{pq-(a-1)^2}{q+1}} \right) dx. \end{aligned}$$

If $\max\{\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1}, \mu(\lambda_2, \gamma, a) \frac{pq-(a-1)^2}{q+1}\} > a(\gamma+1)$, then we get a contradiction with the Hardy inequality. $\square$

To treat the case of critical curves, we need the next proposition.

Proposition 1.9. Assume that $0 < \lambda \leq \Lambda_{N,\gamma,a}$ and consider the following problem

$$\begin{cases} -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} = \frac{c}{|x|^{\mu(\lambda,\gamma,a)(a-1)+a(\gamma+1)}} & \text{in } B_{\frac{1}{2}}(0), \\ w_\lambda = C & \text{in } \partial B_{\frac{1}{2}}(0), \end{cases} \quad (2.15)$$

Then setting

$$w(x) = \begin{cases} |x|^{-\mu} \log \frac{1}{r} & \text{if } a \leq 2, \\ |x|^{-\mu} (\log \frac{1}{r})^{\frac{1}{a-1}} & \text{if } a > 2, \end{cases}$$

Then Aw is a subsolution to (1.9) for A small enough.

Proof. For simplicity of tipping we set $w(x) = |x|^{-\mu} (\log \frac{1}{r})^\beta := \omega(r)$ where $\beta > 0$ to be chosen in each case. Then by a direct computations we reach that

$$\omega'(r) = -r^{-\mu-1} \left(\log \frac{1}{r} \right)^{\beta-1} \left(\mu \log \frac{1}{r} + \beta \right).$$

Writing the main operator in polar coordinate, we deduce that

$$\begin{aligned} -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) &= \frac{-1}{r^{N-1}} \left(r^{N-1-a\gamma} |\omega'|^{a-2} \omega' \right)' \\ &= A^{a-1} (N - a(\gamma+1) - \mu(a-1)) r^{-\mu(a-1)-a(\gamma+1)} (\log \frac{1}{r})^{(a-1)(\beta-1)} \left(\mu \log \frac{1}{r} + \beta \right)^{a-1} \\ &\quad - A^{a-1} \beta(a-1) r^{-\mu(a-1)-a(\gamma+1)} (\log \frac{1}{r})^{(a-1)(\beta-1)-1} \left(\mu \log \frac{1}{r} + \beta - 1 \right) \left(\mu \log \frac{1}{r} + \beta \right)^{a-2}. \end{aligned}$$

Suppose that $a \leq 2$, then $a-1 \leq 1$. Hence it holds that

$$\left(\mu \log \frac{1}{r} + \beta \right)^{a-1} \leq \mu^{a-1} \left(\log \frac{1}{r} \right)^{a-1} + \beta^{a-1}.$$

Choosing $\beta = 1$, it holds that

$$\begin{aligned} -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) &\leq A^{a-1} (N - a(\gamma+1) - \mu(a-1)) \mu^{a-1} r^{-\mu(a-1)-a(\gamma+1)} (\log \frac{1}{r})^{(a-1)} \\ &\quad + A^{a-1} r^{-\mu(a-1)-a(\gamma+1)} - A^{a-1} r^{-\mu(a-1)-a(\gamma+1)} \mu \left(\mu \log \frac{1}{r} + 1 \right)^{a-2}. \end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
& -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \\
& \leq A^{a-1}(N - a(\gamma+1) - \mu(a-1))\mu^{a-1}r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{(a-1)} \\
& - \lambda r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{(a-1)} + A^{a-1}r^{-\mu(a-1)-a(\gamma+1)}
\end{aligned}$$

Taking into consideration the value of μ given in (2.12), it holds that

$$(N - a(\gamma+1) - \mu(a-1))\mu^{a-1} - \lambda = 0,$$

then

$$-\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \leq A^{a-1}|x|^{-\mu(a-1)-a(\gamma+1)},$$

and the result is thereby established in this case. Assume that $a > 2$, then using the fact that

$$\left(\mu \log \frac{1}{r} + \beta\right)^{a-1} - \left(\mu \log \frac{1}{r}\right)^{a-1} \leq (a-1)\beta \left(\mu \log \frac{1}{r} + \beta\right)^{a-2},$$

it holds that

$$\begin{aligned}
& -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \\
& \leq A^{a-1}(N - a(\gamma+1) - \mu(a-1))\mu^{a-1}r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{\beta(a-1)} \\
& - \lambda r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{\beta(a-1)} \\
& + A^{a-1}(a-1)\beta(N - a(\gamma+1)) \\
& - \mu(a-1)r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{(a-1)(\beta-1)}\left(\mu \log(\frac{1}{r}) + \beta\right)^{a-2} \\
& - A^{a-1}\beta(a-1)r^{-\mu(a-1)-a(\gamma+1)}(\log \frac{1}{r})^{(a-1)(\beta-1)-1}\left(\mu \log(\frac{1}{r}) + \beta - 1\right)^{a-2}
\end{aligned}$$

Hence, following the same reasoning as in the first case and applying the definition of μ , we derive that

$$\begin{aligned}
& -\operatorname{div}(|x|^{-a\gamma}|\nabla w|^{a-2}\nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \\
& \leq A^{a-1}(a-1)\beta(N - a(\gamma+1)) \\
& - \mu(a-1)r^{-\mu(a-1)-a(\gamma+1)}(\log(\frac{1}{r}))^{(a-1)(\beta-1)}\left(\mu \log \frac{1}{r} + \beta\right)^{a-2}.
\end{aligned}$$

Since $|x| < \frac{1}{2}$, then $\mu \log(\frac{1}{r}) + \beta \leq C(\mu, \beta) \log \frac{1}{r}$. Hence

$$-\operatorname{div}(|x|^{-a\gamma} |\nabla w|^{a-2} \nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \leq CA^{a-1} |x|^{-\mu(a-1)-a(\gamma+1)} (\log \frac{1}{r})^{\beta(a-1)-1}.$$

Choosing $\beta = \frac{1}{a-1}$, we obtain that

$$-\operatorname{div}(|x|^{-a\gamma} |\nabla w|^{a-2} \nabla w) - \lambda \frac{w^{a-1}}{|x|^{a(\gamma+1)}} \leq CA^{a-1} |x|^{-\mu(a-1)-a(\gamma+1)}.$$

Hence we conclude. Notice that $w \in \mathcal{D}_\gamma^{1,a}(B_{\frac{1}{2}}(0))$ if $\lambda < \Lambda_{N,\gamma,a}$ and $w \in \mathcal{D}_\gamma^{1,a-\varepsilon}(B_{\frac{1}{2}}(0))$ for all $\varepsilon > 0$ if $\lambda = \Lambda_{N,\gamma,a}$. $\square$

We are now able to show the non existence result on the critical curves.

Theorem 1.10. *Assume that p, q are such that*

$$\max \left\{ \mu(\lambda_1, \gamma, a) \frac{pq - (a-1)^2}{p+1}, \mu(\lambda_2, \gamma, a) \frac{pq - (a-1)^2}{q+1} \right\} = a(\gamma+1).$$

It follows that System (2.10) admits a nonpositive solution.

Proof. We begin by proving the first estimate under the first conditions. The other case follows In a similar manner, the proof proceeds through multiple cases.

The first case : $\lambda_1, \lambda_2 < \Lambda_{N,\gamma,a}$ and $\mu(\lambda_1, \gamma, a) \frac{pq - (a-1)^2}{p+1} = a(\gamma+1)$.

Let (u, v) be a non-negative, nontrivial weak supersolution of System (2.10). From the second equation, we obtain

$$-\operatorname{div}(|x|^{-a\gamma} |\nabla v|^{a-2} \nabla v) - \lambda_2 \frac{v^{a-1}}{|x|^{a(\gamma+1)}} \geq u^q \geq |x|^{-q\mu(\lambda_1, \gamma, a)}.$$

Recall that $\mu(\lambda_1, \gamma, a)(pq - (a-1)^2) = a(\gamma+1)(p+1)$, then $q\mu(\lambda_1, \gamma, a) = a(\gamma+1) + \frac{a(\gamma+1) + -\mu(\lambda_1, \gamma, a)(a-1)^2}{p} > a(\gamma+1)$. Hence by Lemma 5.1, therefore $v(x) \geq \frac{c}{|x|^{q\mu(\lambda_1, \gamma, a) - a(\gamma+1)}}$. Returning to the first equation of System (2.10), it follows that

$$-\operatorname{div}(|x|^{-a\gamma} |\nabla u|^{a-2} \nabla u) - \lambda_1 \frac{u^{a-1}}{|x|^{a(\gamma+1)}} \geq v^p \geq \frac{1}{|x|^{p(q\mu(\lambda_1, \gamma, a) - a(\gamma+1))}}.$$

Notice that, by hypothesis $p\left(q\mu(\lambda_1, \gamma, a) - a(\gamma + 1)\right) = \mu(\lambda_1, \gamma, a)(a - 1)^2 + a(\gamma + 1)$. Hence

$$-\operatorname{div}(|x|^{-a\gamma}|\nabla u|^{a-2}\nabla u) - \lambda_1 \frac{u^{a-1}}{|x|^{a(\gamma+1)}} \geq \frac{1}{|x|^{\mu(\lambda_1, \gamma, a)(a-1)^2 + a(\gamma+1)}}.$$

Thus, using Proposition 1.9, we distinguish two main cases. The first one if $a \leq 2$ in which case we have $u \geq w(\lambda_1) = |x|^{-\mu} \log(\frac{\eta}{|x|})$ in $B_\eta(0)$. The second case if $a > 2$, where we obtain $u \geq w(\lambda_1) = |x|^{-\mu} (\log \frac{1}{r})^{\frac{1}{a-1}}$ in $B_\eta(0)$. By the same reasoning as in the proof of Theorem 1.8, for all $\varphi \in \mathcal{C}_0^\infty(B_r(0))$ with $r \ll \eta$. We start by considering the case $a \leq 2$ we arrive to

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx &\geq C(p, q, a) \int_{B_r(0)} |\varphi|^a u^{\frac{pq-(a-1)^2}{p+1}} dx \\ &\geq C \int_{B_r(0)} \varphi^2 |x|^{-\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{p+1}} dx. \end{aligned} \quad (2.16)$$

Since $\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1} = a(\gamma + 1)$, then we get

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} \geq C \int_{B_r(0)} \frac{\varphi^2}{|x|^{a(\gamma+1)}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{p+1}} dx.$$

If $a > 2$, we obtain for all $\varphi \in \mathcal{C}_0^\infty(B_r(0))$ with $r \ll \eta$,

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx &\geq C(p, q, a) \int_{B_r(0)} |\varphi|^a u^{\frac{pq-(a-1)^2}{p+1}} dx \\ &\geq C \int_{B_r(0)} \varphi^2 |x|^{-\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{(a-1)(p+1)}} dx. \end{aligned} \quad (2.17)$$

Since $\mu(\lambda_1, \gamma, a) \frac{pq-(a-1)^2}{p+1} = a(\gamma + 1)$, then we get

$$\int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} \geq C \int_{B_r(0)} \frac{\varphi^2}{|x|^{a(\gamma+1)}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{(a-1)(p+1)}} dx.$$

Hence, the Hardy inequality yields a contradiction in both cases.

The second case : $\lambda_1, \lambda_2 < \Lambda_{N, \gamma}$ and $\mu(\lambda_2, \gamma, a) \frac{pq-(a-1)^2}{q+1} = a(\gamma + 1)$.

Using the first equation in System (2.10), hence, we obtain

$$-\operatorname{div}(|x|^{-a\gamma}|\nabla u|^{a-2}\nabla u) - \lambda_1 \frac{u^{a-1}}{|x|^{a(\gamma+1)}} \geq v^p \geq |x|^{-p\mu(\lambda_2, \gamma, a)}.$$

Similarly to the first case, by using the fact that $\mu(\lambda_2, \gamma, a)(pq - (a-1)^2) = a(\gamma+1)(q+1)$, then $p\mu(\lambda_2, \gamma, a) > a(\gamma+1)$, thus using Lemma 5.1, we deduce that $u(x) \geq \frac{C}{|x|^{p\mu(\lambda_2, \gamma, a) - a(\gamma+1)}}$. Hence we get

$$\begin{aligned} -\operatorname{div}(|x|^{-a\gamma}|\nabla v|^{a-2}\nabla v) - \lambda_2 \frac{v^{a-1}}{|x|^{a(\gamma+1)}} &\geq u^q \geq \frac{1}{|x|^{q(p\mu(\lambda_2, \gamma, a) - a(\gamma+1))}} \\ &\geq \frac{C}{|x|^{\mu(\lambda_1, \gamma, a)(a-1)^2 + a(\gamma+1)}}. \end{aligned} \quad (2.18)$$

Thus, using Proposition 1.9, it holds that $v \geq w(\lambda_1) = \frac{C}{|x|^\mu} \log\left(\frac{\eta}{|x|}\right)$ in $B_\eta(0)$ if $a \leq 2$. It follows, by the argument used in the proof of Theorem 1.8, for $\varphi \in \mathcal{C}_0^\infty(B_r(0))$ with $r \ll \eta$, we reach that

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx &\geq C(p, q, a) \int_{B_r(0)} |\varphi|^a v^{\frac{pq-(a-1)^2}{q+1}} dx \\ &\geq C \int_{B_r(0)} \varphi^2 |x|^{-\mu(\lambda_2, \gamma, a) \frac{pq-(a-1)^2}{q+1}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{q+1}} dx. \end{aligned}$$

If $a > 2$, we have $v \geq w(\lambda_1) = |x|^{-\mu} \left(\log \frac{1}{r} \right)^{\frac{1}{a-1}}$ then we obtain

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^a |x|^{-a\gamma} dx &\geq C(p, q, a) \int_{B_r(0)} |\varphi|^a v^{\frac{pq-(a-1)^2}{q+1}} dx \\ &\geq C \int_{B_r(0)} \varphi^2 |x|^{-\mu(\lambda_2, \gamma, a) \frac{pq-(a-1)^2}{(q+1)}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{(a-1)(q+1)}} dx \\ &\geq C \int_{B_r(0)} \frac{\varphi^2}{|x|^{a(\gamma+1)}} \left(\log\left(\frac{\eta}{|x|}\right) \right)^{\frac{pq-(a-1)^2}{(a-1)(p+1)}} dx, \end{aligned}$$

a contradiction with the Hardy inequality.

The third case : $\lambda_1 = \Lambda_{N, \gamma}$. In this case $\mu(\lambda_1, \gamma) = \frac{N-a(\gamma+1)}{a}$. As above we have $v(x) \geq \frac{C}{|x|^{q \frac{N-a(\gamma+1)}{a} - a(\gamma+1)}}$ in $B_r(0)$. Hence, using the fact that $\int_{B_r(0)} v^p |x|^{-\mu(\Lambda_{N, \gamma}, \gamma, a)} dx < \infty$, it follows that

$$\int_{B_r(0)} \frac{C}{|x|^{p(q \frac{N-a(\gamma+1)}{a} - a(\gamma+1))}} |x|^{-\mu(\Lambda_{N, \gamma}, \gamma, a)} dx < \infty. \quad (2.19)$$

Thus

$$p \left(q \frac{N - a(\gamma + 1)}{a} - a(\gamma + 1) \right) + \mu(\Lambda_{N,\gamma}, \gamma, a) < N.$$

Since $\mu(\Lambda_{N,\gamma}, \gamma, a)(pq - 1) = a(\gamma + 1)(p + 1)$, then $p \left(q \frac{N - a(\gamma + 1)}{a} - a(\gamma + 1) \right) + \mu(\Lambda_{N,\gamma}, \gamma, a) = N$, a contradiction with (2.19). Hence we conclude. $\square$

2 Existence Results

In this section, we address the existence of a positive weak supersolution to System 2.1. We demonstrate that the condition stated in Theorem 2.2 is optimal, in the sense that outside the nonexistence region, a nontrivial supersolution does exist.

Taking into consideration the nonexistence result established in the previous section, we will assume that $\gamma + 1 > 0$. Then we will distinct five regions, defined by

$$A := \left\{ (p, q) \in \left(1, \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)} \right) \times \left(1, \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)} \right) \right\}, \quad (2.20)$$

$$B := \left\{ (p, q) \in \left(1, \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)q - 2(\gamma + 1)} \right) \times \left(\frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)}, \frac{N - \mu(\lambda_2, \gamma)}{\mu(\lambda_1, \gamma)} \right) \right\}, \quad (2.21)$$

$$C := \left\{ (p, q) \in \left(\frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)}, \frac{N - \mu(\lambda_1, \gamma)}{\mu(\lambda_2, \gamma)} \right) \times \left(1, \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)p - 2(\gamma + 1)} \right) \right\}, \quad (2.22)$$

$$D := \left\{ (p, q) : p = \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)}, q \in \left(1, \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)} \right) \right\}, \quad (2.23)$$

$$E := \left\{ (p, q) : p \in \left(1, \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)} \right), q = \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)} \right\}. \quad (2.24)$$

Theorem 2.1. Assume that $\gamma + 1 > 0$. Let $\Omega = B_1(0)$, $0 < \lambda_1, \lambda_2 < \Lambda_{N,\gamma}$ and $p, q > 1$

verify one of the following assumptions:

- i) $1 < p < \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)}$ and $1 < q < \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)}$, or
- ii) $\frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)} \leq q < \frac{N - \mu(\lambda_2, \gamma)}{\mu(\lambda_1, \gamma)}$ and $p < \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)q - 2(\gamma + 1)}$, or
- iii) $\frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)} \leq p < \frac{N - \mu(\lambda_1, \gamma)}{\mu(\lambda_2, \gamma)}$ and $q < \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)p - 2(\gamma + 1)}$,
- iv) $p = \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)}$ and $q < \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)}$,
- v) $q = \frac{\mu(\lambda_2, \gamma) + 2(\gamma + 1)}{\mu(\lambda_1, \gamma)}$ and $p < \frac{\mu(\lambda_1, \gamma) + 2(\gamma + 1)}{\mu(\lambda_2, \gamma)}$,

Then, the system (2.9) has a non-negative weak super-solution.

Proof. The proof is organized into several steps depending on the values of p, q . To simplify the notation, we write $\mu(\lambda_1, \gamma) = \mu(\lambda_1)$ and $\mu(\lambda_2, \gamma) = \mu(\lambda_2)$.

- **Zone A:** $1 < p < \frac{\mu(\lambda_1) + 2(\gamma + 1)}{\mu(\lambda_2)}$ and $1 < q < \frac{\mu(\lambda_2) + 2(\gamma + 1)}{\mu(\lambda_1)}$.

Fix $\beta > 0$ satisfying $\max\{\mu(\lambda_1), \mu(\lambda_2)\} < \beta < \min\{\frac{2(\gamma + 1)}{p - 1}, \frac{2(\gamma + 1)}{q - 1}\}$. Define $(u(x), v(x)) = (c_1|x|^{-\beta}, c_2|x|^{-\beta})$ with c_1, c_2 being positive constants satisfying

$$c_1 \mathcal{P}(\beta) \geq c_2^p \text{ and } c_2 \mathcal{P}(\beta) \geq c_1^q \geq \beta^q, \quad (2.25)$$

with $\mathcal{P}(\beta) = -\beta^2 + (N - 2\gamma - 2)\beta - \lambda$. Then

$$-\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma + 1)}} = c_1 \mathcal{P}(\beta) |x|^{-\beta - 2(\gamma + 1)}.$$

Using the fact that $1 < p < \frac{\mu(\lambda_2) + 2(\gamma + 1)}{\mu(\lambda_1)}$, we obtain a $\beta > \mu(\lambda_2)$ such that $\beta < \frac{2(\gamma + 1)}{p - 1}$. Thus, using (2.25),

$$-\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma + 1)}} \geq v^p.$$

Respect to v , we have

$$-\operatorname{div}(|x|^{-2\gamma} \nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma + 1)}} = c_2 \mathcal{P}(\beta) |x|^{-\beta - 2(\gamma + 1)}.$$

Now, since $1 < q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)}$, choosing $\beta < \frac{2(\gamma+1)}{q-1}$, we obtain that

$$-\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda \frac{v}{|x|^{2(\gamma+1)}} \geq u^q.$$

Thus, hence, the result holds in this case.

• **Zone B :** $\frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)} < q < \frac{N-\mu(\lambda_2)}{\mu(\lambda_1)}$ and $p < \frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_1)q-2(\gamma+1)}$.

Define $\bar{\tau} := \mu(\lambda_1)q - 2(\gamma+1)$ and $\tau_p := (\mu(\lambda_1)q - 2(\gamma+1))p - 2(\gamma+1)$. Since $\frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)} < q < \frac{N-\mu(\lambda_2)}{\mu(\lambda_1)}$, then $\mu(\lambda_2) < \bar{\tau} < \hat{\mu}(\lambda_2)$. Taking into consideration that $p < \frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_1)q-2(\gamma+1)}$, we deduce that $\tau_p < \mu(\lambda_1)$. Now, let

$$u(x) = c_1|x|^{-\mu(\lambda_1)} - c_2|x|^{-\tau_p} \text{ and } v(x) = c_3|x|^{-\bar{\tau}},$$

where c_1, c_2, c_3 are positive constants such that $c_1^q \leq c_3 \mathcal{P}(\bar{\tau})$ and $c_3^p \leq -c_2 \mathcal{P}(\tau_p)$. We establish that (u, v) is a weak supersolution to (2.9). Indeed, using the definition of τ_p and $\mathcal{P}(\cdot)$, it holds that

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} &= -c_2 \mathcal{P}(\tau_p) |x|^{-\tau_p-2(\gamma+1)} = -c_2 \mathcal{P}(\tau_p) |x|^{-p\bar{\tau}} \\ &\geq c_3^p |x|^{-p\bar{\tau}} = v^p. \end{aligned}$$

However,, using the definition of $\bar{\tau}$, we get

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} &= c_3 \mathcal{P}(\bar{\tau}) |x|^{-\bar{\tau}-2(\gamma+1)} = c_3 \mathcal{P}(\bar{\tau}) |x|^{-q\bar{\tau}} \\ &\geq \left| c_1|x|^{-\mu(\lambda_1)} - c_2|x|^{-\tau_p} \right|^q = u^q. \end{aligned}$$

Hence we conclude.

• **Zone C :** $\frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_2)} < p < \frac{N-\mu(\lambda_1)}{\mu(\lambda_2)}$ and $q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_2)p-2(\gamma+1)}$.

We set $\bar{\tau} := p\mu(\lambda_2) - 2(\gamma+1)$ and $\tau_q := q(p\mu(\lambda_2) - 2(\gamma+1)) - 2(\gamma+1) = \bar{\tau}q - 2(\gamma+1)$.

Since $\frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_2)} < p < \frac{N-\mu(\lambda_1)}{\mu(\lambda_2)}$, then $\mu(\lambda_1) < \bar{\tau} < \hat{\mu}(\lambda_1)$. Taking into consideration that $q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_2)p-2(\gamma+1)}$, we have that $\tau_q < \mu(\lambda_2)$. Define

$$u(x) = c_3|x|^{-\bar{\tau}} \text{ and } v(x) = c_1|x|^{-\mu(\lambda_2)} - c_2|x|^{-\tau_q},$$

where c_1, c_2, c_3 are positive constants such that $c_1^p \leq c_3 \mathcal{P}(\bar{\tau})$, $(c_3 \bar{\tau})^q \leq -c_2 \mathcal{P}(\tau_q)$ and $c_1 \geq c_2$.

We show that (u, v) is a weak super-solution to (2.9). Indeed, using the definition of $\bar{\tau}$ and $\mathcal{P}(\cdot)$, it holds that

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} &= c_3 \mathcal{P}(\bar{\tau}) |x|^{-\bar{\tau}-2(\gamma+1)} = c_3 \mathcal{P}(\bar{\tau}) |x|^{-\mu(\lambda_2)p} \\ &\geq c_1^p |x|^{-\mu(\lambda_2)p} \\ &\geq \left| c_1 |x|^{-\mu(\lambda_2)} - c_2 |x|^{-\tau_q} \right|^p = v^p. \end{aligned}$$

However,, using the definition of τ_q , we get

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma} \nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} &= -c_2 \mathcal{P}(\tau_q) |x|^{-\tau_q-2(\gamma+1)} = -c_2 \mathcal{P}(\tau_q) |x|^{-q\bar{\tau}} \\ &\geq c_3^q |x|^{-q\bar{\tau}} = u^q. \end{aligned}$$

Hence we conclude.

• **Zone D :** $p = \frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_2)}$ and $q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)}$.

In this case, we set $\tau' = q\mu(\lambda_1) - 2(\gamma+1) + \varepsilon < \mu(\lambda_2)$, for ε small enough, and define

$$u(x) = |x|^{-\mu(\lambda_1)} \left(-\ln |x| \right) \text{ and } v(x) = c_1 |x|^{-\mu(\lambda_2)} - |x|^{-\tau'},$$

where $c_1^p \leq N - 2(\gamma+1) - 2\mu(\lambda_1)$. Then

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma} \nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} &= (N - 2(\gamma+1) - 2\mu(\lambda_1)) |x|^{-\mu(\lambda_1)-2(\gamma+1)} \\ &= (N - 2(\gamma+1) - 2\mu(\lambda_1)) |x|^{-\mu(\lambda_2)p} \\ &\geq \left(c_1 |x|^{-\mu(\lambda_2)} - c_2 |x|^{-\tau'} \right)^p = v^p, \end{aligned}$$

and

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma} \nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} &= -\mathcal{P}(\tau') |x|^{-\tau'-2(\gamma+1)} = -\mathcal{P}(\tau') |x|^{-\mu(\lambda_1)q-\varepsilon} \\ &\geq \left(|x|^{-\mu(\lambda_1)} \ln |x| \right)^q = u^q. \end{aligned}$$

Thus (u, v) is a weak super-solution.

• **Zone E :** $q = \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)}$ and $p < \frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_2)}$.

Let $u(x) = c_1|x|^{-\mu(\lambda_1)} - c_2|x|^{-\tau''}$ and $v(x) = |x|^{-\mu(\lambda_2)}(-\ln|x|)$ where $\tau'' = p\mu(\lambda_2) - 2(\gamma+1) + \varepsilon < \mu(\lambda_1)$. Here, we assume that $c_1^q \leq N - 2 - 2\mu(\lambda_2)$ and $c_2 > 0$. Then

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma}\nabla u) - \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} &= -c_2 \mathcal{P}(\tau'')|x|^{-\tau''-2(\gamma+1)} = -c_2 \mathcal{P}(\tau'')|x|^{-p\mu(\lambda_2)-\varepsilon} \\ &\geq \left(|x|^{-\mu(\lambda_2)}(-\ln|x|) \right)^p \\ &= v^p. \end{aligned}$$

On other hand,

$$\begin{aligned} -\operatorname{div}(|x|^{-2\gamma}\nabla v) - \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} &= (N - 2(\gamma+1) - 2\mu(\lambda_2))|x|^{-\mu(\lambda_2)-2(\gamma+1)} \\ &= (N - 2(\gamma+1) - 2\mu(\lambda_2))|x|^{-\mu(\lambda_1)q} \\ &\geq \left| c_1|x|^{-\mu(\lambda_1)} - c_2|x|^{-\tau''} \right|^q \\ &= u^q. \end{aligned}$$

Thus (u, v) is a weak super-solution of (2.9). $\square$

In the critical case, we have one of the tree possibilities $\lambda_1 = \lambda_2 = \Lambda_{N,\gamma}$, $\lambda_2 < \lambda_1 = \Lambda_{N,\gamma}$ or $\lambda_1 < \lambda_2 = \Lambda_{N,\gamma}$. In the above cases we will build a suitable supersolution in a small ball. Notice that in this critical situations, the sets B and C are empty.

In the case where $\lambda = \lambda_1 = \lambda_2 = \Lambda_{N,\gamma}$, the system is reduced to

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) &= \Lambda_{N,\gamma} \frac{u}{|x|^{2(\gamma+1)}} + v^p & \text{in } B_\eta(0), \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) &= \Lambda_{N,\gamma} \frac{v}{|x|^{2(\gamma+1)}} + u^q & \text{in } B_\eta(0), \\ u, v > 0 & \text{in } B_\eta(0), \end{cases} \quad (2.26)$$

where $\eta < 1$.

Theorem 2.2. Let $p < \frac{\mu(\lambda)+2(\gamma+1)}{\mu(\lambda)}$ and $q < \frac{\mu(\lambda)+2(\gamma+1)}{\mu(\lambda)}$. Define $w(x) = |x|^{-\mu(\lambda)}(\ln(\frac{1}{|x|}))^\beta$ where $0 < \beta < 1$. Then there exist $c > 0$ such that, setting $(u(x), v(x)) = (cw(x), cw(x))$, then (u, v) is a weak super-solution of (2.26).

Proof. Since $\lambda = \Lambda_{N,\gamma}$, then $\mu(\lambda) = \frac{N-2(\gamma+1)}{2}$. Fix $p < \frac{\mu(\lambda)+2(\gamma+1)}{\mu(\lambda)}$ and $q < \frac{\mu(\lambda)+2(\gamma+1)}{\mu(\lambda)}$, that is the zone A. Define $w(x) = |x|^{-\mu(\lambda)}(\ln(\frac{1}{|x|}))^\beta$ where β denotes a small positive constant to be selected later. Since $\lambda = \Lambda_{N,\gamma}$, then $w \in$

$\mathcal{D}_Y^{1,\sigma}(B_\eta(0))$ for all $\sigma < 2$. By a direct computation we obtain that, for $|x| < \eta < 1$,

$$-\operatorname{div}(|x|^{-2\gamma}\nabla w) - \lambda \frac{w}{|x|^{2(\gamma+1)}} = \frac{\beta(1-\beta)}{|x|^{2(\gamma+1)+\mu(\lambda)}} \left(\ln\left(\frac{1}{|x|}\right) \right)^{\beta-2(\gamma+1)}.$$

Notice that $w^p(x) = |x|^{-p\mu(\lambda)}(\ln(\frac{1}{|x|}))^{p\beta}$, then $w^q(x) = |x|^{-q\mu(\lambda)}(\ln(\frac{1}{|x|}))^{q\beta}$. Setting $(u(x), v(x)) = (cw(x), cw(x))$, with c small enough, we deduce that (u, v) is a weak super-solution of (2.26). Hence we conclude. $\square$

Now, if $\lambda_2 < \lambda_1 = \Lambda_{N,\gamma}$, we have the system

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) &= \Lambda_{N,\gamma} \frac{u}{|x|^{2(\gamma+1)}} + v^p & \text{in } B_\eta(0), \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) &= \lambda_2 \frac{v}{|x|^{2(\gamma+1)}} + u^q & \text{in } B_\eta(0), \\ u, v &> 0 & \text{in } B_\eta(0). \end{cases} \quad (2.27)$$

Theorem 2.3. Assume that $p < \frac{\mu(\lambda_1)+2(\gamma+1)}{\mu(\lambda_2)}$ and $q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda_1)}$. Define $w(x) = |x|^{-\mu(\lambda_1)}(\ln(\frac{1}{|x|}))^\beta$ where $0 < \beta < 1$. Then, there exist $c > 0$ such that, setting $(u(x), v(x)) = (cw(x), c|x|^{-\beta})$, then (u, v) is a weak super-solution of (2.27).

Proof. Since $\lambda = \lambda_1 = \Lambda_{N,\gamma}$, then $\mu(\lambda) = \frac{N-2(\gamma+1)}{2}$. Fix $p < \frac{\mu(\lambda)+2(\gamma+1)}{\mu(\lambda_2)}$ and $q < \frac{\mu(\lambda_2)+2(\gamma+1)}{\mu(\lambda)}$, that is the zone A. Define $w(x) = |x|^{-\mu(\lambda)}(\ln(\frac{1}{|x|}))^\beta$ where β denotes a small positive constant to be selected later. Since $\lambda = \Lambda_{N,\gamma}$, then $w \in \mathcal{D}_Y^{1,\sigma}(B_\eta(0))$ for all $\sigma < 2$. By a direct computation we obtain that, for $|x| < \eta < 1$,

$$-\operatorname{div}(|x|^{-2\gamma}\nabla w) - \lambda \frac{w}{|x|^{2(\gamma+1)}} = \frac{\beta(1-\beta)}{|x|^{2(\gamma+1)+\mu(\lambda)}} \left(\ln\left(\frac{1}{|x|}\right) \right)^{\beta-2(\gamma+1)},$$

Since $c_2^p < \beta(1-\beta)$ and $\beta < \frac{2(\gamma+1)+\mu(\lambda)}{p}$, using the fact $w^q(x) = |x|^{-q\mu(\lambda)}(\ln(\frac{1}{|x|}))^{q\beta}$, then setting $(u(x), v(x)) = (cw(x), cw(x))$, with c small enough, it holds that (u, v) is a weak super-solution of (2.27). Hence we conclude. $\square$

Finally, we have the case $\lambda_2 < \lambda_1 = \Lambda_{N,\gamma}$, as above, we have the next system

$$\begin{cases} -\operatorname{div}(|x|^{-2\gamma}\nabla u) &= \lambda_1 \frac{u}{|x|^{2(\gamma+1)}} + v^p & \text{in } B_\eta(0), \\ -\operatorname{div}(|x|^{-2\gamma}\nabla v) &= \Lambda_{N,\gamma} \frac{v}{|x|^{2(\gamma+1)}} + u^q & \text{in } B_\eta(0), \\ u, v &> 0 & \text{in } B_\eta(0). \end{cases} \quad (2.28)$$

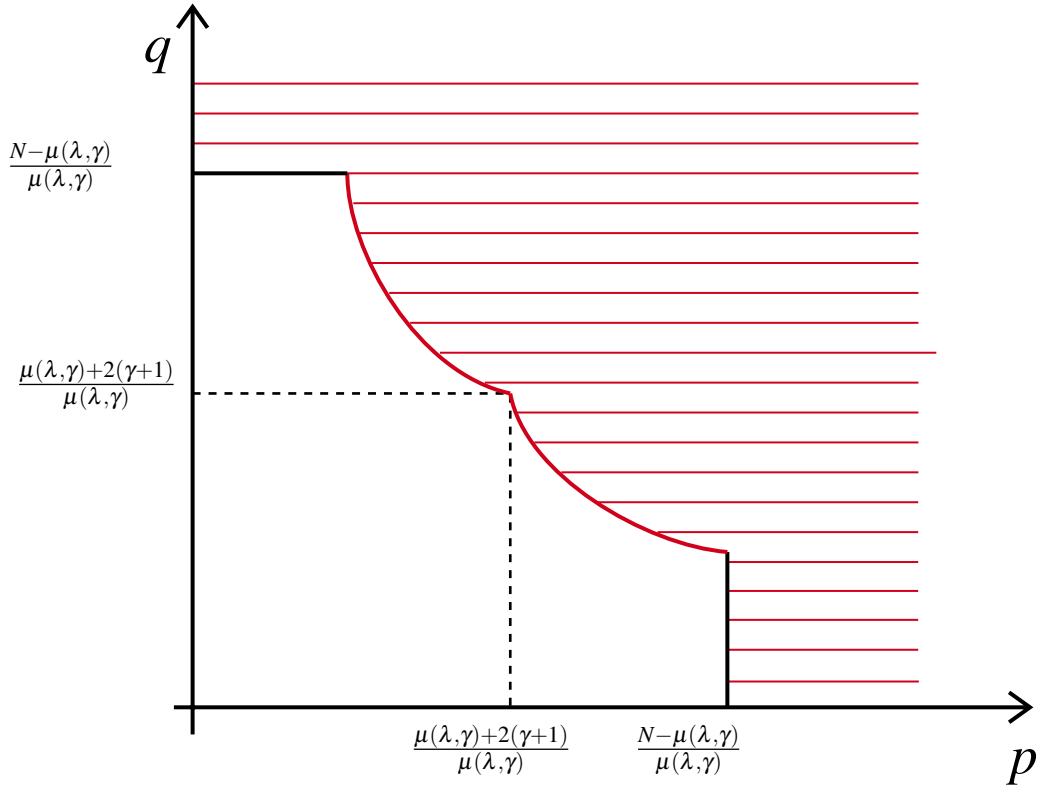


Figure 2.1: Existence and non-existence regions: $\lambda_1 = \lambda_2 = \lambda$.

Then as above, we have the next existence result.

Theorem 2.4. Assume that $p < \frac{\mu(\lambda_1) + 2(\gamma + 1)}{\mu(\lambda)}$ and $q < \frac{\mu(\lambda) + 2(\gamma + 1)}{\mu(\lambda_1)}$. Define $w(x) = |x|^{-\mu(\lambda)} (\ln(\frac{1}{|x|}))^\beta$ where $0 < \beta < 1$. Thus, there is $c > 0$ such that, setting $(u(x), v(x)) = (c|x|^{-\beta}, cw(x))$, we obtain that (u, v) is a weak super-solution of (2.28).

The proof proceeds in the same manner as in the preceding case. Therefore, the existence result is established.

Chapter 3

Cooperative System

This chapter aims to analyze a class of cooperative systems, where the equations are positively coupled. Our aim is to determine the conditions under which such systems admit positive supersolutions.

We begin with the linear case, where the coupling occurs through Hardy-type singular potentials. A sharp nonexistence result is established based on the interaction between the singular coefficients.

We move to the nonlinear case, involving power-type coupling terms. The focus is on how the relation between the exponents affects solvability, leading to the identification of a critical curve in the parameter plane.

This chapter introduces key techniques such as test functions, comparison principles, and the construction of explicit solutions adapted to the cooperative structure of the system

$$\left\{ \begin{array}{ll} -\Delta u &= \lambda_1 \frac{v}{|x|^2} + f(v, \nabla v) \quad \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{u}{|x|^2} + g(u, \nabla u) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega \end{array} \right. \quad (3.1)$$

where $0 < \lambda_1, \lambda_2$ and $f, g \in L^1(\Omega)$.

1 Linear case: estimate of the singularity , existence result and regularity

Our goal in this subsection is to understand how the strength of the singularity, governed by the parameters λ_1 and λ_2 , affects the existence of positive supersolutions. Using a Hardy-type inequality and an appropriate choice of test functions, we establish a sharp condition under which no nontrivial positive supersolution exists.

Theorem 1.1. *Suppose that $\sqrt{\lambda_1 \lambda_2} > \Lambda_N$, then the system (3.1) has no positive supersolution .*

Proof. Suppose that (u, v) is a supersolution to system (3.1). Let $u, v \geq 0$ and $f, g \geq 0$, then by applying the strong maximum principle, we conclude that $u, v > 0$ in Ω . Now, using the Picone inequality in Proposition 1.10, for both equation in $B_r(0) \subset \Omega$, we get $\forall \phi \in \mathcal{C}_0^\infty(B_r(0))$:

$$\begin{aligned} \int_{\Omega} |\nabla \phi|^2 dx &\geq \int_{\Omega} \frac{-\Delta u}{u} \phi^2 dx \geq \lambda_1 \int_{\Omega} \frac{v}{u} \frac{\phi^2}{|x|^2} dx, \\ \int_{\Omega} |\nabla \phi|^2 dx &\geq \int_{\Omega} \frac{-\Delta v}{v} \phi^2 dx \geq \lambda_2 \int_{\Omega} \frac{u}{v} \frac{\phi^2}{|x|^2} dx. \end{aligned}$$

By adding both inequalities we get

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \frac{1}{2} \int_{\Omega} \frac{\phi^2}{|x|^2} \left[\lambda_1 \frac{v}{u} + \lambda_2 \frac{u}{v} \right] dx. \quad (3.2)$$

Let us consider the function $K(a) = \lambda_1 a + \frac{\lambda_2}{a}$ with $a > 0$, then K attains its minimum at the point $a_0 = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{2}}$ and $K(a_0) = 2\lambda_1^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}$. Going back to (3.2), it holds that, $\forall \phi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \sqrt{\lambda_1 \lambda_2} \int_{\Omega} \frac{\phi^2}{|x|^2} dx$$

thereby contradicting the optimality of Λ_N . Thus, the claim is established. $\square$

In what follows we assume that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$.

Before stating the key existence theorem of this section, we proceed to begin by the next proposition which provides a lower bound on the singular behaviour of $u + v$ near the origin

Lemma 1.2. Assume that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$ and let f, g be nonnegative $L^1(\Omega)$ function and (u, v) is a weak positive supersolution solution to System (3.1). Then

$$u + v \geq C|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} \text{ in } B_r(0). \quad (3.3)$$

In addition we have

$$\int_{\Omega} (f + g)|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty.$$

Proof. Suppose that (u, v) is a weak positive supersolution to System (3.1). Consider $c_1, c_2 > 0$ such that $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_2}{\lambda_1}}$. Define $w = c_1 u + c_2 v$, then

$$-\Delta w \geq \frac{\lambda_1 c_1 v + \lambda_2 c_2 u}{|x|^2} + c_1 f + c_2 g \geq \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + c_1 f + c_2 g.$$

Hence using Remark 4.1, we deduce that $w \geq c|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})}$ in $B_r(0)$. Then

$$u + v \geq \bar{c}|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})}.$$

Notice that, as a conclusion we get also that

$$\int_{\Omega} (f + g)|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty.$$

□

We are now prepared to present the principal result of this section.

Theorem 1.3. Let $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$ and f, g nonnegative function such that $f, g \in L^1(\Omega)$. The system (3.1) has a weak solution (u, v) if and only if $\int_{\Omega} (f + g)|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} < \infty$. Moreover $u(x) + v(x) \geq c|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})}$ in some ball $B_r(0) \subset \subset \Omega$. In addition $u, v \in W_0^{1, \sigma}(\Omega) \forall \sigma < \frac{N}{N-1}$.

Proof. Notice that the necessary condition follows from Lemma 1.2. We now prove the sufficient condition. Suppose that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$ and let f, g be non-negative function such that $\int_{\Omega} (f + g)|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} < \infty$. We set $f_n = T_n(f)$ and $g_n = T_n(g)$. Define (u_n, v_n) as the unique positive solution to the system

$$\begin{cases} -\Delta u_n = \frac{\lambda_1}{|x|^2 + \frac{1}{n}} v_n + f_n & \text{in } \Omega, \\ -\Delta v_n = \frac{\lambda_2}{|x|^2 + \frac{1}{n}} u_n + g_n & \text{in } \Omega, \\ u_n = v_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

Notice that the Schauder fixed point theorem guarantees the existence of (u_n, v_n)

Let $c_1, c_2 > 0$ be fixed such that $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_2}{\lambda_1}}$ and define $w_n = c_1 u_n + c_2 v_n$, then w_n solves

$$-\Delta w_n = \frac{\lambda_1 c_1 v_n + \lambda_2 c_2 u_n}{|x|^2 + \frac{1}{n}} + c_1 f_n + c_2 g_n = \sqrt{\lambda_1 \lambda_2} \frac{w_n}{|x|^2 + \frac{1}{n}} + c_1 f_n + c_2 g_n.$$

Setting $h_n = c_1 f_n + c_2 g_n$, $h = c_1 f + c_2 g$ and by the assumptions imposed on f and g , we deduce that $h_n \uparrow h$ strongly in $L^1(\Omega)$ with $\int_{\Omega} h |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty$. Since the sequence $\{w_n\}_n$ is an increasing function, it follows that $w_n \uparrow w$ strongly in $W_0^{1,\sigma}(\Omega)$, $\sigma < \frac{N}{N-1}$ and $\frac{w_n}{|x|^2 + \frac{1}{n}} \uparrow \frac{w}{|x|^2}$ strongly in $L^1(\Omega)$. Thus w solves the problem

$$\begin{cases} -\Delta w = \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + h & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

Thanks to the dominated convergence theorem, we are led to the conclusion that

$$\left\| \frac{u_n}{|x|^2 + \frac{1}{n}} \right\|_{L^1} + \left\| \frac{v_n}{|x|^2 + \frac{1}{n}} \right\|_{L^1} < c \text{ for all } n.$$

Hence by the compactness result, we get the existence of $u, v \in W_0^{1,\sigma}(\Omega)$ for all $\sigma < \frac{N}{N-1}$. Moreover, up to a subsequence, one has $u_n \rightarrow u$ and $v_n \rightarrow v$ strongly in $W_0^{1,\sigma}(\Omega)$. Consequently, possibly extracting another subsequence, we also have $u_n \rightarrow u$ and $v_n \rightarrow v$ a.e. in Ω .

Notice that $\frac{u_n}{|x|^2 + \frac{1}{n}} \rightarrow \frac{u}{|x|^2}$ a.e. in Ω . Since $\frac{u_n}{|x|^2 + \frac{1}{n}} \leq C \frac{w_n}{|x|^2 + \frac{1}{n}}$, thus, the dominated convergence theorem implies that as $n \rightarrow \infty$,

$$\frac{u_n}{|x|^2 + \frac{1}{n}} \rightarrow \frac{u}{|x|^2} \text{ strongly in } L^1(\Omega).$$

Similarly, we obtain

$$\frac{v_n}{|x|^2 + \frac{1}{n}} \rightarrow \frac{v}{|x|^2} \text{ strongly in } L^1(\Omega), \text{ as } n \rightarrow \infty.$$

Therefore, we can pass to the limit in the approximating system (3.4) to get that (u, v) solves System (3.1).

To finish, we show that the solution obtained above is unique. Let $(u_1, v_1), (u_2, v_2)$ be two positive weak solutions to System (3.1). Let $c_1, c_2 > 0$ be such that $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_2}{\lambda_1}}$. Define $w_i = c_1 u_i + c_2 v_i, i = 1, 2$, then w_i solves

$$-\Delta w_i = \sqrt{\lambda_1 \lambda_2} \frac{w_i}{|x|^2} + c_1 f + c_2 g.$$

Since $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$, then $w_1 = w_2$. Thus for all c_1, c_2 such that $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_1}{\lambda_2}}$, we have

$$c_1 u_1 + c_2 v_1 = c_1 u_2 + c_2 v_2.$$

Hence $c_1(u_1 - u_2) = c_2(v_2 - v_1)$. Let $t = \frac{c_2}{c_1} = \sqrt{\frac{\lambda_1}{\lambda_2}}$, then $(u_1 - u_2) = t(v_2 - v_1)$. Define $\bar{u} = u_1 - u_2$ and $\bar{v} = v_1 - v_2$, then

$$-\Delta \bar{u} = \frac{\lambda_1}{|x|^2} (v_1 - v_2) = -\frac{\lambda_1}{t} \frac{\bar{u}}{|x|^2}.$$

Hence

$$-\Delta \bar{u} + \sqrt{\lambda_1 \lambda_2} \frac{\bar{u}}{|x|^2} = 0.$$

Since $\bar{u} \in W_0^{1,\sigma}(\Omega)$ for all $\sigma < \frac{N}{N-1}$, then using Kato type inequality we deduce that $\bar{u}_+ = 0$. In a symmetric way, we obtain that $\bar{u}_- = 0$. Thus $u_1 = u_2$ and the $v_1 = v_2$. $\square$

Remark 1.4. As proved above, we have that $u + v \geq C|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})}$ in $B_r(0)$. This does not allow us to infer how u, v behave separately in a neighborhood of the origin. However we can prove that

$$u, v \geq C \log\left(\frac{\eta}{|x|}\right) \text{ in } B_\eta(0) \subset \subset \Omega.$$

To see that, if (u, v) solves System (3.1), herefore, using the strong maximum principle, we conclude that $u, v \geq c_0$ in $B_\eta(0)$. Thus

$$-\Delta u \geq \lambda_1 \frac{c_0}{|x|^2} \text{ in } B_\eta(0).$$

Let $\vartheta(x) = \varepsilon \log\left(\frac{\eta}{|x|}\right)$, a simple computation yields that

$$-\Delta \vartheta = \varepsilon \frac{c}{|x|^2} \text{ in } B_\eta(0), \vartheta(x) = 0 \text{ if } |x| = \eta.$$

By taking ε sufficiently small, we conclude that $u \geq \vartheta$ in $B_\eta(0)$. The same holds for v . Thus

$$\liminf_{|x| \rightarrow 0} u(x) = \liminf_{|x| \rightarrow 0} v(x) = \infty.$$

1.1 Regularity results

This section is devoted to the analysis of the regularity of solutions to system (3.1)

We suppose that $f, g \in L^q$ and define $w = c_1 u + c_2 v$, then w solves the problem

$$\begin{cases} -\Delta w &= \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + h & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega. \end{cases} \quad \text{where } h \in L^q(\Omega) \quad (3.6)$$

Setting $\tilde{w} = |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} w$, it follows that $\tilde{w}$ solves the problem

$$-\operatorname{div}(|x|^{-2\mu(\sqrt{\lambda_1 \lambda_2})} |\nabla w|) = |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} h(x) := \tilde{h}(x)$$

Theorem 1.5. Assume that f, g satisfy $\int_{B_r(0)} (f+g)|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty$. Let $u, v \in L^1(\Omega)$ denote the unique weak solution of (3.1)

1. If $\beta < 2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1)$ and $q > \frac{(N-\beta)}{2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1) - \beta}$, then $|\nabla u|, |\nabla v| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{\mu(\sqrt{\lambda_1 \lambda_2}) + 1}$. In particular, we have a constant C such that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C \|f|x|^{\beta - \mu(\sqrt{\lambda_1 \lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}$$

$$\|\nabla v\|_{L^p(\mathbb{R}^N)} \leq C \|g|x|^{\beta - \mu(\sqrt{\lambda_1 \lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}$$

2. If $\beta < 2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1)$ and $q = \frac{(N-\beta)}{2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1) - \beta}$, then $|\nabla u|, |\nabla v| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{\mu(\sqrt{\lambda_1 \lambda_2}) + 1}$. Specifically, one can find a positive constant C such that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C \|f|x|^{\beta - \mu(\sqrt{\lambda_1 \lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)},$$

$$\|\nabla v\|_{L^p(\mathbb{R}^N)} \leq C \|g|x|^{\beta - \mu(\sqrt{\lambda_1 \lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}.$$

3. If either $\beta = 2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1)$ or $\beta < 2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1)$ and $\frac{\beta}{\beta - \mu(\sqrt{\lambda_1 \lambda_2})} < q < \frac{(N-\beta)}{2(\mu(\sqrt{\lambda_1 \lambda_2}) + 1) - \beta}$, then $|\nabla u|, |\nabla v| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{qN}{N - \beta - q(\mu(\sqrt{\lambda_1 \lambda_2}) + 1) - \beta}$. Specifically, one can find a positive constant C such that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C \|f|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)},$$

$$\|\nabla v\|_{L^p(\mathbb{R}^N)} \leq C \|g|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}.$$

4. If either $\beta = 2(\mu(\sqrt{\lambda_1\lambda_2}) + 1)$ or $\beta < 2(\mu(\sqrt{\lambda_1\lambda_2}) + 1)$ and $1 < q \leq \frac{\beta}{\beta-\mu(\sqrt{\lambda_1\lambda_2})}$, then $|\nabla u|, |\nabla v| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{qN}{N-q}$. Specifically, one can find a positive constant C such that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C \|f|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)},$$

$$\|\nabla v\|_{L^p(\mathbb{R}^N)} \leq C \|g|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}.$$

Proof. Let $\beta < 2(\mu(\sqrt{\lambda_1\lambda_2}) + 1)$ and $q > \frac{(N-\beta)}{2(\mu(\sqrt{\lambda_1\lambda_2})+1)-\beta}$, then we obtain $w(x) \geq |x|^{-\mu(\sqrt{\lambda_1\lambda_2})}$.

Hence $\frac{w}{|x|^2} \in L^\sigma$ for all $1 \leq \sigma < \frac{N}{\mu(\sqrt{\lambda_1\lambda_2})+2}$, since we are looking for a positive solution in this case we obtain $\frac{v}{|x|^2} \in L^\sigma$ for all $1 \leq \sigma < \frac{N}{\mu(\sqrt{\lambda_1\lambda_2})+2}$. Since $q > \frac{(N-\beta)}{2(\mu(\sqrt{\lambda_1\lambda_2})+1)-\beta}$ Thus, Hölder's inequality allows us to prove the existence of $a > \frac{N}{\mu(\sqrt{\lambda_1\lambda_2})+2}$ such that $\tilde{f} := \frac{v}{|x|^2} + f \in L^\sigma$ for all $1 \leq \sigma < \frac{N}{\mu(\sqrt{\lambda_1\lambda_2})+2}$. Using now the regularity result in [10] it hold that $|\nabla v| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{\mu(\sqrt{\lambda_1\lambda_2})+1}$ and

$$\|\nabla v\|_{L^p(\mathbb{R}^N)} \leq C \|g|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}.$$

Proceeding in a similar manner, it follows that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C \|f|x|^{\beta-\mu(\sqrt{\lambda_1\lambda_2})}\|_{L^q(\Omega, |x|^{-\beta} dx)}.$$

Using the same approach as in the previous case, we obtain the remaining cases. $\square$

2 The potential system: existence and non existence results.

In this section, we focus on the case where $f(x, v, \nabla v) = v^p$ and $g(x, u, \nabla u) = u^q$. that is, we study the system

$$\begin{cases} -\Delta u &= \lambda_1 \frac{v}{|x|^2} + v^p & \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{u}{|x|^2} + u^q & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega \end{cases} \quad (3.7)$$

where $0 < \lambda_1, \lambda_2$ are fixed such that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$. The first non existence result is the following.

Theorem 2.1. *Suppose that $\min\{p, q\} \geq \frac{N - \mu(\sqrt{\lambda_1 \lambda_2})}{\mu(\sqrt{\lambda_1 \lambda_2})}$, then System (3.7) has non weak supersolution.*

Proof. We proceed by contradiction. Suppose that system (3.7) admits a weak positive supersolution (u, v) . Then, by Lemma 1.2, we obtain $\int_{\Omega} (v^p + u^q) |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} < \infty$. Now from Remark 1.4, we know that

$$\liminf_{|x| \rightarrow 0} u(x) = \liminf_{|x| \rightarrow 0} v(x) = \infty.$$

Hence, in a small ball $B_{\eta}(0)$, we have

$$(v^p + u^q) \geq C(u + v)^{\min\{p, q\}}.$$

Thus

$$\int_{B_{\eta}(0)} (u + v)^{\min\{p, q\}} |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty.$$

Recall that $(u + v) \geq C|x|^{-\mu(\sqrt{\lambda_1 \lambda_2})}$, then

$$\int_{B_{\eta}(0)} |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})(1 + \min\{p, q\})} dx < \infty.$$

Therefore we conclude that $\mu(\sqrt{\lambda_1 \lambda_2})(1 + \min\{p, q\}) < N$, which contradicts the main hypothesis. Therefore, we conclude. $\square$

Notice that, $p = q$, then if $p \geq \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}$, then System (3.7) has non weak solution.

We now present the principal nonexistence theorem of this section.

Theorem 2.2. *Let $\lambda_1, \lambda_2 > 0$ be such that $\sqrt{\lambda_1\lambda_2} \leq \Lambda_N$.*

$$1. \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})} < p < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}, \quad q \leq p \text{ and } \mu(\sqrt{\lambda_1\lambda_2}) \frac{pq-1}{p+1} > 2,$$

Or

$$2. \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})} < q < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}, \quad p \leq q \text{ and } \mu(\sqrt{\lambda_1\lambda_2}) \frac{pq-1}{q+1} > 2,$$

then System (3.7) has non weak positive supersolution.

Proof. Taking into consideration the symmetric structure of the System (3.7), we will prove the non existence result under the first set of conditions. The other case follows in a similar way.

We proceed by contradiction and assume that System (3.7) possesses a positive weak supersolution (u, v) . Let $B_r(0) \subset\subset \Omega$ and $\phi \in \mathcal{C}_0^\infty(B_r(0))$. Using Picone inequality for both equations in $B_r(0) \subset \Omega$, we obtain that

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq \int_{B_r(0)} \frac{-\Delta u}{u} \phi^2 dx \geq \int_{B_r(0)} \frac{v^p}{u} \phi^2 dx,$$

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq \int_{B_r(0)} \frac{-\Delta v}{v} \phi^2 dx \geq \int_{B_r(0)} \frac{u^q}{v} \phi^2 dx.$$

Adding both inequalities, it holds that

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq \frac{1}{2} \int_{B_r(0)} \phi^2 \left[\frac{v^p}{u} + \frac{u^q}{v} \right] dx.$$

Let $a = \frac{v}{u^{\frac{1}{p}}}$, then

$$\frac{v^p}{u} + \frac{u^q}{v} = a^p + \frac{\alpha}{a} \text{ with } \alpha = u^{\frac{pq-1}{p}}.$$

Setting $K(a) = a^p + \frac{\alpha}{a}$, then K attains its minimum at $\hat{a} = \alpha^{\frac{1}{p+1}}$ and

$$K(a) \geq K(\hat{a}) = 2\hat{a}^p = 2\alpha^{\frac{p}{p+1}} = 2u^{\frac{pq-1}{p+1}}.$$

Thus

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq C \int_{B_r(0)} \phi^2 u^{\frac{pq-1}{p+1}} dx.$$

In a symmetric way, we get

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq C \int_{B_r(0)} \phi^2 v^{\frac{pq-1}{q+1}} dx.$$

Hence

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq C \int_{B_r(0)} \phi^2 \left(u^{\frac{pq-1}{p+1}} + v^{\frac{pq-1}{q+1}} \right) dx.$$

Using the fact that

$$\liminf_{|x| \rightarrow 0} u(x) = \liminf_{|x| \rightarrow 0} v(x) = \infty,$$

we deduce that

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq C(r) \int_{B_r(0)} \phi^2 (u+v)^{\min\{\gamma_1, \gamma_2\}} dx,$$

where $\gamma_1 = \frac{pq-1}{p+1}$ and $\gamma_2 = \frac{pq-1}{q+1}$. Now using estimate (3.3), we reach that

$$\int_{B_r(0)} |\nabla \phi|^2 dx \geq C \int_{B_r(0)} \phi^2 |x|^{-\mu(\sqrt{\lambda_1 \lambda_2}) \min\{\gamma_1, \gamma_2\}} dx.$$

Since $\mu(\sqrt{\lambda_1 \lambda_2}) \min\{\gamma_1, \gamma_2\} > 2$, then we get a contradiction with the Hardy inequality. Thus we conclude.

Now, assume that $p = q = \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})}$, then as in the proof Theorem 1.3, setting $w = c_1 u + c_2 v$ with $c_1, c_2 > 0$ satisfy $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_2}{\lambda_1}}$, we reach that w solves

$$\begin{aligned} -\Delta w &= \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + c_1 u^p + c_2 v^p \geq \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + C(c_1 u + c_2 v)^p \\ &\geq \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} + Cw^p. \end{aligned}$$

Since $p = \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})}$, hence, a contradiction arises with the nonexistence result derived in [23]. Hence we conclude. $\square$

Remark 2.3. We can be summarized the previous non existence results in the following description:

1. Let $\Upsilon = \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}$, then if $\min\{p, q\} \geq \Upsilon$, there is non positive weak supersolution.
2. If $\min\{p, q\} < \Upsilon$ with $q < p$, then $q < \Upsilon$. Define $h(p) = \frac{2}{\mu(\sqrt{\lambda_1\lambda_2})} + \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{p\mu(\sqrt{\lambda_1\lambda_2})}$. If $h(p) < q < \Upsilon$, then System (3.7) has non positive weak supersolution.
3. In a symmetric way, if $p < q$ and $p < \Upsilon$. Setting $\hat{h}(q) = \frac{2}{\mu(\sqrt{\lambda_1\lambda_2})} + \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{q\mu(\sqrt{\lambda_1\lambda_2})}$. If $\hat{h}(q) < p < \Upsilon$, System (3.7) has non positive weak supersolution.
4. Finally, if $p = q = \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})}$, System (3.7) has non positive weak supersolution.

2.1 Non existence result: The critical curves.

At this stage, we address the case where (p, q) lies on the critical curve. Taking into account the symmetric structure of the system, It will be enough to focus on the case $q < p$ and $q > h(p) = \frac{2}{\mu(\sqrt{\lambda_1\lambda_2})} + \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{p\mu(\sqrt{\lambda_1\lambda_2})}$. We now state the following nonexistence result.

Theorem 2.4. Consider a bounded domain Ω containing the origin. Assume that $\sqrt{\lambda_1\lambda_2} \leq \Lambda_N$. Assume that $q = h(p) = \frac{2}{\mu(\sqrt{\lambda_1\lambda_2})} + \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{p\mu(\sqrt{\lambda_1\lambda_2})}$. Then System (PS) has non positive weak supersolution. In a symmetric way, if $p = \hat{h}(q) = \frac{2}{\mu(\sqrt{\lambda_1\lambda_2})} + \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{q\mu(\sqrt{\lambda_1\lambda_2})}$, then, System (3.7) admits a nonpositive weak supersolution.

2.2 Existence results.

To demonstrate the optimality of the previous nonexistence result, we obtain a weak supersolution in the complementary region. Precisely, the theorem below

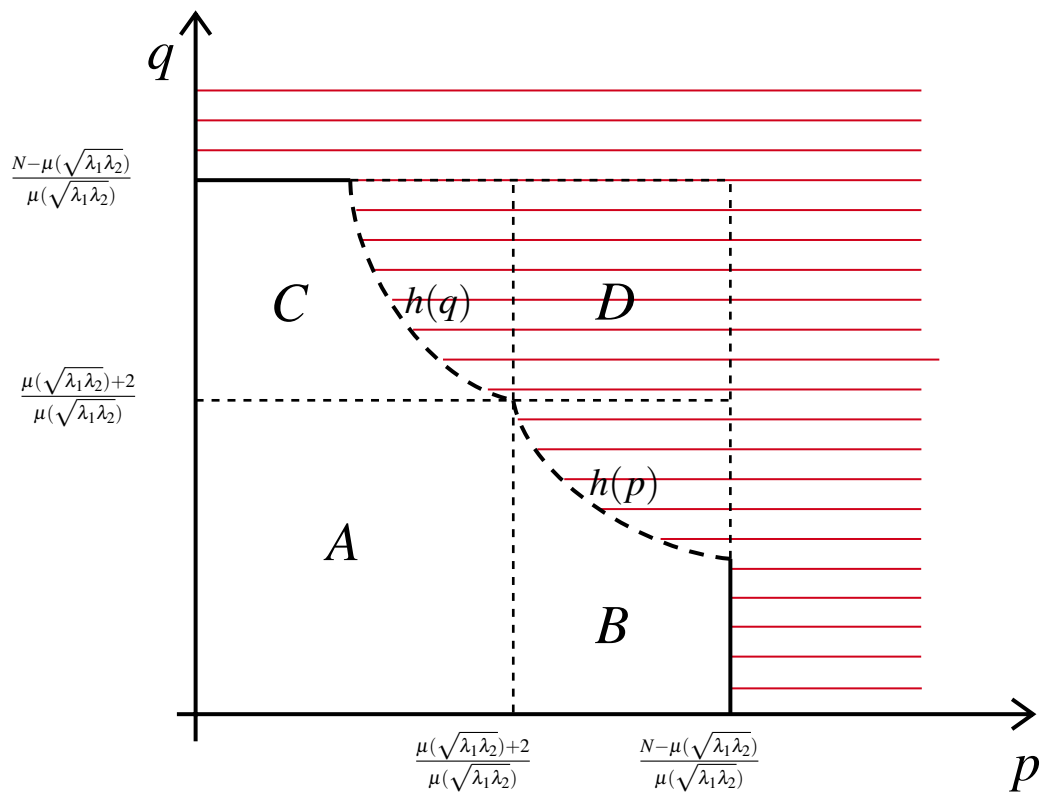


Figure 3.1: Existence and non-existence regions: $f = v^p, g = u^q$.

holds.

Theorem 2.5. *Consider a bounded domain Ω that contains the origin and for which $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$.*

1. *Suppose that $p, q < \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})}$, namely $(p, q) \in A$. Then System (3.7) has a positive supersolution.*

Proof. Suppose that $1 < p, q < \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})}$, then we get the existence of $\theta < \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})}$ such that $p, q < \theta$. Define $w(x) = \hat{C}|x|^{-\gamma}$ with $\gamma = \frac{2}{\theta-1}$. Then we can chose θ as above such that $\gamma(N-2-\gamma) - \sqrt{\lambda_1 \lambda_2} > 0$. Thus

$$-\Delta w - \sqrt{\lambda_1 \lambda_2} \frac{w}{|x|^2} = w^\theta \text{ in } \mathbb{R}^N.$$

Let c_1, c_2 be such that $\frac{c_1}{c_2} = \sqrt{\frac{\lambda_1}{\lambda_2}}$ and define $u = c_1 w, v = c_2 w$, then u, v solves

$$-\Delta u = \frac{\lambda_1 v}{|x|^2} + \frac{c_1}{c_2^\theta} v^\theta \text{ in } \mathbb{R}^N,$$

$$-\Delta v = \frac{\lambda_2 u}{|x|^2} + \frac{c_2}{c_1^\theta} u^\theta \text{ in } \mathbb{R}^N.$$

Notice that $v^\theta \geq \bar{C}_1(\Omega)v^p$ in Ω and $u^\theta \geq \bar{C}_2(\Omega)v^q$ in Ω . Now setting $\tilde{u} = \tilde{A}u$ and $\tilde{v} = \tilde{A}v$, we obtain that

$$-\Delta \tilde{u} = \frac{\lambda_1 \tilde{v}}{|x|^2} + \frac{1}{\tilde{A}^{\theta-1}} \frac{c_1}{c_2^\theta} \bar{C}_1(\Omega)v^p \text{ in } \Omega,$$

and

$$-\Delta \tilde{v} = \frac{\lambda_2 \tilde{v}}{|x|^2} + \frac{1}{\tilde{A}^{\theta-1}} \frac{c_2}{c_1^\theta} \bar{C}_2(\Omega)u^q \text{ in } \Omega.$$

Hence we can choose $\tilde{A}$ so that $(\tilde{u}, \tilde{v})$ forms a supersolution to System (3.7).

□

3 The gradient system: existence and non-existence results.

In the present section, we confine our attention to the gradient–gradient case, and accordingly, the functions are chosen as follows $f(x, v, \nabla v) = |\nabla v|^p$ and $g(x, u, \nabla u) = |\nabla u|^q$

$$\begin{cases} -\Delta u &= \lambda_1 \frac{v}{|x|^2} + |\nabla v|^p & \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{u}{|x|^2} + |\nabla u|^q & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega \end{cases} \quad (3.8)$$

where $0 < \lambda_1, \lambda_2$ are fixed such that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$.

Theorem 3.1. *Suppose that $\min\{p, q\} \geq \frac{N-\mu(\sqrt{\lambda_1 \lambda_2})}{\mu(\sqrt{\lambda_1 \lambda_2})+1}$, then System (3.8) has non supersolution.*

Proof. Assume, by contradiction, that there exists a positive solution. Then, by Lemma 1.2, we obtain that

$$\int_{\Omega} (|\nabla v|^p + |\nabla u|^q) |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty.$$

Hence, the CKN inequality yields

$$\int_{B_r(0)} \left(\frac{v^p}{|x|^p} + \frac{u^q}{|x|^q} \right) |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty. \quad (3.9)$$

If $p \geq q$ we obtain that :

$$\int_{B_r(0)} \left(\frac{v^q}{|x|^q} + \frac{u^q}{|x|^q} \right) |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx < \infty. \quad (3.10)$$

By Lemma 1.2, one can find a positive constant C and a sufficiently small ball $B_r(0) \subset \Omega$ such that $c_1 u(x) + c_2 v(x) \geq C|x|^{-\mu}$ in $B_r(0)$. From (3.10), it follows that

$$\begin{aligned}
\int_{B_r(0)} \left(\frac{u^q}{|x|^q} + \frac{v^q}{|x|^q} \right) |x|^{-\mu(\sqrt{\lambda_1 \lambda_2})} dx &\geq \int_{B_r(0)} \frac{(u+v)^q}{|x|^{q+\mu(\sqrt{\lambda_1 \lambda_2})}} dx \\
&\geq \int_{B_r(0)} \frac{1}{|x|^{q(1+\mu(\sqrt{\lambda_1 \lambda_2})) + \mu(\sqrt{\lambda_1 \lambda_2})}} dx \quad (3.11)
\end{aligned}$$

Thus $q < \frac{N-\mu(\sqrt{\lambda_1 \lambda_2})}{1+\mu(\sqrt{\lambda_1 \lambda_2})} = \bar{q}$, This yields a contradiction with (3.11). Hence, the nonexistence result follows in this case. $\square$

Theorem 3.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing the origin, with $N \geq 3$. Assume that $\sqrt{\lambda_1 \lambda_2} \leq \Lambda_N$, and let $p, q > 0$ satisfy the assumptions stated below:

$$\begin{aligned}
1. \quad q &< \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})+1} < p < \frac{N-\mu(\sqrt{\lambda_1 \lambda_2})}{\mu(\sqrt{\lambda_1 \lambda_2})+1}, \quad q \leq p \text{ and } \frac{2}{\mu(\sqrt{\lambda_1 \lambda_2})} + \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{p\mu(\sqrt{\lambda_1 \lambda_2})} < 0, \\
&\text{or}
\end{aligned}$$

$$2. \quad p < \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{\mu(\sqrt{\lambda_1 \lambda_2})+1} < q < \frac{N-\mu(\sqrt{\lambda_1 \lambda_2})}{\mu(\sqrt{\lambda_1 \lambda_2})+1}, \quad p \leq q \text{ and } \frac{2}{\mu(\sqrt{\lambda_1 \lambda_2})} + \frac{\mu(\sqrt{\lambda_1 \lambda_2})+2}{q\mu(\sqrt{\lambda_1 \lambda_2})} < 0,$$

Then the System (3.8) admits no weak solution that is nonnegative

Proof. Suppose, for the sake of contradiction, that System (3.8) admits a positive solution (u, v) . We obtain

$$\int_{\Omega} \frac{u^q}{|x|^{q+\mu(\sqrt{\lambda_1 \lambda_2})}} dx < \infty \quad \text{and} \quad \int_{\Omega} \frac{v^p}{|x|^{p+\mu(\sqrt{\lambda_1 \lambda_2})}} dx < \infty.$$

We claim that

$$\int_{\Omega} \frac{u}{|x|^{\sigma}} dx < \infty \quad \text{with} \quad \sigma < \frac{N}{q'} + \frac{\mu(\sqrt{\lambda_1 \lambda_2})}{q} + 1. \quad (3.12)$$

By Hölder inequality, we deduce

$$\int_{\Omega} \frac{u}{|x|^{\sigma}} dx \leq \left(\int_{\Omega} \frac{u^q}{|x|^{q+\mu(\sqrt{\lambda_1 \lambda_2})}} dx \right)^{\frac{1}{q}} \left(\int_{\Omega} \frac{1}{|x|^{q'(\sigma-1-\frac{\mu(\sqrt{\lambda_1 \lambda_2})}{q})}} dx \right)^{\frac{1}{q'}} \leq c \left(\int_{\Omega} \frac{1}{|x|^{q'(\sigma-1-\frac{\mu(\sqrt{\lambda_1 \lambda_2})}{q})}} dx \right)^{\frac{1}{q'}}.$$

If $\sigma < \frac{N}{q'} + \frac{\mu}{q} + 1$, then $q'(\sigma - 1 + \frac{\mu}{q}) < N$. Thus $\int_{\Omega} \frac{1}{|x|^{q'(\sigma - 1 + \frac{\mu(\sqrt{\lambda_1 \lambda_2})}{q})}} dx < \infty$

and the claim follows.

Define ψ the unique solution to the problem

$$\begin{cases} -\Delta \psi - \lambda \frac{\psi}{|x|^2} = \frac{1}{|x|^{\sigma}} & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Near the origin, we have $\psi(x) \asymp |x|^{2-\sigma}$. Taking ψ as a test function in the first equation yields $\int_{\Omega} |\nabla v|^p \psi dx < \infty$. Hence $\int_{\Omega} \frac{|\nabla v|^p}{|x|^{\sigma-2}} dx < \infty$. Furthermore, by Lemma (1.13), we obtain

$$\int_{\Omega} \frac{v^p}{|x|^{p-\sigma-2}} dx < \int_{\Omega} \frac{|\nabla v|^p}{|x|^{\sigma-2}} dx < \infty. \quad (3.14)$$

Now, we look for some $v > 2$ such that

$$\int_{\Omega} \frac{v}{|x|^v} dx < \infty. \quad (3.15)$$

Using the same argument as before, by Hölder inequality and (3.14), we obtain

$$\int_{\Omega} \frac{v}{|x|^{v_0}} dx \leq \left(\int_{\Omega} \frac{v^p}{|x|^{p+\sigma_0-2}} dx \right)^{\frac{1}{p}} \left(\int_{\Omega} \frac{1}{|x|^{p'(\sigma_0 - \frac{p+\sigma_0-2}{p})}} dx \right)^{\frac{1}{p'}} \leq c \left(\int_{\Omega} \frac{1}{|x|^{p'(\sigma_0 - \frac{p+\sigma_0-2}{p})}} dx \right)^{\frac{1}{p'}}.$$

To find (3.15), we need

$$v_0 < \frac{N}{p'} + \frac{\sigma_0 - 2}{p} + 1. \quad (3.16)$$

Let us fix σ_1 , and consider $\theta \asymp \frac{1}{|x|^{\sigma_1-2}}$ the solution corresponding to the following problem

$$\begin{cases} -\Delta \theta - \lambda \frac{\theta}{|x|^2} = \frac{1}{|x|^{v_0}} & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.17)$$

Using θ as a test function in the second equation we find ,

$$\int_{\Omega} \frac{|\nabla u|^q}{|x|^{v_0-2}} dx \leq \int_{\Omega} \left(|\nabla u|^q + \lambda \frac{u}{|x|^2} \right) \theta dx = \int_{\Omega} v(-\Delta \theta) dx = \int_{\Omega} \frac{v}{|x|^{v_0}} dx + \lambda \int_{\Omega} \frac{v\theta}{|x|^2} dx < \infty.$$

Then

$$\int_{\Omega} \frac{u^q}{|x|^{q+v_0-2}} dx < \infty. \quad (3.18)$$

Now, looking for some $\sigma_1 > \sigma_0$ such that

$$\int_{\Omega} \frac{u}{|x|^{\sigma_1}} dx < \infty. \quad (3.19)$$

By Hölder Inequality and (3.18), we get

$$\int_{\Omega} \frac{u}{|x|^{\sigma_1}} dx \leq c \left(\int_{\Omega} \frac{1}{|x|^{p'(\sigma_1-1-\frac{v_0-2}{p})}} dx \right)^{\frac{1}{p'}}.$$

We find another condition to acquire (3.19)

$$\sigma_1 < \frac{N}{p'} + \frac{v_0-2}{p} + 1. \quad (3.20)$$

Let us fix $\sigma_1 \geq \sigma_0$ such that (3.19) is satisfied, and define $\xi \asymp \frac{1}{|x|^{\sigma_1-2}}$, the solution corresponding to the following problem

$$\begin{cases} -\Delta \xi - \lambda \frac{\xi}{|x|^2} = \frac{1}{|x|^{\sigma_1}} & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.21)$$

By taking ξ as a test function in the first equation we get:

$$c \int_{\Omega} \frac{v^p}{|x|^{p+\sigma_1-2}} dx \leq c \int_{\Omega} \frac{|\nabla v|^p}{|x|^{\sigma_1-2}} dx \leq c \int_{\Omega} \left(|\nabla v|^p + \lambda \frac{v}{|x|^2} \right) \xi dx = \int_{\Omega} u(-\Delta \xi) dx \leq \int_{\Omega} \frac{u}{|x|^{\sigma_1}} dx < \infty.$$

Repeating these arguments, An increasing sequence is constructed by induction on n , $(\sigma_n)_n$ and $(v_n)_n$ defined by

$$\begin{cases} \sigma_{n+1} < \frac{N}{q'} + \frac{v_n-2}{q} + 1, \\ v_n < \frac{N}{p'} + \frac{\sigma_n-2}{p} + 1. \end{cases} \quad (3.22)$$

Where

$$\mu + 2 < \sigma_0 < \frac{N}{q'} + \frac{\mu}{q} + 1. \quad (3.23)$$

We use also the following estimations:

$$\int_{\Omega} \frac{u^q}{|x|^{q+\sigma_n-2}} < \infty \quad \text{and} \quad \int_{\Omega} \frac{v^p}{|x|^{p+v_n-2}} < \infty. \quad (3.24)$$

In order to treat the extremal case

$$\begin{cases} \bar{\sigma}_{n+1} &= \frac{N}{q'} + \frac{\bar{v}_n - 2}{q} + 1, \\ \bar{v}_n &= \frac{N}{p'} + \frac{\bar{\sigma}_n - 2}{p} + 1. \end{cases} \quad (3.25)$$

We suppose

$$l = \lim_{n \rightarrow +\infty} \sigma_n \quad \text{and} \quad \bar{l} = \lim_{n \rightarrow +\infty} v_n \quad (3.26)$$

From (3.24), taking the limit yields:

$$\int_{\Omega} \frac{u^q}{|x|^{q+l-2}} dx \leq \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{u^q}{|x|^{q+\sigma_n-2}} dx < \infty, \quad (3.27)$$

and

$$\int_{\Omega} \frac{v^p}{|x|^{p+\bar{l}-2}} dx \leq \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{v^p}{|x|^{p+v_n-2}} dx < \infty. \quad (3.28)$$

Then by adding 3.27 and 3.28 and using Lemma 1.2, we obtain

$$\int_{B_r(0)} \frac{1}{|x|^{q(\mu+1)+\min\{l,\bar{l}\}-2}} dx \leq \int_{B_r(0)} \frac{u^q}{|x|^{q+l-2}} dx + \int_{B_r(0)} \frac{v^q}{|x|^{q+\bar{l}-2}} dx < \infty.$$

Then if $q \leq p$, we obtain $q(\mu+1) + l - 2 < N$.

□

The above result presented by the following curve.

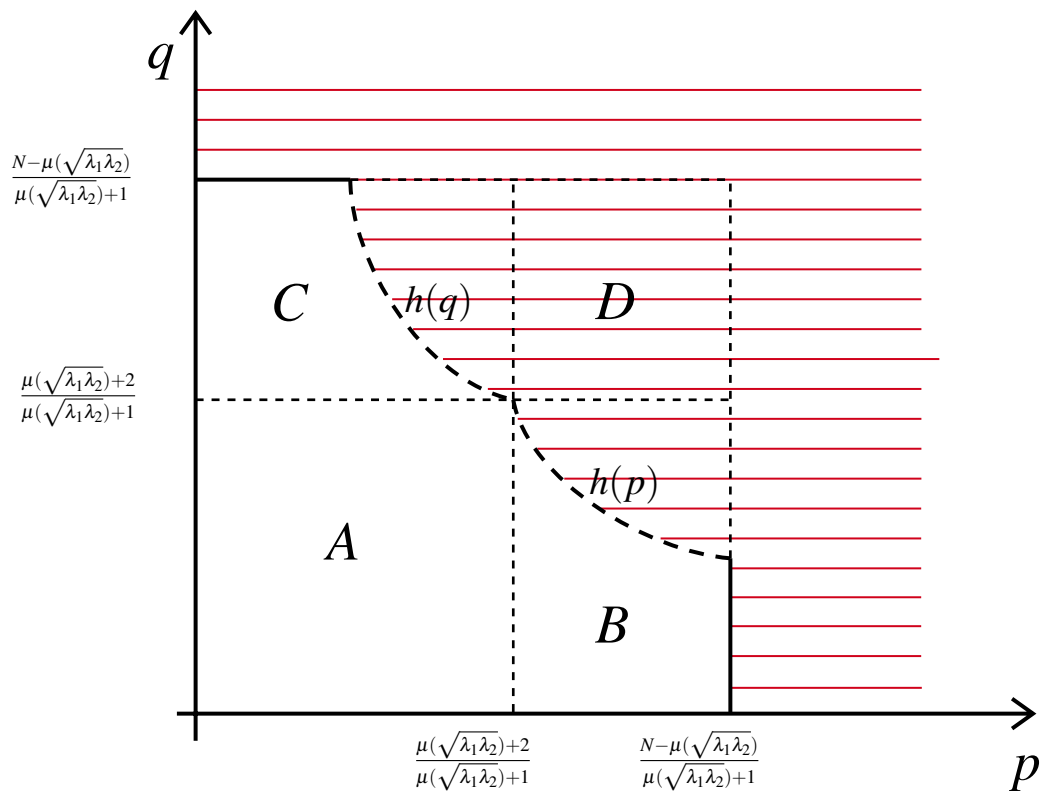


Figure 3.2: Existence and non-existence regions: $f = |\nabla v|^p, g = |\nabla u|^q$.

Part II

Parabolic Case

Chapter 4

Classical System

In this chapter, we investigate the existence and nonexistence of positive solutions to the system

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + f(v, \nabla v), & \text{in } \Omega_T, \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + g(u, \nabla u), & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0, x) = u_0(x), & \text{in } \Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega, \end{cases} \quad (4.1)$$

where $\Omega \subseteq \mathbb{R}^N$ with $N \geq 3$ is either a bounded Lipschitz domain containing 0, or the whole space $\mathbb{R}^N$. We denote $\Omega_T := \Omega \times (0, T)$ for some $T > 0$.

Our analysis focuses on the following types of nonlinearities:

$$\text{i) } f = v^p, g = u^q, \quad \text{ii) } f = |\nabla v|^p, g = |\nabla u|^q, \quad \text{iii) } f = v^p, g = |\nabla u|^q. \quad (4.2)$$

In addition, in the case $f = v^p$ and $g = u^q$, we also address the question of existence or nonexistence of global solutions for the associated Cauchy system.

1 Existence and Nonexistence of Local Supersolutions

In this section, we establish both existence and nonexistence results for very weak supersolutions corresponding to the nonlinearities f and g described in 5.2. The analysis is carried out in a bounded cylinder $\Omega \times (0, T)$, and the conclusions depend on the values of the exponents p and q .

1.1 The system with power nonlinearities

In this subsection we focus on the particular choice of nonlinearities $f = v^p$ and $g = u^q$. With this setting, the system under consideration becomes

$$\begin{cases} u_t - \Delta u = \lambda \frac{u}{|x|^2} + v^p, & \text{in } \Omega_T, \\ v_t - \Delta v = \lambda \frac{v}{|x|^2} + u^q, & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0, x) = u_0(x), & \text{in } \Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega, \end{cases} \quad (4.3)$$

where the exponents satisfy $p, q > 1$.

Our first aim is to show the existence of entropy solutions to (4.3). To achieve this, we assume the availability of a very weak supersolution for the same system with general nonlinearities of the type described in (5.2). The construction relies on a sequence of approximate problems together with a comparison procedure. Such an approach is justified by the fact that the functions $\sigma \mapsto \sigma^p$ and $\sigma \mapsto \sigma^q$ are monotone on $[0, +\infty)$. Moreover, classical regularity arguments, combined with the method developed in [4, Lemma 2.3], allow us to establish the following result.

Proposition 1.1. *Let $\lambda_1, \lambda_2 \in (0, \lambda_N]$ and let (u, v) be a nonnegative very weak supersolution of (4.3). Assume that $u_0, v_0 \in L^1_{\text{loc}}(\Omega)$. Then, for every open set $\Omega' \Subset \Omega$, one can construct two monotone sequences $\{\hat{u}_n\}_{n \in \mathbb{N}}$ and $\{\hat{v}_n\}_{n \in \mathbb{N}}$ in $L^\infty(\Omega'_T)$ such that*

$$\hat{u}_n \leq u, \quad \hat{v}_n \leq v \quad \text{for all } n.$$

Moreover, the initial data satisfy

$$\hat{u}_n(\cdot, 0) = T_n(u_0) \leq u_0, \quad \hat{v}_n(\cdot, 0) = T_n(v_0) \leq v_0 \quad \text{a.e. in } \Omega'.$$

Each function $\hat{u}_n$ and $\hat{v}_n$ satisfies

$$\hat{u}_n, \hat{v}_n \in L^2((0, T); W^{2,2}(\Omega')) \cap L^\infty((0, T); W_0^{1,2}(\Omega')) \cap C([0, T]; L^2(\Omega')),$$

and their time derivatives belong to $(\hat{u}_n)_t, (\hat{v}_n)_t \in L^2((0, T); L^2(\Omega'))$.

Moreover, the sequences converge strongly as follows:

$$\hat{u}_n \rightarrow \tilde{u}, \quad \hat{v}_n \rightarrow \tilde{v} \quad \text{in } L^r((0, T); W_0^{1,r}(\Omega')) \quad \text{for every } r < \frac{N+2}{N+1},$$

where $(\tilde{u}, \tilde{v})$ is an entropy solution of (4.3) in $\Omega' \times (0, T)$.

Furthermore, each pair $(\hat{u}_n, \hat{v}_n)$ satisfies the approximate system

$$\begin{cases} (\hat{u}_n)_t - \Delta \hat{u}_n = \lambda_1 \frac{\hat{u}_{n-1}}{|x|^2 + \frac{1}{n}} + \hat{v}_{n-1}^p, & \text{in } \Omega'_T, \\ (\hat{v}_n)_t - \Delta \hat{v}_n = \lambda_2 \frac{\hat{v}_{n-1}}{|x|^2 + \frac{1}{n}} + \hat{u}_{n-1}^q, & \text{in } \Omega'_T, \\ \hat{u}_n(x, t) = \hat{v}_n(x, t) = 0, & \text{on } \partial\Omega' \times (0, T), \\ \hat{u}_n(0, x) = T_n(u_0(x)), & \text{in } \Omega', \\ \hat{v}_n(0, x) = T_n(v_0(x)), & \text{in } \Omega', \end{cases} \quad (4.4)$$

for any $n \in \mathbb{N}$, $n \geq 1$ while

$$\begin{cases} (\hat{u}_0)_t - \Delta \hat{u}_0 = 0, & \text{in } \Omega'_T, \\ (\hat{v}_0)_t - \Delta \hat{v}_0 = 0, & \text{in } \Omega'_T, \\ \hat{u}_0(x, t) = \hat{v}_0(x, t) = 0, & \text{on } \partial\Omega' \times (0, T), \\ \hat{u}_0(0, x) = T_1(u_0(x)), & \text{in } \Omega', \\ \hat{v}_0(0, x) = T_1(v_0(x)), & \text{in } \Omega'. \end{cases} \quad (4.5)$$

Remark 1.2. The previous proposition provides a useful framework for proving the nonexistence of very weak solutions in the case of potential-type nonlinearities. Specifically, it enables us to work with an entropy solution that lies dominated by a given very weak supersolution.

We can now establish the main nonexistence result for this setting. We start by considering the first case.

Proposition 1.3. Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$. If either

$$p \geq \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu(\lambda_2)}{\mu(\lambda_1)}, \quad (4.6)$$

where $\mu(\lambda_i)$ is defined as in (1.5) for $i = 1, 2$, then any non-negative very weak supersolution (u, v) of (4.3) must be trivial.

Proof. Let $r > 0$ be such that $B_r(0) \subset\subset \Omega$ and let (u, v) be a very weak supersolution of (4.3). Then $u^p, v^q \in L^1_{loc}(\Omega_T)$. Now, we deduce that $u(x, t) \geq C|x|^{-\mu(\lambda_1)}$ and $v(x, t) \geq C|x|^{-\mu(\lambda_2)}$ a.e. in $B_r(0) \times (t_1, t_2)$ where $\mu(\lambda_i)$ is defined in (1.5). Thus $u^p(x, t) \geq C|x|^{-p\mu(\lambda_1)}$ and $v^q(x, t) \geq C|x|^{-q\mu(\lambda_2)}$ a.e. in $B_r(0) \times (t_1, t_2)$. Now, using Remark 4.2, we deduce that

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu(\lambda_1)} v^p dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} \int_{B_r(0)} |x|^{-\mu(\lambda_2)} dx dt < +\infty.$$

Hence

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu(\lambda_1)-p\mu(\lambda_2)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu(\lambda_2)-q\mu(\lambda_1)} dx dt < +\infty.$$

Thus

$$p < \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)} \quad \text{and} \quad q < \frac{N - \mu(\lambda_2)}{\mu(\lambda_1)}.$$

A contradiction with (4.6). Hence we conclude. $\square$

Notice that the previous result does not provide the optimal threshold. Exploiting Proposition 1.1, together with the quantitative blow-up behaviour of very weak supersolutions we can actually identify the precise critical curves that distinguish the region where solutions exist from the region where they do not. More precisely, we obtain the following result.

Theorem 1.4. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$ and assume that*

$$\frac{p(q+1)}{p+1} \geq 1 + \frac{2}{\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} \geq 1 + \frac{2}{\mu(\lambda_2)}, \quad (4.7)$$

where the quantities $\mu(\lambda_i)$, $i = 1, 2$, are given in (1.5). Under these conditions, the system (4.3) does not admit any non-trivial nonnegative very weak supersolution; the only possible one is the trivial pair $(u, v) \equiv (0, 0)$.

Proof. Assume, in order to reach a contradiction, that (u, v) is a nonnegative very weak supersolution of (4.3). Choose $R > 0$ such that $B_R(0) \subset \Omega$. By Proposition 1.1, we can construct an entropy solution of (4.3) inside $B_R(0) \times (0, T)$, which we

still denote by (u, v) for simplicity. Consequently, (u, v) satisfies

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + v^p, & \text{in } B_R(0) \times (0, T), \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + u^q, & \text{in } B_R(0) \times (0, T), \\ u(x, t) > 0, v(x, t) > 0, & \text{in } B_R(0) \times (0, T), \\ u = v = 0, & \text{on } \partial B_R(0) \times (0, T), \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{in } B_R(0). \end{cases} \quad (4.8)$$

From the quantitative lower bound for supersolutions, there exist $0 < r < R$ and times $0 < T_1 < T_2 \leq T$, together with a constant $C > 0$ depending only on N, R, T_1, T_2 , such that

$$u(x, t) \geq C|x|^{-\mu(\lambda_1)}, \quad v(x, t) \geq C|x|^{-\mu(\lambda_2)} \quad \text{a.e. in } B_r(0) \times (T_1, T_2). \quad (4.9)$$

Since $u, v \in L^1_{\text{loc}}(B_R(0) \times (0, T))$, this also implies that $\ln(u)$ and $\ln(v)$ belong to $L^\theta(B_r(0) \times (T_1, T_2))$ for any exponent $\theta \in [1, \infty]$.

Let $\phi \in C_c^\infty(B_r(0))$. By a standard density argument, we may use $\frac{\phi^2}{u}$ as test function in the first equation. This yields

$$\int_{T_1}^{T_2} \int_{B_r(0)} \frac{v^p}{u} \phi^2 dx dt \leq \int_{T_1}^{T_2} \int_{B_r(0)} \frac{\phi^2}{u} (u_n)_t dx dt + \int_{T_1}^{T_2} \int_{B_r(0)} (-\Delta u) \frac{\phi^2}{u} dx dt. \quad (4.10)$$

Using Picone's identity (see Proposition 1.10), we obtain the estimate

$$\int_{T_1}^{T_2} \int_{B_r(0)} \frac{v^p}{u} \phi^2 dx dt \leq \int_{B_r(0)} |\ln(u(x, T_2))| \phi^2 dx + \int_{T_1}^{T_2} \int_{B_r(0)} |\nabla \phi|^2 dx dt.$$

Thus

$$\int_{T_1}^{T_2} \int_{B_r(0)} \frac{v^p}{u} \phi^2 dx dt \leq \int_{B_r(0)} |\ln(u(x, T_2))| \phi^2 dx + (T_2 - T_1) \int_{B_r(0)} |\nabla \phi|^2 dx. \quad (4.11)$$

By the Hölder and Sobolev inequalities,

$$\begin{aligned} \int_{B_r(0)} |\ln(u(x, T_2))| \phi^2 dx &\leq \left(\int_{B_r(0)} |\phi|^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{B_r(0)} |\ln(u(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \\ &\leq \mathcal{S} \left(\int_{B_r(0)} |\ln(u(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \int_{B_r(0)} |\nabla \phi|^2 dx, \end{aligned} \quad (4.12)$$

where $\mathcal{S}$ is the optimal constant in the Sobolev inequality. Hence, the critical relation obtained above leads directly to

$$\left((T_2 - T_1) + \mathcal{S} \left(\int_{B_r(0)} |\ln(u(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \right) \int_{B_r(0)} |\nabla \phi|^2 dx \geq \int_{T_1}^{T_2} \left(\int_{B_r(0)} \frac{v^p}{u} \phi^2 dx \right) dt. \quad (4.13)$$

Applying the same type of estimate to the second equation, we derive that

$$\left((T_2 - T_1) + \mathcal{S} \left(\int_{B_r(0)} |\ln(v(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \right) \int_{B_r(0)} |\nabla \phi|^2 dx \geq \int_{T_1}^{T_2} \left(\int_{B_r(0)} \frac{u^q}{v} \phi^2 dx \right) dt, \quad (4.14)$$

and so

$$\begin{aligned} & \left[2(T_2 - T_1) + \mathcal{S} \left(\int_{B_r(0)} |\ln(u(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} + \mathcal{S} \left(\int_{B_r(0)} |\ln(v(x, T_2))|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \right] \int_{B_r(0)} |\nabla \phi|^2 dx \\ & \geq \int_{T_1}^{T_2} \left(\int_{B_r(0)} \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \phi^2 dx \right) dt. \end{aligned} \quad (4.15)$$

Setting $a := vu^{-1/p}$ and $\alpha := u^{\frac{pq-1}{p}}$, we introduce, for $a > 0$, the function $K(a) := a^p + \frac{\alpha}{a}$. It is straightforward to check that K attains its minimum at $\bar{a} = p^{-\frac{1}{p+1}} \alpha^{\frac{1}{p+1}}$. Therefore,

$$K(a) \geq K(\bar{a}) \geq p^{-\frac{p}{p+1}} \alpha^{\frac{p}{p+1}} = p^{-\frac{p}{p+1}} u^{\frac{pq-1}{p+1}}. \quad (4.16)$$

We conclude that,

$$\begin{aligned} \int_{t_1}^{t_2} \left(\int_{B_r(0)} \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \phi^2 dx \right) dt & \geq p^{-\frac{p}{p+1}} \int_{t_1}^{t_2} \left(\int_{B_r(0)} u^{\frac{pq-1}{p+1}} \phi^2 dx \right) dt \\ & \geq (t_2 - t_1) p^{-\frac{p}{p+1}} C \int_{B_r(0)} \frac{\phi^2}{|x|^{\mu(\lambda_1) \frac{pq-1}{p+1}}} dx \end{aligned} \quad (4.17)$$

By an analogous computation, we also obtain that

$$\begin{aligned} \int_{t_1}^{t_2} \left(\int_{B_r(0)} \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \phi^2 dx \right) dt & \geq p^{-\frac{q}{q+1}} \int_{t_1}^{t_2} \left(\int_{B_r(0)} v^{\frac{pq-1}{q+1}} \phi^2 dx \right) dt \\ & \geq (t_2 - t_1) q^{-\frac{q}{q+1}} C \int_{B_r(0)} \frac{\phi^2}{|x|^{\mu(\lambda_2) \frac{pq-1}{q+1}}} dx. \end{aligned} \quad (4.18)$$

Therefore, if either

$$\mu(\lambda_1) \frac{pq-1}{p+1} > 2 \quad \text{or} \quad \mu(\lambda_2) \frac{pq-1}{q+1} > 2,$$

and $(u, v) \neq (0, 0)$, we reach a contradiction with the optimality of the Hardy potential, in view of (4.15). $\square$

We now turn our attention to the critical case. $\mu(\lambda_1) \frac{pq-1}{p+1} = 2$, hence $p(q\mu(\lambda_1) - 2) = \mu(\lambda_1) + 2$. So, v is a supersolution of

$$v_t - \Delta v - \lambda_2 \frac{v}{|x|^2} \geq u^q \geq C|x|^{-q\mu(\lambda_1)} \quad (4.19)$$

in $B_r(0) \times (t_1, t_2)$. Moreover, for $(x, t) \in B_\eta(0) \times (t_1, t_2)$, we define $\tilde{v}(x, t) := \varepsilon(t - t_1)|x|^{-(q\mu(\lambda_1)-2)}$. Then, for any $\varepsilon > 0$, it holds that $\tilde{v}(x, t_1) = 0$ and

$$\tilde{v}_t - \Delta \tilde{v} - \lambda_2 \frac{\tilde{v}}{|x|^2} = \varepsilon|x|^{-(q\mu(\lambda_1)-2)} - \varepsilon P_{\lambda_2}(q\mu(\lambda_1) - 2)|x|^{-q\mu(\lambda_1)} \leq \varepsilon \tilde{C}|x|^{-q\mu(\lambda_1)}, \quad (4.20)$$

for a certain positive constant $\tilde{C} > 0$. Therefore, by taking $\varepsilon > 0$ sufficiently small and applying the comparison principle, we deduce that $v \geq \varepsilon|x|^{-(q\mu(\lambda_1)-2)}$ in $B_r(0) \times (t_1, t_2)$. Returning to the equation for u , it then follows that

$$u_t - \Delta u - \lambda_1 \frac{u}{|x|^2} \geq v^p \geq \varepsilon C \frac{1}{|x|^{p(q\mu(\lambda_1)-2)}} = \varepsilon C \frac{1}{|x|^{\mu(\lambda_1)+2}}, \quad (4.21)$$

in $B_r(0) \times (t_1, t_2)$. In particular, if we define $\tilde{w}(x, t) := (t - t_1)w$, where w is as previously defined,

$$\tilde{w}_t - \Delta \tilde{w} - \lambda_1 \frac{\tilde{w}}{|x|^2} = \frac{C}{|x|^{\mu(\lambda_1)+2}} + w(x) \leq \frac{\bar{C}}{|x|^{\mu(\lambda_1)+2}}, \quad (4.22)$$

for some positive constant $\bar{C} > 0$. Since $\tilde{w}(x, t_1) = 0$, applying the comparison principle yields $u \geq C_1|x|^{-\mu(\lambda_1)}(\ln|r| - \ln|x|)$ in $B_r(0) \times (t_1, t_2)$, for some small constant $C_1 > 0$.

Now, combining the estimates (4.15) and (4.17), and taking $\rho \ll r$ sufficiently small, we obtain that for all $\phi \in \mathcal{C}_0^\infty(B_\rho(0))$,

$$C(t_1, t_2, r, \rho, u) \int_{B_\rho(0)} |\nabla \phi|^2 dx \geq C \int_{B_\rho(0)} \frac{\phi^2}{|x|^2} \left(\ln \left(\frac{1}{|x|} \right) \right)^{\frac{2}{\mu(\lambda_1)}} dx.$$

Since $\ln\left(\frac{1}{|x|}\right) \rightarrow \infty$ as $|x| \rightarrow 0$, this leads to a contradiction with the optimality of the Hardy constant.

In the case where $\mu(\lambda_2) \frac{pq-1}{q+1} = 2$, a similar argument can be applied.

Results on the Existence of Supersolutions

We establish the sharpness of the previous nonexistence results by showing that positive supersolutions do indeed exist when the conditions in (4.7) are not met. In particular, we construct positive radial supersolutions in the ball $B_r(0)$ for any radius $r > 0$, which ensures the existence of positive supersolutions on any bounded domain $\Omega \times (0, T)$. Let

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{N - \mu(\lambda_2)}{\mu(\lambda_1)}\right) : \right. \\ \left. \frac{p(q+1)}{p+1} < 1 + \frac{2}{\mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{2}{\mu(\lambda_2)} \right\}. \quad (4.23)$$

We aim to demonstrate the existence of a nontrivial supersolution for every $(p, q) \in F_{\lambda_1, \lambda_2}$. To this end, we partition F_{λ_1, λ_2} into five distinct regions. A straightforward computation then shows that

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}\right) \right\}, \quad (4.24)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(\frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}\right) \right\}, \quad (4.25)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(\frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{\mu(\lambda_1)}\right) \right\}, \quad (4.26)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, q \in \left(\frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{\mu(\lambda_1)}\right) \right\}, \quad (4.27)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}\right), q = \frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)} \right\}. \quad (4.28)$$

Proposition 1.5. *For any $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist positive constants ε_0 and δ_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the pair (u, v) defined by*

$$\varepsilon(|x|^{-\gamma_1}, |x|^{-\gamma_2}), \quad \gamma_i \in (\mu(\lambda_i), \mu^+(\lambda_i)), \quad \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \quad (4.29)$$

$$\varepsilon\left(|x|^{-(p\mu(\lambda_2)-2)}, |x|^{-\mu(\lambda_2)} - \delta|x|^{-(pq\mu(\lambda_2)-2q-2)}\right), \quad \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \quad (4.30)$$

$$\varepsilon\left(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(pq\mu(\lambda_1)-2p-2)}, |x|^{-(q\mu(\lambda_1)-2)}\right), \quad \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \quad (4.31)$$

$$\varepsilon\left(-|x|^{-\mu(\lambda_1)} \ln(|x|/r), |x|^{-\mu(\lambda_2)} - \delta|x|^{-(q\mu(\lambda_1)-2+\varepsilon)}\right), \quad \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \quad (4.32)$$

$$\varepsilon\left(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(p\mu(\lambda_2)-2+\varepsilon)}, -|x|^{-\mu(\lambda_2)} \ln(|x|/r)\right), \quad \text{if } (p, q) \in E_{\lambda_1, \lambda_2}, \quad (4.33)$$

provides a positive supersolution of (4.3) for sufficiently small initial data.

Proof. We divide the proof into three cases for clarity. Let P_λ be as defined in (1.6).

Case 1: Zone A_{λ_1, λ_2} . Let (u, v) satisfies (4.3) in $B_r(0)$ if and only if

$$\begin{cases} -P_{\lambda_1}(\gamma_1) \geq \varepsilon^{p-1}|x|^{\gamma_1+2-p\gamma_2}, & x \in B_r(0), \\ -P_{\lambda_2}(\gamma_2) \geq \varepsilon^{q-1}|x|^{\gamma_2+2-q\gamma_1}, & x \in B_r(0). \end{cases} \quad (4.34)$$

By taking $\varepsilon > 0$ sufficiently small, the inequalities in (4.34) are satisfied thanks to (4.24).

Case 2: Zones B_{λ_1, λ_2} and C_{λ_1, λ_2} . A direct computation shows

$$p\mu(\lambda_2) - 2 \in (\mu(\lambda_1), \mu^+(\lambda_1)), \quad pq\mu(\lambda_2) - 2q - 2 \in (-\infty, \mu(\lambda_2)).$$

In particular, $P_{\lambda_1}(p\mu(\lambda_2) - 2) < 0$ and $P_{\lambda_2}(pq\mu(\lambda_2) - 2q - 2) > 0$. Choosing $\delta > 0$ small enough ensures $|x|^{-\mu(\lambda_2)} \geq \delta|x|^{-(pq\mu(\lambda_2)-2q-2)}$ for all $x \in B_r(0)$. Hence, (u, v) satisfies (4.3) in $B_r(0)$ if and only if

$$\begin{cases} -P_{\lambda_1}(p\mu(\lambda_2) - 2) \geq \varepsilon^{p-1} \left(1 - \delta r^{-pq\mu(\lambda_2)+2q+2+\mu(\lambda_2)}\right)^p, & x \in B_r(0), \\ P_{\lambda_2}(pq\mu(\lambda_2) - 2q - 2) \geq \varepsilon^{q-1}, & x \in B_r(0). \end{cases} \quad (4.35)$$

Choosing ε sufficiently small, these inequalities hold. The argument for C_{λ_1, λ_2} is analogous.

Case 3: Zones D_{λ_1, λ_2} and E_{λ_1, λ_2} . Without loss of generality, let $r = 1$. So (u, v) satisfies (4.3) in $B_r(0)$ if and only if

$$\begin{cases} N - 2 - 2\mu(\lambda_1) \geq \varepsilon^{p-1} \left(1 - \delta|x|^{\mu(\lambda_2)-q\mu(\lambda_1)+2-\varepsilon}\right), & x \in B_r(0), \\ P_{\lambda_2}(q\mu(\lambda_1) - 2 + \varepsilon) \geq -\varepsilon^{q-1}|x|^\varepsilon \ln(|x|)^q, & x \in B_r(0). \end{cases} \quad (4.36)$$

Since $q\mu(\lambda_1) - 2 + \varepsilon < \mu(\lambda_2)$ for small $\varepsilon > 0$, we have $P_{\lambda_2}(q\mu(\lambda_1) - 2 + \varepsilon) > 0$. Also, $\mu(\lambda_2) - q\mu(\lambda_1) + 2 - \varepsilon > 0$ for small ε , and $N - 2 - 2\mu(\lambda_1) > 0$ for any $\lambda_1 \in (0, \Lambda_N)$. Therefore, (4.36) holds with a suitably small choice of ε .

The argument for E_{λ_1, λ_2} is similar, showing that (u, v) is a supersolution in that case as well. $\square$

At this point, we can immediately deduce the following result.

Corollary 1.6. *Let $(p, q) \in F_{\lambda_1, \lambda_2}$. Suppose that $\Omega \subset B_r(0)$ and let the initial data $u_0, v_0 \in L^1(\Omega)$ satisfy $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, and*

$$u_0 \leq \hat{u}, \quad v_0 \leq \hat{v},$$

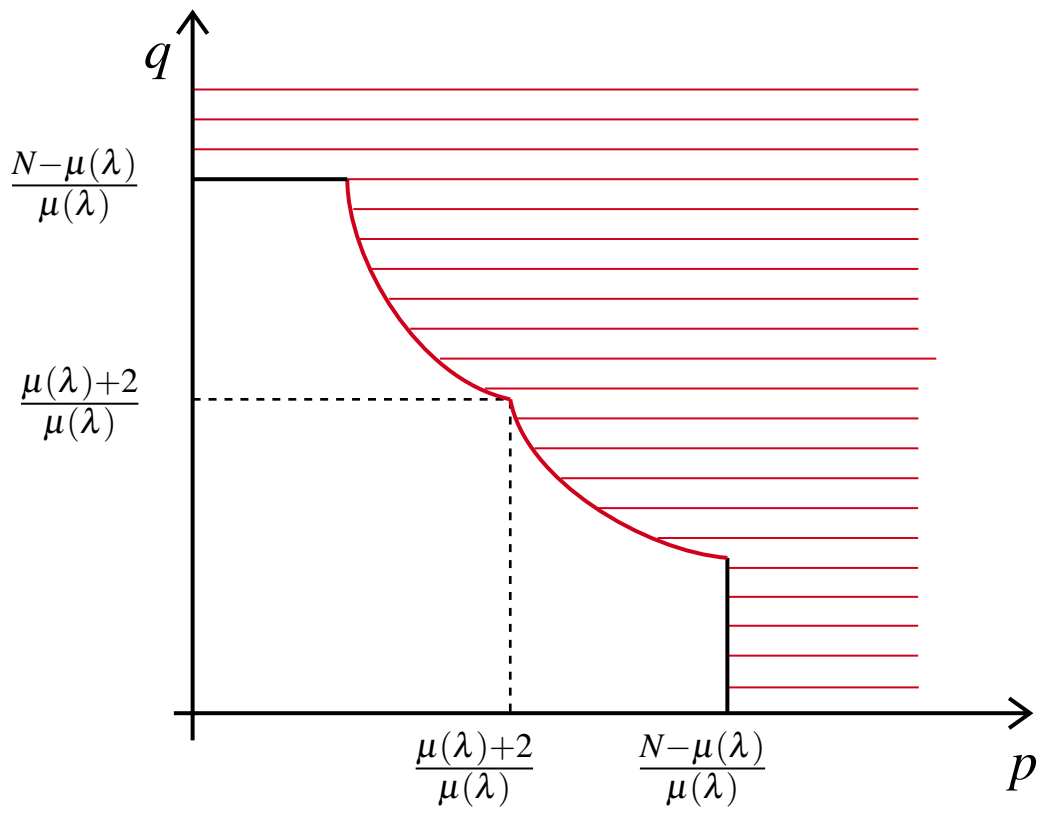


Figure 4.1: $f = v^p, g = u^q$ and $\lambda_1 = \lambda_2 = \lambda$.

where $(\hat{u}, \hat{v})$ denotes the radial supersolution constructed below. Then, for any $T > 0$, there exists a positive, non-trivial entropy solution (u, v) of (4.3) corresponding to the initial data (u_0, v_0) .

1.2 The System with Gradient Nonlinearities

In this subsection, we focus on the case where $f = |\nabla v|^p$ and $g = |\nabla u|^q$, leading to the following system:

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + |\nabla v|^p, & \text{in } \Omega_T, \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + |\nabla u|^q, & \text{in } \Omega_T, \\ u(x, t) \geq 0, \quad v(x, t) \geq 0, & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (4.37)$$

with $p, q > 1$.

Following the approach of Subsection 1.1, we begin with a basic result derived from Remark 4.2.

Proposition 1.7. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$. If either*

$$p \geq \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}, \quad (4.38)$$

with $\mu(\lambda_i)$ as in (1.5) for $i = 1, 2$, then the only non-negative very weak supersolution (u, v) of (4.37) is the trivial solution $u \equiv 0$ and $v \equiv 0$.

Proof. Assume, by contradiction, that (u, v) is a non-trivial non-negative very weak supersolution of (4.37). By Remark 4.2, for any $B_r(0) \subset\subset \Omega$ and $(t_1, t_2) \subset (0, T)$, we have

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla v|^p |x|^{-\mu(\lambda_1)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-\mu(\lambda_2)} dx dt < +\infty. \quad (4.39)$$

Since $p, q \in (1, N)$ and $\mu(\lambda_1)/p \in (0, (N-p)/p)$, $\mu(\lambda_2)/q \in (0, (N-q)/q)$ by (4.38), implies

$$\int_{t_1}^{t_2} \int_{B_r(0)} v^p |x|^{-p-\mu(\lambda_1)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} u^q |x|^{-q-\mu(\lambda_2)} dx dt < +\infty. \quad (4.40)$$

It follows that

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-p\mu(\lambda_2)-p-\mu(\lambda_1)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-q\mu(\lambda_1)-q-\mu(\lambda_2)} dx dt < +\infty. \quad (4.41)$$

Hence we must have $p\mu(\lambda_2) + p + \mu(\lambda_1) < N$ and $q\mu(\lambda_1) + q + \mu(\lambda_2) < N$, which contradicts (4.38).

Thus, the only possible solution is the trivial one: $u \equiv 0$ and $v \equiv 0$. $\square$

The next non-existence result is established using an iterative approach introduced in [20].

Theorem 1.8. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$ and assume that*

$$\frac{p(q+1)}{p+1} > 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} > 1 + \frac{1}{1+\mu(\lambda_2)}, \quad (4.42)$$

where $\mu(\lambda_i)$, $i = 1, 2$, are defined as in (1.5). Then the only non-negative very weak supersolution (u, v) of (4.37) is the trivial solution $u \equiv 0$ and $v \equiv 0$.

Proof. Let (u, v) be a non-negative very weak supersolution of (4.37) with exponents p, q satisfying (4.42). We aim to show that $u \equiv 0$ and $v \equiv 0$.

By Proposition 1.7, we may assume

$$p < \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)}, \quad q < \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}.$$

Let $B_r(0) \subset\subset \Omega$ and $t_1 < t_2 \in (0, T)$, and define

$$a_0 := 2 + \mu(\lambda_2), \quad b_0 := 2 + \mu(\lambda_1).$$

For $n \geq 1$, define recursively

$$\begin{cases} a_n := 1 + \frac{N}{p'} + \frac{b_{n-1}-2}{p} - \frac{1}{n}, \\ b_n := 1 + \frac{N}{q'} + \frac{a_{n-1}-2}{q} - \frac{1}{n}. \end{cases} \quad (4.43)$$

We claim that for all $n \in \mathbb{N}$,

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty. \quad (4.44)$$

We proceed by induction. The claim holds for $n = 0$ as in Proposition 1.7. Suppose it holds for $n - 1 \geq 0$; we show it holds for n .

Then,

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-a_{n-1}+2} dx dt < +\infty.$$

If $a_{n-1} - 2 \geq N - q$, then $u \equiv 0$ and hence $v \equiv 0$; otherwise, using the Caffarelli-Kohn-Nirenberg inequality and Hölder's inequality, one obtains

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty.$$

Similarly, applying Remark 4.2 to v and using the same reasoning, we deduce

$$\int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty.$$

Hence the claim (4.44) is established. Letting

$$\bar{a} := \limsup_{n \rightarrow \infty} a_n = N + 1 - \frac{q+1}{pq-1}, \quad \bar{b} := \limsup_{n \rightarrow \infty} b_n = N + 1 - \frac{p+1}{pq-1},$$

we find that either $u \equiv 0$ and $v \equiv 0$, or

$$\mu(\lambda_2) + \bar{a} \leq N, \quad \mu(\lambda_1) + \bar{b} \leq N,$$

which contradicts (4.42). Therefore, the only possibility is $u \equiv 0$ and $v \equiv 0$. $\square$

Existence of Supersolutions.

Following the approach in Subsection 1.1, we establish the existence of positive supersolutions.

Let

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right) \times \left(1, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) : \right. \\ \left. \frac{p(q+1)}{p+1} < 1 + \frac{1}{1 + \mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{1}{1 + \mu(\lambda_2)} \right\}. \quad (4.45)$$

We will show that there exists a non-trivial supersolution for any $(p, q) \in F_{\lambda_1, \lambda_2}$. It is not hard to see that

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.46)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(\frac{2 + \mu(\lambda_1)}{1 + \mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.47)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right) \times \left(\frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.48)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2 + \mu(\lambda_1)}{1 + \mu(\lambda_2)}, q \in \left(\frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.49)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2 + \mu(\lambda_1)}{1 + \mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right), q = \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)} \right\}. \quad (4.50)$$

Following the same approach as in Subsection 1.1, we can establish the existence of supersolutions as stated below.

Proposition 1.9. *For any pair $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist positive constants ε_0 and*

δ_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the functions (u, v) defined by

$$\varepsilon(|x|^{-\gamma_1}, |x|^{-\gamma_2}), \quad \text{with } \gamma_i \in (\mu(\lambda_i), \mu^+(\lambda_i)) \text{ for } i = 1, 2, \quad \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \quad (4.51)$$

$$\varepsilon(|x|^{-(p\mu(\lambda_2)-1)}, |x|^{-\mu(\lambda_2)} - \delta|x|^{-(pq\mu(\lambda_2)-2q-1)}), \quad \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \quad (4.52)$$

$$\varepsilon(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(pq\mu(\lambda_1)-2p-1)}, |x|^{-(q\mu(\lambda_1)-1)}), \quad \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \quad (4.53)$$

$$\varepsilon(-|x|^{-\mu(\lambda_1)} \ln(|x|/r), |x|^{-\mu(\lambda_2)} - \delta|x|^{-(q\mu(\lambda_1)-1+\varepsilon)}), \quad \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \quad (4.54)$$

$$\varepsilon(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(p\mu(\lambda_2)-1+\varepsilon)}, -|x|^{-\mu(\lambda_2)} \ln(|x|/r)), \quad \text{if } (p, q) \in E_{\lambda_1, \lambda_2}, \quad (4.55)$$

provides a positive supersolution of (4.37) in the ball $B_r(0)$.

Proof. We divide the proof into three cases. Let P_λ be defined by (1.6).

Case 1: Zone A_{λ_1, λ_2} .

Let $u = \varepsilon|x|^{-\gamma_1}$, $v = \varepsilon|x|^{-\gamma_2}$, with $\gamma_i \in (\mu(\lambda_i), \mu^+(\lambda_i))$. Since

$$|\nabla|x|^{-\gamma}| = \gamma|x|^{-\gamma-1},$$

we obtain

$$|\nabla v|^p = (\varepsilon\gamma_2)^p |x|^{-p(\gamma_2+1)},$$

and similarly for $|\nabla u|^q$.

Hence (u, v) is a supersolution of (4.37) if and only if

$$\begin{cases} -P_{\lambda_1}(\gamma_1) \geq \varepsilon^{p-1} \gamma_2^p |x|^{\gamma_1+1-p(\gamma_2+1)}, \\ -P_{\lambda_2}(\gamma_2) \geq \varepsilon^{q-1} \gamma_1^q |x|^{\gamma_2+1-q(\gamma_1+1)}. \end{cases} \quad (4.56)$$

By choosing $\varepsilon > 0$ sufficiently small, the inequalities follow from the definition of A_{λ_1, λ_2} .

Case 2: Zones B_{λ_1, λ_2} and C_{λ_1, λ_2} . A direct computation yields

$$p\mu(\lambda_2) - 1 \in (\mu(\lambda_1), \mu^+(\lambda_1)),$$

and $pq\mu(\lambda_2) - 2q - 1 < \mu(\lambda_2)$, which implies

$$P_{\lambda_1}(p\mu(\lambda_2) - 1) < 0, \quad P_{\lambda_2}(pq\mu(\lambda_2) - 2q - 1) > 0.$$

Choosing $\delta > 0$ sufficiently small, we have

$$|x|^{-\mu(\lambda_2)} \geq \delta |x|^{-(pq\mu(\lambda_2) - 2q - 1)} \text{ in } B_r(0).$$

Consequently, (u, v) satisfies (4.37) provided

$$\begin{cases} -P_{\lambda_1}(p\mu(\lambda_2) - 1) \geq \varepsilon^{p-1} (1 - \delta r^{-pq\mu(\lambda_2) + 2q + 1 + \mu(\lambda_2)})^p, \\ P_{\lambda_2}(pq\mu(\lambda_2) - 2q - 1) \geq \varepsilon^{q-1}, \end{cases} \quad (4.57)$$

which holds for sufficiently small ε . The proof for C_{λ_1, λ_2} is identical.

Case 3: Zones D_{λ_1, λ_2} and E_{λ_1, λ_2} .

Without loss of generality, let $r = 1$.

Using $|\nabla(-|x|^{-\mu} \ln |x|)| \leq C|x|^{-\mu-1}(1 + |\ln |x||)$,

we obtain that (u, v) satisfies (4.37) provided

$$\begin{cases} N - 2 - 2\mu(\lambda_1) \geq \varepsilon^{p-1} (1 - \delta |x|^{\mu(\lambda_2) - q\mu(\lambda_1) + 1 - \varepsilon}), \\ P_{\lambda_2}(q\mu(\lambda_1) - 1 + \varepsilon) \geq -C\varepsilon^{q-1} |x|^\varepsilon |\ln |x||^q. \end{cases} \quad (4.58)$$

Since $q\mu(\lambda_1) - 1 + \varepsilon < \mu(\lambda_2)$, for sufficiently small ε , we have

$$P_{\lambda_2}(q\mu(\lambda_1) - 1 + \varepsilon) > 0.$$

Moreover, $\mu(\lambda_2) - q\mu(\lambda_1) + 1 - \varepsilon > 0$, and $N - 2 - 2\mu(\lambda_1) > 0$.

Therefore, choosing ε sufficiently small proves that (u, v) is a supersolution in D_{λ_1, λ_2} .

The proof for E_{λ_1, λ_2} is completely analogous. □

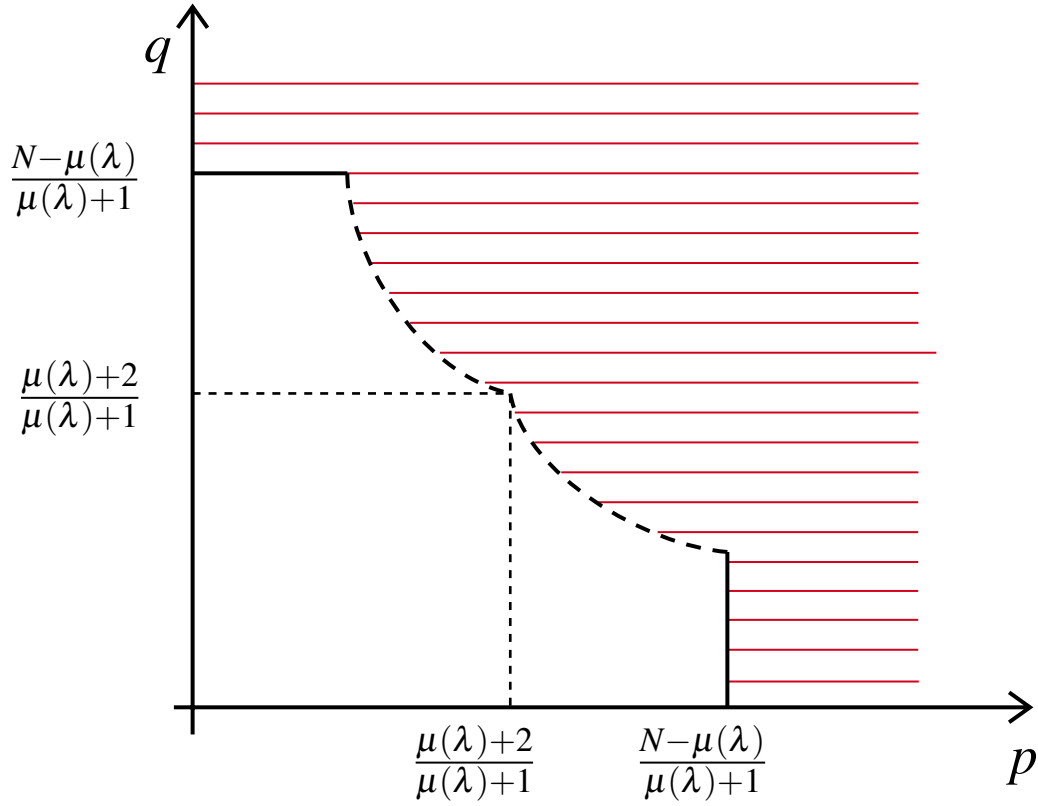


Figure 4.2: $f = |\nabla v|^p, g = |\nabla u|^q$ when $\lambda_1 = \lambda_2 =: \lambda$.

An analogue of Corollary 1.6 cannot be established because Proposition 1.1 depends on the monotonicity of the functions σ^p and σ^q for $\sigma \geq 0$, which does not hold in the case of gradient terms. Moreover, since no comparison principle is available for (4.37), to the best of the authors' knowledge, the existence of supersolutions does not generally guarantee the existence of actual solutions.

1.3 The system with mixed nonlinearities

Our aim in this subsection is to analyze the system

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + v^p & \text{in } \Omega_T, \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + |\nabla u|^q & \text{in } \Omega_T, \\ u(x, t) \geq 0, \quad v(x, t) \geq 0 & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (4.59)$$

with $p, q > 1$.

As in Subsections 1.1 and 1.2, we begin with a basic nonexistence statement.

Proposition 1.10. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$. Suppose that either*

$$p \geq \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}, \quad (4.60)$$

where $\mu(\lambda_i)$, $i = 1, 2$, is defined in (1.5). Then every nonnegative very weak supersolution (u, v) of system (4.59) is identically $(0, 0)$.

Proof. If $p \geq \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}$, the nonexistence follows directly. Assume now that $q \geq \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}$. By Remark 4.2, we have

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-\mu(\lambda_2)} dx dt < +\infty. \quad (4.61)$$

Using (4.38), we deduce that $q \in (1, N)$ and $\mu(\lambda_2)/q \in \left(0, \frac{N-q}{q}\right)$. Therefore,

$$\int_{t_1}^{t_2} \int_{B_r(0)} u^q |x|^{-q - \mu(\lambda_2)} dx dt < +\infty. \quad (4.62)$$

we get

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-q\mu(\lambda_1) - q - \mu(\lambda_2)} dx dt < +\infty. \quad (4.63)$$

Following the same contradiction argument as in the proof of Proposition 1.3, we conclude that such an integral cannot be finite. Hence $u \equiv 0$ and $v \equiv 0$. $\square$

Using the same iterative scheme employed in Subsection 1.2, we obtain the following nonexistence result.

Theorem 1.11. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$ and assume that*

$$\frac{p(q+2)}{2p+1} > 1 + \frac{1}{1+\mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)+1}{q+2} > 1 + \frac{1}{\mu(\lambda_2)}, \quad (4.64)$$

where $\mu(\lambda_i)$, $i = 1, 2$, is given by (1.5). Then every nonnegative very weak supersolution (u, v) of (4.59) is identically $(0, 0)$.

Proof. Let (u, v) be a nonnegative very weak supersolution of (4.59) with (p, q) satisfying (4.64). We show that $u \equiv 0$ and $v \equiv 0$. By Proposition 1.10, we may assume that

$$p < \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)} \quad \text{and} \quad q < \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}.$$

Fix $B_r(0) \subset \subset \Omega$ and $t_1 < t_2$ in $(0, T)$, and define

$$a_0 := 2 + \mu(\lambda_2) \quad \text{and} \quad b_0 := 2 + \mu(\lambda_1) \quad (4.65)$$

and for any $n \in \mathbb{N}$, $n \geq 1$

$$\begin{cases} a_n := \frac{N}{p'} + \frac{b_{n-1}-2}{p} - \frac{1}{n}, \\ b_n := 1 + \frac{N}{q'} + \frac{a_{n-1}-2}{q} - \frac{1}{n}. \end{cases} \quad (4.66)$$

We claim that, for all n , we have

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty \quad \text{and} \quad \int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty. \quad (4.67)$$

We proceed by induction. As in Proposition 1.10, the statement is valid for a_0 and b_0 . Assume now that (4.67) holds for the index $n-1$ with $n \geq 1$, we will show that it also holds for n .

So ,

$$\int_{t_1}^{t_2} \int_{B_r(0)} v^p |x|^{-b_{n-1}+2} dx dt < +\infty. \quad (4.68)$$

By applying Hölder's inequality, we deduce that

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt \\
& \leq (t_2 - t_1)^{\frac{1}{p'}} \left(\int_{t_1}^{t_2} \int_{B_r} v^p |x|^{-b_{n-1}+2} dx \right)^{\frac{1}{p}} \left(\int_{B_r(0)} |x|^{(b_{n-1}-2-a_n)p'-b_{n-1}+2} dx \right)^{\frac{1}{p'}}.
\end{aligned} \tag{4.69}$$

Hence if

$$(a_n - b_{n-1} + 2)p' + b_{n-1} - 2 < N, \tag{4.70}$$

or, in equivalent form

$$a_n < \frac{N}{p'} + \frac{b_{n-1} - 2}{p}. \tag{4.71}$$

Then

$$\int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty. \tag{4.72}$$

Similarly,

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-a_{n-1}+2} dx dt < +\infty. \tag{4.73}$$

If $a_{n-1} - 2 \geq N - q$, proceeding as in the previous arguments, we deduce that $u \equiv 0$ and $v \equiv 0$. Conversely, if $a_{n-1} - 2 < N - q$, then,

$$\int_{t_1}^{t_2} \int_{B_r(0)} |u|^q |x|^{-a_{n-1}+2-q} dx dt < +\infty. \tag{4.74}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt \\
& \leq (t_2 - t_1)^{\frac{1}{q'}} \left(\int_{t_1}^{t_2} \int_{B_r} v^q |x|^{-a_{n-1}+2-q} dx \right)^{\frac{1}{p}} \left(\int_{B_r(0)} |x|^{(a_{n-1}-1-b_n)q'-a_{n-1}+2} dx \right)^{\frac{1}{q'}}.
\end{aligned} \tag{4.75}$$

Consequently, if $(b_n - a_{n-1} + 1)q' + a_{n-1} - 2 < N$, equivalently, $b_n < 1 + \frac{N}{q'} + \frac{a_{n-1}-2}{q}$, then

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty. \tag{4.76}$$

Thus, by (4.66), the claim (4.67) is established.

Moreover, we have shown that either $u \equiv 0$ and $v \equiv 0$ or $a_n < N$ and $b_n < N$. Therefore, we may assume that the sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are bounded. A straightforward computation then yields

$$\bar{a} := \limsup_{n \rightarrow \infty} a_n = N + 1 - \frac{2p+1}{pq-1}, \quad \text{and} \quad \bar{b} := \limsup_{n \rightarrow \infty} b_n = N - \frac{q+2}{pq-1}. \quad (4.77)$$

It then follows from (4.67) that either $u \equiv 0$ and $v \equiv 0$, or, for any $\varepsilon > 0$,

$$\int_{B_r} |x|^{-\mu(\lambda_2) - \bar{a} + \varepsilon} < +\infty \quad \text{and} \quad \int_{B_r} |x|^{-\mu(\lambda_1) - \bar{b} + \varepsilon} < +\infty, \quad (4.78)$$

which is equivalent to

$$\mu(\lambda_2) + \bar{a} \leq N \quad \text{and} \quad \mu(\lambda_1) + \bar{b} \leq N, \quad (4.79)$$

which that contradicts (4.64). $\square$

Existence of Supersolutions.

Analogously to Subsections 1.1 and 1.2, we establish the existence of supersolutions.

We begin by defining

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) : \right. \\ \left. \frac{p(q+2)}{2p+1} < 1 + \frac{1}{1 + \mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)+1}{q+2} < 1 + \frac{1}{\mu(\lambda_2)} \right\}. \quad (4.80)$$

We aim to show the existence of a non-trivial supersolution for every $(p, q) \in F_{\lambda_1, \lambda_2}$. Observe that

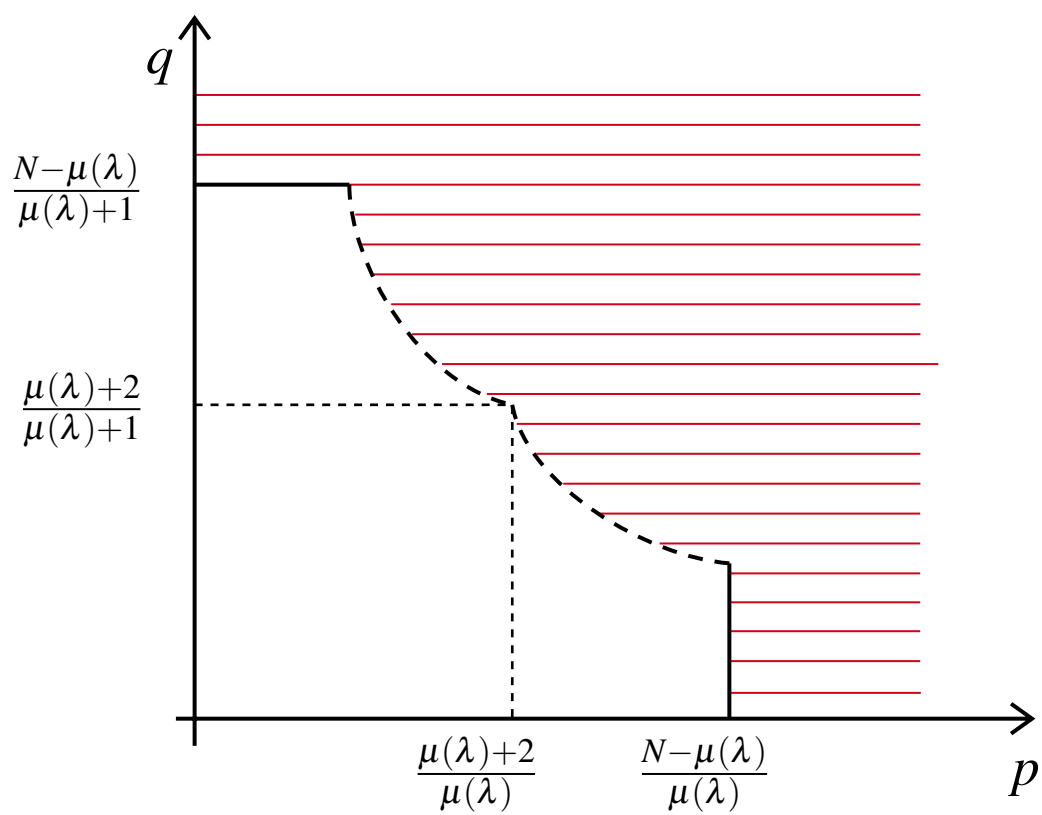


Figure 4.3: $f = v^p, g = |\nabla u|^q$ for $\lambda_1 = \lambda_2 =: \lambda$.

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.81)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(\frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(1, \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.82)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : (p, q) \in \left(1, \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}\right) \times \left(\frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.83)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, q \in \left(\frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)}, \frac{N - \mu(\lambda_2)}{1 + \mu(\lambda_1)}\right) \right\}, \quad (4.84)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, \frac{N - \mu(\lambda_1)}{1 + \mu(\lambda_2)}\right), q = \frac{2 + \mu(\lambda_2)}{1 + \mu(\lambda_1)} \right\}. \quad (4.85)$$

Following the same approach as in Subsection 1.1, we can establish the following existence result.

Proposition 1.12. *For any $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the pair (u, v) defined by*

$$\begin{aligned} \varepsilon(|x|^{-\eta}, |x|^{-\eta}), & \quad \gamma_i \in (\mu(\lambda_i), \mu^+(\lambda_i)), \quad i = 1, 2, & \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-(p\mu(\lambda_2)-2)}, |x|^{-\mu(\lambda_2)} - \delta|x|^{-(pq\mu(\lambda_2)-2q-1)}), & & \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(pq\mu(\lambda_1)-p-2)}, |x|^{-(q\mu(\lambda_1)-1)}), & & \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \\ \varepsilon(-|x|^{-\mu(\lambda_1)} \ln(|x|/r), |x|^{-\mu(\lambda_2)} - \delta|x|^{-(q\mu(\lambda_1)-1+\varepsilon)}), & & \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(p\mu(\lambda_2)-2+\varepsilon)}, -|x|^{-\mu(\lambda_2)} \ln(|x|/r)), & & \text{if } (p, q) \in E_{\lambda_1, \lambda_2}, \end{aligned}$$

defines a positive supersolution of (4.59) in $B_r(0)$.

2 The Cauchy Problem: Determination of the Fujita Exponent

Consider the system

$$\begin{cases} u_t - \Delta u = \lambda_1 \frac{u}{|x|^2} + v^p, & \text{in } \mathbb{R}^N \times (0, T), \\ v_t - \Delta v = \lambda_2 \frac{v}{|x|^2} + u^q, & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, t) > 0, \quad v(x, t) > 0, & \text{in } \mathbb{R}^N \times (0, T), \\ u(0, x) = u_0(x), & \text{in } \mathbb{R}^N, \\ v(0, x) = v_0(x), & \text{in } \mathbb{R}^N. \end{cases} \quad (4.86)$$

In this section, we investigate the finite-time blow-up or global existence of positive solutions to (4.86), depending on the values of p and q .

To begin, we specify the precise class of solutions for (4.86) that will be considered in our analysis.

Definition 2.1. Let $u_0, v_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$. We say that a pair (u, v) is a very weak supersolution (resp. subsolution) of (4.86) if

$$u, v \in C([0, T], L^1_{\text{loc}}(\mathbb{R}^N)), \quad \frac{u}{|x|^2}, \frac{v}{|x|^2} \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T)), \quad u^q, v^p \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T)),$$

and for every test function $\phi, \psi \in C_c^\infty(\mathbb{R}^N \times [0, T])$ with $\phi, \psi \geq 0$, the following inequalities hold:

$$\begin{cases} - \int_{\mathbb{R}^N} u_0(x) \phi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta \phi) u dx dt \geq (\leq) \int_0^T \int_{\mathbb{R}^N} (\lambda_1 \frac{u}{|x|^2} + v^p) \phi dx dt, \\ - \int_{\mathbb{R}^N} v_0(x) \psi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta \psi) v dx dt \geq (\leq) \int_0^T \int_{\mathbb{R}^N} (\lambda_2 \frac{v}{|x|^2} + u^q) \psi dx dt. \end{cases} \quad (4.87)$$

Observe that $u(\cdot, 0) \rightarrow u_0$ (resp. $v(\cdot, 0) \rightarrow v_0$) strongly in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \rightarrow 0$. If (u, v) is simultaneously a supersolution and a subsolution, we then refer to (u, v) as a solution of (4.86).

As very weak solutions of (4.86) can be unbounded, it is necessary to clarify the notion of finite-time blow-up, as discussed in [4].

Definition 2.2. A positive very weak solution (u, v) of system (4.86) blow up in finite time if, for every $r > 0$, there exists a time $T^* > 0$ such that

$$\lim_{t \rightarrow (T^*)^-} \int_{B_r(0)} |x|^{-\mu(\lambda_1)} u(x, t) dx = +\infty \quad \text{or} \quad \lim_{t \rightarrow (T^*)^-} \int_{B_r(0)} |x|^{-\mu(\lambda_2)} v(x, t) dx = +\infty. \quad (4.88)$$

Theorem 2.3. If $u_0, v_0 \in L^1_{loc}(\mathbb{R}^N)$ are non-negative,

$$\frac{p(q+1)}{p+1} \leq 1 + \frac{2}{N - \mu(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} \leq 1 + \frac{2}{N - \mu(\lambda_2)}, \quad (4.89)$$

then every positive very weak solution (u, v) of (4.86) blow-up as defined in Definition 2.2.

Proof. Let (u, v) be a positive very weak solution to the system (4.86) and assume, by contradiction, that there exists some $r > 0$ such that, for all $t > 0$,

$$\int_{B_r(0)} |x|^{-\mu(\lambda_1)} u(x, t) dx < +\infty \quad \text{and} \quad \int_{B_r(0)} |x|^{-\mu(\lambda_2)} v(x, t) dx < +\infty. \quad (4.90)$$

Due to the regularity of u and v , we infer that (4.90) holds for any radius $r > 0$ and any fixed $t > 0$.

For clarity, we structure the argument in three steps.

Step 1. For $\varepsilon > 0$ define $\rho_{\varepsilon, i}(x) := (|x| + \varepsilon)^{-\mu(\lambda_i)}$, where $\mu(\lambda_i)$ is as in (1.5). Let $\phi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$ be non-negative. Testing the first equation of (2.1) with $\tilde{\phi}_{\varepsilon, 1} := \rho_{\varepsilon, 1} \phi$ gives

$$\begin{aligned} - \int_{\mathbb{R}^N} u_0(x) \tilde{\phi}_{\varepsilon, 1}(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} \left(-(\tilde{\phi}_{\varepsilon, 1})_t - \Delta \tilde{\phi}_{\varepsilon, 1} - \frac{\lambda_1 \tilde{\phi}_{\varepsilon, 1}}{|x|^2} \right) u dx dt \\ = \int_0^{+\infty} \int_{\mathbb{R}^N} v^p \tilde{\phi}_{\varepsilon, 1} dx dt. \end{aligned} \quad (4.91)$$

A straightforward computation using (1.5) yields

$$-\Delta \tilde{\phi}_{\varepsilon, 1} - \frac{\lambda_1}{|x|^2} \tilde{\phi}_{\varepsilon, 1} = -\Delta \phi \rho_{\varepsilon, 1} - 2\mu(\lambda_1) \nabla \phi \cdot \frac{x}{|x|} (|x| + \varepsilon)^{-\mu(\lambda_1)-1}. \quad (4.92)$$

Thus, applying the Dominated Convergence Theorem together with (4.90) and the

initial condition $u_0 = u(\cdot, 0)$, we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \left[- \int_{\mathbb{R}^N} u_0(x) \tilde{\phi}_{\varepsilon,1}(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} \left(-(\tilde{\phi}_{\varepsilon,1})_t - \Delta \tilde{\phi}_{\varepsilon,1} - \frac{\lambda_1 \tilde{\phi}_{\varepsilon,1}}{|x|^2} \right) u dx dt \right] \\ &= - \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)} u_0(x) \phi(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} \left(-|x|^{-\mu(\lambda_1)} \phi_t - |x|^{\mu(\lambda_1)} \operatorname{div}(|x|^{-2\mu(\lambda_1)} \nabla \phi) \right) u dx dt, \end{aligned} \quad (4.93)$$

where

$$-\Delta \phi |x|^{-\mu(\lambda_1)} - 2\mu(\lambda_1) \nabla \phi \cdot \frac{x}{|x|} |x|^{-\mu(\lambda_1)-1} = -|x|^{\mu(\lambda_1)} \operatorname{div}(|x|^{-2\mu(\lambda_1)} \nabla \phi). \quad (4.94)$$

Using Fatou's Lemma and (4.90), we deduce

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)} v^p \phi dx dt < +\infty. \quad (4.95)$$

Let us set

$$\tilde{u}(x, t) := |x|^{\mu(\lambda_1)} u(x, t), \quad \tilde{v}(x, t) := |x|^{\mu(\lambda_2)} v(x, t). \quad (4.96)$$

Then, for any non-negative $\phi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$, we have

$$\begin{aligned} & - \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)} u_0(x) \phi(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} \left(-|x|^{-2\mu(\lambda_1)} \phi_t - \operatorname{div}(|x|^{-2\mu(\lambda_1)} \nabla \phi) \right) \tilde{u} dx dt \\ &= \int_0^{+\infty} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_2)p - \mu(\lambda_1)} \tilde{v}^p \phi dx dt, \end{aligned} \quad (4.97)$$

and similarly

$$\begin{aligned} & - \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_2)} v_0(x) \phi(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} \left(-|x|^{-2\mu(\lambda_2)} \phi_t - \operatorname{div}(|x|^{-2\mu(\lambda_2)} \nabla \phi) \right) \tilde{v} dx dt \\ &= \int_0^{+\infty} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)q - \mu(\lambda_2)} \tilde{u}^q \phi dx dt. \end{aligned} \quad (4.98)$$

In particular, for any $R > 0$,

$$\int_0^R \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_2)p - \mu(\lambda_1)} \tilde{v}^p dx dt < +\infty, \quad (4.99)$$

$$\int_0^R \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)q - \mu(\lambda_2)} \tilde{u}^q dx dt < +\infty. \quad (4.100)$$

Step 2. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ outside $B_2(0)$. For $R \geq 1$, define

$$\psi_R(x, t) := \phi \left(\frac{t^2 + |x|^4}{R^2} \right), \quad \mathcal{Q}_R := \{(x, t) \in \mathbb{R}^N \times (0, \infty) : R^2 \leq t^2 + |x|^4 \leq 2R^2\}. \quad (4.101)$$

Let p', q' be the conjugate exponents of p and q , and choose $m \in \mathbb{N}$ such that $m > \max\{p', q'\}$. Define

$$A(R) := \int_0^{2R} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_2)p - \mu(\lambda_1)\tilde{v}^p} \psi_R^m dx dt, \quad (4.102)$$

$$B(R) := \int_0^{2R} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda_1)q - \mu(\lambda_2)\tilde{u}^q} \psi_R^m dx dt. \quad (4.103)$$

We assert that

$$\lim_{R \rightarrow +\infty} A(R) < +\infty \quad \text{or} \quad \lim_{R \rightarrow +\infty} B(R) < +\infty. \quad (4.104)$$

Testing (4.97) with ψ_R^m , using (4.99), the properties of ψ_R , and Hölder's inequality, we obtain the estimates leading to

$$A(R) \leq C_1 R^{\frac{q(N-\mu(\lambda_1))+\mu(\lambda_2)-N-2}{2q}} B(R)^{1/q}, \quad (4.105)$$

$$B(R) \leq C_2 R^{\frac{p(N-\mu(\lambda_2))+\mu(\lambda_1)-N-2}{2p}} A(R)^{1/p}, \quad (4.106)$$

for some constants $C_1, C_2 > 0$ independent of R .

From these inequalities, one deduces

$$A(R) \leq C_1^{\frac{pq}{pq-1}} C_2^{\frac{p}{pq-1}} R^{\frac{N-\mu(\lambda_1)}{2} - \frac{p+1}{pq-1}}, \quad (4.107)$$

$$B(R) \leq C_1^{\frac{q}{pq-1}} C_2^{\frac{pq}{pq-1}} R^{\frac{N-\mu(\lambda_2)}{2} - \frac{q+1}{pq-1}}. \quad (4.108)$$

Passing to the limit as $R \rightarrow +\infty$ and using (4.89) proves (4.104). If either inequality in (4.89) is strict, we conclude

$$\lim_{R \rightarrow +\infty} A(R) = 0 \quad \text{or} \quad \lim_{R \rightarrow +\infty} B(R) = 0, \quad (4.109)$$

which contradicts the positivity of v or u , respectively.

Step 3. In the remaining case, assume that neither inequality in (4.89) is strict, i.e.,

$$\frac{2(p+1)}{pq-1} \leq N - \mu(\lambda_1), \quad \frac{2(q+1)}{pq-1} = N - \mu(\lambda_2), \quad (4.110)$$

or

$$\frac{2(q+1)}{pq-1} \leq N - \mu(\lambda_2), \quad \frac{2(p+1)}{pq-1} = N - \mu(\lambda_1). \quad (4.111)$$

Suppose (4.110) holds. By the previous estimates, there exist constants $C_3, C_4 > 0$ such that

$$\int_0^{2R} \int_{\mathbb{R}^N} |x|^{-2\mu(\lambda_1)} |(\psi_R^m)_t| \tilde{u} dx dt \leq C_3 \int_{Q_R} |x|^{-\mu(\lambda_1)q - \mu(\lambda_2)} \tilde{u}^q dx dt, \quad (4.112)$$

$$\int_0^{2R} \int_{\mathbb{R}^N} |x|^{-2\mu(\lambda_1)} |\operatorname{div}(|x|^{-2\mu(\lambda_1)} \nabla \psi_R^m)| \tilde{u} dx dt \leq C_4 \int_{Q_R} |x|^{-\mu(\lambda_1)q - \mu(\lambda_2)} \tilde{u}^q dx dt. \quad (4.113)$$

Using (4.104) and letting $R \rightarrow +\infty$, we obtain

$$\lim_{R \rightarrow +\infty} A(R) = 0, \quad (4.114)$$

which is a contradiction with the positivity of v . A similar argument for (4.111) yields

$$\lim_{R \rightarrow +\infty} B(R) = 0, \quad (4.115)$$

contradicting the positivity of u . This completes the proof. $\square$

In the next Proposition, we prove the existence of a family of global supersolution to (4.86) on $\mathbb{R}^N \times (0, +\infty)$ for suitable (p, q) .

Proposition 2.4. *Assume that*

$$\frac{p(q+1)}{p+1} > 1 + \frac{2}{N - \mu(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} > 1 + \frac{2}{N - \mu(\lambda_2)}, \quad (4.116)$$

$$p < \frac{2 + \mu(\lambda_1)}{\mu(\lambda_2)}, \quad \text{and} \quad q < \frac{2 + \mu(\lambda_2)}{\mu(\lambda_1)}. \quad (4.117)$$

For any $\varepsilon > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and $a > 0$, let

$$u_{\varepsilon, \gamma_1, a}(x, t) := \varepsilon(1+t)^{\frac{p+1}{1-pq} - \gamma_1} |x|^{-\gamma_1} e^{-a \frac{|x|^2}{4(1+t)}}, \quad (4.118)$$

$$v_{\varepsilon, \gamma_2, a}(x, t) := \varepsilon(1+t)^{\frac{q+1}{1-pq} - \gamma_2} |x|^{-\gamma_2} e^{-a \frac{|x|^2}{4(1+t)}}. \quad (4.119)$$

There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\gamma_1 \in (\mu(\lambda_1), \mu(\lambda_1) + \varepsilon_0)$, $\gamma_2 \in (\mu(\lambda_2), \mu(\lambda_2) + \varepsilon_0)$, and $a \in (1 - \varepsilon_0, 1)$, the pair $(u_{\varepsilon, \gamma_1, a}, v_{\varepsilon, \gamma_2, a})$ is a positive supersolution of (4.86) in $\mathbb{R}^N \times (0, \infty)$.

Proof. We define the functions

$$u(x, t) := \varepsilon(1+t)^\alpha \psi((1+t)^\sigma r), \quad v(x, t) = \varepsilon(1+t)^\beta \phi((1+t)^\sigma r), \quad (4.120)$$

where $r = |x|$, and α, β, σ are parameters to be determined. Direct computation gives

$$u_t = \varepsilon \sigma (1+t)^{\alpha+\sigma-1} r \psi'((1+t)^\sigma r) + \varepsilon \alpha (1+t)^{\alpha-1} \psi((1+t)^\sigma r), \quad (4.121)$$

$$\Delta u = \varepsilon (1+t)^{\alpha+2\sigma} \psi''((1+t)^\sigma r) + \varepsilon \sigma (1+t)^{\alpha+\sigma} \frac{N-1}{r} \psi'((1+t)^\sigma r). \quad (4.122)$$

Choosing $\sigma = -\frac{1}{2}$ and introducing the new variable $s = (1+t)^\sigma r$, the system (4.86) reduces to

$$\begin{cases} (1+t)^{\alpha-1} \left[-\psi''(s) - \left(\frac{N-1}{s} + \frac{s}{2} \right) \psi'(s) - \left(-\alpha + \frac{\lambda_1}{s^2} \right) \psi(s) \right] \geq \varepsilon^{p-1} (1+t)^{\beta p} \phi^p(s), \\ (1+t)^{\beta-1} \left[-\phi''(s) - \left(\frac{N-1}{s} + \frac{s}{2} \right) \phi'(s) - \left(-\beta + \frac{\lambda_1}{s^2} \right) \phi(s) \right] \geq \varepsilon^{q-1} (1+t)^{\alpha q} \psi^q(s). \end{cases} \quad (4.123)$$

By setting

$$\alpha = \frac{p+1}{1-pq}, \quad \beta = \frac{q+1}{1-pq}, \quad (4.124)$$

the inequalities (4.123) can be rewritten as

$$\begin{cases} -\psi''(s) - \left(\frac{N-1}{s} + \frac{s}{2} \right) \psi'(s) - \left(-\frac{p+1}{1-pq} + \frac{\lambda_1}{s^2} \right) \psi(s) - \phi^p(s) \geq 0, \\ -\phi''(s) - \left(\frac{N-1}{s} + \frac{s}{2} \right) \phi'(s) - \left(-\frac{q+1}{1-pq} + \frac{\lambda_1}{s^2} \right) \phi(s) - \psi^q(s) \geq 0. \end{cases} \quad (4.125)$$

Next, we choose

$$\psi(s) := \varepsilon s^{-\gamma_1} e^{-a \frac{s^2}{4}}, \quad \phi(s) := \varepsilon s^{-\gamma_2} e^{-a \frac{s^2}{4}}, \quad (4.126)$$

with positive constants $\gamma_1, \gamma_2, a, \varepsilon$ to be specified. A straightforward computation shows that the first inequality becomes

$$\begin{aligned} \left[\frac{a}{4} - \frac{a^2}{4} \right] s^2 - [\gamma_1^2 - (N-2)\gamma_1 + \lambda_1] s^{-2} - a\gamma_1 + \frac{\gamma_1}{2} + \frac{a}{2}N + \frac{p+1}{1-pq} \\ - \varepsilon^{p-1} s^{-\gamma_2 p + \gamma_1} e^{(-p+1)a \frac{s^2}{4}} \geq 0. \end{aligned} \quad (4.127)$$

By taking $a \leq \frac{1}{4}$, the coefficient of s^2 is non-negative. Let us define

$$G(a, \gamma_1) := -a\gamma_1 + \frac{\gamma_1}{2} + \frac{a}{2}N + \frac{p+1}{1-pq}. \quad (4.128)$$

From (4.116), we have $G(1, \mu(\lambda_1)) > 0$. Hence, choosing γ_1 sufficiently close to $\mu(\lambda_1)$ and a slightly less than 1, we ensure $G(a, \gamma_1) > 0$. Moreover, by (1.5), for $\gamma_1 > \mu(\lambda_1)$ near $\mu(\lambda_1)$, the term $\gamma_1^2 - (N-2)\gamma_1 + \lambda_1 < 0$. Similarly, (4.117) implies $\gamma_2 p - \gamma_1 < 2$ for γ_2 close to $\mu(\lambda_2)$. Finally, selecting $\varepsilon > 0$ sufficiently small guarantees that the first inequality in (4.125) holds for all $s \geq 0$. The second inequality can be verified analogously. $\square$

Building upon Proposition 1.12 and employing arguments similar to those in [4, Sections 6.3–6.4], we can deduce the following existence result.

Corollary 2.5. *Let the condition (4.117) be satisfied. Consider $u_0, v_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, and assume that*

$$u_0 \leq u_{\varepsilon, \gamma_1, a}, \quad v_0 \leq v_{\varepsilon, \gamma_2, a},$$

where $u_{\varepsilon, \gamma_1, a}$ and $v_{\varepsilon, \gamma_2, a}$ are the supersolutions constructed in Proposition 2.4. Then there exists a positive entropy solution (u, v) of system (4.3) associated with the initial data (u_0, v_0) .

Chapter 5

Fractional System

In this chapter, we investigate the existence and nonexistence of positive solutions for the following fractional system

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + f(v, \nabla v), & \text{in } \Omega_T, \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + g(u, \nabla u), & \text{in } \Omega_T, \end{cases} \quad (5.1)$$

where $\Omega \subseteq \mathbb{R}^N$ with $N \geq 3$ is either a bounded Lipschitz domain containing the origin, or the whole space $\mathbb{R}^N$. We denote $\Omega_T := \Omega \times (0, T)$ for some $T > 0$.

The main focus of our analysis is on several classes of nonlinear terms, namely:

$$\text{i) } f = v^p, g = u^q, \quad \text{ii) } f = |\nabla v|^p, g = |\nabla u|^q, \quad \text{iii) } f = v^p, g = |\nabla u|^q. \quad (5.2)$$

Furthermore, in the case $f = v^p$ and $g = u^q$, we also address the question of global existence versus nonexistence of solutions for the corresponding fractional Cauchy problem.

1 Potential-potential nonlinearities

In this subsection we consider the fractional counterpart of the system with power-type nonlinearities. More precisely, we take $f = v^p, g = u^q$, with exponents $p, q > 1$.

The system under consideration becomes

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + v^p, & \text{in } \Omega_T, \\ v_t + (-\Delta)^s v = \lambda \frac{v}{|x|^{2s}} + u^q, & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(0, x) = u_0(x), & \text{in } \Omega, \\ v(0, x) = v_0(x), & \text{in } \Omega, \end{cases} \quad (5.3)$$

where $0 < s < 1$.

Our first goal is to establish the existence of entropy solutions to (5.3). To this end, we assume the existence of a very weak supersolution for the same system with general nonlinearities of the form described in (5.2).

The construction is based on an approximation scheme combined with a comparison argument. This approach is justified by the monotonicity of the mappings

$$\sigma \mapsto \sigma^p, \quad \sigma \mapsto \sigma^q \quad \text{on } [0, +\infty).$$

Moreover, using fractional regularity estimates together with arguments analogous to those developed in [4, Lemma 2.3], we obtain the following result.

Proposition 1.1. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s}]$ and let (u, v) be a nonnegative very weak supersolution of (5.3). Assume that $u_0, v_0 \in L^1_{\text{loc}}(\Omega)$.*

Then, for every open set $\Omega' \Subset \Omega$, there exist two monotone sequences $\{\widehat{u}_n\}_{n \in \mathbb{N}}$ and $\{\widehat{v}_n\}_{n \in \mathbb{N}}$ such that $u_n \leq u$, $\widehat{v}_n \leq v$ for all n . Moreover,

$$\widehat{u}_n, \widehat{v}_n \in L^2((0, T); H^{2s}(\Omega')) \cap L^\infty((0, T); H^s_0(\Omega')) \cap C([0, T]; L^2(\Omega')),$$

and

$$(\widehat{u}_n)_t, (\widehat{v}_n)_t \in L^2((0, T); L^2(\Omega')).$$

Furthermore,

$$\widehat{u}_n \rightarrow \widetilde{u}, \quad \widehat{v}_n \rightarrow \widetilde{v} \quad \text{in } L^r((0, T); W^{s,r}_0(\Omega')) \quad \text{for every } r < \frac{N+2s}{N+1},$$

where $(\widetilde{u}, \widetilde{v})$ is an entropy solution of (5.3) in $\Omega' \times (0, T)$.

In addition, $(\hat{u}_n, \hat{v}_n)$ solves the approximate system

$$\begin{cases} (\hat{u}_n)_t + (-\Delta)^s \hat{u}_n = \lambda_1 \frac{\hat{u}_{n-1}}{|x|^{2s + \frac{1}{n}}} + \hat{v}_{n-1}^p, & \text{in } \Omega'_T, \\ (\hat{v}_n)_t + (-\Delta)^s \hat{v}_n = \lambda_2 \frac{\hat{v}_{n-1}}{|x|^{2s + \frac{1}{n}}} + \hat{u}_{n-1}^q, & \text{in } \Omega'_T, \\ \hat{u}_n = \hat{v}_n = 0, & \text{in } (\mathbb{R}^N \setminus \Omega') \times (0, T), \\ \hat{u}_n(0, x) = T_n(u_0(x)), \quad \hat{v}_n(0, x) = T_n(v_0(x)), & \text{in } \Omega', \end{cases} \quad (5.4)$$

for all $n \geq 1$, while the initial step is given by

$$\begin{cases} (\hat{u}_0)_t + (-\Delta)^s \hat{u}_0 = 0, \\ (\hat{v}_0)_t + (-\Delta)^s \hat{v}_0 = 0, \\ \hat{u}_0 = \hat{v}_0 = 0, \\ \hat{u}_0(0, x) = T_1(u_0(x)), \quad \hat{v}_0(0, x) = T_1(v_0(x)). \end{cases} \quad (5.5)$$

Remark 1.2. The previous result allows us to work with entropy solutions lying below a given very weak supersolution, which is a key ingredient in proving nonexistence results in the fractional setting.

We are now in a position to derive the principal nonexistence result in the fractional framework. We begin with a first elementary case.

Proposition 1.3. Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$. Assume that

$$p \geq \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu_s(\lambda_2)}{\mu_s(\lambda_1)}, \quad (5.6)$$

where $\mu_s(\lambda_i)$, $i = 1, 2$, is defined as in (1.21). Then any nonnegative very weak supersolution (u, v) of (5.3) is necessarily trivial, that is, $(u, v) \equiv (0, 0)$.

Proof. Let $r > 0$ be such that $B_r(0) \Subset \Omega$, and let (u, v) be a nonnegative very weak supersolution of (5.3). In particular, we have $u^p, v^q \in L^1_{\text{loc}}(\Omega_T)$.

By the lower estimates for supersolutions, it follows that there exist $t_1 < t_2$ and a constant $C > 0$ such that $u(x, t) \geq C|x|^{-\mu_s(\lambda_1)}$ and $v(x, t) \geq C|x|^{-\mu_s(\lambda_2)}$ almost everywhere in $B_r(0) \times (t_1, t_2)$. Consequently,

$$u^p(x, t) \geq C|x|^{-p\mu_s(\lambda_1)}, \quad v^q(x, t) \geq C|x|^{-q\mu_s(\lambda_2)}$$

. Using Remark 7.3, we infer that

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu_s(\lambda_1)} v^p dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu_s(\lambda_2)} u^q dx dt < +\infty.$$

Combining these estimates with the previous lower bounds yields

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu_s(\lambda_1) - p\mu_s(\lambda_2)} dx dt < +\infty,$$

and

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\mu_s(\lambda_2) - q\mu_s(\lambda_1)} dx dt < +\infty.$$

These integrability conditions imply

$$p < \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \quad q < \frac{N - \mu_s(\lambda_2)}{\mu_s(\lambda_1)},$$

which contradicts (5.6). The conclusion follows. $\square$

Observe that the above condition is not optimal. By combining Proposition 1.1 with the quantitative singular behaviour of very weak supersolutions, one can derive the exact critical curves separating the existence and nonexistence regimes. We make this statement precise in the next result.

Theorem 1.4. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$ and assume that*

$$\frac{p(q+1)}{p+1} \geq 1 + \frac{2s}{\mu_s(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} \geq 1 + \frac{2s}{\mu_s(\lambda_2)}, \quad (5.7)$$

where $\mu(\lambda_i)$, $i = 1, 2$, is defined in (1.21). Then the fractional system (5.3) admits no nontrivial nonnegative very weak supersolution. In other words, $(u, v) \equiv (0, 0)$ is the only possible solution.

Proof. We argue by contradiction. Assume that (u, v) is a nonnegative very weak supersolution of (5.3).

Let $R > 0$ be such that $B_R(0) \subset \Omega$. By Proposition 1.1, we may construct an entropy solution in $B_R(0) \times (0, T)$, still denoted by (u, v) . Thus (u, v) satisfies

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + v^p, & \text{in } B_R(0) \times (0, T), \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + u^q, & \text{in } B_R(0) \times (0, T), \\ u, v > 0, & \text{in } B_R(0) \times (0, T), \\ u = v = 0, & \text{in } (\mathbb{R}^N \setminus B_R(0)) \times (0, T). \end{cases} \quad (5.8)$$

From the quantitative lower bounds for supersolutions, there exist $0 < r < R$, $0 < T_1 < T_2 \leq T$, and $C > 0$ such that

$$u(x, t) \geq C|x|^{-\mu_s(\lambda_1)}, \quad v(x, t) \geq C|x|^{-\mu_s(\lambda_2)} \quad \text{in } B_r(0) \times (T_1, T_2).$$

Let $\phi \in C_c^\infty(B_r(0))$. Using a standard approximation argument, we test the first equation with $\frac{\phi^2}{u}$. Combining this with the fractional Picone-type inequality (1.19), we obtain

$$\int_{T_1}^{T_2} \int_{B_r(0)} \frac{v^p}{u} \phi^2 dx dt \leq C \int_{T_1}^{T_2} \|\phi\|_{H^s(\mathbb{R}^N)}^2 dt + \int_{B_r(0)} |\ln(u(x, t_2))| \phi^2 dx. \quad (5.9)$$

A similar estimate holds for the second equation, yielding

$$C(T_1, T_2, u, v) \|\phi\|_{H^s(\mathbb{R}^N)}^2 \geq \int_{T_1}^{T_2} \int_{B_r(0)} \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \phi^2 dx dt. \quad (5.10)$$

Arguing as in the local case, we introduce

$$a := v u^{-1/p}, \quad \alpha := u^{\frac{pq-1}{p}},$$

and consider the function $K(a) = a^p + \frac{\alpha}{a}$. Minimizing K gives

$$\frac{v^p}{u} + \frac{u^q}{v} \geq C u^{\frac{pq-1}{p+1}}.$$

Using the lower bound on u , we deduce

$$\int_{T_1}^{T_2} \int_{B_r(0)} \left[\frac{v^p}{u} + \frac{u^q}{v} \right] \phi^2 dx dt \geq C \int_{B_r(0)} \frac{\phi^2}{|x|^{\mu_s(\lambda_1) \frac{pq-1}{p+1}}} dx. \quad (5.11)$$

A similar estimate involving v also holds.

Therefore, if

$$\mu_s(\lambda_1) \frac{pq-1}{p+1} > 2s \quad \text{or} \quad \mu_s(\lambda_2) \frac{pq-1}{q+1} > 2s,$$

we obtain a contradiction with the fractional Hardy inequality, which yields

$$\|\phi\|_{H^s(\mathbb{R}^N)}^2 \geq C \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^{2s}} dx.$$

This contradiction proves the result. $\square$

1.1 Existence of Supersolutions

We now prove that the nonexistence results obtained above are sharp by showing that positive supersolutions do exist whenever condition (5.7) fails. More precisely, we construct explicit positive radial supersolutions in the ball $B_r(0)$ for any $r > 0$. This construction immediately yields the existence of supersolutions in any bounded domain $\Omega \times (0, T)$.

Define

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right) \times \left(1, \frac{N - \mu_s(\lambda_2)}{\mu_s(\lambda_1)}\right) : \right. \\ \left. \frac{p(q+1)}{p+1} < 1 + \frac{2s}{\mu_s(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{2s}{\mu_s(\lambda_2)} \right\}. \quad (5.12)$$

We shall prove that for every $(p, q) \in F_{\lambda_1, \lambda_2}$ there exists a nontrivial supersolution. To this end, we decompose F_{λ_1, λ_2} into five regions:

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2s + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right) \times \left(1, \frac{2s + \mu_s(\lambda_2)}{\mu_s(\lambda_1)}\right) \right\}, \quad (5.13)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p \in \left(\frac{2s + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right), q \in \left(1, \frac{2s + \mu_s(\lambda_2)}{\mu_s(\lambda_1)}\right) \right\}, \quad (5.14)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p \in \left(1, \frac{2s + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right), q \in \left(\frac{2s + \mu_s(\lambda_2)}{\mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{\mu_s(\lambda_1)}\right) \right\}, \quad (5.15)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2s + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, q \in \left(\frac{2s + \mu_s(\lambda_2)}{\mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{\mu_s(\lambda_1)}\right) \right\}, \quad (5.16)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2s + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right), q = \frac{2s + \mu_s(\lambda_2)}{\mu_s(\lambda_1)} \right\}. \quad (5.17)$$

Proposition 1.5. *For any $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the pair (u, v) defined by*

$$\varepsilon(|x|^{-\gamma_1}, |x|^{-\gamma_2}), \quad \gamma_i \in (\mu(\lambda_i), \mu^+(\lambda_i)), \quad \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \quad (5.18)$$

$$\varepsilon\left(|x|^{-(p\mu(\lambda_2)-2s)}, |x|^{-\mu(\lambda_2)} - \delta|x|^{-(pq\mu(\lambda_2)-2qs-2s)}\right), \quad \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \quad (5.19)$$

$$\varepsilon\left(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(pq\mu(\lambda_1)-2ps-2s)}, |x|^{-(q\mu(\lambda_1)-2s)}\right), \quad \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \quad (5.20)$$

$$\varepsilon\left(-|x|^{-\mu(\lambda_1)} \ln(|x|/r), |x|^{-\mu(\lambda_2)} - \delta|x|^{-(q\mu(\lambda_1)-2s+\varepsilon)}\right), \quad \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \quad (5.21)$$

$$\varepsilon\left(|x|^{-\mu(\lambda_1)} - \delta|x|^{-(p\mu(\lambda_2)-2s+\varepsilon)}, -|x|^{-\mu(\lambda_2)} \ln(|x|/r)\right), \quad \text{if } (p, q) \in E_{\lambda_1, \lambda_2}, \quad (5.22)$$

is a positive supersolution of (5.3) for sufficiently small initial data.

Proof. **Case 1: Region A_{λ_1, λ_2} .** The verification reduces to checking

$$-P_{\lambda_1}(\gamma_1) \geq \varepsilon^{p-1}|x|^{\gamma_1+2s-p\gamma_2}, \quad -P_{\lambda_2}(\gamma_2) \geq \varepsilon^{q-1}|x|^{\gamma_2+2s-q\gamma_1}.$$

These inequalities hold for ε sufficiently small.

Case 2: Regions B_{λ_1, λ_2} and C_{λ_1, λ_2} . A direct computation shows that

$$p\mu(\lambda_2) - 2s \in (\mu(\lambda_1), \mu^+(\lambda_1)),$$

and the correction terms are of lower order. Choosing δ and ε sufficiently small ensures that the supersolution inequalities are satisfied.

Case 3: Regions D_{λ_1, λ_2} and E_{λ_1, λ_2} . In the critical case, logarithmic corrections are introduced. Using that

$$q\mu(\lambda_1) - 2s < \mu(\lambda_2),$$

and the positivity of the fractional Hardy operator, one verifies that the constructed pair satisfies the supersolution inequalities for small ε .

The proof is thus complete. □

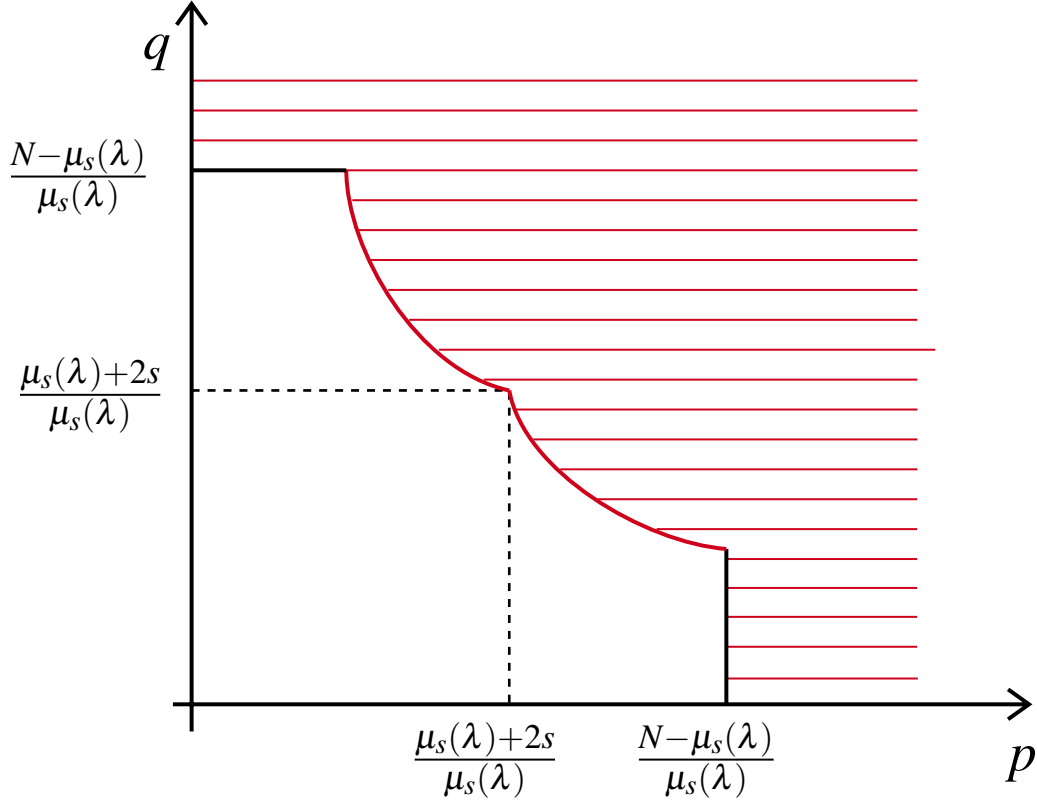


Figure 5.1: $f = v^p, g = u^q$ and $\lambda_1 = \lambda_2 = \lambda$.

Corollary 1.6. *Let $(p, q) \in F_{\lambda_1, \lambda_2}$ and assume that $\Omega \subset B_r(0)$. Let $u_0, v_0 \in L^1(\Omega)$ be nontrivial initial data such that*

$$u_0 \leq \widehat{u}, \quad v_0 \leq \widehat{v},$$

where $(\widehat{u}, \widehat{v})$ is the supersolution constructed above.

Then, for any $T > 0$, there exists a positive nontrivial entropy solution (u, v) of (5.3).

2 Gradient-gradient nonlinearities

In this part, we investigate the situation where the nonlinearities depend on the gradients, namely $f = |\nabla v|^p$ and $g = |\nabla u|^q$. In the fractional framework, the

system takes the form

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + |\nabla v|^p, & \text{in } \Omega_T, \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + |\nabla u|^q, & \text{in } \Omega_T, \\ u(x, t) \geq 0, \quad v(x, t) \geq 0, & \text{in } \Omega_T, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (5.23)$$

where $p, q > 1$ and $\frac{1}{2} < s < 1$.

Following the strategy developed in Subsection 1.1, we begin with a preliminary nonexistence result derived from Remark 7.3.

Proposition 2.1. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_N)$. Assume that either*

$$p \geq \frac{N - \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}, \quad (5.24)$$

where $\mu_s(\lambda_i)$ is defined in (1.21) for $i = 1, 2$. Then any nonnegative very weak supersolution (u, v) of (5.23) is necessarily trivial, that is, $u \equiv 0$ and $v \equiv 0$.

Proof. We argue by contradiction. Suppose that (u, v) is a nontrivial nonnegative very weak supersolution of (5.23). According to Remark 7.3, for any ball $B_r(0) \subset\subset \Omega$ and any time interval $(t_1, t_2) \subset (0, T)$, one has

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla v|^p |x|^{-\mu_s(\lambda_1)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-\mu_s(\lambda_2)} dx dt < +\infty. \quad (5.25)$$

Then,

$$\int_{t_1}^{t_2} \int_{B_r(0)} v^p |x|^{-p-\mu_s(\lambda_1)} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} u^q |x|^{-q-\mu_s(\lambda_2)} dx dt < +\infty. \quad (5.26)$$

Combining these estimates with the lower bounds induced by the singular behavior of supersolutions, we obtain

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-p\mu_s(\lambda_2)-p-\mu_s(\lambda_1)} dx dt < +\infty, \quad (5.27)$$

and similarly

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-q\mu_s(\lambda_1)-q-\mu_s(\lambda_2)} dx dt < +\infty. \quad (5.28)$$

This forces the conditions

$$p\mu_s(\lambda_2) + p + \mu_s(\lambda_1) < N, \quad q\mu_s(\lambda_1) + q + \mu_s(\lambda_2) < N,$$

which contradict (5.24).

Therefore, no nontrivial nonnegative supersolution can exist, and the proof is complete. $\square$

The next nonexistence result will be obtained through an iterative argument inspired by [20].

Theorem 2.2. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$ and assume that*

$$\frac{p(q+1)}{p+1} > 1 + \frac{2s}{1+\mu_s(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} > 1 + \frac{2s}{1+\mu_s(\lambda_2)}, \quad (5.29)$$

where $\mu_s(\lambda_i)$, $i = 1, 2$, are defined as in (1.21) and $s \in (0, 1)$. Then any nonnegative very weak supersolution (u, v) of (5.23) is necessarily trivial, that is, $u \equiv 0$ and $v \equiv 0$.

Proof. Let (u, v) be a nonnegative very weak supersolution of (5.23), with exponents p, q satisfying (5.29). We show that (u, v) must vanish identically.

In view of Proposition 1.7, we may restrict ourselves to the range

$$p < \frac{N - \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}, \quad q < \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}.$$

Fix $B_r(0) \subset \subset \Omega$ and $0 < t_1 < t_2 < T$. Set

$$a_0 := 2s + \mu_s(\lambda_2), \quad b_0 := 2s + \mu_s(\lambda_1).$$

For $n \geq 1$, define recursively

$$\begin{cases} a_n := 1 + \frac{N}{p'} + \frac{b_{n-1} - 2s}{p} - \frac{1}{n}, \\ b_n := 1 + \frac{N}{q'} + \frac{a_{n-1} - 2s}{q} - \frac{1}{n}. \end{cases} \quad (5.30)$$

We claim that, for every $n \in \mathbb{N}$,

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty. \quad (5.31)$$

The statement is true for $n = 0$ by Proposition 1.7. Assume it holds for some $n - 1 \geq 0$. We prove it for n .

From the supersolution structure and Remark 7.3, we obtain

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-a_{n-1}+2s} dx dt < +\infty.$$

If $a_{n-1} - 2s \geq N - q$, then necessarily $u \equiv 0$, which in turn implies $v \equiv 0$. Otherwise, applying Caffarelli–Kohn–Nirenberg inequality together with Hölder’s inequality, we deduce

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty.$$

Proceeding in the same way for the second equation yields

$$\int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty.$$

This proves (5.31) by induction.

Passing to the limit, we obtain

$$\bar{a} := \limsup_{n \rightarrow \infty} a_n = N + s - \frac{q+1}{pq-1}, \quad \bar{b} := \limsup_{n \rightarrow \infty} b_n = N + s - \frac{p+1}{pq-1}.$$

If (u, v) is not identically zero, then necessarily

$$\mu_s(\lambda_2) + \bar{a} \leq N, \quad \mu_s(\lambda_1) + \bar{b} \leq N.$$

However, these inequalities contradict condition (5.29). Hence, no nontrivial nonnegative supersolution can exist, and we conclude that $u \equiv 0$ and $v \equiv 0$. $\square$

2.1 Existence of Supersolutions

In analogy with Subsection 1.1, we now prove the existence of positive supersolutions for the gradient system.

Define

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}\right) \times \left(1, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) : \right. \\ \left. \frac{p(q+1)}{p+1} < 1 + \frac{2s}{1 + \mu_s(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)}{q+1} < 1 + \frac{2s}{1 + \mu_s(\lambda_2)} \right\}. \quad (5.32)$$

Our goal is to show that for every $(p, q) \in F_{\lambda_1, \lambda_2}$, there exists a nontrivial positive supersolution. As in the potential case, the set F_{λ_1, λ_2} can be decomposed as

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2s + \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}\right) \times \left(1, \frac{2s + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.33)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p \in \left(\frac{2s + \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}\right), q \in \left(1, \frac{2s + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.34)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p \in \left(1, \frac{2s + \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}\right), q \in \left(\frac{2s + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.35)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2s + \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}, q \in \left(\frac{2s + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.36)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2s + \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{1 + \mu_s(\lambda_2)}\right), q = \frac{2s + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)} \right\}. \quad (5.37)$$

Proceeding as in the proof of the potential case, one can construct explicit radial supersolutions in each region.

Proposition 2.3. *For any $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the pair (u, v) defined below is a positive*

supersolution of (4.37) in $B_r(0)$:

$$\varepsilon(|x|^{-\gamma_1}, |x|^{-\gamma_2}), \quad \gamma_i \in (\mu_s(\lambda_i), \mu_s^+(\lambda_i)), \quad \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \quad (5.38)$$

$$\varepsilon(|x|^{-(p\mu_s(\lambda_2)-s)}, |x|^{-\mu_s(\lambda_2)} - \delta|x|^{-(pq\mu_s(\lambda_2)-2qs-s)}), \quad \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \quad (5.39)$$

$$\varepsilon(|x|^{-\mu_s(\lambda_1)} - \delta|x|^{-(pq\mu_s(\lambda_1)-2ps-s)}, |x|^{-(q\mu_s(\lambda_1)-s)}), \quad \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \quad (5.40)$$

$$\varepsilon(-|x|^{-\mu_s(\lambda_1)} \ln(|x|/r), |x|^{-\mu_s(\lambda_2)} - \delta|x|^{-(q\mu_s(\lambda_1)-s+\varepsilon)}), \quad \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \quad (5.41)$$

$$\varepsilon(|x|^{-\mu_s(\lambda_1)} - \delta|x|^{-(p\mu_s(\lambda_2)-s+\varepsilon)}, -|x|^{-\mu_s(\lambda_2)} \ln(|x|/r)), \quad \text{if } (p, q) \in E_{\lambda_1, \lambda_2}. \quad (5.42)$$

Unlike the potential case, an analogue of Corollary 1.6 cannot be obtained here. Indeed, Proposition 1.1 relies crucially on the monotonicity of the mappings $\sigma \mapsto \sigma^p$ and $\sigma \mapsto \sigma^q$ for $\sigma \geq 0$, a property that is no longer available when dealing with gradient-dependent nonlinearities.

Furthermore, due to the lack of a suitable comparison principle for system (4.37), the existence of supersolutions does not, in general, ensure the existence of genuine solutions.

3 Potential-gradient nonlinearities

We consider the fractional counterpart of the mixed system:

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + v^p & \text{in } \Omega_T, \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + |\nabla u|^q & \text{in } \Omega_T, \\ u(x, t) \geq 0, \quad v(x, t) \geq 0 & \text{in } \Omega_T, \\ u = v = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (5.43)$$

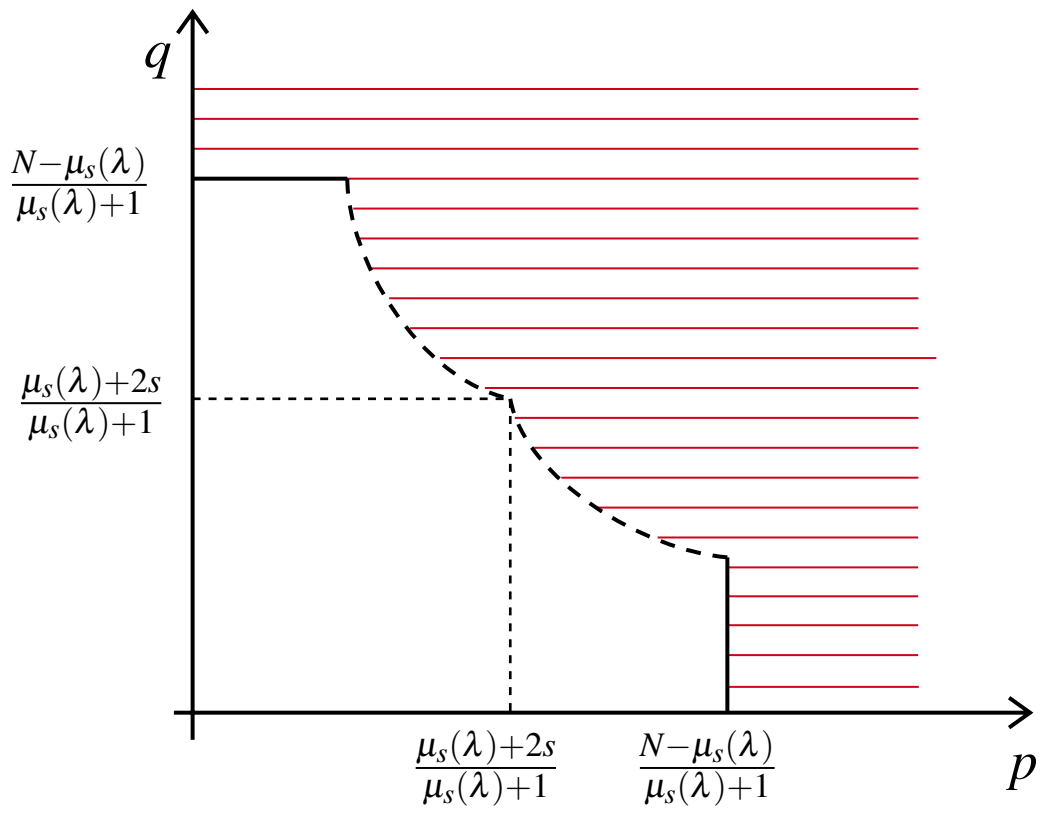


Figure 5.2: $f = |\nabla v|^p, g = |\nabla u|^q$ when $\lambda_1 = \lambda_2 =: \lambda$.

with $p, q > 1$ and $\frac{1}{2} < s < 1$.

As in Subsections 1 and 2, we begin with a basic nonexistence result in the fractional setting.

Proposition 3.1. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$. Suppose that either*

$$p \geq \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)} \quad \text{or} \quad q \geq \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}, \quad (5.44)$$

where $\mu_s(\lambda_i)$, $i = 1, 2$, denotes the fractional Hardy exponent. Then every nonnegative very weak supersolution (u, v) of the fractional system (5.43) is necessarily trivial, that is, $u \equiv 0$, $v \equiv 0$.

Proof. If

$$p \geq \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)},$$

then the conclusion follows by the same scaling argument as in the potential case.

Assume now that

$$q \geq \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}.$$

By the necessary integrability condition for very weak supersolutions in the fractional setting (see Remark 7.3), we obtain

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-\mu_s(\lambda_2)} dx dt < +\infty. \quad (5.45)$$

Using the fractional exponent relation together with (5.44), we deduce that $q \in (1, N)$ and

$$\frac{\mu_s(\lambda_2)}{q} \in \left(0, \frac{N - q}{q}\right).$$

Hence,

$$\int_{t_1}^{t_2} \int_{B_r(0)} u^q |x|^{-q - \mu_s(\lambda_2)} dx dt < +\infty. \quad (5.46)$$

Combining this with the lower bound for u , we obtain

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-q\mu_s(\lambda_1) - q - \mu_s(\lambda_2)} dx dt < +\infty. \quad (5.47)$$

Arguing exactly as in Proposition 1.3 in the fractional framework, such an integrability condition is impossible. We conclude that $u \equiv 0$ and $v \equiv 0$. $\square$

Theorem 3.2. *Let $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$ and assume that*

$$\frac{p(q+1)}{p+1} > 1 + \frac{2s}{1 + \mu_s(\lambda_1)} \quad \text{or} \quad \frac{q(p+1)}{q+1} > 1 + \frac{2s}{\mu_s(\lambda_2)}, \quad (5.48)$$

where $\mu_s(\lambda_i)$ defined as in (1.21). Then system (5.43) admits no nontrivial non-negative very weak supersolution; hence

$$u \equiv 0, \quad v \equiv 0.$$

Proof. Let (u, v) be a nonnegative very weak supersolution of (5.43). We prove that $(u, v) \equiv (0, 0)$.

Assume that condition (5.48) holds. We may assume

$$p < \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \quad q < \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}.$$

Let $B_r(0) \Subset \Omega$ and $0 < t_1 < t_2 < T$. Define the initial weights

$$a_0 := 2s + \mu_s(\lambda_2), \quad b_0 := 2s + \mu_s(\lambda_1).$$

For $n \geq 1$, define recursively

$$\begin{cases} a_n := 1 + \frac{N}{p'} + \frac{b_{n-1}-2s}{p} - \frac{1}{n}, \\ b_n := 1 + \frac{N}{q'} + \frac{a_{n-1}-2s}{q} - \frac{1}{n}. \end{cases} \quad (5.49)$$

We claim that for all $n \in \mathbb{N}$,

$$\int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty, \quad \int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty. \quad (5.50)$$

The proof follows by induction. The case $n = 0$ is obtained from the basic integrability property of very weak supersolutions.

Assume the claim holds at step $n - 1$. Testing the second equation with suitable fractional cut-off weights together with Caffarelli–Kohn–Nirenberg type inequalities, we obtain

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla u|^q |x|^{-a_{n-1}+2s} dx dt < +\infty.$$

If $a_{n-1} - 2s \geq N - q$, then $u \equiv 0$ and consequently $v \equiv 0$. Otherwise, the fractional weighted inequalities imply

$$\int_{t_1}^{t_2} \int_{B_r(0)} v |x|^{-a_n} dx dt < +\infty.$$

A symmetric argument applied to the first equation yields

$$\int_{t_1}^{t_2} \int_{B_r(0)} u |x|^{-b_n} dx dt < +\infty.$$

Hence the induction is complete.

Letting

$$\bar{a} = \limsup_{n \rightarrow \infty} a_n = N + 2s - \frac{q+1}{pq-1}, \quad \bar{b} = \limsup_{n \rightarrow \infty} b_n = N + 2s - \frac{p+1}{pq-1},$$

we deduce that either $(u, v) \equiv (0, 0)$, or the limiting relations contradict the fractional Hardy optimality, namely

$$\mu_s(\lambda_2) + \bar{a} \leq N, \quad \mu_s(\lambda_1) + \bar{b} \leq N.$$

This contradicts (5.48). Therefore,

$$u \equiv 0, \quad v \equiv 0.$$

□

3.1 Existence of supersolutions .

Analogously to sections 1 and 2, we now construct positive supersolutions for the mixed fractional system.

We introduce the admissible region of exponents

$$F_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)} \right) \times \left(1, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)} \right) : \right. \\ \left. \frac{p(q+2)}{2p+1} < 1 + \frac{1}{1 + \mu_s(\lambda_1)} \quad \text{and} \quad \frac{q(p+1)+1}{q+2} < 1 + \frac{1}{\mu_s(\lambda_2)} \right\}. \quad (5.51)$$

We aim to show that for every $(p, q) \in F_{\lambda_1, \lambda_2}$, the system admits a nontrivial positive supersolution. As in the previous subsections, the set F_{λ_1, λ_2} can be decomposed as

$$F_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \cup B_{\lambda_1, \lambda_2} \cup C_{\lambda_1, \lambda_2} \cup D_{\lambda_1, \lambda_2} \cup E_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \left\{ (p, q) \in \left(1, \frac{2 + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right) \times \left(1, \frac{2 + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.52)$$

$$B_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p \in \left(\frac{2 + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right), q < \frac{2 + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)} \right\}, \quad (5.53)$$

$$C_{\lambda_1, \lambda_2} := \left\{ (p, q) \in F_{\lambda_1, \lambda_2} : p < \frac{2 + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, q \in \left(\frac{2 + \mu_s(\lambda_2)}{\mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.54)$$

$$D_{\lambda_1, \lambda_2} := \left\{ (p, q) : p = \frac{2 + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, q \in \left(\frac{2 + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}, \frac{N - \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)}\right) \right\}, \quad (5.55)$$

$$E_{\lambda_1, \lambda_2} := \left\{ (p, q) : p \in \left(\frac{2 + \mu_s(\lambda_1)}{\mu_s(\lambda_2)}, \frac{N - \mu_s(\lambda_1)}{\mu_s(\lambda_2)}\right), q = \frac{2 + \mu_s(\lambda_2)}{1 + \mu_s(\lambda_1)} \right\}. \quad (5.56)$$

Following the same strategy as in Subsection 1, we obtain the following result.

Proposition 3.3. *For any $(p, q) \in F_{\lambda_1, \lambda_2}$, there exist constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, the pair (u, v) defined by*

$$\begin{aligned} \varepsilon(|x|^{-\gamma_1}, |x|^{-\gamma_2}), \quad \gamma_i \in (\mu_s(\lambda_i), \mu_s^+(\lambda_i)), \quad i = 1, 2, & \quad \text{if } (p, q) \in A_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-(p\mu_s(\lambda_2)-2s)}, |x|^{-\mu_s(\lambda_2)} - \delta|x|^{-(pq\mu_s(\lambda_2)-2q-2s)}), & \quad \text{if } (p, q) \in B_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-\mu_s(\lambda_1)} - \delta|x|^{-(pq\mu_s(\lambda_1)-2p-2s)}, |x|^{-(q\mu_s(\lambda_1)-2s)}), & \quad \text{if } (p, q) \in C_{\lambda_1, \lambda_2}, \\ \varepsilon(-|x|^{-\mu_s(\lambda_1)} \ln(|x|/r), |x|^{-\mu_s(\lambda_2)} - \delta|x|^{-(q\mu_s(\lambda_1)-2s+\varepsilon)}), & \quad \text{if } (p, q) \in D_{\lambda_1, \lambda_2}, \\ \varepsilon(|x|^{-\mu_s(\lambda_1)} - \delta|x|^{-(p\mu_s(\lambda_2)-2s+\varepsilon)}, -|x|^{-\mu_s(\lambda_2)} \ln(|x|/r)), & \quad \text{if } (p, q) \in E_{\lambda_1, \lambda_2}, \end{aligned}$$

defines a positive supersolution of the mixed system (5.43) in $B_r(0)$.

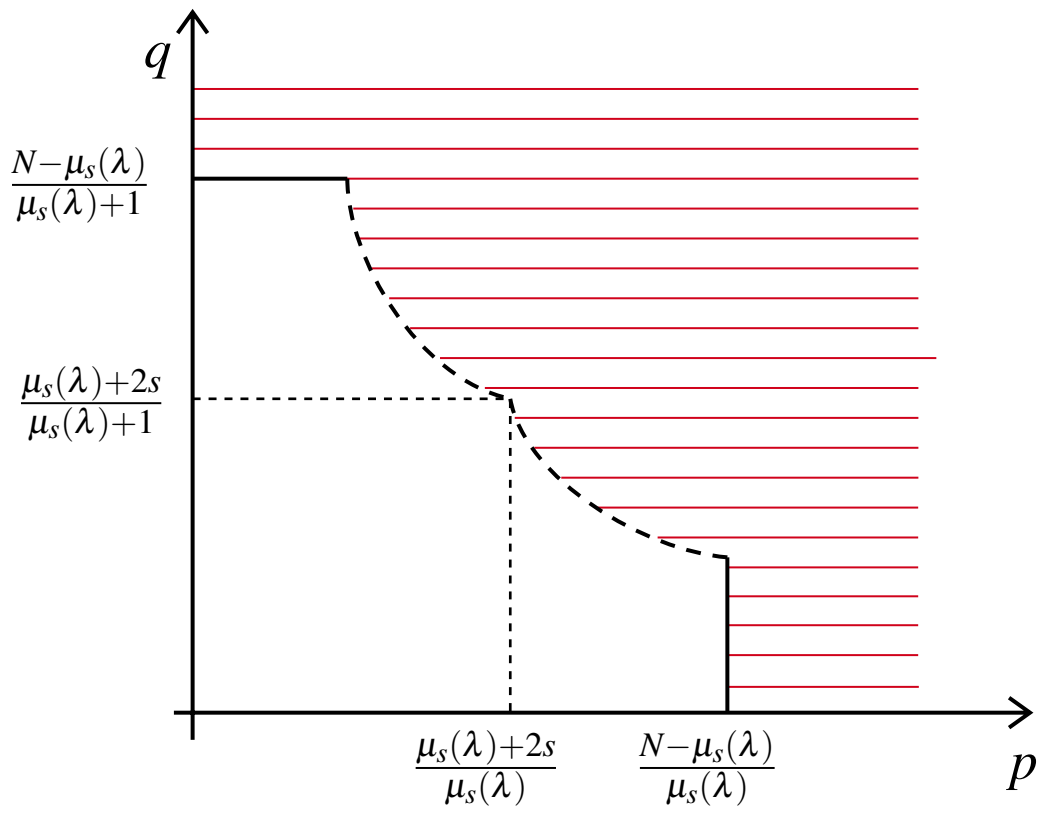


Figure 5.3: $f = v^p, g = |\nabla u|^q$ for $\lambda_1 = \lambda_2 =: \lambda$.

4 Fujita exponent

Consider the following fractional Hardy–Lane–Emden system:

$$\begin{cases} u_t + (-\Delta)^s u = \lambda_1 \frac{u}{|x|^{2s}} + v^p, & \text{in } \mathbb{R}_+^N, \\ v_t + (-\Delta)^s v = \lambda_2 \frac{v}{|x|^{2s}} + u^q, & \text{in } \mathbb{R}_+^N, \\ u(t, x) > 0, \quad v(t, x) > 0, & \text{in } \mathbb{R}_+^N, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & \text{on } \mathbb{R}^N, \end{cases} \quad (5.57)$$

where $\lambda_1, \lambda_2 > 0$ and $\mathbb{R}_+^N := \mathbb{R}^N \times (0, +\infty)$. Our objective is to investigate the existence of a *Fujita-type curve* $F(p, q)$ characterizing the critical exponents (p, q) for which global in time solutions exist.

A key tool in this analysis is the fractional Hardy inequality, which asserts that

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \geq \Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} u^2(x) dx, \quad (5.58)$$

where the optimal constant is given by

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2\left(\frac{N+2}{4}\right)}{\Gamma^2\left(\frac{N-2}{4}\right)},$$

and it is well-known that this constant is never attained.

On a bounded smooth domain $\Omega \subset \mathbb{R}^N$ containing the origin, the inequality takes the localized form

$$\Lambda_{N,s} \int_{\Omega} \frac{\phi^2(x)}{|x|^{2s}} dx \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy, \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega), \quad (5.59)$$

where $D_{\Omega} := \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2$.

This inequality provides a crucial estimate for the potential terms and allows for a rigorous analysis of existence and nonexistence of positive solutions to (5.57).

Definition 4.1. Let $u_0, v_0 \in L_{\text{loc}}^1(\mathbb{R}^N)$. We say that (u, v) is a supersolution (resp. subsolution) of (5.57) if

$$u, v \in C([0, +\infty), L_{\text{loc}}^1(\mathbb{R}^N)), \quad \frac{u}{|x|^{2s}}, \frac{v}{|x|^{2s}} \in L_{\text{loc}}^1(\mathbb{R}_+^N), \quad u^q, v^p \in L_{\text{loc}}^1(\mathbb{R}_+^N),$$

and for all nonnegative test functions $\phi, \psi \in C_c^\infty(\mathbb{R}^N \times [0, +\infty))$, we have

$$\begin{cases} -\int_{\mathbb{R}^N} u_0(x) \phi(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} (-\phi_t + (-\Delta)^s \phi) u dx dt \geq (\leq) \int_0^{+\infty} \int_{\mathbb{R}^N} \left(\lambda_1 \frac{u}{|x|^{2s}} + v^p \right) \phi dx dt, \\ -\int_{\mathbb{R}^N} v_0(x) \psi(x, 0) dx + \int_0^{+\infty} \int_{\mathbb{R}^N} (-\psi_t + (-\Delta)^s \psi) v dx dt \geq (\leq) \int_0^{+\infty} \int_{\mathbb{R}^N} \left(\lambda_2 \frac{v}{|x|^{2s}} + u^q \right) \psi dx dt. \end{cases} \quad (5.60)$$

Furthermore, we require the initial condition $u(\cdot, 0) = u_0$ and $v(\cdot, 0) = v_0$. If (u, v) is both a supersolution and a subsolution, we call (u, v) a solution.

Definition 4.2. A positive weak solution (u, v) of (5.57) is said to blow up in finite time if for any $r > 0$ there exists $T^* \in [0, +\infty)$ such that

$$\lim_{t \rightarrow (T^*)^-} \int_{B_r(0)} |x|^{-\mu(\lambda_1)} u(x, t) dx = +\infty \quad \text{or} \quad \lim_{t \rightarrow (T^*)^-} \int_{B_r(0)} |x|^{-\mu(\lambda_2)} v(x, t) dx = +\infty. \quad (5.61)$$

Before proving the main Theorem of this section, we need some analysis related to the fractional heat equation.

Let $h(x, t)$ be the fractional heat kernel, namely,

$$\begin{cases} h_t + (-\Delta)^s h = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ h(x, 0) = \delta_0. \end{cases} \quad (5.62)$$

There is no known closed form for $h(x, t)$ in real variables. However, in Fourier variables it is given by

$$\mathcal{F}(h)(t, \xi) = e^{-t|2\pi\xi|^{2s}}.$$

The properties of the kernel h were studied in [52] for $N = 1$ and in [17] for all dimensions. More precisely, $h(x, t)$ can be written as

$$h(x, t) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^{2s} t} d\xi, \quad (5.63)$$

where $0 < s < 1$ and $N \geq 2s$. Moreover, there exists a constant $C > 1$ such that

$$\frac{1}{C(1 + |x|^{N+2s})} \leq h(x, 1) \leq \frac{C}{1 + |x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N. \quad (5.64)$$

A direct proof of these estimates can be obtained using real analysis techniques without relying on Bessel functions, based on Bernstein's characterization of completely monotone functions via Laplace transforms (see Section 12.5 of [50]).

We note that

$$h(x, t) = t^{-\frac{N}{2s}} H \left(\frac{|x|}{t^{1/(2s)}} \right),$$

where H is a decreasing function satisfying

$$H(\sigma) \approx \frac{1}{(1 + \sigma^2)^{\frac{N+2s}{2}}}, \quad |H'(\sigma)| \leq \frac{C}{(1 + \sigma^2)^{\frac{N+2s+1}{2}}}, \quad (5.65)$$

and

$$2s(-\Delta)^s H = NH + rH'. \quad (5.66)$$

We refer to [28, 59] for more details.

Now, define

$$\widehat{h}(x, t) = |x|^{-\mu(\lambda)} h(x, t) = |x|^{-\mu(\lambda)} t^{-\frac{N}{2s}} H \left(\frac{|x|}{t^{1/(2s)}} \right).$$

We recall the elementary identity

$$(-\Delta)^s(wv) = v(-\Delta)^s w + w(-\Delta)^s v - \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+2s}} dy. \quad (5.67)$$

Applying this with $w(x) = |x|^{-\mu(\lambda)}$ and $v(x) = h(x, t)$, for $t > 0$ we obtain

$$\begin{aligned} (-\Delta)^s \widehat{h}(x, t) &= |x|^{-\mu(\lambda)} (-\Delta)^s h(x, t) + h(x, t) (-\Delta)^s |x|^{-\mu(\lambda)} \\ &\quad - \int_{\mathbb{R}^N} \frac{(|x|^{-\mu(\lambda)} - |y|^{-\mu(\lambda)})(h(x, t) - h(y, t))}{|x - y|^{N+2s}} dy. \end{aligned}$$

Observe that

$$(|x|^{-\mu(\lambda)} - |y|^{-\mu(\lambda)})(h(x, t) - h(y, t)) \geq 0,$$

which implies

$$(-\Delta)^s \widehat{h}(x, t) \leq |x|^{-\mu(\lambda)} (-\Delta)^s h(x, t) + h(x, t) (-\Delta)^s |x|^{-\mu(\lambda)}.$$

Using the equation satisfied by h , we get

$$\begin{aligned} (-\Delta)^s \widehat{h}(x, t) &\leq |x|^{-\mu(\lambda)} (-h_t(x, t)) + \lambda \frac{h(x, t)}{|x|^{2s}} |x|^{-\mu(\lambda)} \\ &= -\partial_t \widehat{h}(x, t) + \lambda \frac{\widehat{h}(x, t)}{|x|^{2s}}. \end{aligned}$$

More precisely, using the explicit scaling of h , we obtain

$$(-\Delta)^s \widehat{h}(x, t) \leq \frac{N}{2s} \frac{\widehat{h}(x, t)}{t} + \lambda \frac{\widehat{h}(x, t)}{|x|^{2s}},$$

where we used that $H' \leq 0$.

Therefore,

$$-(-\Delta)^s \widehat{h}(x, t) + \lambda \frac{\widehat{h}(x, t)}{|x|^{2s}} \geq -\frac{N}{2s} \frac{\widehat{h}(x, t)}{t}. \quad (5.68)$$

We are now in a position to state the main theorem of this section, in the particular case $p = q$ and $\lambda_1 = \lambda_2 = \lambda$.

Theorem 4.3. *Assume that $1 < p < F(\lambda, s) := 1 + \frac{2s}{N - \mu_s(\lambda)}$, and let (u, v) be a positive solution of (5.57). Then the quantity $u + v$ blows up in finite time. More precisely, there exists $T^* > 0$ such that*

$$\lim_{t \rightarrow T^*} \int_{\mathbb{R}^N} (u(x, t) + v(x, t)) |x|^{-\mu_s(\lambda)} dx = +\infty.$$

Proof. Let (u, v) be a positive solution of (5.57). For $\eta > 0$, consider the test function

$$\psi_\eta(x) := \widehat{h}\left(x, \frac{1}{\eta}\right) = \eta^{\frac{N}{2s} - \frac{\mu(\lambda)}{2s}} H\left(\eta^{\frac{|x|}{2s}}\right),$$

where H is defined below. It satisfies

$$(-\Delta)^s \psi_\eta(x) + \lambda \frac{\psi_\eta(x)}{|x|^{2s}} \geq -\frac{N}{2s} \eta \psi_\eta(x). \quad (5.69)$$

Moreover, we have the scaling property

$$\int_{\mathbb{R}^N} \psi_\eta(x) dx = C \eta^{-\frac{\mu(\lambda)}{2s}}.$$

Testing the first equation in (5.57) with ψ_η , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} u \psi_\eta dx = \int_{\mathbb{R}^N} v^p \psi_\eta dx + \int_{\mathbb{R}^N} \left((-\Delta)^s \psi_\eta + \lambda \frac{\psi_\eta}{|x|^{2s}} \right) u dx.$$

Using (5.75), it follows that

$$\frac{d}{dt} \int_{\mathbb{R}^N} u \psi_\eta dx + \frac{N}{2s} \eta \int_{\mathbb{R}^N} u \psi_\eta dx \geq \int_{\mathbb{R}^N} v^p \psi_\eta dx.$$

Applying Jensen's inequality yields

$$\int_{\mathbb{R}^N} v^p \psi_\eta dx \geq C \eta^{\frac{(p-1)\mu(\lambda)}{2s}} \left(\int_{\mathbb{R}^N} v \psi_\eta dx \right)^p.$$

Hence,

$$\frac{d}{dt} \int_{\mathbb{R}^N} u \psi_\eta dx + \frac{N}{2s} \eta \int_{\mathbb{R}^N} u \psi_\eta dx \geq C \eta^{A_p} \left(\int_{\mathbb{R}^N} v \psi_\eta dx \right)^p,$$

where $A_p = \frac{(p-1)\mu(\lambda)}{2s}$.

A completely analogous argument applied to the second equation gives

$$\frac{d}{dt} \int_{\mathbb{R}^N} v \psi_\eta dx + \frac{N}{2s} \eta \int_{\mathbb{R}^N} v \psi_\eta dx \geq C \eta^{A_p} \left(\int_{\mathbb{R}^N} u \psi_\eta dx \right)^p.$$

We now introduce the damped functionals

$$Y(t) := e^{\frac{N}{2s}\eta t} \int_{\mathbb{R}^N} u(x,t) \psi_\eta(x) dx, \quad Z(t) := e^{\frac{N}{2s}\eta t} \int_{\mathbb{R}^N} v(x,t) \psi_\eta(x) dx.$$

Then there exists a constant $C > 0$ such that

$$\begin{cases} Y'(t) \geq C \eta^{A_p} e^{-a_p \eta t} Z^p(t), \\ Z'(t) \geq C \eta^{A_p} e^{-a_p \eta t} Y^p(t), \end{cases} \quad (5.70)$$

where

$$A_p = \frac{(p-1)\mu(\lambda)}{2s}, \quad a_p = (p-1) \frac{N}{2s}.$$

Finally, setting

$$W(t) := Y(t) + Z(t),$$

Summing the two inequalities in (5.70), yields

$$W'(t) \geq C \eta^{\frac{(p-1)\mu(\lambda)}{2s}} e^{-(p-1)\frac{N}{2s}\eta t} (Y^p(t) + Z^p(t)),$$

for some $C > 0$.

Using the convexity inequality, we deduce

$$W'(t) \geq C \eta^{\frac{(p-1)\mu(\lambda)}{2s}} e^{-(p-1)\frac{N}{2s}\eta t} W^p(t). \quad (5.71)$$

Integrating the previous differential inequality, we obtain

$$\begin{aligned} \frac{1}{p-1} \left(\frac{1}{W^{p-1}(0)} - \frac{1}{W^{p-1}(t)} \right) &\geq C \eta^{(p-1)\frac{\mu(\lambda)}{2s}} \frac{2s}{(p-1)N} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right) \\ &\geq C \frac{2s}{N(p-1)} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right). \end{aligned}$$

Therefore,

$$W^{p-1}(t) \geq \frac{1}{\frac{1}{W^{p-1}(0)} - C \frac{2s}{N} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right)}.$$

It is clear that, if for some $T < \infty$, we have

$$\frac{1}{W^{p-1}(0)} \leq C \frac{2s}{N} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta T} \right), \quad (5.72)$$

then $W(T) = \infty$.

Since

$$1 - e^{-(p-1)\frac{N}{2s}\eta T} \rightarrow 1 \quad \text{as } T \rightarrow \infty,$$

condition (5.79) holds if

$$W^{p-1}(0) > \frac{1}{C} \frac{N}{2s} \eta^{1-(p-1)\frac{\mu(\lambda)}{2s}}.$$

Hence,

$$\eta^{(p-1)\left(\frac{N}{2s} - \frac{\mu(\lambda)}{2s}\right)} \left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{p-1} > \frac{1}{C} \frac{N}{2s} \eta^{1-(p-1)\frac{\mu(\lambda)}{2s}}.$$

and then

$$\left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{p-1} > \frac{1}{C} \frac{N}{2s} \eta^{-(p-1)\left(\frac{N}{2s} - \frac{\mu(\lambda)}{2s}\right)+1}. \quad (5.73)$$

It is clear that (5.80) holds for η small if and only if

$$-(p-1) \left(\frac{N}{2s} - \frac{\mu(\lambda)}{2s} \right) + 1 > 0,$$

which yields $p < F(\lambda, s)$.

Since

$$\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} dx > C_0,$$

and using the fact that H is bounded, there exists $\eta > 0$ such that

$$\left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{p-1} \geq \frac{2s}{N} C \eta^{-(p-1)\left(\frac{N}{2s} - \frac{\mu(\lambda)}{2s}\right) + 1}. \quad (5.74)$$

Hence the result follows. □

The case $p > q$ is treated in the next part, while the case $q > p$ can be derived by symmetry arguments.

Theorem 4.4. *Let (u, v) be a positive solution of (5.57). Then the quantity $u + v$ blows up in finite time. More precisely, there exists $T^* > 0$ such that*

$$\lim_{t \rightarrow T^*} \int_{\mathbb{R}^N} (u(x, t) + v(x, t)) |x|^{-\mu_s(\lambda)} dx = +\infty.$$

Proof. Let (u, v) be a positive solution of (5.57), and define, for $\eta > 0$, the test function

$$\psi_\eta(x) := \hat{h}\left(x, \frac{1}{\eta}\right) = \eta^{\frac{N}{2s} - \frac{\mu(\lambda)}{s}} H\left(\eta^{\frac{|x|}{2s}}\right),$$

where H is defined below, it holds that

$$(-\Delta)^s \psi_\eta(x) + \lambda \frac{\psi_\eta(x)}{|x|^{2s}} \geq -\frac{N}{2s} \eta \psi_\eta(x). \quad (5.75)$$

Notice also that

$$\int_{\mathbb{R}^N} \psi_\eta(x) dx = \eta^{-\frac{\mu(\lambda)}{2s}}.$$

Testing the first equation of (5.57) with ψ_η , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} u \psi_\eta dx = \int_{\mathbb{R}^N} v^p \psi_\eta dx + \int_{\mathbb{R}^N} ((-\Delta)^s \psi_\eta + \lambda_1 \frac{\psi_\eta}{|x|^{2s}}) u dx.$$

Applying (5.75) and Jensen's inequality yields

$$\frac{d}{dt} \int_{\mathbb{R}^N} u \psi_\eta dx + \frac{N}{2s} \eta \int_{\mathbb{R}^N} u \psi_\eta dx \geq \eta^{(p-1)\frac{\mu(\lambda_1)}{2s}} \left(\int_{\mathbb{R}^N} v \psi_\eta dx \right)^p.$$

Similarly, for the second equation,

$$\frac{d}{dt} \int_{\mathbb{R}^N} v \psi_\eta dx + \frac{N}{2s} \eta \int_{\mathbb{R}^N} v \psi_\eta dx \geq \eta^{(q-1)\frac{\mu(\lambda_2)}{2s}} \left(\int_{\mathbb{R}^N} u \psi_\eta dx \right)^q.$$

Define the damped functionals

$$Y(t) := e^{\frac{N}{2s}\eta t} \int_{\mathbb{R}^N} u(x,t) \psi_\eta(x) dx, \quad Z(t) := e^{\frac{N}{2s}\eta t} \int_{\mathbb{R}^N} v(x,t) \psi_\eta(x) dx.$$

Then there exist constants $C_1, C_2 > 0$ such that

$$\begin{cases} Y'(t) \geq C_1 \eta^{A_p} e^{-a_p \eta t} Z(t)^p, \\ Z'(t) \geq C_2 \eta^{A_q} e^{-a_q \eta t} Y(t)^q, \end{cases} \quad (5.76)$$

with

$$A_p = \frac{(p-1)\mu(\lambda_1)}{2s}, \quad A_q = \frac{(q-1)\mu(\lambda_2)}{2s}, \quad a_p = (p-1)\frac{N}{2s}, \quad a_q = (q-1)\frac{N}{2s}.$$

Let

$$W(t) := Y(t) + Z(t).$$

Since $p > q$ and $Z(t) \geq Z(0) > 0$, we have

$$Z^p(t) \geq Z(0)^{p-q} Z^q(t).$$

Therefore,

$$Y'(t) \geq C_1 Z(0)^{p-q} \eta^{A_p} e^{-a_p \eta t} Z^q(t).$$

Recalling that

$$a_p = (p-1)\frac{N}{2s}, \quad a_q = (q-1)\frac{N}{2s},$$

and since $p > q$, we obtain $a_p > a_q$. Hence,

$$Z'(t) \geq C_2 \eta^{A_p} e^{-a_p \eta t} Y^q(t).$$

Adding the two inequalities yields

$$W'(t) \geq C \eta^{\frac{(p-1)\mu(\lambda)}{2s}} e^{-a_p \eta t} (Y^q(t) + Z^q(t)),$$

for some constant $C > 0$.

Using the convexity inequality,

$$Y^q(t) + Z^q(t) \geq 2^{1-q}(Y(t) + Z(t))^q,$$

we deduce

$$W'(t) \geq C\eta^{\frac{(p-1)\mu(\lambda)}{2s}} e^{-(p-1)\frac{N}{2s}\eta t} W^q(t). \quad (5.77)$$

Using the convexity inequality, we deduce

$$W'(t) \geq C\eta^{\frac{(p-1)\mu(\lambda)}{2s}} e^{-(p-1)\frac{N}{2s}\eta t} W^q(t). \quad (5.78)$$

Integrating the previous differential inequality, we obtain

$$\begin{aligned} \frac{1}{q-1} \left(\frac{1}{W^{q-1}(0)} - \frac{1}{W^{q-1}(t)} \right) &\geq C\eta^{(p-1)\frac{\mu(\lambda)}{2s}} \int_0^t e^{-(p-1)\frac{N}{2s}\eta \tau} d\tau \\ &= C\eta^{(p-1)\frac{\mu(\lambda)}{2s}} \frac{2s}{(p-1)N} \eta^{-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right) \\ &\geq C \frac{2s}{N(p-1)} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right). \end{aligned}$$

Therefore,

$$W^{q-1}(t) \geq \frac{1}{\frac{1}{W^{q-1}(0)} - C \frac{2s}{N(p-1)} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta t} \right)}.$$

It is clear that, if for some $T < \infty$, we have

$$\frac{1}{W^{q-1}(0)} \leq C \frac{2s}{N(p-1)} \eta^{(p-1)\frac{\mu(\lambda)}{2s}-1} \left(1 - e^{-(p-1)\frac{N}{2s}\eta T} \right), \quad (5.79)$$

then $W(T) = \infty$.

Since

$$1 - e^{-(p-1)\frac{N}{2s}\eta T} \rightarrow 1 \quad \text{as } T \rightarrow \infty,$$

condition (5.79) holds if

$$W^{q-1}(0) > \frac{1}{C} \frac{N(p-1)}{2s} \eta^{1-(p-1)\frac{\mu(\lambda)}{2s}}.$$

Hence,

$$\eta^{(q-1)\left(\frac{N}{2s}-\frac{\mu(\lambda)}{2s}\right)} \left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{q-1} > \frac{1}{C} \frac{N(p-1)}{2s} \eta^{1-(p-1)\frac{\mu(\lambda)}{2s}}.$$

and then

$$\left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{q-1} > \frac{1}{C} \frac{N(p-1)}{2s} \eta^{1-(p-1)\frac{\mu(\lambda)}{2s} - (q-1)\left(\frac{N}{2s}-\frac{\mu(\lambda)}{2s}\right)}. \quad (5.80)$$

It is clear that (5.80) holds for η small if and only if

$$1 - (p-1)\frac{\mu(\lambda)}{2s} - (q-1)\left(\frac{N}{2s}-\frac{\mu(\lambda)}{2s}\right) > 0,$$

that is,

$$(q-1)(N - \mu(\lambda)) < 2s - (p-1)\mu(\lambda).$$

Since

$$\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} dx > C_0,$$

and using the fact that H is bounded, there exists $\eta > 0$ such that

$$\left(\int_{\mathbb{R}^N} (u_0(x) + v_0(x)) |x|^{-\mu(\lambda)} H\left(\eta^{\frac{1}{2s}} |x|\right) dx \right)^{q-1} \geq \frac{2s}{N(p-1)} C \eta^{1-(p-1)\frac{\mu(\lambda)}{2s} - (q-1)\left(\frac{N}{2s}-\frac{\mu(\lambda)}{2s}\right)}. \quad (5.81)$$

Hence the result follows.

□

Perspectives

The results obtained in this thesis naturally suggest several directions for future research.

A first perspective concerns the study of fractional systems involving nonlocal gradient nonlinearities. More precisely, it would be interesting to investigate the existence, nonexistence, and qualitative properties of solutions to systems driven simultaneously by the fractional Laplacian, Hardy-type singular potentials, and fractional gradient terms. The nonlocal nature of both the diffusion and the gradient introduces additional analytical difficulties and requires the development of new techniques.

$$\begin{cases} (-\Delta)^s u - \lambda_1 \frac{u}{|x|^{2s}} = |\nabla^s v|^p, & \text{in } \Omega, \\ (-\Delta)^s v - \lambda_2 \frac{v}{|x|^{2s}} = |\nabla^s u|^q, & \text{in } \Omega, \\ u, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (5.82)$$

Another important research direction concerns the regularity theory for elliptic cooperative systems with Hardy potentials. While the existence and nonexistence regions have been characterized in this thesis, the regularity of weak or entropy solutions in the existence region remains widely open. In particular, it would be interesting to establish local and global regularity results, as well as optimal estimates near the singularity, for pairs of exponents (p, q) belonging to the existence region.

$$\begin{cases} -\Delta u = \lambda_1 \frac{v}{|x|^2} + v^p + f(x), & \text{in } \Omega, \\ -\Delta v = \lambda_2 \frac{u}{|x|^2} + u^q + g(x), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \quad (5.83)$$

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Abstract:

This thesis investigates systems of nonlinear, singular, local and nonlocal partial differential equations, focusing on elliptic and fractional parabolic systems. The first part addresses existence, non-existence, and regularity of solutions for elliptic systems with singular potentials. The second part examines parabolic systems, establishing a critical curve separating global existence from finite-time blow-up. These results provide original contributions to the qualitative study of coupled systems with singular and nonlocal terms.

Keywords: PDEs, coupled systems, singularities, nonlocal operators, α -Laplacian, blow-up, Fujita critical curve, weak solutions.

Résumé :

Cette thèse étudie des systèmes d'équations aux dérivées partielles non linéaires, singulières et non locales, en considérant les systèmes elliptiques et paraboliques fractionnaires. La première partie traite de l'existence, de la non-existence et de la régularité des solutions pour des systèmes elliptiques avec potentiel singulier. La seconde partie analyse les systèmes paraboliques, définissant une courbe critique séparant existence globale et blow-up en temps fini. Ces travaux apportent des contributions originales pour l'étude qualitative de systèmes couplés avec termes singuliers et non locaux.

Mots-clés : EDP, systèmes couplés, singularités, opérateurs non locaux, α -laplacien, blow-up, courbe critique de Fujita, solutions faibles.

المخلص :

تتناول هذه الرسالة أنظمة المعادلات التفاضلية الجزئية غير الخطية، والمتفردة، وغير المحلية، مع التركيز على الأنظمة الإهليلجية والبارابولية الكسرية. يتناول الجزء الأول وجود أو عدم وجود وتنظيم الحلول للأنظمة الإهليلجية ذات الإمكانات المتفردة. أما الجزء محددًا منحنى حرج يفصل بين الوجود الكلي والانفجار في زمن محدود. توفر هذه النتائج مساهمات الثاني فيحلل أنظمة بارابولية، أصلية في دراسة النوعية للأنظمة المزدوجة ذات الحدود المتفردة و غير المحلية .

الكلمات المفتاحية: المعادلات التفاضلية الجزئية، الأنظمة المزدوجة، التفرد، العوامل غير المحلية، لابلاس ، الانفجار، منحنى فوجيتا الحرج، الحلول الضعيفة