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Sur le thème

Some Banach Autoregressive Estimation Problems

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Dedication

“Dedicated to the memory of My defunct supervisor,**TaharMourid**, who always believed in my ability to be successful in the academic arena. You are gone but your belief in me has made this journey possible.”

Declaration

I declare that I had accomplished this doctoral thesis independently using only the cited sources, literature and other experienced sources. It has not been used to acquire another or the same degree. I am aware of the fact that my dissertation relates to the rights and obligations under the Act n°98-254, n° 10-231, n° 153, n° 704 and n°1082

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Contents

The Functional Autoregressive Model of order 1 (FAR1) generalizes to random elements with values in an infinite dimensional space the classical AR(1) model belonging to the celebrated class of ARMA process, widely used in time series analysis. This model was introduced by Bosq [9], then studied by several authors. Several chapters in Bosq [10] are dedicated to a thorough study of this strictly stationary process

Introduction

The present-day age is notorious for being an data-rich era where big data loom large in the probe of scientific issues and supply abundant information for statistical inference. Their analyse outfits challenges in various research fields, including economics, finance, medicine, biology, and energy and then compels essential to develop advanced techniques to find out substantial amounts of data and evoking the rudimentary information in an adaptable approach. In addition, profound dependence structures, big sample sizes and high dimensions in big data wreak another rank of statistical and computational concerns that cannot be fixed using traditional statistical methods. It has been very important to build sophisticated and compromising methods to analyze rigourous structures in big data and grasp the basic motive.

Recently several models and methodologies such as in [RD91] have been developed for handling big data sets as being naturally represented as a series of dependent functions or curves. In the high-dimensional area, these fact arise innately in many cases such as electricity price curves studied in [CL17] and in [CMZ19], natural gas flow curves in [CCK18] and plenty of similar examples in several works. This is due to the fact that data, what ever the applied science field they become from are measured at number of successive time points or over a continuous time interval and then they are described as realizations of random curves and analysed through functional data analysis FDA which is now-a-day present a major topic in statistics. For practical purposes, these curves are commonly gatherd as vectors of high-dimension with extremely correlated inputs. Hence, appears the necessity of the dimension reduction techniques that take cognizance of the continuous nature of the data. A guiding light of functional data examples is provided in the monograph [RS02]. New achievements in functional data analysis (FDA) have empowered streamlined methods for analyzing big data with specific characteristics. various books by different authors, such as [HK12] later came to upgrade this topic. Meanwhile, a huge number of articles that deal with Functional Data Analysis, are published as overview papers and surveys [Cue14; GV16]. Specific issues on FDA emerged in Computational Statistics & Data Analysis (vol. 51, no. 10, 2007), Multivariate Analysis (vol. 101, no. 2, 2010), and in Journal of Statistica Sinica (vol. 14, no. 3, 2004).

In FDA, the observed time series are viewed as discrete realisations of a continuous function or curve. With a parsimonious and obvious functional representation, the high-dimensional data is modulated to a series of curves and can then be analyzed with advanced efficiency and accuracy. For more details on independent functional data analyses, we can cite the book of Ferraty and Vieu [FV06] in which the authors provide an innovative approach of nonparametric statistical methods for functional data analysis, the article of Müller and Stadtmüller [MS05] where they introduce the generalised functional linear models and Wang et al.[WCM16] in which authors overview statistical concepts on FDA.

Since functional data are collected over time, it goes without saying that they set out serial dependence that mismatch with the independent identically distributed i.i.d. assumption commonly used. functional regressions provide a good illustrative example. This stimulates the study of functional time series analysis, that focuses on the comprehension of the serial dependence between the curves, the modeling of their dynamics over time, and the performance of statistical prediction. Nevertheless, the analysis of functional time series become a fast developing field by the emerging of the so called functional autoregressive (FAR). Bosq [Bos91] was pionneer in introducing the first version of them, the autoregressive Hilbertian process(ARH), a model under Hilbert space. His work is likely the most popular groundbreaking one and has played a major role in this context. A functional autoregressive process being a natural extension of the scalar and vector-valued AR process introduce by Brockwell and Davis in [BD91] to the infinite-dimensional space is statistically and mathematically susceptible to be used as a matter of fact for modeling and prediction of continuous time series so that many applications have been successfully performed. Examples stretch from road traffic, air pollution concentration levels, electricity consumption forecast to El-Nino temperature.

A painstaking research has been yet expanded involving the Banach-valued time series structure such as the results of strong-consistency presented in [**(Pumo92;Pum98)**] on the estimation of an $ARB(1)$, the Banach-valued autoregressive process of order where $B = C([0,1])$ denoted by $ARC(1)$. After that, Mourid extended it to order greater than one [MOU93] and he characterised a class of real continous time processes by an $ARC(1)$ process, such as the Ornstein-Uhlenbeck process or the one owing a seasonal component [Mou95].

Functional autoregressive process contemplate chronological correlation between the functional observations and models it as a linear operator on a functional space. Bosq offered in his monograph [Bos00] a comprehensive theory of the general linear functional process involving the limit theorem of linear process in both Hilbert and Banach spaces. forasmuch, Guillas developed a non-causality approach for Banach-valued stochastic processes (see[Gui01]), while Benyelles and Mourid established relevant presentation for the periodicity of an Banach autoregressive process. Mas in [Mas07]supplemented this theoretical study by tackling the issue of weak convergence for estimation in functional autoregressive model, which went to show the appearance of common effects about weak convergence in nonparametric models.

Several phenomenon in real life show a behavior with basic patterns that repeat themselves within a certain period. Their modelisation using functional time series put on display seasonal variations that can be seen as the presence of periodic oscillations. The mastery of the seasonal changes in curves intensify the forecast preciseness. In the article [CMZ19], authors presented the $WFAR$ model (Warping Functional Autoregressive model) to synchronically explain the phase and amplitude variations of functional time series with seasonality by separating the amplitude changes in the functional curves from seasonal phase variations using seasonal adjustment methods (warping), and then the seasonally-adjusted curves are used in Functional Autoregressive framework. In [Zam+22], authors claim that the standard formulation functional autoregressive models comes to grief in handling seasonal behaviour in functional time series data. On that account, they introduced seasonal functional autoregressive time series models and proved their merits using simulation studies and via an application to hourly pedestrian counts.

In this Thesis we study a seasonality perturbed by a real continuous time process admitting a Functional Autoregressive representation.
Strictly speaking we consider a real process $(Y(t), t \in \mathbb{R})$ defined by :

$$Y(t) = m(t) + \eta(t), \qquad t \in \mathbb{R}$$

where $(\eta(t), t \in \mathbb{R})$ a zero mean continuous time process admitting a Functional Autoregressive representation in the Banach space $C_{[0,\delta]}$ of continuous functions on $[0, \delta]$, $\delta > 0$, and where $m(t)$ is a periodic continuous real function of period δ. An example of random noise $(\eta(t), t \in \mathbb{R})$ is the Ornstein–Uhlenbeck process [Bos00].
Notice that the process $(Y(t), t \in \mathbb{R})$ as well admits a Functional Autoregressive representation in the same space $C_{[0,\delta]}$ but it is not necessarily stationary even if the process $(\eta(t), t \in \mathbb{R})$ is so.
The real process mean estimation has been investigated by a lot of researchers in various situations. Without entirety, one can cite: [BL14] and [BL16] where authors, considering the equidistant case and Gaussian of errors has investigated the impact of long and short-range dependence in the random noise on the estimation of the mean $EY(t)$ and provided functional limit theorems for kernel estimators and kernel covariance $C(s, t)$. Sooner, the case of independent identically distributed (i.i.d.) noise and randomly distributed time points was examined in [Yao07] and in [Yao+03]. Nonparametric estimation of the mean $\mu(t) = EX(t)$ in the case of repeated time series is further contemplated in [HW86; LC00] among many others. Kernel estimation of $\mu(t) = EX(t)$ in the case of repeated series assuming that errors are strongly dependent is discussed in [Gho01].

The current work is dedicated to study statistical results of the seasonality m addressed in the context of functional autoregressive processes. getting hands on limit theorems such as strong law of large numbers and compact iterated logarithm law for functional autoregressive processes, we derive limit results for the seasonality m. In his papers [Ant82; Ant88], the author Anestis Antoniadis elaborated certain estimation problems and tests on the mean for infinite dimensional gaussian models in the case of independent identically distributed (i.i.d.) with confidence balls deriving from the law of compact iterated logarithm applied for Banach space valued random variables. In the light of this work and in the context of non i.i.d. case and not necessarily gaussian random noise functional autoregressive processes, we construct an estimator and confidence balls for the seasonality m in the function space $C_{[0,\delta]}$ using the compact iterated logarithm law. In addition, when the seasonality belongs to a known finite dimensional space of (dimensional reduction), we study an estimator on the seasonality m giving its asymptotic properties. Finally, the dimension of this space is unknown we also study an estimator of its dimension. We follow the same steps by [Ant82]. furthermore, we illustrate the asymptotic results of the estimators through numerical simulations.

The manuscript is structured as follows:

The preliminary chapter presents an overview of random variables and random processes in Banach space. There is a wide range of literature concerning such topic such us [LT91], [Bha72] and many others. An ample amount of this chapter is a summary of well-known results, it provides the required background for the specific results in the succeeding chapters.

In chapter two, are garnered small amount of fundamental facts concerning order one autoregressive processes in a separable Banach space. It points out the most appropriate properties, enumerates different limit laws theorems and discuss the autoregressive representation of continuous processes

The third one deals with a special case where the Banach space is the space of continuous functions defined on a closed interval. More precisely, it pertain to the situation where a seasonality is perturbed by a continuous process owing an autoregressive representation in a such space. This chapter comes to terms with our published article entitled "*functinal autoregressive process with seasonality*" [BM22]. It gives an estimation of a seasonality, provides the almost sure convergence, asymptotic normality and compact iterated logarithm law and builds confidence balls from compact iterated logarithm law. Within the context of dimensional reduction, it discuss asymptotic properties of its estimator when the seasonality belongs to a finite dimensional space in both cases of known and unknown dimension. In the eventual cause an examination of an estimator of the dimension of this space holds.

Numerical simulations illustrating the asymptotic results of the estimators and the results behind are exposed in the last fourth chapter.

Chapter 1

Preliminaries

Whereas classical probability theory is concerned primarily with real-valued random variables and processes. The need to study random variables and processes with values in general topological spaces was pointed out in 1947 by Frichet [Fré48] and in 1953, Mourier published her fundamental paper [Mou53] which initiated the systematic study of random variables with values in a Banach space. In this chapter we will deal without details with the notion of Banach space-valued random variables in section 1. In section 2 we deal with convergence concepts and the third one is devoted to introduce the theory of integration for Banach space-valued random variables. Then, in section 4, we introduce Banach Spaces valued stochastic processes. Section 5 deals with the Karhunen-Loève Expansion and is succeeded by the sixth one referring to the introduction of bilateral standard Wiener.
It is appropriate to mention here that almost all results are from the monograph of Bosq [Bos00] otherwise the reference is indicated.

1.1 Banach space-valued random variables

In all that follows, we design by $(\Omega, \mathcal{A}, P)$ the complete probability space where Ω is the set where occurs the random experiment, $\mathcal{A}$ is its σ-algebra and P is a probability measure over $\mathcal{A}$.
Denote by B a real separable Banach space equipped with its Borel σ-algebra $\mathcal{B}$ and its norm $\|.\|$. We use B^* to denote the dual space of B (i.e., the set of all continuous linear functionals defined on B). The natural uniform norm on B^* is also denoted by $\|.\|$ and is defined by

$$\|x^*\| = \sup_{\substack{x \in B \\ \|x\| \leq 1}} |x^*(x)|, \quad x^* \in B^*.$$

Denote by $L(B)$ the Banach algebra of bounded linear operators defined over B equipped with the uniform norm $\|.\|_L$.

Definition 1.1. *A B-valued random variable (or B-random variable) defined on* $(\Omega, \mathcal{A}, P)$ *is an $\mathcal{A}$-$\mathcal{B}$ measurable mapping*

Theorem 1.1. *[LT91] A mapping $X : (\Omega, \mathcal{A}) \to (B, \mathcal{B})$ is a B-valued random variable if* *and only if $x^*(X)$ is a real random variable for every $x^* \in B^*$*

Example 1.1. *Let $C[0,1]$ be the space of continuous real functions defined on $[0,1]$. It is a separable Banach space when equipped with the sup-norm*

$$\|x\| = \sup_{t\in[0,1]} |x|, \quad x \in C[0,1],$$

According to Riesz theorem, its dual is the space $\mathcal{M}[0,1]$ of bounded signed measures over $([0,1], \mathcal{B}_{[0,1]})$. From the theorem 1.1, we can affirm that a real process with continuous sample paths is a $C[0,1]$-random variable.

1.2 Integration of Random variables in Banach Spaces

Definition 1.2. *A B-random variable X is said to be weakly integrable if $x^*(X)$ is integrable for all x^* in B^* and if there exists an element of B, denoted by $E(X)$, such that*

$$E(x^*(X)) = x^*(E(X)), \quad x^* \in B^*.$$

$E(X)$ is called the weak expectation or weak integral of X.

Definition 1.3. *A B-random variable X is said to be integrable (strongly integrable) if $E\|X\| < \infty$.*

Theorem 1.2. *Let X be an integrable B-random variable. Then, there exists a sequence (X_n) of simple B-random variables such that*

$$\lim_{n\to\infty} E(\|X_n - X\|) = 0 \tag{1.1}$$

Moreover, X is weakly integrable

Definition 1.4. *If X is integrable, $E(X)$ is called the integral or expectation of X. It is also denoted by $\int XdP$. Finally, we set*

$$\int_A XdP = \int \mathbb{1}_{\{A\}} XdP, \qquad A \in \mathcal{A}.$$

Example 1.2. *Let $\eta = (\eta_t, \quad 0 \leq t \leq 1)$ be a real random process with continuous sample paths. As it was seen in example(1.1), it defines a $C[0,1]$-random variable, and if $E(\sup\limits_{0\leq t\leq 1} |\eta_t|) < \infty$, it is integrable and*

$$E(\eta)(t) = E[\eta(t)], \quad t \in [0,1].$$

Properties[Bos00]

1. The space L^1_B of equivalence classes of integrable B-random variables (with respect to the equivalence relation $X = Y$ **a.s.**) is a Banach space with respect to the norm

$$\|X\|_{1,B} = E(\|X\|).$$

2. E defines a continuous linear operator from L^1_B to B, which satisfies the contractive property $\|E(X)\| \leq E(\|X\|)$.

3. Let B_1 and B_2 be two separable Banach spaces and let $\ell \in \mathcal{L}(B_1, B_2)$ (the space of continuous linear operators from B_1 to B_2. If $X \in L^1_{B_1}(P)$, then $\ell(X) \in L^1_{B_2}(P)$ and

$$E[\ell(X)] = \ell[E(X)].$$

4. **Dominated convergence:** If $X_n \to X$ a.s. in B and $\|X_n\| \leq Y$ a.s., where $n \geq 1$ and Y is an integrable real random variable, then $X_n \in L^1_B(P), n \geq 1, X \in L^1_B(P)$ and $E(\|X_n - X\| \underset{n\to\infty}{\to} 0$.

5. The space $L^p_B(P)$, where $1 < p < \infty$, of classes of B-random variables X such that $E(\|X\|^p) < \infty$, is a Banach space with respect to the norm $\|X\|_{p,B} = [E(\|X\|^p)]^{\frac{1}{p}}$.

6. The space $L^\infty_B(P)$ of essentially bounded B-random variables (i.e., where there exists b such that $P(\|X\| < b) = 1$) is a Banach space with respect to the norm

$$\|X\|_{\infty,B} = \inf\{b > 0 : P(\|X\| < b) = 1\}$$

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1.3 Covariance and Cross-Covariance Operators in Banach spaces

Covariance Operator

Let X be a centred random variable in $\mathcal{L}^2_B(\Omega, \mathcal{A}, P)$.

Definition 1.5. *The covariance operator C_X of X is the bounded linear operator from B^* to B, defined by*

$$C_X(x^*) = E[x^*(X)X], \quad x^* \in B^* \tag{1.2}$$

Remark 1.1. • *If X is not centred, we set $C_X = C_{X-E(X)}$*

- *C_x is completely determined by the covariance function of X, defined as*

$$c_X(x^*, y^*) = y^*[C_X(x^*)] = E[x^*(X)y^*(X)] = Cov(x^*(X)y^*(X)); \quad x^*, y^* \in B^*$$

Example 1.3. *1. In $\mathbb{R}^n$, the covariance matrix is given by*

$$C_x(x_1, \cdots, x_n) = E\left[\left(\sum_{i=1}^{n} x_i\right) X\right],$$

where $x^* = (x_1, \cdots, x_n)$ and $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. In this case, c_X is given by

$$c_X[(x_1, \cdots, x_n), (y_1, \cdots, y_n)] = \sum_{i=1}^{n}\sum_{i=1}^{n} x_i E(X_i X_j) y_j,$$

2. Let X be a random variable with values in $C[0,1]$ such that $E\left(\sup_{0\le t\le 1} |X_t|^2\right) < \infty$, then

$$c_X(\mu, \nu) = \int_0^1 \int_0^1 E(X_s X_t) d\mu(s) d\nu(t); \qquad \mu, \quad \nu \in \mathcal{M}([0,1]) \tag{1.3}$$

Now, we state a condition for a mapping to be a covariance operator.

Theorem 1.3. *If C is a mapping from B^* to B, then there exists a B-random variable X such that $C = C_X$ if and only if for each fixed sequence (x_i) in B such that $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ we have*

$$C(x^*) = \sum_{i=1}^{\infty} x^*(x_i)x_i, \qquad x^* \in B^*, \tag{1.4}$$

Cross-Covariance Operators

Let B_1 and B_2 be two separable Banach spaces and let X and Y two centred random variables with values respectively in $L^2_{B_1}(P)$ and $L^2_{B_2}(P)$.

Definition 1.6. *The cross-covariance operators of X and Y are bounded linear operators defined by*

$$\begin{array}{rcccl} C_{X,Y} & : & B_1^* & \rightarrow & B_2 \\ & & x^* & \mapsto & E[x^*(X)Y] \end{array}$$

and

$$\begin{array}{rcccl} C_{Y,X} & : & B_2^* & \rightarrow & B_1 \\ & & y^* & \mapsto & E[y^*(Y)X] \end{array}$$

It is easy to see that

$$y^*[C_{X,Y}(x^*)] = x^*[C_{Y,X}(y^*)] = E[x^*(X)y^*(Y)], \quad x^* \in B_1^*, y^* \in B_2^*.$$

Hence, the cross-covariance function is given by

$$c_{X,Y}(x^*, y^*) = E[x^*(X)y^*(Y)], \quad x^* \in B_1^*, y^* \in B_2^*.$$

1.4 Random Variables Convergence in Banach Spaces

Let $(X, X_n, n \geq 1)$ be a family of B-valued random variables defined on the probability space $(\Omega, \mathcal{A}, P)$

Definition 1.7. *we say that the sequence X_n converges to X almost surely and we write $X_n \xrightarrow{a.s.} X$, if*

$$P\{\omega \in \Omega : \|X_n(\omega) - Y(\omega)\| \underset{n\to+\infty}{\longrightarrow} 0\} = 1$$

Definition 1.8. *we say that the sequence X_n converges to X in probability and we write $X_n \xrightarrow{p} X$, if for each $\varepsilon > 0$, we have*

$$P(\|X_n - X\| > \varepsilon) \underset{n\to+\infty}{\longrightarrow} 0.$$

$X_n \xrightarrow{\text{a.s.}} X$ implies that $\xrightarrow{\text{p}} X$ and the converse fails.

Definition 1.9. *we say that the sequence X_n converges to X in $L^2_B(\Omega, \mathcal{A}, P)$ and we write $X_n \xrightarrow{L^2_B(P)} X$, if*

$$E(\|X_n - X\|^2) \underset{n\to+\infty}{\longrightarrow} 0.$$

$X_n \xrightarrow{L^2_B(P)} X$ implies that $\xrightarrow{\text{p}} X$ and the converse fails.

Let $(\mu, \mu_n, n \geq 1)$ a family of probability measures over $(B, \mathcal{B}_B)$.

Lemma 1.4. *The five following conditions are equivalent:*

1. *$\mu_n(A) \longrightarrow \mu(A)$ for all $A \in \mathcal{B}_B$ such that $\mu(\partial A) = 0$, where ∂A denotes the boundary of A*
2. *$\limsup \mu_n(A) \leq \mu(A)$ for all closed set A.*
3. *$\liminf \mu_n(A) \geq \mu(A)$ for all open set A.*
4. *$\int f d\mu_n \longrightarrow \int f d\mu$ for all f in the class of bounded continuous real functions on B*
5. *$\int f d\mu_n \longrightarrow \int f d\mu$ for all bounded uniformly continuous real f*

Definition 1.10. *We say that a sequence $(\mu_n, n \geq 1)$ converges weakly to the measure μ and we write $\mu_n \xrightarrow{w} \mu$ if any of the five above equivalent conditions holds.*

Definition 1.11. *we say that the sequence X_n converges to X in distribution or weakly and we write $X_n \xrightarrow{\boldsymbol{d}} X$, if*

$$P_{X_n} \xrightarrow{\boldsymbol{w}} P_X$$

where P_{X_n} and P_X are the respective distribution of X_n and X.

$X_n \xrightarrow{\mathbf{p}} X$ implies that $X_n \xrightarrow{\mathbf{d}} X$ and the converse fails.

Theorem 1.5 (Dudley-Skorokhod Theorem)**.** *If $X_n \xrightarrow{\boldsymbol{d}} X$, there exists a family of B-random variables $(T, T_n, n \geq 1)$ defined on some probability space $(\Omega', \mathcal{A}', P')$ such that*

$$P_T = P_X, \qquad P_{T_n} = P_{X_n}, \qquad \textit{for all} \quad n \geq 1$$

and

$$T_n(\omega') \longrightarrow T(\omega'), \qquad \omega' \in \Omega'.$$

Theorem 1.6 (Geffroy-Ito-Nisio (GIN) Theorem)**.** *Let $(X_n, n \geq 1)$ be a sequence of B-valued independent random variables. Set $S_n = X_1 + \cdots + X_n, \quad n \geq 1$. The following conditions are equivalent:*

1. *(S_n) converges weakly.*
2. *(S_n) converges in probability.*
3. *(S_n) converges almost surely*

1.4.1 Strong Law of Large Number

Let $(X_n, n \geq 1)$ be a sequence of i.i.d. centred integrable B-random variables.

Let associate to each X_i and each $\varepsilon > 0$ a simple random variable

$$X_{i,\varepsilon} = \sum_{i=1}^{N_\varepsilon} x_i \mathbb{1}_{\{X_i \in B_i\}}$$

such that

$$E(\|X_i - X_{i,\varepsilon}\|) < \varepsilon$$

where $(B_i, i \geq 1)$ is a partition of B in Borel sets of diameter less than $\frac{\varepsilon}{2}$.

By the scalar law of large numbers, we have

$$\limsup_{n\to\infty} \| \frac{S_n}{n} - \frac{\sum_{i=1}^{n} X_{i,\varepsilon}}{n} \| \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|X_i - X_{i,\varepsilon}\| < \varepsilon \qquad a.s. \tag{1.5}$$

In an other hand, we have

$$\frac{1}{n}\sum_{j=1}^{n} X_{i,\varepsilon} = \sum_{i=1}^{N_\varepsilon} x_j \left(\sum_{i=1}^{n} \mathbb{1}_{\{X_i \in B_j\}} \right), \quad n \geq 1$$

Notice that $\mathbb{1}_{\{X_i \in B_i\}}$ are i.i.d. Bernoulli random variables which satisfy the scalar law of large numbers, hence

$$\frac{1}{n}\sum_{j=1}^{n} X_{i,\varepsilon} \xrightarrow{\textbf{a.s.}} \sum_{i=1}^{N_\varepsilon} x_j P\left(X_i \in B_j\right) = E\left(X_{1,\varepsilon}\right).$$

Using (1.5) and the fact that $\|E\left(X_{1,\varepsilon}\right)\| < \varepsilon$, we get

$$\limsup \| \frac{S_n}{n} \| < 2\varepsilon, \quad \textbf{a.s.}, \quad \varepsilon > 0,$$

which leads to law of large numbers for (X_n)

$$\| \frac{S_n}{n} \| \xrightarrow{\textbf{a.s.}} 0$$

Now, consider that the random variables X_n are not centered but of an expectation m, then if we put $Y_n = X_n - m$, we get

$$\| \frac{S'_n}{n} \| \xrightarrow{\textbf{a.s.}} 0$$

where $S'_n = \sum_{i=1}^{n} Y_i$. Hence

$$\frac{S_n}{n} = \frac{S'_n}{n} + m.$$

which the law of large numbers for i.i.d B-valued random variables(X_n) of mean equals to m

$$\|\frac{S_n}{n}\| \xrightarrow{\textbf{a.s.}} m \tag{1.6}$$

In order to get a convergence rate of the law of large numbers, let enunciate some large deviation inequalities

Lemma 1.7. *Suppose that there exists $c > 0$ such that $E(e^{c\|X_i\|}) < \infty, \quad 1 \leq i \leq n$. Then, for all $x_1, \cdots, x_n$ in B we have*

$$Var(\|S_n\|) \leq \sum_{i=1}^{n} E(\|X_i - x_i\|^2), \tag{1.7}$$

$$E(e^{c\|S_n\|}) \leq \exp\left[cE(\|S_n\|) + \sum_{i=1}^{n} E\left(e^{c\|X_i - x_i\|} - 1 - c\|X_i - x_i\|\right)\right] \tag{1.8}$$

and

$$E(e^{c\|S_n\|}) \leq e^{cE(\|S_n\|)} \prod_{i=1}^{n} E\left(e^{c\|X_i - x_i\|} - c\|X_i - x_i\|\right) \tag{1.9}$$

Proof. Let $\mathcal{A}_i$ be the σ-algebra generated by $(X_1, \cdots, X_i), \qquad i \geq 1$ and $\mathcal{A}\prime = \{\emptyset, \Omega\}$. We put

$$T_i = E^{\mathcal{A}_i}(\|S_n\|) - E^{\mathcal{A}_{i-1}}(\|S_n\|), \qquad 1 \leq i \leq n1$$

and

$$Z_i = E^{\mathcal{A}_i}(\|S_n\| - \|S_n - (X_i - x_i)\|), \qquad 1 \leq i \leq n.$$

It is easy to see that

$$Z_i \leq \|X_i - x_i\| \qquad (a.s.) \tag{1.10}$$

Notice that since $S_n - (X_i - x_i)$ is independent from X_i, we have

$$T_i = Z_i - E^{\mathcal{A}_i}(Z_i). \tag{1.11}$$

Combining (1.10) and (1.11) we get

$$E^{\mathcal{A}_{i-1}}(T_i^2) \leq E(\|X_i - x_i\|^2.$$

Thus

$$\sum_{i=1}^{n} E^{\mathcal{A}_{i-1}}(T_i^2) \leq \sum_{i=1}^{n} E(\|X_i - x_i\|^2),$$

which proves (1.7). Now using again (1.10) and (1.11) and the elementary inequality

$$e^s - s \leq e^t - t, \qquad |s| \leq t$$

we get successively

$$E^{\mathcal{A}_{i-1}}(e^{T_i}) = e^{-E^{\mathcal{A}_{i-1}}(Z_i)} \left[E^{\mathcal{A}_{i-1}}(e^{Z_i} - Z_i) + E^{\mathcal{A}_{i-1}}(Z_i)\right]$$

and

$$E^{\mathcal{A}_{i-1}}(e^{T_i}) \leq e^{-\beta_i}(\alpha_i + \beta_i) where \alpha_i = E(e^{\|X_i - x_i\|} - \|X_i - x_i\|). \tag{1.12}$$

Since $e^{-\beta_i}(\alpha_i + \beta_i) \leq e^{-\alpha_i - 1}$, we obtain

$$E^{\mathcal{A}_{i-1}}(e^{T_i}) \leq e^{-\alpha_i - 1}. \tag{1.13}$$

In an other hand, we have

$$\sum_{i=1}^{n} T_i = \|S_n\| - E(\|S_n\|).$$

So

$$E(e^{\|S_n\|}) = e^{E(\|S_n\|)} E\left(e^{\sum_{i=1}^{n} T_i}\right).$$

From (1.13), we obtain

$$\sum_{i=1}^{n} T_i \leq \exp\left(E(\|S_n\|) + \sum_{i=1}^{n} \alpha_i - n\right)$$

and by replacing X_i by cX_i, we get (1.8) Now, let prove (1.9). As above let define $\mathcal{B}_i$ the σ-algebra generated by $(X_k, k \in \{1, \cdots, n\} - \{i\})$. Using the Jenson inequality and the fact that a norm is convex, we get

$$\begin{aligned} E^{\mathcal{A}_{i-1}}(\|S_n\|) &= E^{\mathcal{A}_{i-1}}(E^{\mathcal{B}_i}(\|S_n\|)) \\ &\geq E^{\mathcal{A}_{i-1}}(\|E^{\mathcal{B}_i}(S_n)\|) \\ &= E^{\mathcal{A}_{i-1}}(\|S_n - X_i + E(X_i)\|)). \end{aligned}$$

So

$$\beta_i = E^{\mathcal{A}_{i-1}}(Zi) = E^{\mathcal{A}_{i-1}}(\|S_n\|) - E^{\mathcal{A}_{i-1}}(\|S_n - X_i + E(X_i)\|)) \geq 0.$$

Combining (1.12) and the following elementary inequality

$$e^{-b}(a+b) \leq a, \qquad b \geq 0,$$

we get

$$E^{\mathcal{A}_{i-1}}(e^{Ti}) \leq \alpha_i,$$

and then we obtain

$$E(e^{\|S_n\|}) \leq e^{E(\|S_n\|)} \prod_{i=1}^{n} \alpha_i,$$

which is not but (1.9) when replacing cX_i by X_i. □

Theorem 1.8. *Let $X_1, \cdots, X_n$ be independent B-random variables and $x_1, \cdots, x_n$ be elements of B. Suppose that there exists positive constants a and b such that*

$$\sum_{i=1}^{n} E(\|X_i - x_i\|^k) \leq \frac{k!}{2} b^2 a^{k-2}; \qquad k = 2, 3, \cdots. \tag{1.14}$$

Then

$$P(|\|S_n\| - E(\|S_n\|)| \geq t) \leq 2 \exp\left(-\frac{t^2}{2b^2 + 2at}\right) \tag{1.15}$$

Proof. Notice that for $c < \frac{1}{a}$ we have

$$\sum_{i=1}^{n} E\left(e^{c\|X_i - x_i\|} - 1 - c\|X_i - x_i\|\right) \leq \frac{c^2 b^2}{2(1-ca)}. \tag{1.16}$$

And from the classical Bernstein bound we have

$$P(\|S_n\| \geq \alpha) \leq e^{-c\alpha} E\left(e^s \|S_n\|\right), \qquad \alpha > 0. \tag{1.17}$$

Then for $\alpha = t + E(\|S_n\|)$ and $c = \frac{t}{b^2+at}$ we use (1.8) to obtain

$$P(|\|S_n\| - E(\|S_n\|)| \geq t) \leq \exp\left(-\frac{t^2}{2b^2 + 2at}\right).$$

This last inequality is true for $-X_i, \quad 1 \leq i \leq n$, hence (1.15) holds. □

Corollary 1.1. *If* $\|X_i - x_i\| \leq a \quad 1 \leq i \leq n$ *then*

$$P(|\|S_n\| - E(\|S_n\|)| \geq t) \leq 2\exp\left(-\frac{t^2}{2c_n^2 + (2/3)at}\right), t > 0 \tag{1.18}$$

where $c_n^2 = \sum_{i=1}^{n} \|X_i - x_i\|^2$.

Proof. Using the fact that the function $\frac{e^u - 1 - u}{u^2}$ is increasing one and $\|X_i - x_i\| \leq a$ we get

$$\sum_{i=1}^{n} E\left(e^{c\|X_i - x_i\|} - 1 - c\|X_i - x_i\|\right) \leq \frac{e^{ca} - 1 - ca}{a^2} c_n^2.$$

Combining (1.16),(1.17),(1.7) and (1.8) and taking $c = \frac{1}{a}\log\left(1 + \frac{ta}{c_n^2}\right)$ we obtain (1.18) □

The independence assumption considered above is often not suitable in practice. So we need a more general law of large numbers.

Theorem 1.9. *Let* $(X_i, i \geq 1)$ *be a sequence of zero-mean B-random variables Suppose that there exist two real sequences* $(f(p) : p \geq 0)$ *and* $(g(m) : m \geq 0)$ *such that*

1. $E\|X_n + \cdots + X_{n+p-1}\|^2 \leq f(p), \quad p \geq 1, \quad n \geq 1,$
2. $f(p) = o(p^2)$ *as* $p \to \infty$,
3. $(g(m))$ *is an increasing sequence of integers and* $g(0) = 0$,

4. $\displaystyle\sum_{m=1}^{\infty}\left[\frac{g(m+1)}{g(m)}-1\right]^2<\infty$ *and*

5. $\displaystyle\sum_{m=1}^{\infty}\frac{f(g(m))}{g^2(m)}-1<\infty$

Then

$$\frac{S_n}{n}\to 0$$

almost sure and in $L_B^2(P)$

Proof. From the first and second items we have

$$\limsup_{n\to\infty}E(\|\frac{S_n}{n}\|^2)\leq\limsup_{n\to\infty}\frac{f(n)}{n^2}=0$$

which prove the convergence in $L_B^2(P)$. Now, let apply Tchebychev inequality and the first item. We get

$$P\left(\|\frac{S_{g(m)}}{g(m)}\|\geq\varepsilon\right)\leq\frac{E\left(\|S_{g(m)}\|^2\right)}{\varepsilon^2g^2(m)}\leq\frac{1}{\varepsilon^2}\frac{f(g(m))}{g^2(m)},\quad\varepsilon>0.$$

Combining the last item with the Borel-Cantelli Lemma we obtain

$$\|\frac{S_{g(m)}}{g(m)}\|\xrightarrow{\textbf{a.s.}}0\text{ as }m\to\infty.\tag{1.19}$$

We put

$$M_m=\max_{g(m)<n\leq g(m+1)}\|\frac{S_n-S_{g(m)}}{n}\|^2,\quad m\geq 1.$$

Notice that

$$M_m\leq\frac{1}{g^2(m)}\left(\|X_{g(m)+1}\|+\cdots+\|X_{g(m+1)}\|\right)^2,$$

then we get

$$E(M_m)\leq\frac{1}{g^2(m)}\left[\sum_{i=1}^{g(m+1)-g(m)}\left(E(\|X_{g(m)+i}\|^2)\right)^{\frac{1}{2}}\right]^2.$$

For $p=1$ it follows that

$$E(M_m)\leq\frac{1}{g^2(m)}\left[g(m+1)-g(m)\right]^2f(1).$$

Using Markov inequality and Bore-Cantelli Lemma with association of the fourth item we get

$$M_m \xrightarrow{\text{a.s.}} 0 \text{ as } m \to \infty.$$

According to the third item, for each integer $n \geq 1$ and m_n such that $g(m_n) < n \leq g(m_n + 1)$ we have

$$M_{m_n} \xrightarrow{\text{a.s.}} 0 \text{ as } m \to \infty, \tag{1.20}$$

since $n \to \infty$ entails $m_n \to 0$.

Let now consider the relation

$$\frac{S_n}{n} = \frac{S_n - S_{g(m_n)}}{n} + \frac{S_{g(m_n)}}{g(m_n)} \frac{g(m_n)}{n}$$

which leads to

$$\|\frac{S_n}{n}\| \leq M_{m_n}^{\frac{1}{2}} + \|\frac{S_{g(m_n)}}{g(m_n)}\|$$

We deduce fro (1.19) an (1.20) that

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0,$$

hence our proof is ended. □

Corollary 1.2. *et $(X_i, i \geq 1)$ be a sequence of zero-mean B-random variables such that*

$$E\|X_n + \cdots + X_{n+p-1}\|^2 \leq cp^\gamma, \quad p \geq 1, \quad n \geq 1 \tag{1.21}$$

where $c > 0$ and $\gamma \in]0, 2[$ are constants.

Then, for all $\beta > \frac{1}{2}$,

$$\frac{n^{(2-\gamma)/4}}{(\log n)^\beta} \|\frac{S_n}{n}\| \xrightarrow{a.s.} 0. \tag{1.22}$$

Proof. From (1.21) and Tchebychev inequality it follows that, for all $\varepsilon > 0$,

$$P\left(\frac{g_m^{(2-\gamma)/4}}{(\log g_m)^\beta} \|\frac{S_{g_m}}{g_m}\| \geq \varepsilon\right) = O\left(\frac{1}{m(\log m)^2\beta}\right).$$

where $g_m = [m^{2/(2-\gamma)}]$, $m \geq 1$

So, if we take $\beta > \frac{1}{2}$, and apply the Borel-Cantelli lemma, we get

$$\frac{g_m^{(2-\gamma)/4}}{(\log g_m)^\beta} \| \frac{S_{g_m}}{g_m} \| \xrightarrow{\textbf{a.s.}} 0 \text{ as } m \to \infty. \tag{1.23}$$

Let set

$$M_m = \max_{g_m < n \leq g_{m+1}} \| \frac{S_n - S_{g_m}}{n^{(2+\gamma)/2} (\log n)^2 \beta} \|^2, \quad m \geq 1.$$

where $g_{m+1} > g_m$ for m large enough.

In the same way as in the proof of Theorem 1.9, we have the bound

$$E(M_m) \leq \frac{c(g_{m+1} - g_m)^2}{g_m^{(2+\gamma)/2} (\log g_m)^{2\beta}},$$

and then

$$E(M_m) \leq \frac{c' m}{(\log g_m)^{2\beta}} \left(\frac{g_{m+1}}{g_m} - 1 \right)^2,$$

where c' is a constant. Notice that for large enough m

$$\frac{g_{m+1}}{g_m} - 1 \leq \frac{4}{m(2-\gamma)}.$$

Consequently

$$E(M_m) = O\left(\frac{1}{m (\log m)^{2\beta}} \right)$$

which ensure the convergence almost sure of M_m to zero as m goes to infinity. Now, if $n \to \infty$, we have $M_{m(n)} \xrightarrow{\textbf{a.s.}} 0$ where $m(n)$ is such that $g_{m(n)} < n \leq g_{m(n)+1}$. So

$$\frac{S_n - S_{g_{m(n)}}}{n^{(2+\gamma)/4} (\log n)^\beta} \xrightarrow{\textbf{a.s.}} 0$$

which combined with (1.23) entails (1.22) □

1.4.2 Central Limit Theorem CLT

A sequence of independant random variables with values in a separable Banach space B is said to satisfy the central limit theorem (CLT) in B if the sequence $(S_n/\sqrt{n})$ converges weakly in B. It is well-known that on the line a random variable X satisfies the CLT if and only if $E(X^2) < \infty$. The sufficiency of the condition extends to the case where X takes values in a finite dimensional space and also to Hilbert space. However, the situation is more intricate in a general Banach space since it depends on some geometrical properties of B, mentioned below

Type and Cotype of Banach space

[Bos00]

Definition 1.12. *A banach space B is said to be of type p where p is a number in $]1,2[$ if there exists a strictly positive constant c such that*

$$E(\|\sum_{i=1}^{n} X_i\|^p) \leq c\sum_{i=1}^{n} E(\|X_i\|^p) \tag{1.24}$$

where X_i are independent centred B-random variables such that $E(\|X_i\|^p) < \infty$

Definition 1.13. *A banach space B is said to be of cotype p where p is a number in $]1,2[$ if there exists a strictly positive constant c such that*

$$E(\|\sum_{i=1}^{n} X_i\|^p) \geq c\sum_{i=1}^{n} E(\|X_i\|^p) \tag{1.25}$$

where X_i are independent centred B-random variables such that $E(\|X_i\|^p) < \infty$

In the following we provide central limit theorems in some Banach spaces (see [LT91]).

Theorem 1.10. *Let B a banach space of type 2 and $X = (X_i, i \geq 1)$be a sequence of B-valued i.i.d. random variables. If $E(\|X_i\|^2) < \infty$, then X satisfies the CLT*

Definition 1.14. *A b-valued random variable is said to be pregaussian if there exists a B-valued Gaussian random variable Y such that $C_y = C_X$.*

Theorem 1.11. *Let $X = (X_i, i \geq 1)$be a sequence of B-valued i.i.d. random variables. If B is of cotype 2 and if X_1 is pregaussian, then X satisfies the CLT*

Theorem 1.12. *Let $X = (X_i, i \geq 1)$be a sequence of $C[0,1]$-valued i.i.d. random vari-*

ables. X satisfies the CLT if

$$|X_1(t) - X_1(s)| \leq M_1|t-s|, 0 \leq s, t \leq 1, \tag{1.26}$$

where M_1 is a positive random variable such that $E(M_1^2) < \infty$,

Example 1.4. *Let X defined by*

$$X_i = a_i \cos 2\pi t + b_i \sin 2\pi t \qquad 0 \leq t \leq 1, i \geq 1$$

where a_i and b_i formed together a family of I.i.d. real random variables with finite variance. It is easy to see that X_i is a sequence of i.d.d. $C[0,1]$-random variables and that

$$|X_1(t) - X_1(s)| \leq M_1|t-s|, 0 \leq s, t \leq 1,$$

where $M_1 = 2\pi(|a_i| + |b_i|)$. So (1.26) is satisfied and the CLT holds.

1.4.3 Compact Law of the Iterated Logarithm (CLIL)

Finally let state the CLIL in a Banach space. We recall the definition of a distance:

$$d(x, A) = \inf(\|x - y\|, y \in A), x \in B, A \subset B$$

and we denote by $c(x_n)$ the set of limit points of the sequence (x_n).
We say that a sequence of centred random variables $X = (X_n, n \in \mathbb{Z})$ satisfies the compact law of the iterated logarithm CLIL if:

$$P\left\{\left(\frac{S_n}{\sqrt{2n \log\log n}}\right)_{n\geq 3} \text{is relatively compact in} (B, \|.\|)\right\} = 1$$

Theorem 1.13. *Suppose that B is a separable Banach space of type 2.*
Let $X = (X_i, i \geq 1)$be a sequence of centred B-valued i.i.d. random variables, such that $E\left(\frac{\|X_1\|^2}{\log\log\|X_1\|}\right) < \infty$ and that the random variable $x^*(X_1), x* \in B^*, \|x^*\| \leq 1$ are uniformly

integrable. Then there exists a compact convex symmetric set K such that

$$\lim_{n\to\infty} d(\frac{S_n}{\sqrt{2n\log\log n}}, K) = 0 \quad a.s. \tag{1.27}$$

and

$$c(\frac{S_n}{\sqrt{2n\log\log n}}) = K \quad a.s. \tag{1.28}$$

where

$$K = x \in B : x = E(\xi X_1, \xi \in L^2(\Omega, \mathcal{A}, P), E(\xi) = 0, E(\xi^2) \leq 1$$

with $c(A)$ is the set of adherant points of a set A

1.5 Stochastic Processes in Banach Spaces

Continuous sample paths processes are observed in several fields such as biology, economy, meteorology, and plenty of other area. To be able of constructing an appropriate random model to such a situation, it is natural to use Banach spaces whose elements are regular functions. If (X_n) is a given sequence of real random variables such that $\sup_n \|X_n\| < \infty$ almost surely, and if for instance we ask for the integrability properties behavior of this supremum, we are clearly faced with a random element of infinite dimension. That is to say, it would be convenient to have a notion of random variable with values in l_∞. Furthermore, it is known that every separable Banach space is isometric to a closed subspace of l_∞. Notice also that there exists another type of infinite dimensional random elements which are the random functions or stochastic processes.

Definition 1.15. *Let T be a (infinite) index set which will be usually assumed to be a metric space (T, d). A random function or process $X = (X_t, t \in T)$ indexed by T is a collection of real valued random variables $X_t, t \in T$.*
By the distribution or law of X we mean the distribution on $\mathbb{R}^T$, equipped with the cylindrical σ-algebra generated by the cylinder sets, determined by the collection of all marginal distributions of the finite dimensional random vectors $(X_{t_1}, \cdots, X_{t_N}), t_i \in T$.

Definition 1.16. *A random process $X = (X_t, t \in T)$ is almost surely bounded or con-*

tinuous, or has almost all its trajectories or sample paths bounded or continuous, if, for almost all ω, the path $t \to X_t(\omega)$ is bounded or continuous.

Remark 1.2. *[LT91]*

1. *If $X = (X_t, t \in T)$ is an almost surely continuous process on (T, d) assumed to be compact, it defines a random variable in the separable Banach space $C(T)$ of all continuous functions on T.*

2. *Notice that for this generalized notion of random variable with values in a Banach space B, almost sure, in distribution and weak convergence of a sequence (X_n) makes sense similarly.*

1.6 Karhunen-Loève Expansion

The Karhunen-Loève expansion can be viewed as an extending form of the Fourier analysis in deterministic functions to random function so to stochastic processes. The Karhunen-Loeve expansion is a representation where the process is decomposed into a series of orthogonal functions and the way of calculating the coefficients is like that of the Fourier analysis, which relays on minimising the mean squared error of the finite representation. The Karhunen-Loève Expansion provides an explicit form of the random variable associated with a second order process. It is interesting in itself and will be used in the construction of a Wiener process (see section1.7).

Lemma 1.14. *Let c be a covariance function continuous over $[0,1]^2$. Then there exists a sequence (f_n) of continuous functions and a decreasing sequence (λ_n) of positive numbers such that*

$$\int_0^1 c(s,t) f_n(s) ds = \lambda_n f_n(t), \qquad t \in [0,1], \quad n \in \mathbb{N}, \tag{1.29}$$

and

$$\int_0^1 f_k(s) f_l(s) ds = \delta_{k,l}, \qquad k, l \in \mathbb{N}, \tag{1.30}$$

Moreover

$$c(s,t)=\sum_{n=0}^{\infty}\lambda_n f_n(s)f_n(t), \qquad t\in[0,1], \quad n\in\mathbb{N}, \tag{1.31}$$

$$\sum_{n=0}^{\infty}\lambda_n=\int_0^1 c(s,s)ds<\infty \tag{1.32}$$

Theorem 1.15. *Let $X=(X_t, 0\le t\le 1)$ be a second order zero-mean measurable process with continuous covariance function c. Then*

$$X_t=\sum_{n=0}^{\infty}\lambda_n f_n(t), \qquad t\in[0,1], \tag{1.33}$$

where (λ_n) is a sequence of real zero-mean random variables such that

$$E(Z_k Z_l)=\lambda_k\delta_{k,l}, \qquad k,l\in\mathbb{N}, \tag{1.34}$$

and where the sequence $(\lambda_n, , f_n)$ is defined in the Mercer lemma. The series in (1.33) converges uniformly with respect to the $L^2(\Omega, \mathrm{A}, P)$-norm.
Moreover, X defines a $L^2([0,1], \mathrm{B}_{[0,1]}, \lambda)$ random variable via (1.33).

1.7 bilateral standard Wiener

A Wiener process (or Brownian motion process) $W=(W_t, t\ge 0)$ is a centered Gaussian process such that

$$c(s,t)=\sigma^2 min(s,t)$$

, where σ^2 is a strictly positive constant.
In this case we have

$$e_i(t) = \sqrt{2}\sin\left[\left(i-\frac{1}{2}\right)\pi t\right] \tag{1.35}$$

$$\mu_i = \frac{1}{\left(i-\frac{1}{2}\right)^2\pi^2} \tag{1.36}$$

as the eigne elements of the integral operator

$$(Af)(t) = \int_0^1 c(t,s)f(s)ds$$

By taking $Z_i^* = \dfrac{Z_i}{\sqrt{\mu_i}}$ in (1.33), we get

$$W_t = \sqrt{2}\sum_{i=1}^{\infty} Z_i^* \frac{\sin\left[\left(i-\frac{1}{2}\right)\pi t\right)\right]}{\left(i-\frac{1}{2}\right)\pi}$$

Remark 1.3.

- *If* $\sigma^2 = 1$, W *is said to be standard.*
- *A Wiener process has independent stationary increments that is to say:*

$$P_{W_{t_1+h}-W_{t_0+h},\cdots,W_{t_k+h}-W_{t_{k-1}+h}} = P_{W_{t_1}-W_{t_0}} \otimes \cdots \otimes P_{W_{t_k}-W_{t_{k-1}}};$$

  where $k \geq 2$, $0 \leq t_0 < \cdots < t_k$ *and* $h \geq 0$.

Chapter 2

First Order Autoregressive Process in Banach spaces *ARB(1)*

"*Observation of processes with continuous or differentiable sample takes place in physics, chemistry, finance, meteorology, and many other fields. In order to construct a random model adapted to such a situation,it is natural to use Banach spaces whose elements are regular functions,instead of general Hilbert spaces.*" (Denis Bosq, 2000, p 147)[Bos00]

In this chapter we collect in rather an informal way some basic facts about autoregressive processes of order one with values in a separable Banach space processes. We give conditions of existence and provide limit theorems.
The material that we present actually only appears as the necessary background for the subsequent analysis developed in the next chapters. Only a few proofs are given and many important results are only just mentioned. Here again it is appropriate to mention that almost all results are from either the monograph of Bosq [Bos00] or the thesis of Mourid [MOU93] otherwise the reference is indicated.

2.1 Definitions and Properties

Definition 2.1. *A B-valued white noise (ε_n) is a sequence of random variables defined on $(\Omega, \mathcal{A}, P)$ and with values in B such that*

$$E\varepsilon_n = 0, \qquad 0 < E\|\varepsilon_n\|^2 = \sigma_\varepsilon^2 < \infty \text{ and } C_\varepsilon := C_{\varepsilon_n}$$

If in addition, the variables are independent and identically distributed, (ε_n) is said to be a strong white noise

Example 2.1. *Let $B = C_{[0,1]}$. For a bilateral Wiener process W, we set*

$$\varepsilon_n^{(\varphi)}(t) = \int_n^{n+1} \varphi(n+t-s)dW(s), \qquad 0 \le t \le 1, n \in \mathbb{Z} \tag{2.1}$$

where φ is a square integrable real function such that $\int_0^1 \varphi^2(t)dW(t) > 0$. Then a continuous version of $(\varepsilon_n^{(\varphi)})$ defines a $C_{[0,1]}$-valued strong white noise.

Let now define the Banach autoregressive process of order one $ARB(1)$.

Definition 2.2. *A B-valued sequence $(X_n,\ n \in \mathbb{Z})$ defined on $(\Omega, \mathcal{A}, P)$ is a Banach autoregressive process of order one ($ARB(1)$) if there exists a linear operator ρ in $\mathrm{L}(B)$ and a B-valued white noise ε such that*

$$X_n - \mu = \rho(X_{n-1} - \mu) + \varepsilon_n. \tag{2.2}$$

Existence and Stationarity

Let set the following condition:

(**A0**): There exists $j_0 < 1$ such that $\|\rho_0^j\|_L < 1$, and let prove the following result

Lemma 2.1. *the condition (**A0**) is equivalent to the following one:*

$$(\textbf{\textit{A1}}): \quad \textit{There exists } a > 0 \quad \textit{and} \quad 0 < b < 1 \quad \textit{such that} \quad \|\rho^j\|_{\mathrm{L}} < ab^j, j \geq 0$$

Proof. It is clear that (**A1**) leads to (**A0**), so it remains to prove that (**A0**) implies (**A1**). For that we just need to show (**A1**) is true for $j > j_0$ and $\|\rho_0^j\|_L < 1$. Let write the result of the euclidian division of such a j by j_0:

$$j = qj_0 + r$$

where $q \geq 1$ and $0 \leq r < j_0$. We get

$$\|\rho^j\|_L \leq \|\rho_0^j\|_L^q |\rho^r\|_L.$$

It is sufficient, therefore, to take $a = \|\rho_0^j\|_L^{-1} \max_{0 \leq r < j_0} |\rho^r\|_L$ and $b = \|\rho_0^j\|^{\frac{1}{j_0}}$. □

The next theorem states the existence and uniqueness of X.

Theorem 2.2. *If the condition (**A0**) is verified, then (2.2) has a unique strictly stationary solution given by*

$$X_n = \mu + \sum_{j=0}^{\infty} \rho^j(\varepsilon_{n-j}), n \in \mathbb{Z} \tag{2.3}$$

the series being convergent in $\mathrm{L}_B^2(\Omega, \mathcal{A}, P)$ *and almost sure.*

Proof. We have, for any integer m and m' such that $m < m'$

$$E\|\sum_{j=m}^{m'} \rho^j(\epsilon_{n-j})\|^2 \leq \sum_{m \leq j,j' \leq m'} \|rho^j\|_L \|rho^{j'}\|_L E(\|\epsilon_{n-j})\| \|\epsilon_{n-j'}\|).$$

Then, by the Cauchy-Schwartz inequality we get

$$\begin{aligned} E\|\sum_{j=m}^{m'}\rho^j(\epsilon_{n-j})\|^2 &\leq \sum_{j=m}^{m'}\|\rho^j\|_L(E\|\epsilon_{n-j}\|^2)^{\frac{1}{2}}]^2 \\ &\leq \sigma_\varepsilon^2(E\|\rho^j\|_L)^2 \end{aligned}$$

According to the previous lemma we have from (**A0**)

$$\|\rho^j\|_L < ab^j \tag{2.4}$$

with $a > 0$ and $0 < b < 1$, which gives the convergence of the series $\sum_{j=0}^{\infty}\rho^j(\epsilon_{n-j})$ in $L_B^2(\Omega, \mathcal{A}, P)$, and the Geffroy-Ito-Nisio (GIN)Theorem (which asserts that various modes of convergence of sums of independent symmetric B-valued random variables are equivalent, see theorem 1.6), assures the almost sure convergence. Now, we consider the stationary process

$$Y_n = \sum_{j=0}^{\infty}\rho^j(\epsilon_{n-j}), n \in \mathbb{Z}.$$

Due to the fact that ρ is bounded, one can write

$$\begin{aligned} Y_n - \rho(Y_{n-1}) &= \sum_{j=0}^{\infty}\rho^j(\epsilon_{n-j}) - \sum_{j=0}^{\infty}\rho^{j+1}(\epsilon_{n-1-j}), \\ &= \varepsilon, \qquad n \in \mathbb{Z}. \end{aligned}$$

which means that (Y_n) is a solution of (2.2) Furthermore, it is easy to see that if X_n is a stationary solutions for (2.2), we have:

$$X_n = \sum_{j=0}^{k}\rho^j(\epsilon_{n-j}) + \rho^{k+1}(X_{n-k-1}), k \geq 1,$$

which implies that

$$E\|X_n - \sum_{j=0}^{k}\rho^j(\epsilon_{n-j})\| \leq \|\rho^{k+1}\|_L^2.$$

where $E\|X_{n-k-1}\|^2$ is constant by stationarity.

According to lemma(2.1), we have $\|\rho^{k+1}\|_L^2 \to 0$ as $k \to \infty$.

So

$$X_n = \sum_{j=0}^{\infty} \rho^j(\epsilon_{n-j}), n \in \mathbb{Z}.$$

Which prove the uniqueness of the solution. □

Properties

Under the condition (**A0**) we have:

1. X_0 is integrable with $E(X_0) = \mu$ and $X_0 \in L_B^2(\Omega, \mathcal{A}, P)$,

2. If we suppose that $\mu = 0$ and define the covariance operator of X_0 by

$$C(x^*) = E(x^*(X_0)X_0), \qquad x^* \in B^*$$

   and the cross covariance operator of X_0 by

$$D(x^*) = E(x^*(X_0)X_0), \qquad x^* \in B^*,$$

   Then

$$D \;=\; \rho C \tag{2.5}$$

   and than ρ is called the autocorrelation operator of X.

3. Let B_n to be the σ-algebra generated by the set $(X_i, \quad i \leq n)$. We have

$$E^{B_{n-1}}(X_n) = \rho(X_{n-1}), \quad n \in \mathbb{Z}. \tag{2.6}$$

$$\varepsilon_{n-1} = X_n - E^{B_{n-1}}(X_n), \quad n \in \mathbb{Z}. \tag{2.7}$$

   We say that (ε_n) is the innovation process of (X_n)

4. $(x^*(X_n - \mu), \quad n \in \mathbb{Z})$ is an real autoregressive process of order one $AR(1)$ which

satisfies

$$x^*(X_n - \mu) = \lambda[x^*(X_{n-1} - \mu)] + x^*(\varepsilon_n), \quad n \in \mathbb{Z},$$

where $x^* \in B^*$ is the eigenvector of the adjoint ρ^* of ρ associated with the eigenvalue $\lambda \in]-1, +1[$

Remark 2.1. *The autoregressive process of order one defined in (2.2) is a Markov chain ([AM02] with transition probability given by*

$$\forall A \in \mathcal{B}; P(x, A) = P(X_1 \in A / X_0 = x) = P(\varepsilon_1 + \rho x \in A)$$

2.2 Limit Theorems for ARB(1) processes

Let $X = (X_n,\ n \in \mathbb{Z})$ be an $(ARB(1))$ of mean μ with associated ρ and white noise ε As of now, we suppose that the condition (**A0**) is satisfied.

2.2.1 Strong Law of Large Numbers SLLN

Let set $S_n = \sum_{i=1}^{n} X_i$, we consider that $\mu = 0$ Notice that we can write

$$I - \rho^n = (I - \rho)(I + \rho + \rho^2 + \cdots + \rho^{n-1})$$

According to the condition (**A0**) $(I - \rho)^{-1}$ exists and is bounded. We get then

$$(I + \rho + \rho^2 + \cdots + \rho^{n-1}) = (I - \rho)(I - \rho^n)$$

In the other hand from (2.3) we have for $k \leq 1$

$$X_k = \sum_{i=0}^{k-1} \rho^i \epsilon_{k-1} + \rho^k X_0 \tag{2.8}$$

So

$$\begin{aligned}
\frac{S_n}{n} &= \sum_{k=1}^{n}\sum_{i=0}^{k-1}\rho^i\epsilon_{k-i}+\sum_{k=1}^{n}\rho^k X_0\\
&= \rho^0\varepsilon_1+(\rho^0\varepsilon_2+\rho^1\varepsilon_1)+\cdots+(\rho^0\varepsilon_n+\rho^1\varepsilon_{n-1}+\cdots+\rho^{n-1}\varepsilon_1)+\sum_{k=1}^{n}\rho^k X_0\\
&= (I+\rho^0+\rho^1+\cdots+\rho^{n-1})\varepsilon_1+(I+\rho^0+\cdots+\rho^{n-2})\varepsilon_2+\cdots+(I+\rho)\varepsilon_{n-1}+\varepsilon_n)+\sum_{k=1}^{n}\rho^k X_0\\
&= (I-\rho)^{-1}\left[\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n)-(\rho^n\varepsilon_1+\rho^{n-1}\varepsilon_2+\cdots+\rho\varepsilon_n)\right]+\sum_{k=1}^{n}\rho^k X_0
\end{aligned}$$

hence

$$S_n=(I-\rho)^{-1}n\overline{\varepsilon}_n-(I-\rho)^{-1}(\rho^n\varepsilon_1+\rho^{n-1}\varepsilon_2+\cdots+\rho\varepsilon_n)+\sum_{k=1}^{n}\rho^k X_0. \tag{2.9}$$

Since (ε_n) satisfies the law of large numbers in B (section1.4.1), we have $n\overline{\varepsilon}_n \longrightarrow 0, \quad a.s.$ as n goes to the infinity. Using Tchebychev and Minkowski inequalities we get for $\alpha>0$

$$\begin{aligned}
P\left(\|\frac{\rho^n\varepsilon_1+\cdots+\rho\varepsilon_n}{n}\|>\alpha\right) &\le \frac{1}{n^2\alpha^2}E(\|\rho^n\varepsilon_1+\cdots+\rho\varepsilon_n\|^2)\\
&\le \frac{1}{n^2\alpha^2}\left[E^{\frac{1}{2}}(\|\rho^n\varepsilon_1\|^2)+\cdots+E^{\frac{1}{2}}(\|\rho\varepsilon_n\|^2)\right]\\
&\le \frac{1}{n^2\alpha^2}\left[\|\rho^n\|E^{\frac{1}{2}}(\|\varepsilon_1\|^2)+\|\rho^{n-1}\|E^{\frac{1}{2}}(\varepsilon_2\|^2)+\cdots+\|\rho\|E^{\frac{1}{2}}(\varepsilon_n\|^2)\right]\\
&\le \frac{1}{n^2\alpha^2}\sigma_\varepsilon^2\left(\sum_{i=1}^{n}\|\rho^i\|\right)^2
\end{aligned}$$

and then according to the Borel-Cantelli,we obtain

$$\frac{\rho^n\varepsilon_1+\cdots+\rho\varepsilon_n}{n}\underset{n\to\infty}{\longrightarrow}0, \quad a.s.$$

Now it is easy to see that

$$\|\frac{1}{n}\sum_{k=1}^{n}\rho^k X_0\|\le\frac{1}{n}\|X_0\|\sum_{k=1}^{n}\|\rho^k\|\underset{n\to\infty}{\longrightarrow}0, \quad a.s.$$

So we can claim that

Theorem 2.3. *Under **(A0)** and if the random variables $(\varepsilon_n, n\in\mathbb{Z})$ are centred i.i.d.*

with $E(\|\varepsilon\|^2) < \infty$, *we have*

$$\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} \mu, \quad a.s.$$

2.2.2 Strong Law of Large Numbers of upper order

Theorem 2.4. *Under* ***(A0)*** *and if the random variables* $(\varepsilon_n, n \in \mathbb{Z})$ *are i.i.d., if* b *is a Banach space of type* α, $0 < \alpha \leq 2$, *we have:*

$$E\left|\frac{S_n}{n} - \mu\right|^{\alpha} = C_1 \left(\frac{r}{n}\right)^{\alpha} [nE(\|\varepsilon_0\|^{\alpha} + E(\|X_0 - \mu\|)^{\alpha}]$$

where C_1 *is a constant and* $r := \sum_{i=0}^{\infty} \|\rho^i\|$

Proof. Let put $Y_n = X_n - \mu$ and C_1 a constant. From (2.8) we have

$$\begin{aligned} \overline{Y}_n &= \frac{1}{n}\sum_{i=0}^{n-1} \rho^i \varepsilon_{n-i} + \frac{1}{n}\sum_{i=1}^{n} \rho^i Y_0 \\ &= \frac{1}{n}\left(\sum_{i=0}^{n-1} \rho^i \varepsilon_{n-i} + (\sum_{i=1}^{n} \rho^i) Y_0\right) \end{aligned} \tag{2.10}$$

Since the Banach space B is of type α, we get

$$\begin{aligned} E(\|\varepsilon_\alpha\|^\alpha) &= \frac{1}{n^\alpha} E\left(\|\sum_{i=0}^{n-1} \rho^i \varepsilon_{n-i} + (\sum_{i=1}^{n} \rho^i) Y_0\|_\alpha^\alpha\right) \\ &\leq \frac{C_1}{n^\alpha}\left(\sum_{i=0}^{n-1} E(\|\rho^i \varepsilon_{n-i}\|_\alpha^\alpha) + E(\|\sum_{i=1}^{n} \rho^i) Y_0\|_\alpha^\alpha)\right) \\ &\leq \frac{C_1}{n^\alpha}(nr^\alpha E(\|\varepsilon_0\|_\alpha^\alpha) + r^\alpha E(\|Y_0\|_\alpha^\alpha)) \\ &\leq C_1 \left(\frac{r}{n}\right)^\alpha (nE(\|\varepsilon_0\|_\alpha^\alpha) + E(\|Y_0\|_\alpha^\alpha)) \end{aligned}$$

hence, it is sufficient yo replace $Y_n$by $X_n - \mu$ and Y_0 by $X_0 - \mu$. $\square$

Suppose now that the random variables $(\varepsilon_n)$are centred and independent and satisfy for all $k \geq 2$:

a)

$$E(\|\varepsilon_i\|^k) \leq \frac{k!}{2} b_i^j H^{k-2}, \qquad \forall i$$

where $H > 0$,(b_i) is a bounded sequence of positive reals and $B_n := \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$

b)

$$E(\|\varepsilon_1 + \cdots + \varepsilon_n\|) \leq \beta_n$$

where (β_i) is a sequence of positive reals. Then, under the condition (**A0**) we have the following result

Theorem 2.5. *For all $r > 0$ we have*

$$\lim_{n\to\infty} E(\|\frac{S_n}{n} - \mu\|^r) = 0.$$

2.2.3 Central limit theorem

We say that an $ARB(1)$ process X satisfies the central limit theorem CLT in B if the sequence $(\frac{S_n - n\mu}{\sqrt{n}}, n \in \mathbb{N})$ converges in law to a gaussian random variable in B.
Let introduce the following technical result

Theorem 2.6. *Suppose that the random variables (ε_n) are i.i.d.and **A0** is satisfied. Then the process X satisfies the CLT if and only if the (ε_n) does. In that case*

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow \mathfrak{N}$$

where $\mathfrak{N}$ is a centred gaussian distribution owing $(I-\rho)^{-1}C_\varepsilon(I-\rho)^{*-1}$ as covariance operator.

Proof. Suppose that the sequence (ε_n) satisfies the CLT i.e.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i \Rightarrow Z$$

where Z is a centred gaussian with covariance operator (C_ε). Consequently (see [Bil13]

$$(I-\rho)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i \Rightarrow (I-\rho)^{-1}Z$$

where $(I-\rho)^{-1}Z$ is a centred gaussian with covariance operator $((I-\rho)^{-1}C_\varepsilon(I-\rho)^{*-1})$. In the decomposition (2.9), we have seen in the proof of Theorem 2.3 that the second and the third terms goes to zero almost sure so in distribution.

So by the continuity of the sum we conclude that

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow (I-\rho)^{-1}Z$$

of covariance operator $((I-\rho)^{-1}C_\varepsilon(I-\rho)^{*-1})$. Now, if we conversely suppose that x satisfies the CLT it will be the same for ε. It is sufficient to use the difference in (2.9)

$$(I-\rho)^{-1}n\overline{\varepsilon}_n = S_n - (I-\rho)^{-1}(\rho^n\varepsilon_1 + \rho^{n-1}\varepsilon_2 + \cdots + \rho\varepsilon_n) + \sum_{k=1}^{n}\rho^k X_0.$$

and the same arguments as above. □

Corollary 2.1. *If the variables (ε_n) are i.i.d. of order two, then each of the following condition is sufficient for the $ARB(1)$, X to satisfies the CLT:*

1. *B is a Banach space of type 2.*
2. *B is a Banach space of cotype 2 and ε_0 is pregaussian.*
3. *B is a the space $C[0,1]$ and there exists a positive random variable M with $E(M_1^2) < \infty$ such that*

$$|\varepsilon_0(t) - \varepsilon_0(s)| \leq M|t-s|, 0 \leq s,t \leq 1.$$

Proof. to proof of first and second points is given by the association of Theorem 2.6, and Theorem1.10, Theorem1.11 and Theorem 1.12 respectively. □

2.2.4 Law of the compact iterated logarithm CLIL

Under **A0** we have

Theorem 2.7. *Suppose that the random variables Suppose that the random variables (ε_n) are i.i.d. and that there exist $\delta > 0$ such that $E(\|\varepsilon_0\|^{2+\delta}) < \infty$. Then the process X satisfies the CLIL if and only if the (ε_n) does. In that case there exists a compact convex* *symmetric set K such that*

$$\lim_{n\to\infty} d\left(\frac{S_n}{\sqrt{2n\log\log n}}, (I-\rho)^{-1}K\right) = 0 \quad a.s. \tag{2.11}$$

and

$$c\left(\frac{S_n}{\sqrt{2n\log\log n}}\right) = (I-\rho)^{-1}K \quad a.s. \tag{2.12}$$

where

$$K = \{x \in B : x = E(\xi\varepsilon_0), \xi \in L^2(\Omega, \mathcal{A}, P), E(\xi) = 0, E(\xi^2) \leq 1\}$$

and $c(A)$ is the set of adherent point of the set A.

Proof. From the decomposition (2.9), we have for $n \geq 3$:

$$\frac{S_n}{a_n} = (I-\rho)^{-1}\frac{n\bar{\varepsilon}_n}{a_n} - (I-\rho)^{-1}\frac{(\rho^n\varepsilon_1 + \rho^{n-1}\varepsilon_2 + \cdots + \rho\varepsilon_n)}{a_n} + \frac{1}{a_n}\sum_{k=1}^{n}\rho^k X_0$$

where $a_n = \sqrt{2n\log\log n}$.

Applying successively the Markov and Minkowski inequalities for $\alpha > 0$, we get

$$\begin{aligned}
P\left(\|\frac{(\rho^n\varepsilon_1 + \rho^{n-1}\varepsilon_2 + \cdots + \rho\varepsilon_n)}{a_n}\| > \alpha\right) &\leq \frac{1}{\alpha^{2+\delta}a_n^{2+\delta}}E\left(\sum_{i=1}^{n}\|\rho^{n+1-i}\varepsilon_i\|\right)^{2+\delta} \\
&\leq \frac{1}{\alpha^{2+\delta}a_n^{2+\delta}}\sum_{i=1}^{n}\left(\|\rho^{n+1-i}\|(E(\|\varepsilon_0\|^{2+\delta}))^{\frac{1}{2+\delta}}\right)^{2+\delta} \\
&\leq \left(\frac{R}{\alpha a_n}\right)^{2+\delta} E(\|\varepsilon_0\|^{2+\delta}).
\end{aligned}$$

Then by the Borel-Cantelli

$$\frac{(\rho^n\varepsilon_1 + \rho^{n-1}\varepsilon_2 + \cdots + \rho\varepsilon_n)}{a_n} \underset{n\to\infty}{\longrightarrow} 0, \quad a.s.$$

and we have

$$\frac{1}{a_n}\sum_{k=1}^{n}\rho^k X_0 \underset{n\to\infty}{\longrightarrow} \mu, \quad a.s.$$

Hence

$$\frac{S_n}{a_n} = (I-\rho)^{-1}\frac{n\bar{\varepsilon}_n}{a_n} + o(n) \qquad a.s.$$

So, if (ε_n) satisfies the CLIL we have

$$\lim_{n\to\infty} d(\frac{n\overline{\varepsilon}_n}{a_n}, K) = 0 \quad a.s.$$

and

$$c\left\{\left(\frac{n\overline{\varepsilon}_n}{a_n}\right)\right\} = K \qquad a.s.$$

And by the continuity of $(I-\rho)^{-1}$ we get the desired result. □

Corollary 2.2. *If the random variables (ε_n) are i.i.d. and $E(\|\varepsilon_0\|^{2+\delta}) < \infty$ for a $\delta > 0$ and if the random variables$\left(\frac{1}{\sqrt{2n\log\log n}}\sqrt{n}\overline{\varepsilon}_n\right)$ are bounded. Then the $ARB(1)$ process X satisfies the CLIL and (2.11) and (2.12) hold.*

Corollary 2.3. *If the variables (ε_n) are centred i.i.d. such that $E(\|\varepsilon_0\|^{2+\delta}) < \infty$ for a $\delta > 0$, then each of the following condition is sufficient for the $ARB(1)$, X to satisfies the CLIL:*

1. *B is a Banach space of type 2.*
2. *$B = C[0,1]$ and there existe a positive random variable M with $E(M_1^2) < \infty$ such that*

$$|\varepsilon_0(t) - \varepsilon_0(s)| \leq M|t-s|, 0 \leq s,t \leq 1.$$

2.3 Autoregressive Representation of Continuous Processes

When observing processes with continuous sample paths and in the aim of constructing an associated random model; it is common to use Banach spaces whose elements are regular functions. In addition, many continuous-time real processes have $ARB(1)$ representations in appropriate Banach spaces.

Definition 2.3. *We say that the process $\zeta = (\zeta(t), t \in \mathbb{R})$ admits a functional autoregressive representation of order one in the function space B ($ARB(1)$) if there exists a bounded linear operator ρ on B and a B- valued white noise ε such that the rv's*

$Z_n(t) := \zeta(n\delta + t),\ 0 \le t \le \delta,\ n \in \mathbb{Z}$ *satisfy*

$$Z_n = \rho(Z_{n-1}) + \varepsilon_n. \tag{2.13}$$

where the function space B holds for Hilbert space or Banach space (Bosq,2000, see Chap.3 and 6).

Several continuous-time real processes admit an $ARB(1)$ representation in an adequate Banach space. In this section we state some classical real continuous-time processes that may be associated with particular autoregressive models in appropriate Banach spaces.

Example 2.2. *Let* $\xi = (\xi_t, t \in \mathbb{R})$ *be a Ornstein-Uhlenbeck with process variance equals to* 1 *defined by*

$$\xi_t = \int_{-\infty}^{t} e^{-c(t-s)} dW(s), \qquad t \in \mathbb{R},$$

where θ *is a positive parameter and* W *is a bilateral standard Wiener process.* ξ *is a real process. To construct en* $ARB(1)$ *from* ξ *, one can choose* $B = C[0, \delta]$ *and set*

$$X_n(t) = \xi_{n+t}, \quad 0 \le t \le 1, \quad n \in \mathbb{Z}.$$

Since

$$E(\xi_{n+t}/\xi_s, \quad s \le n) = e^{-\theta t}\xi_n, \quad 0 \le t \le 1$$

we can set

$$\rho(f)(t) = e^{-\theta t} f(1), \quad 0 \le t \le 1, f \in C[0, \delta]$$

and

$$\begin{aligned} \varepsilon_n(t) &= \int_{n}^{n+t} e^{-c(n+t-s)} dW(s) \\ &= \int_{0}^{t} e^{-c(t-s)} dW(n+s), \quad 0 \le t \le 1, \quad n \in \mathbb{Z}. \end{aligned}$$

Also

$$\begin{aligned}\varepsilon_n(t) &= \int_{-\infty}^{n+t} e^{-c(n+t-s)}dW(s) - \int_{-\infty}^{n} e^{-c(n-s)}dW(s)\\ &= \xi_{n+t} - e^{-ct}\xi_n\\ &= X_n(t) - \rho(X_{n-1})(t), \quad 0 \leq t \leq 1, \quad n \in \mathbb{Z}.\end{aligned}$$

Since (W_s) *is of independent increments,* (ε_n) *is a white noise.*

In addition we have

$$\|\rho^n\| = \sup_{\|f\|=1} \|\rho^n f\| = e^{-\theta(n-1)}.$$

Which leads to

$$\sum_{n\geq 0} \|\rho^n\| < \infty.$$

Consequently (X_n) *is* $ARC(1)$

Example 2.3. *Let consider a real process of the form*

$$\eta_t = a(t) + \xi_t, \quad t \in \mathbb{R}$$

where (ξ_t) *is continuous sample path centred process owing an* $ARB(1)$ *representation with* $B = C[0,h]$.

We suppose that a *is a non constant periodic deterministic continuous function. Then* (η_t) *admits* $ARB(1)$ *representation with* $B = C[0,h]$ *where* h *is the period of* a.

This example is the cornerstone of the following chapter

Chapter 3

Functional Autoregressive Process with Seasonality

In this chapter, We deal with the estimation of a seasonality perturbed by a continuous time process owning an autoregressive representation in $C[0,\delta]$; the Banach space of continuous functions defined on $[0,\delta]$ where $\delta > 0$. Afterwards, we supply the almost sure convergence, asymptotic normality and compact iterated logarithm law. Then and in the same space $C[0,\delta]$ we construct confidence balls for the seasonality. And we give an estimation of the seasonality if it belongs to a finite dimensional space and state the asymptotic properties of such estimator. Eventually, when the dimension of this space is unknown we provide an estimator for it.

3.1 Motivation

For great number of real life situations we need to seek information on the evolution of a continuous-time stochastic process $\eta = (\eta(t),\ t \in \mathbb{R})$ in the future.
Having observed η on an interval $[0,T]$, it is interested to forecast its behaviour on the interval $[T, T+h]$ where $h > 0$.
An appropriate approach to this problem is to divide the interval $[0,T]$ into subintervals $[(i-1)h, ih], i = 1,\cdots,n$ with $h = \frac{T}{n}$, thus the observations of η on this n successive intervals can be interpreted as random variables $X_1,\cdots,X_n$ with values in a function space such that

$$X_n(t) := \eta(ih+t), \quad 0 \le t \le h,\ n \in \mathbb{Z}. \tag{3.1}$$

This representation is especially fruitful if the process η has a seasonal component a with period h.
If the process X in (3.1) is an zero-mean Banach valued autoregressive process of order one; $ARB(1)$, the best prediction of X_{n+1} given its past history $(X_n, X_{n-1},\cdots)$ is then (Bosq[1], chap6) obtained by

$$\begin{aligned} X^*_{n+1} &= E(X_{n+1}/X_n, X_{n-1},\cdots) \\ &= \rho(X_n), \quad n \in \mathbb{Z}, \end{aligned}$$

where ρ is a bounded linear operator associated with the $ARB\,(1)$ process. However, the process η is not generally centred. One can suppose that its mean is the periodic B-valued function a with period h. Thus the centred stochastic process $\xi = (\xi_n = \eta_n - a)$ is an $ARB\,(1)$ process, entailing that the best predictor of X_{n+1} is given by

$$\begin{aligned} X^*_{n+1} &= E\left(X_{n+1}/X_n, X_{n-1}, \cdots\right) \\ &= a + \rho\left(X_n - a\right), \quad n \in \mathbb{Z}. \end{aligned} \tag{3.2}$$

Therefore if we estimate the period function a say by $\widehat{a}$, and the operator ρ say by $\widehat{\rho}$, given $X_1, \cdots X_n$, then the statistical predictor of X_{n+1} based on (3.2) is

$$X^*_{n+1} = \widehat{a} + \widehat{\rho}\left(X_n - \widehat{a}\right), \quad n \in \mathbb{Z} \tag{3.3}$$

3.2 seasonality perturbed by an autoregressive process

Let $C[0,\delta]$ be the Banach space of continuous functions on the interval $[0,\delta]$, $\delta > 0$, equipped with the sup-norm $\|.\|_\infty$.
We consider the real process $(Y(t), t \in \mathbb{R})$ defined by

$$Y(t) = m(t) + \eta(t),\ t \in \mathbb{R} \tag{3.4}$$

where $m(t)$ is a δ-periodic continuous real function and $(\eta(t),\ t \in \mathbb{R})$ is a zero mean process admitting a $C[0,\delta]$- valued autoregressive representation (2.13) with operator ρ and white noise ε i.e. the $C[0,\delta]$- valued rv's

$$Z_n(t) := \eta(n\delta + t),\ 0 \le t \le \delta, n \in \mathbb{Z}$$

verify the equation (2.13).

Proposition 3.1. *The process $(Y(t) - m(t), t \in \mathbb{R})$ has an $C[0,\delta]$- valued autoregressive representation with the same associated linear operator and white noise.*

Proof. For $0 \leq t \leq \delta$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned} Y_n(t) &:= Y(n\delta + t) \\ &= m(n\delta + t) + \eta(n\delta + t) \\ &= m(t) + Z_n(t) \\ &= m(t) + \rho(Z_{n-1}) + \varepsilon_n \end{aligned}$$

Recall that

$$Z_n(t) = Y_n(t) - m(t)$$

So

$$Y_n(t) = m(t) + \rho(Y_{n-1} - m(t)) + \varepsilon_n$$

which leads to

$$Y_n(t) - m(t) = \rho(Y_{n-1} - m(t)) + \varepsilon_n,$$

and then we can see that the process $(Y(t) - m(t), t \in \mathbb{R})$ effectively has an $C[0, \delta]$- valued autoregressive representation □

Notice that the process $(Y(t), t \in \mathbb{R})$ has also a functional autoregressive representation with mean m in the functions space $C[0, \delta]$. Just recall that $Y_n(t) := Y(n\delta + t)$.

For the model

$$Y_n = \rho(Y_{n-1}) + \varepsilon_n, \tag{3.5}$$

we make the following condition:

A0: $\|\rho\|_L^{j_0} < 1, \quad j_0 \geq 1.$

Observing the process $(Y(t), t \in [0, T])$ in model (3.5), one can define the estimator

$$\overline{Y}_T^* = \frac{1}{T} \int_0^T Y(t) dt. \tag{3.6}$$

Its observations on n successive intervals give the empirical mean

$$\bar{Y}_n(t) := \frac{1}{n}\sum_{i}^{n} Y(i\delta + t), t \in [0, T])$$

or

$$\bar{Y}_n = \frac{1}{n}\sum_{i}^{n} Y(i\delta + .)$$

which is an unbiased estimator of m.

Indeed

$$\begin{aligned} E(\bar{Y}_n) &= E\left[\frac{1}{n}\sum_{i}^{n} Y(i\delta + .)\right] \\ &= \frac{1}{n}\sum_{i}^{n} E(Y(i\delta + .)) \\ &= \frac{1}{n}\sum_{i}^{n} m \\ &= m \end{aligned}$$

The following statement provides the strong law of large numbers for both $\overline{Y}^*_T$ and $\bar{Y}_n$.

Proposition 3.2. *Under the condition **A0**, we get*

$$\lim_{T\to+\infty} \overline{Y}^*_T = \frac{1}{\delta}\int_0^{\delta} m(t)dt \qquad a.s.$$

and

$$\lim_{n\to+\infty} \bar{Y}_n = m \qquad a.s.$$

Proof. Let $T \in \mathbb{R}^+$ and $\delta > 0$. There exist an integer $n = n_{T,\delta}$ such that $n\delta \leq T \leq (n+1)\delta$. From the definition (3.6) of $\overline{Y}^*_T$, we have

$$\begin{aligned} \overline{Y}^*_T &= \frac{1}{T}\int_0^T Y(t)dt \\ &= \frac{1}{T}\int_0^{n\delta} Y(t)dt + \frac{1}{T}\int_{n\delta}^{T} Y(t)dt \\ &=: A_n + B_n \end{aligned}$$

Recall that $Y_j(t) := Y(j\delta + t), \quad t \in [0, \delta]$, so we can write:

$$\begin{aligned}
A_n &:= \frac{1}{T}\int_0^{n\delta} Y(t)dt \\
&= \frac{1}{T}\sum_{j=0}^{n-1}\int_{j\delta}^{(j+1)\delta} Y(t)dt \\
&= \frac{1}{T}\left(\int_0^{\delta} Y(t)dt + \int_{\delta}^{2\delta} Y(t)dt + \cdots + \int_{(n-1)\delta}^{n\delta} Y(t)dt\right) \\
&= \frac{1}{T}\sum_{j=0}^{n-1}\int_0^{\delta} Y(j\delta + t)dt \\
&= \frac{n}{T}\int_0^{\delta}\left(\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t)\right)dt
\end{aligned}$$

From theorem 2.3, we have

$$\begin{aligned}
\underset{0\leq t\leq\delta}{sup}\left|\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t) - m(t)\right| &=: \left\|\frac{1}{n}\sum_{j=0}^{n-1} Y_j - m\right\| \\
&= \left\|\frac{S_n}{n} - m\right\| \underset{n\to\infty}{\longrightarrow} 0 \qquad a.s.
\end{aligned}$$

Now for $n \geq n_0$, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t)\right| \\
&\qquad = \left|\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t) - m(t) + m(t)\right| \\
&\qquad \leq \left|\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t) - m(t)\right| + |m(t)| \\
&\qquad \leq C + |m(t)| \\
&\qquad =: g(t)
\end{aligned}$$

where g is an integrable function.

Consequently, using the dominated convergence theorem and the fact that

$$\frac{n}{T} \underset{n\to\infty}{\longrightarrow} \frac{1}{\delta}$$

we get

$$A_n := \frac{1}{T}\int_0^{n\delta} Y(t)dt \underset{n\to\infty}{\longrightarrow} \frac{1}{\delta}\int_0^{\delta} m(t)dt$$

In the author hand we have

$$\begin{aligned} B_n &= \frac{1}{T}\int_{n\delta}^{T} Y(t)dt \\ &\leq \frac{1}{T}\underset{n\delta\leq t\leq T}{sup}|Y(t)| \\ &= \frac{1}{T}\underset{0\leq s\leq \delta}{sup}|Y(s+n\delta)| \\ &= \frac{1}{T}\|Y_n\| \end{aligned}$$

Using Markov inequality and the stationarity of (Y_n) we get

$$\begin{aligned} P\left(\left\|\frac{Y_n}{T}\right\| > \alpha\right) &\leq \frac{1}{T^2\alpha^2}E(\|X_n\|^2) \\ &= \frac{1}{T^2\alpha^2}E(\|X_0\|^2) \end{aligned}$$

Hence

$$\int_1^{\infty} P\left(\left\|\frac{Y_n}{T}\right\| > \alpha\right) dT \leq \frac{E(\|X_0\|^2)}{\alpha^2}\int_1^{\infty}\frac{1}{T^2}dT < \infty.$$

And by the lemma of Borel-Cantelli, we obtain

$$\frac{Y_n}{T} \underset{n\to\infty}{\longrightarrow} 0 \qquad a.s.$$

Then we deduce that

$$B_n := \frac{1}{T}\int_{n\delta}^{T} Y(t)dt \underset{T\to\infty}{\longrightarrow} 0 \qquad a.s.$$

So, combining the convergence result of A_n with the one of B_n we get the desired result

$$\underset{n\to+\infty}{lim}\overline{Y}_T^* = \frac{1}{\delta}\int_0^{\delta} m(t)dt$$

Notice that for the empirical mean $\bar{Y}_n$ and since the variables ε_i are i.i.d., we apply the theorem 2.3 to $S_n = \dfrac{1}{n}\sum_{i}^{n} Y(i\delta + .)$. Then we get

$$\lim_{n\to+\infty} \frac{S_n}{n} = m \qquad a.s.$$

That is to say

$$\lim_{n\to+\infty} \bar{Y}_n = m \qquad a.s.$$

$\square$

Let now state the convergence of both $\overline{Y}_T^*$ and $\bar{Y}_n$ in L^2 .

Proposition 3.3. *Assume* ***A0****. We have*

$$\lim_{n\to+\infty} \overline{Y}_T^* = \frac{1}{\delta}\int_0^\delta m(t)dt \qquad in \quad L^2$$

and

$$\lim_{n\to+\infty} \bar{Y}_n = m \qquad in \quad L^2.$$

Proof.

For $T \in \mathbb{R}^+$ and $\delta > 0$, there exist an integer $n = n_{T,\delta}$ such that $n\delta \leq T \leq (n+1)\delta$.

Notice that

$$\frac{n}{T} \underset{n\to\infty}{\longrightarrow} \frac{1}{\delta}$$

By definition, we have

$$\begin{aligned}
\overline{Y}_T^* &= \frac{1}{T}\int_0^T Y(t)dt \\
&= \frac{1}{T}\int_0^{n\delta} Y(t)dt + \frac{1}{T}\int_{n\delta}^T Y(t)dt
\end{aligned}$$

So let put

$$A_n := \frac{1}{T}\int_0^{n\delta} Y(t)dt$$

and

$$B_n := \frac{1}{T}\int_{n\delta}^{T} Y(t)dt$$

From the autoregressive representation of (Y_t); $Y_j(t) := Y(j\delta + t), \quad t \in [0, \delta]$, we get

$$\begin{aligned} A_n &:= \frac{1}{T}\int_0^{n\delta} Y(t)dt \\ &= \frac{1}{T}\sum_{j=0}^{n-1}\int_{j\delta}^{(j+1)\delta} Y(t)dt \\ &= \frac{1}{T}\sum_{j=0}^{n-1}\int_0^{\delta} Y(j\delta + t)dt \\ &= \frac{n}{T}\int_0^{\delta}\left(\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t)\right)dt \end{aligned}$$

and

$$\begin{aligned} B_n &= \frac{1}{T}\int_{n\delta}^{T} Y(t)dt \\ &\leq \frac{1}{T}\sup_{n\delta\leq t\leq T}|Y(t)| \\ &= \frac{1}{T}\sup_{0\leq s\leq \delta}|Y(s+n\delta)| \\ &= \frac{1}{T}\|Y_n)\| \end{aligned}$$

From theorem 2.5 and for $r = 2$, we have $\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} m \qquad \text{in} \quad L^2$

$$\begin{aligned} \left|A_n - \frac{1}{\delta}\int_0^{\delta} m(t)dt\right|^2 &= \left|\frac{n}{T}\int_0^{\delta}\left(\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t)\right)dt - \frac{1}{\delta}\int_0^{\delta} m(t)dt\right|^2 \\ &= \left|\frac{n}{T}\int_0^{\delta}\left(\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t)\right)dt - \frac{n}{T}\int_0^{\delta} m(t)dt + \frac{n}{T}\int_0^{\delta} m(t)dt - \frac{1}{\delta}\int_0^{\delta} m(t)dt\right|^2 \\ &= \left|\frac{n}{T}\int_0^{\delta}[\frac{1}{n}\sum_{j=0}^{n-1} Y_j(t) - m(t)]dt + (\frac{n}{T} - \frac{1}{\delta})\int_0^{\delta} m(t)dt\right|^2 \\ &\leq \left|\frac{n}{T}\delta\|\frac{S_n}{n} - m\| + (\frac{n}{T} - \frac{1}{\delta})\delta\|m(t)\|\right|^2 \end{aligned}$$

Now since $\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} m \qquad \text{in} \quad L^2$ and using the fact that $\frac{n}{T} \underset{n\to\infty}{\longrightarrow} \frac{1}{\delta}$ we get

$$A_n := \frac{1}{T}\int_0^{n\delta} Y(t)dt \underset{n\to\infty}{\longrightarrow} \frac{1}{\delta}\int_0^{\delta} m(t)dt$$

In the other hand we have

Using Markov inequality and the stationarity of (Y_n) we get

$$\begin{aligned} P\left(\left\|\frac{Y_n}{T}\right\| > \alpha\right) &\leq \frac{1}{T^2\alpha^2}E(\|X_n\|^2) \\ &= \frac{1}{T^2\alpha^2}E(\|X_0\|^2) \end{aligned}$$

Hence

$$\int_1^\infty P\left(\left\|\frac{Y_n}{T}\right\| > \alpha\right) dT \leq \frac{E(\|X_0\|^2)}{\alpha^2}\int_1^\infty \frac{1}{T^2}dT < \infty.$$

And by the lemma of Borel-Cantelli, we obtain

$$\frac{Y_n}{T} \underset{n\to\infty}{\longrightarrow} 0 \qquad a.s.$$

Then we deduce that

$$B_n := \frac{1}{T}\int_{n\delta}^{T} Y(t)dt \underset{T\to\infty}{\longrightarrow} 0 \qquad a.s.$$

So, combining the convergence result of A_n with the one of B_n we get the desired result

$$\underset{n\to+\infty}{lim}\ \overline{Y}_T^* = \frac{1}{\delta}\int_0^\delta m(t)dt$$

□

The following result establishes the central limit theorem (CLT) for $\bar{Y}_n$.

Proposition 3.4. *Assume **A0**. If there exists a positive square integrable random variable M such that*

$$|\varepsilon_0(\omega, s) - \varepsilon_0(\omega, t)| \leq M(\omega)|s-t|, \quad s,t \in [0,\delta] \tag{3.7}$$

then

$$\sqrt{n}(\bar{Y}_n - m) \Rightarrow \mathcal{N}(0, (I-\rho)^{-1}C_{\varepsilon_0}(I-\rho)^{*-1}) \tag{3.8}$$

where $\mathcal{N}$ is a zero mean $C[0,\delta]$-valued Gaussian random variable and where C_{ε_0} is the covariance operator of ε_0.

Proof. According to the corollary 2.6 the condition in (3.7) implies that the process (ε_n) satisfies the CLT which is as stated by 2.3, equivalent to say that the process X itself satisfies the CLT. $\square$

The following result gives a compact iterated logarithm law (CILL) for (Y_n).

Theorem 3.1. *Assume* ***A0****. The sequence $(Y(n\delta + .))$ satisfies a compact iterated logarithm law in $C[0,\delta]$ if and only if (ε_n) satisfies it in $C[0,\delta]$. In this case we have*

$$\limsup d\left(\frac{\bar{Y}_n - m}{\sqrt{\frac{2\log\log(n)}{n}}} \, , \; (I-\rho)^{-1}K \right) = 0 \quad a.s.$$

$$\text{and} \quad A\left\{ \left(\frac{\bar{Y}_n - m}{\sqrt{\frac{2\log\log(n)}{n}}} \right)_{n\geq 3} \right\} = (I-\rho)^{-1}K \quad a.s.$$

where K is the unit ball of the reproducing kernel space associated to the covariance of ε_0, $d(x,E) := \inf_{\{y\in E\}} \|x-y\|$ if $x \in C[0,\delta]$, $E \subset C[0,\delta]$ and $A\{(x_n)\}$ is the set of the limits points in $C[0,\delta]$ of the sequence (x_n). Moreover we have

$$\limsup_{n\to\infty} \left\| \frac{\bar{Y}_n - m}{\sqrt{\frac{2\log\log(n)}{n}}} \right\|_\infty = \sup_{x\in K} \|(I-\rho)^{-1}x\|_\infty$$

Proof. We have

$$S_n := \frac{1}{n}\sum_i^n Y_i = n\bar{Y}_n.$$

Then we apply the theorem 2.6 with $S_n - nm$ and we obtain the first result.

For the second result, the definition of the set A and taking the sup-norm, the first result gives the result $\square$

Remark 3.1. *Theorems 2.8 and Theorems 2.10 in [Bos00] give sufficient conditions for CLT and CILL for the iid (ε_n), Banach space and $C[0,\delta]$ valued rv's.*

3.3 Seasonality in Finite Dimension

In all what follows we consider the Banach space $(C[0,\delta], \|.\|_\infty)$ equipped with $(e_i, i \in \mathbb{N})$ the Schauder basis, and denote by $B(x_0, r)$ the ball centered in x_0 with radius $r > 0$ in $(C[0,\delta], \|.\|_\infty)$.

Let define the subspace E_k of $C[0,\delta]$ generated by the first k functions $(e_1, \dots, e_k)$, where $k \in \{1, 2, \dots\}$.

The projector operator P_k of the subspace E_k is defined by:

$$\begin{array}{rcccl} P_k & : & C[0,\delta] & \to & E_k \\ & & x & \mapsto & P_k(x) = \sum_{i=1}^{k} a_i e_i \end{array}$$

The operators $(P_k, k \in \mathbb{N})$ Suppose that the seasonality m when it belongs to a subspace E_{k_0} of $C[0,\delta]$ with a known finite dimension k_0. In ordrer to estimate it, we introduce the following assumption on m:

A1: *$m = \sum_{i=1}^{k_0} f_i(m) e_i$ where $f_i, \quad i = 1, \dots, k_0$, are functionals coefficients with respect to the basis (e_i) of the space $C[0,\delta]$ such that $f_{k_0}(m) \neq 0$.*

The condition **A1** includes classical regression model of $(\eta(t))$ where the trend m is expressed as a linear form on regressors functions $e_1, \cdots, e_{k_0}$ with a perturbation process $\xi(t)$:

$$\eta(t) = \sum_{i=1}^{k_0} f_i(m)\, e_i(t) + \xi(t)$$

The following result provides confidence balls for seasonality m in the space of continuous functions $C[0,\delta]$. Firstly we suppose that

A2: *the operators ρ and C_{ε_0} of the model 3.5 are known.*

Theorem 3.2. *Assume **A2** and that the white noise $(\varepsilon_n, n \in \mathbb{Z})$ satisfies the compact iterated logarithm law in the space $C[0,\delta]$. Then there exists $\Omega_0 \subset \Omega$, such that $P(\Omega_0) = 1$*

and $\forall \omega \in \Omega_0$*:*

i) $\forall \beta > 0, \exists N(\beta, \omega) \in \mathbb{N},\ \forall n \geq N(\beta, \omega)$

$$m \in B\left(\overline{Y}_n(\omega), \sqrt{\frac{2\log\log n}{n}}\left(\beta + \left\|(I-\rho)^{-1}\right\|\ \|C_{\varepsilon_0}\|_L^{1/2}\right)\right).$$

where $B(x, r)$ *is the ball in the space* $(C_{[0,\delta]}, \|.\|_\infty)$ *centered at point* x *with radius* r*.*

ii) Assume ***A2*** *and that the seasonality* m *satisfies the condition* ***A1****, then* $P_{k_0}\left(\overline{Y}_n\right)$ *is an almost sure consistent unbiased estimator of* m *and we have*

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log\log n}}\left\|\overline{Y}_n - P_{k_0}\left(\overline{Y}_n\right)\right\|_\infty \leq |1 + C_1|\left\|(I-\rho)^{-1}\right\|\ \|C_{\varepsilon_0}\|_L^{1/2}\ a.s.,$$

where $C_1 = \sup_{k\in\mathbb{N}}\|P_k\|$ *is the basic constant of the space* $C[0,\delta]$*.*

Proof. i) First observe that for each $x \in B$ we have:

$$\begin{aligned}
\left\|\bar{Y}_n - m\right\| &= \left\|\frac{1}{n}\sum_{i=1}^{n}(Y_i - m)\right\| \\
&= \sqrt{\frac{2\log\log n}{n}}\left\|\frac{\sum_{i=1}^{n}(Y_i -)}{\sqrt{2n\log\log n}}\right\| \\
&\leq \sqrt{\frac{2\log\log n}{n}}\left(\left\|\frac{\sum_{i=1}^{n}(Y_i - m)}{\sqrt{2n\log\log n}} - x\right\| + \|x\|\right).
\end{aligned}$$

Then we get

$$\left\|\widehat{Y}_n - m\right\| \leq \sqrt{\frac{2\log\log n}{n}}\left(\inf_{x\in}\left\|\frac{\sum_{i=1}^{n}(Y_i - m)}{\sqrt{2n\log\log n}} - x\right\| + \sup_{x\in K_1}\|x\|\right) \tag{3.9}$$

where

$$\begin{aligned}
K_1 &= (I-\rho)^{-1} K \\
K &= \left\{x \in B/\ x = E(\xi\varepsilon_0),\ \xi \in L^2(\Omega, \mathcal{A}, P),\ E\left(\xi^2\right) \leq 1\right\}.
\end{aligned}$$

where K is the unit ball of the reproducing kernel space associated to the covariance of ε_0.

Now

$$\begin{aligned}\sup_{x\in K_1}\|x\| &= \sup_{y\in K}\left\|(I-\rho)^{-1}y\right\| \\ &\leq \left\|(I-\rho)^{-1}\right\|\sup_{y\in K}\|y\|.\end{aligned}$$

Since $\sup_{y\in K}\|y\| = \sqrt{\|R_{\varepsilon_0}\|}$, we get

$$\sup_{x\in K_1}\|x\| \leq \left\|(I-\rho)^{-1}\right\|\sqrt{\|R_{\varepsilon_0}\|}. \tag{3.10}$$

On the other hand we have

$$\lim_{n\to\infty} d\left(\frac{\sum_{i=1}^{n}(Y_i-m)}{\sqrt{2n\log\log n}}, K_1\right) = 0 \qquad p.s.$$

By assumption the sequence $(Y_n, n\in)$ satisfies the compact law of iterated logarithm. Consequently, there exists $\forall\Omega_0\in\mathcal{A}$ such that $P(\Omega_0)=1$ and for each $\omega\in\Omega_0$:

$$\lim_{n\to\infty} d\left(\frac{\sum_{i=1}^{n}(Y_i(\omega)-m)}{\sqrt{2n\log\log n}}, K_1\right) = 0.$$

Moreover

$$\forall\beta>0,\ \exists N := N(\beta,\omega)\,/\quad \forall n>N,\ \ d\left(\frac{\sum_{i=1}^{n}(Y_i-m)}{\sqrt{2n\log\log n}}, K_1\right) \leq \beta. \tag{3.11}$$

Now, by definition we have

$$\lim_{n\to\infty} d\left(\frac{\sum_{i=1}^{n}(Y_i-m)}{\sqrt{2n\log\log n}}, K_1\right) = \inf_{x\in K_\&}\left\|\frac{\sum_{i=1}^{n}(Y_i-m)}{\sqrt{2n\log\log n}} - x\right\|.$$

So using (3.9), (3.10) and (3.11) we get

$$\forall N>N(\beta,\omega),\ d\left(\frac{\sum_{i=1}^{n}(Y_i-m)}{\sqrt{2n\log\log n}}, K_1\right) \leq \sqrt{\frac{2\log\log n}{n}}\left(\beta+\left\|(I-\rho)^{-1}\right\|\|R_{\varepsilon_0}\|^{\frac{1}{2}}\right)$$

which entails the desired result.

ii) The estimator $P_{k_0}\left(\overline{Y}_n\right)$ is an unbiased estimator. Indeed

$$E\left(P_{k_0}\left(\overline{Y}_n\right)\right) = P_{k_0}\left(E\left(\overline{Y}_n\right)\right) = P_{k_0}\left(m\right) = m.$$

Since (Y_n) satisfies the strong law of large numbers by Proposition 1 and P_{k_0} is a continuous operator on $C[0,\delta]$, we get

$$P_{k_0}\left(\overline{Y}_n\right) \underset{n\to\infty}{\longrightarrow} m \qquad a.s.$$

In order to proof the inequality in ii); observe that since $m \in B_{k_0}$ and P_{k_0} is continuous we may write

$$\begin{aligned}
\left\|\overline{Y}_n - P_{k_0}\left(\overline{Y}_n\right)\right\| &\leq \left\|\overline{Y}_n - m\right\| + \left\|m - P_{k_0}\left(\overline{Y}_n\right)\right\| \\
&\leq \left\|\overline{Y}_n - m\right\| + \left\|P_{k_0}\left(m\right) - P_{k_0}\left(\overline{Y}_n\right)\right\| \\
&\leq \left\|\overline{Y}_n - m\right\| + \left\|P_{k_0}\left(m - \overline{Y}_n\right)\right\| \\
&\leq \left\|\overline{Y}_n - m\right\| + \left\|P_{k_0}\right\| \left\|m - \overline{Y}_n\right\|.
\end{aligned}$$

Now $C_1 := \sup_k \left\|P_{k_0}\right\| < \infty$. We obtain then

$$\left\|\overline{Y}_n - P_{k_0}\left(\overline{Y}_n\right)\right\| \leq \left(1 + C_1\right) \left\|\overline{Y}_n - m\right\|.$$

Finally, using similar arguments as in the proof of a), we obtain:

$$\forall N > N\left(\beta,\omega\right),\ \left\|\overline{Y}_n - P_{k_0}\left(\overline{Y}_n\right)\right\| \leq \left(1 + C_1\right) \sqrt{\frac{2\log\log n}{n}}\left(\beta + \alpha\right)$$

where $\alpha = \left\|\left(I-\rho\right)^{-1}\right\| \left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}$.

This completes the proof of Theorem 2. □

Remark 3.2. *1. The result i) provides confidence ball for the seasonality m in the space $\left(C_{[0,\delta]}, \|.\|_\infty\right)$ choosing small β with large enough $N\left(\beta,\omega\right)$.*
2. By result ii), we may define an estimator of m by projecting $\left(\overline{Y}_n\right)$ on subspace E_{k_0}. However, the estimator $P_{k_0}\left(\overline{Y}_n\right)$ needs more calculus for its functional coefficients in the

basis $(e_1, \cdots, e_{k_0})$.

The following result gives a limiting law for the projecting estimator.

Proposition 3.5. *Assume **A0**, **A1** and the conditions of Proposition 2. Then we have:*

$$\sqrt{n}\left(P_{k_0}\left(\overline{Y}_n\right) - m\right) \underset{n\to\infty}{\Longrightarrow} P_{k_0}\left(\mathcal{N}\right)$$

where $P_{k_0}(\mathcal{N})$ *is a zero-mean Gaussian r.v. in* $\mathbb{R}^{k_0}$ *with covariance operator*
$P_{k_0}(I-\rho)^{-1} R_{\varepsilon_0} (I-\rho)^{*-1} P_{k_0}$

and $\mathcal{N}$ *is a zero mean* $C[0,\delta]$*-valued Gaussian random variable of Proposition 2.*

Proof. From Proposition 2 we have

$$\sqrt{n}\left(\overline{Y}_n - m\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}$$

where $\mathcal{N}$ is a zero-mean Gaussian rv given in Proposition 2 wit covariance operator

$$(I-\rho)^{-1} R_{\varepsilon_0} (I-\rho)^{*-1}$$

By the continuity of the operator P_{k_0} we get

$$P_{k_0}\left(\sqrt{n}\left(\overline{Y}_n - m\right)\right) \underset{n\to\infty}{\Longrightarrow} P_{k_0}\left(\mathcal{N}\right)$$

where $P_{k_0}(\mathcal{N})$ is a zero-mean Gaussian r.v. in $\mathbb{R}^{k_0}$ with covariance operator
$P_{k_0}(I-\rho)^{-1} R_{\varepsilon_0} (I-\rho)^{*-1} P_{k_0}$.

This ends the proof. □

3.4 Dimension Estimation

Now, we suppose that the dimension k_0 of the subspace E_{k_0} is unknown and that $0 \leq k_0 \leq K_0$, where K_0 is a given positive integer. The above result (ii) of Theorem 2 allows us to define an estimator of the dimension k_0. We suppose that the operators ρ and C_{ε_0}

are known.

We define an estimator of k_0 setting :

$$\widehat{k}_n \quad : \quad = \min \{k \in \{1, \cdots, K_0\} / \tag{3.12}$$

$$\left. \left\| \overline{Y}_n - P_k\left(\overline{Y}_n\right) \right\|_\infty \leq (1 + C_1) \left\| C_{\varepsilon_0} \right\|^{1/2} \left\| (I - \rho)^{-1} \right\| \sqrt{\frac{2 \log \log n}{n}} \right\} \tag{3.13}$$

Remark 3.3. ***1.*** *The existence of $\widehat{k}_n$ is derived from the result (ii) of Theorem 2 and we have: for large enough n, $\widehat{k}_n \leq k_0$. For small values of n, we may take $\widehat{k}_n = 1$ when it is not defined.*

2. *If the parameters ρ and C_{ε_0} are unknown, we may consider consistent estimators of these parameters and define an estimator $\widehat{\widehat{k}}_n$ in the same way.*

The following result gives the almost sure convergence of $\widehat{k}_n$:

Theorem 3.3. *Assume **A0**, **A1** and conditions of Theorem 2. Then we have:*

$$\lim_{n \to \infty} \widehat{k}_n = k_0 \quad a.s.$$

Proof. From Remark 3 we know that for large enough n, we have $\widehat{k}_n \leq k_0 \quad a.s.$ So $\limsup \widehat{k}_n \leq k_0 \quad a.s.$

Now, we proceed by the absurd. We suppose by assumption that $\liminf \widehat{k}_n \leq k_0 \quad a.s.$.

Hence for all n enough large, there exists $n_0 \geq n$ such that $\widehat{k}_{n_0} < k_0$ $a.s.$. By the definition of the subspaces E_k, we get

$$E_{\widehat{k}_{n_0}} \subset E_{k_0} \quad a.s.$$

which entails

$$\inf_{x\in E_{\widehat{k}_{n_0}}} \left\|\overline{X}_{n_0}-x\right\| \leq \inf_{x\in E_{k_0}} \left\|\overline{X}_{n_0}-y\right\| + \inf_{\substack{y\in B_{k_0},\\ x\in E_{\widehat{k}_{n_0}}}} \|y-x\|.$$

Now

$$\inf_{\substack{y\in E_{k_0},\\ x\in B_{\widehat{k}_{n_0}}}} \|y-x\|=0.$$

Since $P_{k_0}\left(\overline{X}_{k_0}\right)\in B_{k_0}$, this gives that

$$\inf_{x\in E_{\widehat{k}_{n_0}}} \left\|\overline{X}_{n_0}-x\right\| \leq \left\|\overline{X}_{n_0}-P_{k_0}\left(\overline{X}_{k_0}\right)\right\|.$$

By the second part of Proposition 3 and for large enough n_0 we obtain:

$$\inf_{x\in B_{\widehat{k}_{n_0}}} \left\|\overline{X}_{n_0}-x\right\| \leq (1+C_1)\left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}\left\|(I-\rho)^{-1}\right\|\sqrt{\frac{2\log\log n_0}{n_0}}.$$

On the other hand we have

$$\inf_{x\in E_{\widehat{k}_{n_0}}} \|m-x\| \leq \left\|m-\overline{X}_{n_0}\right\| + \inf_{x\in E_{\widehat{k}_{n_0}}} \left\|\overline{X}_{n_0}-x\right\|.$$

Hence

$$\inf_{x\in B_{\widehat{k}_{n_0}}} \|m-x\| \leq \left\|\overline{X}_{n_0}-m\right\| + (1+C_1)\left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}\left\|(I-\rho)^{-1}\right\|\sqrt{\frac{2\log\log n_0}{n_0}}.$$

Notice now that since $m\in E_{k_0},\ f_{k_0}(m)\neq 0$ and $\widehat{k}_{n_0}<k_0$, it follows that there exists a constant $C_2>0$, such that

$$\inf_{x\in B_{\widehat{k}_{n_0}}} \|m-x\| > C_2 > 0.$$

Consequently;

$$0<C_2<\left\|\overline{X}_{n_0}-m\right\| + (1+C_1)\left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}\left\|(I-\rho)^{-1}\right\|\sqrt{\frac{2\log\log n_0}{n_0}}.$$

Furthermore

$$\begin{aligned} C_2\sqrt{\frac{n_0}{2\log\log n_0}} &< \sqrt{\frac{2\log\log n_0}{n_0}}\left\|\overline{X}_{n_0}-m\right\| + (1+C_1)\left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}\left\|(I-\rho)^{-1}\right\| \\ &\leq \left\|\frac{\sum\limits_{i=1}^{n_0}(X_i-m)}{\sqrt{2\log\log n_0}}\right\| + (1+C_1)\left\|R_{\varepsilon_0}\right\|^{\frac{1}{2}}\left\|(I-\rho)^{-1}\right\|. \end{aligned}$$

and from Theorem 1 we have

$$\limsup_{n_0}\left\|\frac{\sum\limits_{i=1}^{n_0}(X_i-m)}{\sqrt{2\log\log n_0}}\right\| = \sup_{s\in K}\left\|(I-\rho_1)^{-1}x\right\| < \infty \quad a.s.$$

But $\sqrt{\frac{n_0}{2\log\log n_0}}$ tends to the infinity and then

$$\limsup_{n_0}\left\|\frac{\sum\limits_{i=1}^{n_0}(X_i-m)}{\sqrt{2\log\log n_0}}\right\| = \sup_{s\in K}\left\|(I-\rho_1)^{-1}x\right\| = \infty \quad a.s.$$

This entails a contradiction and consequently $\liminf \widehat{k}_n \geq k_0 \quad a.s.$ Finally we obtain $\widehat{k}_n = k_0 \;\; p.s.$, which concludes the proof □

The preceding result allows us to define an estimator of the seasonality m.

Proposition 3.6. *Assume* ***A0***, ***A1*** *and conditions of Theorem 2. Then we have*

$$P_{\widehat{k}_n}\left(\overline{Y}_n\right) \underset{n\to\infty}{\longrightarrow} m \qquad \textit{in probability}$$

Proof. Notice that we have for all ε

$$\begin{aligned} P((\left\|P_{\widehat{k}_n}\left(\overline{X}_n\right)-m\right\| > \varepsilon) &\leq P\left(\left\|P_{\widehat{k}_n}\left(\overline{X}_n\right)-m\right\| > \varepsilon/\widehat{k}_n = k_0\right)P\left(\widehat{k}_n = k_0\right) \\ &+P\left(\left\|P_{\widehat{k}_n}\left(\overline{X}_n\right)-a\right\| > \varepsilon/\widehat{k}_n \neq k_0\right)P\left(\widehat{k}_n \neq k_0\right) \end{aligned}$$

Each terms of the second part of the last inequality tends to zero according to the part ii) of Theorem 2 and Theorem 3 respectively and we have the result. □

Chapter 4

Numerical Simulations

Our aim in this part is to construct confidence balls provided by the Compact theorem of iterated logarithm (Theorem 3.1). Numerical simulations are carried out using the package 'far' of the software R developed by Damons and Guillas (Modelization for FAR processes Package:far Version: 0.6-5 License: LGPL-2.1 version R 4.0.2). generally speaking, simulations require a lot of attention because errors can cause problems either in carrying out the process or in estimating the model parameters. On the one hand, these errors are linked with the fact that we actually produce pseudo-random numbers and on the other hand with calculation errors.

4.1 Wiener Process Simulation

The function simul.wiener() gives such Wiener processes trajectories (Figure:4.1). This function use the Karhunen-Loève expansion of Wiener processes on the interval $[0, 1]$ to simulate observations of such a process:

$$W_t = \sum_{j=1}^{\infty} N_j \frac{\sin(j - \frac{1}{2})\pi t}{j - \frac{1}{2}}$$

where N_j are i.i.d. standard Gaussian random variables (see section 1.7). The function has three argument; simul.wiener(p,n,$m2$), where $p$is the number of discretization points, n is the number of observations and $m2$ is the Length of the Karhunen-Loève expansion ($2p$ by default)[Pum92].

Inspiring for this function of package FAR, we have produce our own function to simulate a wiener process

```
wiener.process.sim<-function(n=30,m=50)
{
 m2 <- 2 * m
cst1<-0.05
  delta<- 2*pi
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
```

```
T<- matrix(0, nrow=m, ncol=n)
T[,1] <- u
for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
        m2 <- 2 * m
 if (ncol(d.rho) > m2)
 d.rho<- d.rho[, 1:m2, drop = FALSE]
 if (nrow(d.rho) > m2)
 d.rho<- d.rho[1:m2, , drop = FALSE]
 if (ncol(d.rho) < m2) {
            d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
 d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
 d.rho<- d2.rho
       }
 nn<-1
 mm<-m2
 Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
 if (!is.matrix(Y))
            Y<- as.matrix(Y)
 nn<- nrow(Y)
       cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
 prod1 <- outer(T[1:length(T)], cst11, "*")
 res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
 res1<- as.fdata(res, dates = 1:ncol(Y))
  multplot.fdata(res1,  whole=TRUE, main="example of a Wiener Process ")
mtext(bquote(p==.(m)),adj=0.2,col="red",line=0.2,cex=0.9)
mtext(bquote(n==.(n)),adj=0.4,col="red",line=0.2,cex=0.9)
}
```

```
library(far)
wiener.process.sim(10,300)
```

Using this code, one could produce such a following figure wich displays ten observations of a Wiener process with 300 points of discritisation

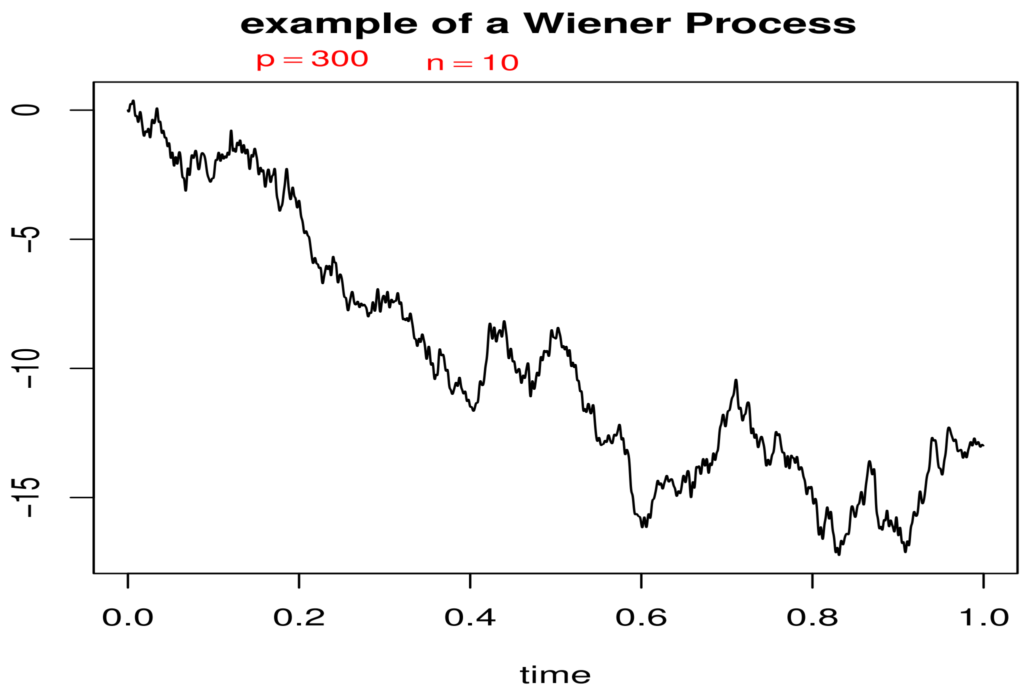

Figure 4.1: An example of Wiener process trajectory of 10 replications with 600 points of discritisation

In order to obtain figure, we have used the following instruction:

library(far)

wiener.process.sim(10,300)

4.2 White Noise Process Simulation

From the simulated Wiener process above, we construct a white noise (Figure:4.2) $\epsilon = (\epsilon_i(t);\ i = 0, 1, \ldots,\ t \in [0, 1])$ using the Wiener process increments as

$$\epsilon_n(t) = W_{t+n} - W_n$$

and we have write a program for that within the following function:

```
  white.noise.sim<-function(n=30,m=50)
{
 m2 <- 2 * m
cst1 <-0.05
 delta<- 2*pi
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
        m2 <- 2 * m
 if (ncol(d.rho) > m2)
 d.rho<- d.rho[, 1:m2, drop = FALSE]
 if (nrow(d.rho) > m2)
 d.rho<- d.rho[1:m2, , drop = FALSE]
 if (ncol(d.rho) < m2) {
           d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
 d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
 d.rho<- d2.rho
        }
```

```
library(far)
 nn<-1
 mm<-m2
 Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
  if (!is.matrix(Y))
            Y<- as.matrix(Y)
 nn<- nrow(Y)
      cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
 prod1 <- outer(T[1:length(T)], cst11, "*")
 res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
 res1<- as.fdata(res, dates = 1:ncol(Y))
 WU<- res1[[1]]
 WU1<- matrix(WU, nrow=m, ncol=n)
 bruit<-matrix(0, nrow = m, ncol = n)
 bruit[1,1]<- WU1[1,1]
 for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
 for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
 bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

 Noise<- bruit1[[1]]    #Noise[1]:Noise[n*m]
 noises<- matrix(Noise, nrow=m, ncol=n)
 multplot.fdata(bruit1, whole=TRUE, main="Example of a white noise")
mtext(bquote(p==.(m)),adj=0.2,col="red",line=0.2,cex=0.9)
mtext(bquote(n==.(n)),adj=0.4,col="red",line=0.2,cex=0.9)}
library(far)
white.noise.sim(10,300)
```

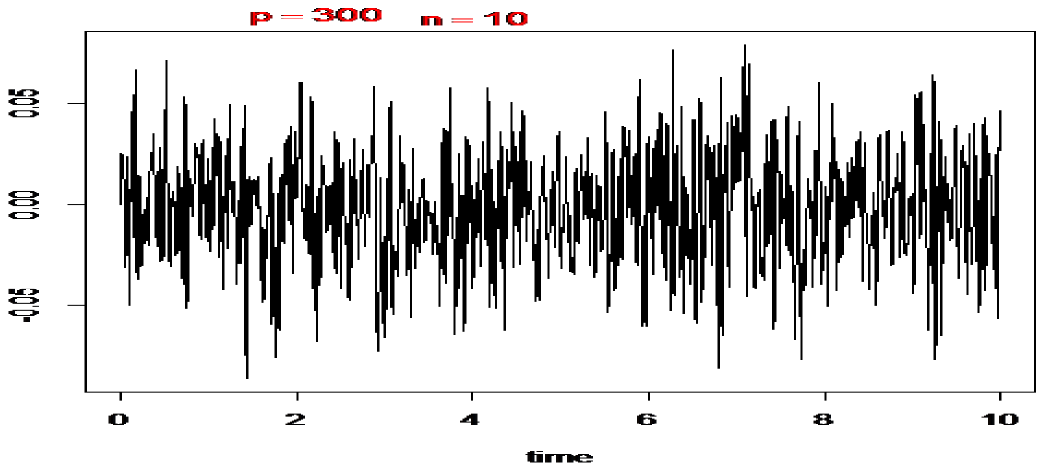

Figure 4.2: A white noise trajectory

4.3 ARC(1) with seasonality

To simulate trajectories of an ARC(1) process $X = (X(t), t \in [0, T])$, with $T = n$ (a multiple of the period $\delta = 1$), we use the relation

$$Y_n(t) := X(n + t), t \in [0, 1]$$

, where Y_n satisfies

$$Y_n - m = \rho(Y_{n-1} - m) + \epsilon_n; n = 1, 2, \ldots$$

with initial value $Y_0 = \epsilon_0$.

The mean is choosen to be

$$m(t) = \sin(2\pi t)$$

and ρ is a kernel operator with the symmetric kernel

$$K(s, t) = \cos(t - s)$$

. Eventually others choices of the mean m and kernel $K(s, t)$ are possible. We first simulate a centered autoregressive process which will be used as a perturbation to the seasonality m.

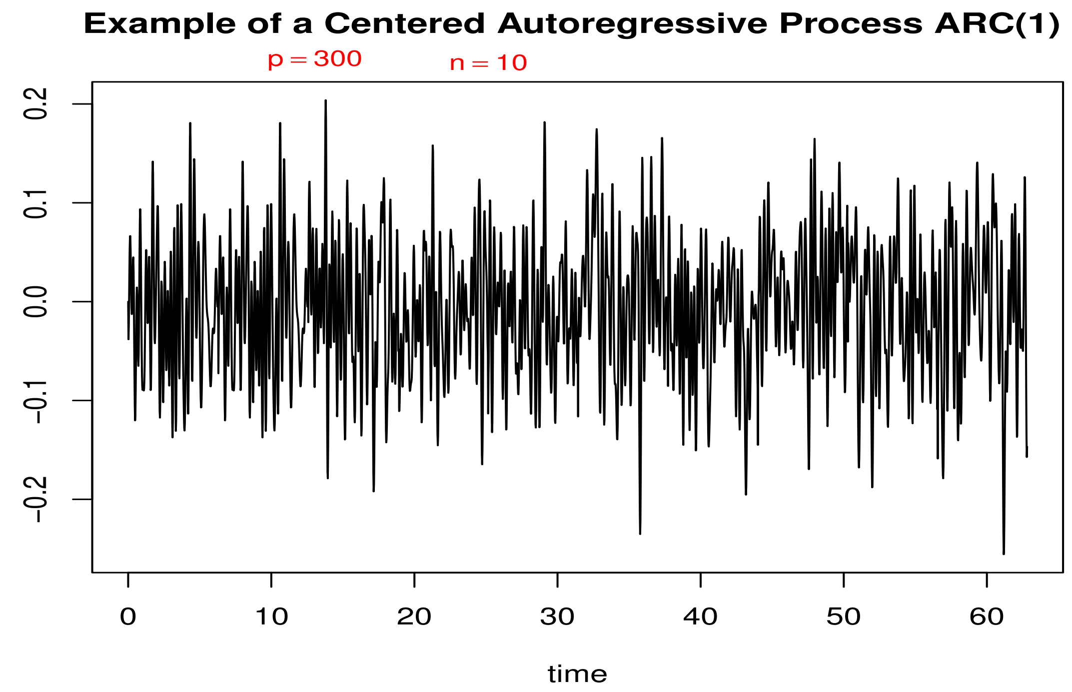

Figure 4.3: Simulation of 10 observations of a centred ARC(1) with 300 discritisation points

To do so, we have use the following R code

```
centered.autoregressive.process.sim<-function(n=30,m=50)
{
 m2 <- 2 * m
cst1 <-0.05
 delta<- 2*pi
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
         m2 <- 2 * m
 if (ncol(d.rho) > m2)
```

```
d.rho<- d.rho[, 1:m2, drop = FALSE]
if (nrow(d.rho) > m2)
d.rho<- d.rho[1:m2, , drop = FALSE]
if (ncol(d.rho) < m2) {
        d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
d.rho<- d2.rho
    }
nn<-1
mm<-m2
Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
if (!is.matrix(Y))
        Y<- as.matrix(Y)
nn<- nrow(Y)
    cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
prod1 <- outer(T[1:length(T)], cst11, "*")
res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
res1<- as.fdata(res, dates = 1:ncol(Y))
WU<- res1[[1]]
WU1<- matrix(WU, nrow=m, ncol=n)
bruit<-matrix(0, nrow = m, ncol = n)
bruit[1,1]<- WU1[1,1]
for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

Noise<- bruit1[[1]]    #Noise[1]:Noise[n*m]
noises<- matrix(Noise, nrow=m, ncol=n)
 x <- matrix(0, nrow = m, ncol = n)
```

```
x[, 1] <- noises[, 1]
for (i in (2:n)) x[, i] <-d.rho %*%x[, i - 1]+ noises[, i - 1]
 x1<- as.fdata(x, dates=1:ncol(x))
multplot.fdata(x1,xval=c(T[1:m,1:n]),whole=TRUE,main="Exemple of a
 Centered ARC(1)")
}
centered.autoregressive.process.sim(10,300)
```

and by adding the following lines to the above code we get a simulation of an autoregressive process with sesonality

```
a<=matrix(nrow=m, ncol=n)
a<=sin(T)
y<=x+a
x1<= as.fdata(y, dates=1:ncol(x))
```

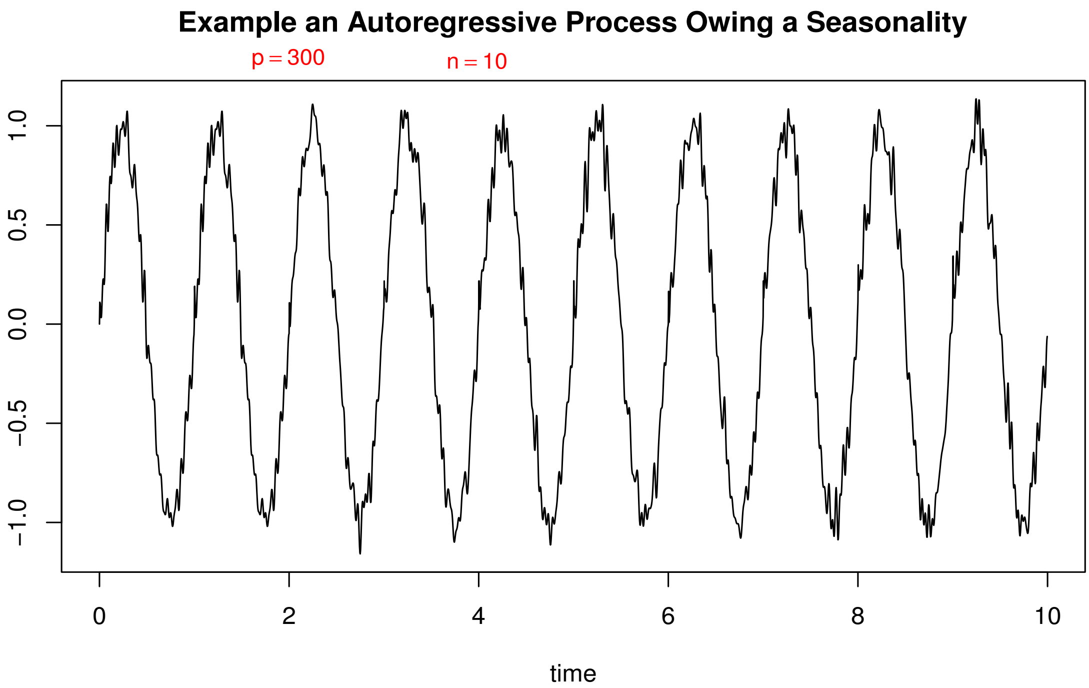

Figure 4.4: Simulation of 10 observations of an ARC(1)perturbing a seasonality with 300 discritisation points

The following figures display the empirical mean and the true mean function $m(t) = \sin 2\pi t$ with effect of the number of discretization points and the sample size (the empirical mean $\bar{X}_n$ (in red) and the seasonality m (in blue)).

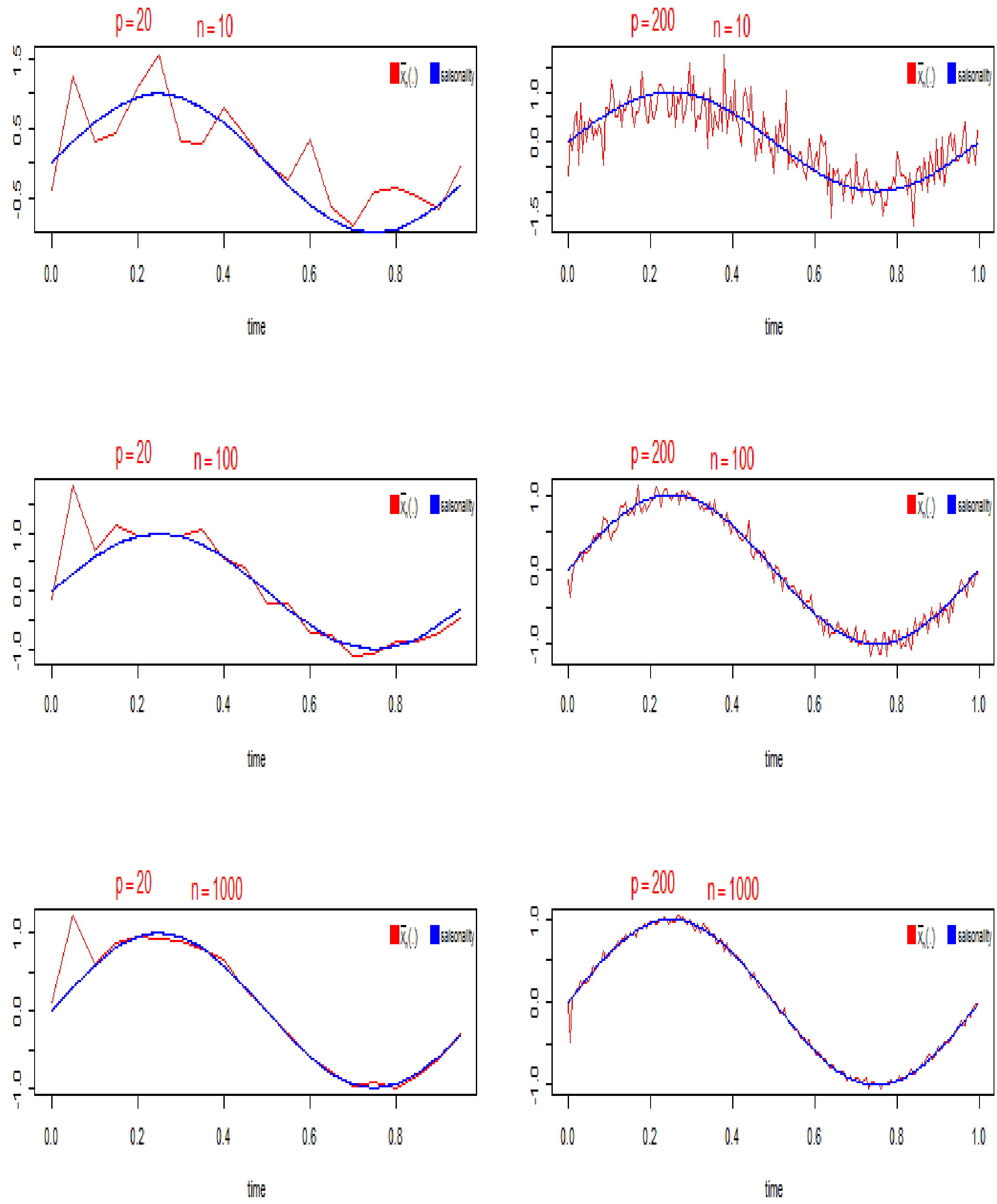

Figure 4.5: Empirical mean and seasonality m

The following code gives one of the above drawwings

```
saicvg<-function(n=30,m=50)
{
 m2 <- 2 * m
cst1<-0.05
 delta<- 2*pi
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
        m2 <- 2 * m
 if (ncol(d.rho) > m2)
 d.rho<- d.rho[, 1:m2, drop = FALSE]
 if (nrow(d.rho) > m2)
 d.rho<- d.rho[1:m2, , drop = FALSE]
 if (ncol(d.rho) < m2) {
          d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
 d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
 d.rho<- d2.rho
        }
 nn<-1
 mm<-m2
 Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
 if (!is.matrix(Y))
          Y<- as.matrix(Y)
 nn<- nrow(Y)
```

```
      cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
 prod1 <- outer(T[1:length(T)], cst11, "*")
 res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
 res1<- as.fdata(res, dates = 1:ncol(Y))
 WU<- res1[[1]]
 WU1<- matrix(WU, nrow=m, ncol=n)
 bruit<-matrix(0, nrow = m, ncol = n)
 bruit[1,1]<- WU1[1,1]
 for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
 for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
 bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

 Noise<- bruit1[[1]]   #Noise[1]:Noise[n*m]
 noises<- matrix(Noise, nrow=m, ncol=n)

  x <- matrix(0, nrow = m, ncol = n)
 x[, 1] <- noises[, 1]
 for (i in (2:n)) x[, i] <-d.rho %*% x[, i - 1]+ noises[, i - 1]
  x1<- as.fdata(x, dates=1:ncol(x))
a<-matrix(nrow=m, ncol=n)
  a<-sin(T) #ajout de parenth ses n*delta
y<-x+a
po<-seq(0, delta, by=delta/m)[-(m+1)]
mat<-matrix(c(a[,1],rowMeans(y)),nrow=m)
matplot(po,mat,type="l",col=c("red","blue"),xval=c(T[1 :m,1 :n]),
xlab="temps",ylab="moyenne vs perturbation")
}
```

To further contrast the effect of sampling frequency on estimating m, we now fix the number of discritization points at $p = 50$ and increasing the sample size till 100. For each size, data sets were simulated following the same mechanism as before. Figure:4.6 gives the estimation error averaged over the one hundred data sets. It clearly shows that phenomenon is in agreement with our theoretical results developed in the earlier section. The following R code permits to show the convergence in the sens of squared mean of the empirical mea to the seasonality:

```
library(far)
mq<-function(n=30,m=50)
{
delta<- 2*pi
moy<-function(m,n)
{
 m2 <- 2 * m
cst1 <-0.05
 #delta<- 2*pi
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
        m2 <- 2 * m
 if (ncol(d.rho) > m2)
 d.rho<- d.rho[, 1:m2, drop = FALSE]
 if (nrow(d.rho) > m2)
 d.rho<- d.rho[1:m2, , drop = FALSE]
 if (ncol(d.rho) < m2) {
```

```
        d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
d.rho<- d2.rho
     }
nn<-1
mm<-m2
Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
if (!is.matrix(Y))
        Y<- as.matrix(Y)
nn<- nrow(Y)
     cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
prod1 <- outer(T[1:length(T)], cst11, "*")
res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
res1<- as.fdata(res, dates = 1:ncol(Y))
WU<- res1[[1]]
WU1<- matrix(WU, nrow=m, ncol=n)
bruit<-matrix(0, nrow = m, ncol = n)
bruit[1,1]<- WU1[1,1]
for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

Noise<- bruit1[[1]]    #Noise[1]:Noise[n*m]
noises<- matrix(Noise, nrow=m, ncol=n)

 x <- matrix(0, nrow = m, ncol = n)
x[, 1] <- noises[, 1]
for (i in (2:n)) x[, i] <-d.rho %*% x[, i - 1]+ noises[, i - 1]
 x1<- as.fdata(x, dates=1:ncol(x))
```

```
a<-matrix(nrow=m, ncol=n)
  a<-sin(T)  #ajout de parenth ses  n*delta
y<-x+a
return(rowMeans(y))
}
moy1<-matrix(0,m,n)
for (i in 1:n) moy1[,i]<-moy(m,i+1)


u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
 a<-sin(T)
qm<-sqrt(apply((moy1-a)^2,2,sum))
plot(c(T),c(moy1), type ="n", xlab ="", ylab = "",
main=" quadratic error convergence to zero ")
lines(c(T), c(moy1), col = "blue")
lines(c(T), c(sin(T)), col = "red")
mtext(bquote(m==.(m)),adj=0.2,col="red",line=0.2,cex=1.2)
mtext(bquote(n==.(n)),adj=0.4,col="red",line=0.2,cex=1.2)
mtext(expression(a(t)==sin(2*pi*t)),adj=0.9,col="red",line=0.2,cex=1.2)
legend("bottomright", legend = c("emperical mean", "saisonality"),
col = c("red", "blue"), pch = 15, bty = "n", pt.cex = 2, cex = 0.8,
text.col = "black", horiz = TRUE, inset = c(0.01, 0.01))
x11()
plot(qm,type="l")
abline(h=0,col="red")
}
```

```
mq(100,50)
```

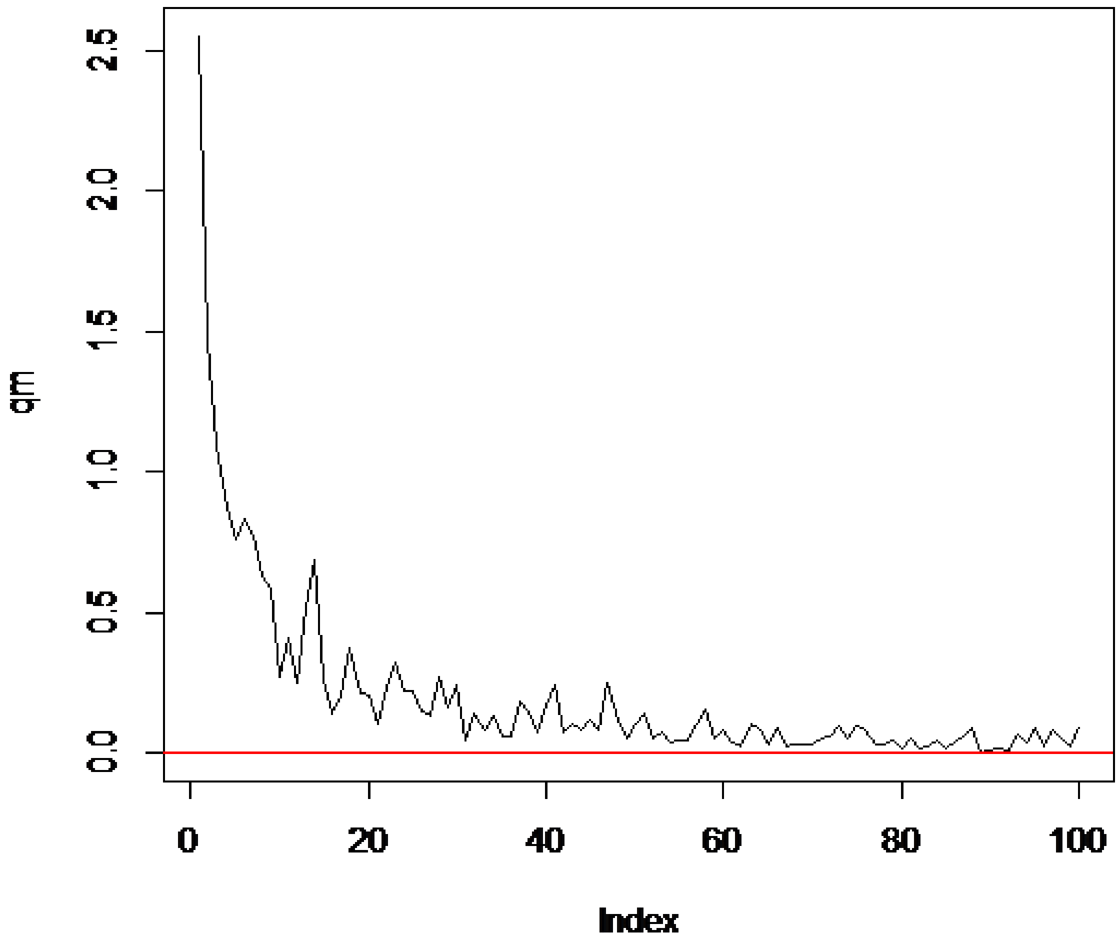

Figure 4.6: quadratic error convergence to zero

4.4 Confidence balls

We provide confidence balls from Theorem 3.1 with different values of n.

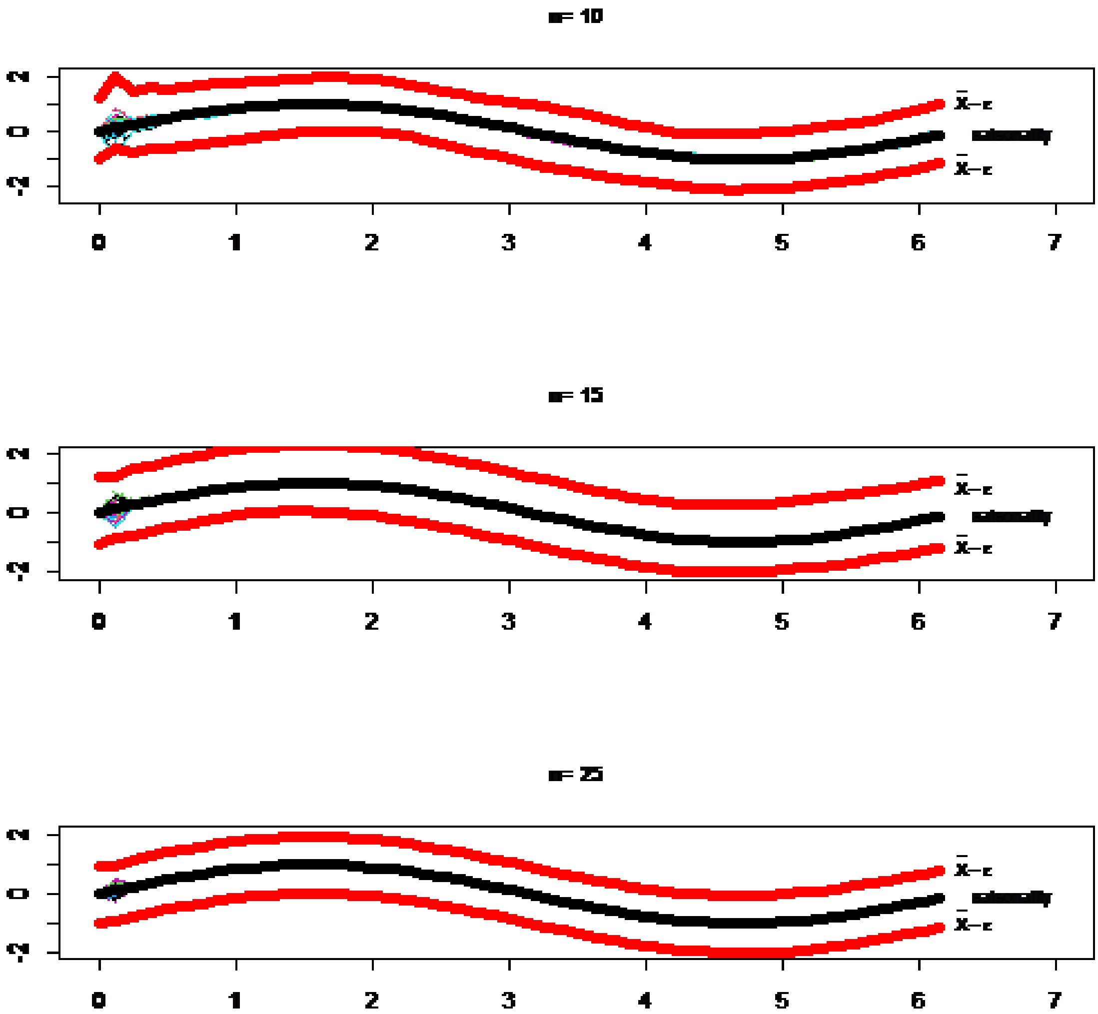

Figure 4.7: Confidence Ball for Different values of the Sample Size

These figures display confidence balls from Theorem 3.1 and show some large balls than provided by the Central Limit Theorem because the existence of extra factor (giving that the factor $\sqrt{n}$ is the same in both theorems).
Notice that this figures can be obtained using the two following functions:

```
empericalmean<-function(n=30,m=50,beta=1)
{
```

```
m2 <- 2 * m
cst1 <-0.05
delta<- 2*pi
u<-seq(0, delta, by=delta/m)
u<- u[-(m+1)]
T<- matrix(0, nrow=m, ncol=n)
T[,1] <- u
for(i in 2:n) T [,i] <- (delta)+T[,i-1]
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
if (is.null(m2))
        m2 <- 2 * m
if (ncol(d.rho) > m2)
d.rho<- d.rho[, 1:m2, drop = FALSE]
if (nrow(d.rho) > m2)
d.rho<- d.rho[1:m2, , drop = FALSE]
if (ncol(d.rho) < m2) {
           d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
d.rho<- d2.rho
       }
nn<-1
mm<-m2
Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
if (!is.matrix(Y))
           Y<- as.matrix(Y)
nn<- nrow(Y)
       cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
prod1 <- outer(T[1:length(T)], cst11, "*")
res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
```

```
 res1<- as.fdata(res, dates = 1:ncol(Y))
 WU<- res1[[1]]
 WU1<- matrix(WU, nrow=m, ncol=n)
 bruit<-matrix(0, nrow = m, ncol = n)
 bruit[1,1]<- WU1[1,1]
 for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
 for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
 bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

 Noise<- bruit1[[1]]   #Noise[1]:Noise[n*m]
 noises<- matrix(Noise, nrow=m, ncol=n)
 ##### calcul du rayon de la boule de confiance
I<-diag(rep(1,m*m),m,m)
H<-solve(I-d.rho)
N<-max(rowSums(abs(H)))
lambda1<-((eigen(cov(bruit)))$values)[1]
eps<-sqrt(2*log(log(n))/n)*(N*sqrt(lambda1))

  x <- matrix(0, nrow = m, ncol = n)
 x[, 1] <- noises[, 1]
 for (i in (2:n)) x[, i] <-d.rho %*% x[, i - 1]+ noises[, i - 1]
  x1<- as.fdata(x, dates=1:ncol(x))
a<-matrix(nrow=m, ncol=n)
  a<-sin(T)  #ajout de parenth ses  n*delta
y<-x+a
return(list("delta"=delta,"theoricalmean"=a[,1],
"empiricalmean"=rowMeans(y),"ball radius"=eps))
}
```

```
confidentialball<-function(n=30,m=50,N=100,beta=1)
{

int<-matrix(0,nrow=m,ncol=N)
int[,1]<-empericalmean(n,m,beta)[[2]]
for(i in 2:N)
int[,i]<-empericalmean(n,m,beta)[[3]]

po<-seq(0, empericalmean(n,m,beta)[[1]],
  by=empericalmean(n,m,beta)[[1]]/m)[-(m+1)]
matplot(po,int[,2:N],type="l",xlab="temps",ylab="",xlim=c(0,7),
ylim=c(-1-empericalmean(n,m,beta)[[4]]-0.2,
1+empericalmean(n,m,beta)[[4]]+0.2))
title(main=paste("Ball of Confidence of the Saisonality","\n",sep="")
,cex.main=0.9)
title(main=paste("\n","\n",c("beta=","n="),c(beta,n),sep=""),
cex.main=0.9)
lines(po,empericalmean(n,m,beta)[[2]],lwd=5)
text(cbind(6.7, sin(6.4)-0.2), "saisonality",
col= "black", cex = 0.7 )
lines(po,empericalmean(n,m,beta)[[3]]+empericalmean(n,m,beta)[[4]],
col="red",lwd=5)
text(cbind(6.4, sin(6.4)+empericalmean(n,m,beta)[[4]]-0.2),
expression(bar(X)-epsilon) , col= 1:3, cex = 0.7)
lines(po,empericalmean(n,m,beta)[[3]]-empericalmean(n,m,beta)[[4]],
col="red",lwd=5)
text(cbind(6.4, sin(6.4)-empericalmean(n,m,beta)[[4]]-0.2),
expression(bar(X)-epsilon) , col= 1:3, cex = 0.7)
 }
```

4.5 Dimension Estimation

We first compute the values of the k first functional of the Schauder basis $(e_i, i = 1, \dots, k)$ of $C[0,1]$ at m . Recall that the basis is constructed as follows(ref!!!):

$$
\begin{aligned}
e_1(t) &= t \qquad t \in [0,1] \\
e_2(t) &= \begin{cases} 2\text{t} & \text{if } t \in [0, \frac{1}{2}] \\ 2\text{(1-t)} & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \\
e_n(t) &= e_{j,k}(t) = e_2(2^j t - k)
\end{aligned}
$$

with $n = 2^j t + k, j \in \mathbb{N}$ and $0 \leq k < 2^j$. Racall that Schauder basis or countable basis is similar to the usual (Hamel) basis of a vector space; the difference is that Hamel bases use linear combinations that are finite sums, while for Schauder bases they may be infinite sums. This makes Schauder bases more suitable for the analysis of infinite-dimensional topological vector spaces including Banach spaces.

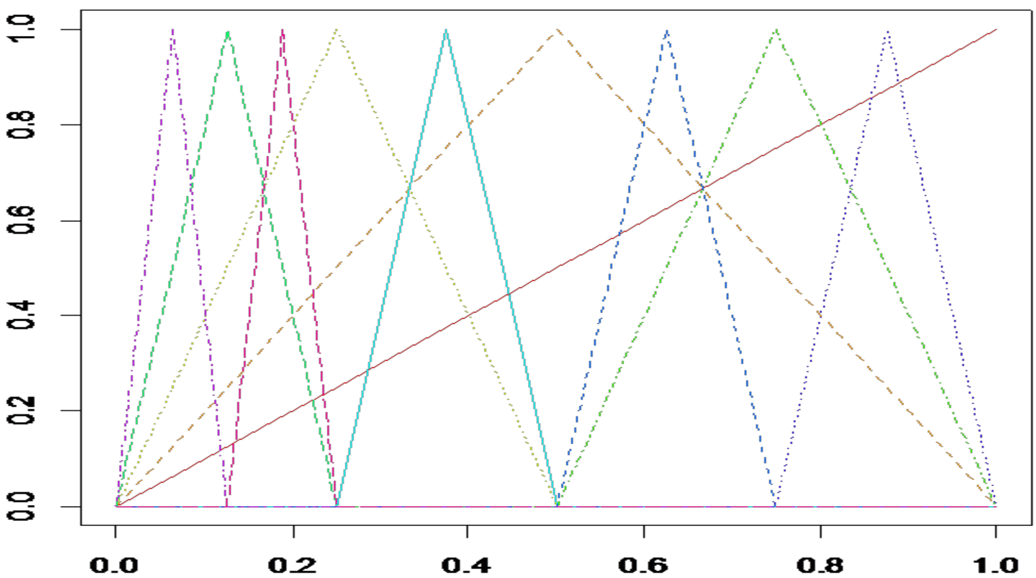

Figure 4.8: simulation of 10 first axis of a Schauder basis

To provide a such illustration of the first ten vectors of the Schauder basis above, one may use the following R program:

```
schauderbasis<-function(K0)
{

dec<-function(l)
{
 j<-0
 repeat
 {
  m<-2^j
  k<-l-m
  if (k>=m) j<-j+1
  else
     break
 }
# print(j)
 #print(k)
return(c(j,k))
}
j0<-dec(K0)[1]
k0<-dec(K0)[2]
 t<-seq(0,1,by=1/(2^(j0+1)))
n<-length(t)
T<-matrix(rep(0,n*K0),n,K0)
for (i in 1:K0) T[,i]<-t
 e<-matrix(rep(0,n*K0),n,K0)
e[,1]<-T[,1]
for (i in 2:K0)
{
```

```
    j<-dec(i-1)[1]
    k<-dec(i-1)[2]
    a<-k/(2^j)
    b<-(1+2*k)/(2^(j+1))
    c<-(1+k)/(2^j)
  t1<-t[(t<=b)]
  t2<-t1[(t1>=a)]
  e1<-((2^(j+1))*t2)-(2*k)
  t3<-t[(t>b)]
  t4<-t3[(t3<=c)]
  e2<-(2*(k+1))-((2^(j+1))*t4)
  t5<-t[(t<a)]
  t6<-t3[(t>c)]
  e3<-rep(0,length(t5))
  e4<-rep(0,length(t6))
  e[,i]<-c(e3,e1,e2,e4)
}
matplot(T,e,type="l",col=rainbow(K0),ylab="",xlab="")
}
```

Now by fixing an integer k_0, we express the coordinates of the seasonality m in the Schauder basis generated by the k_0 first vectors. So we get a projection of the seasonality on a subspace of dimension k_0 that we denote by $P_{k_0}(m)$. Using the same scheme above, we simulate n observations of an $ARC(1)$ defined by:

$$X_n(t) - P_{k_0}(m(t)) = \rho(X_{n-1}(t) - P_{k_0}(m(t))) + \epsilon_n.$$

The mean of this n observations is then projected successively on spaces generated by the $k = 1, \ldots, K_0$ first vectors of the Schauder basis ($1 < k_0 < K_0$) Finally we get an estimate $\widehat{k}_n$ of the real dimension k_0 expressed by

$$\begin{aligned} \widehat{k}_n \quad := \quad & \min\left\{k \in \{1, \cdots, K_0\} \,/\right. \\ & \left. \left\|\overline{Y}_n - P_k\left(\overline{Y}_n\right)\right\|_\infty \leq (1 + C_1)\left\|C_{\varepsilon_0}\right\|^{1/2}\left\|(I-\rho)^{-1}\right\|\sqrt{\frac{2\log\log n}{n}}\right\} \end{aligned}$$

. A fact that is shown in the followingfFigure:4.9:

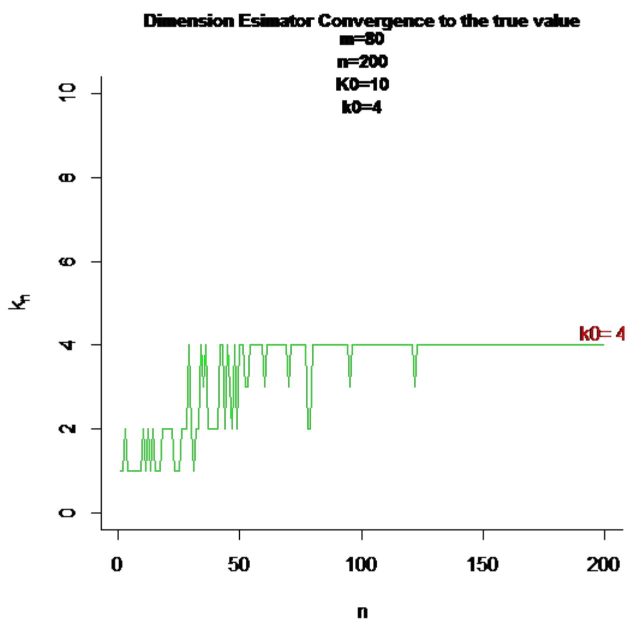

Figure 4.9: example of convergence of the estimate to the real deimension

This figure is the rsult of the following code:

```
proj<-function(K0,k0,n,m)
```

```
{
library(far)
  m2 <- 2 * m
cst1 <-0.05
 delta<- 1 # periode
#### vecteur des points de discitisation
 u<-seq(0, delta, by=delta/m)
 u<- u[-(m+1)]
 T<- matrix(0, nrow=m, ncol=n)
 T[,1] <- u
 for(i in 2:n)  T [,i] <- (delta)+T[,i-1]
### l'operateur d'autocorrelation (pumo)
d.rho = diag(c(0.45, 0.9, 0.34, 0.45))
 if (is.null(m2))
        m2 <- 2 * m
 if (ncol(d.rho) > m2)
 d.rho<- d.rho[, 1:m2, drop = FALSE]
 if (nrow(d.rho) > m2)
 d.rho<- d.rho[1:m2, , drop = FALSE]
 if (ncol(d.rho) < m2) {
          d2.rho <- diag(cst1/((1:m)^2) + (1 - cst1)/exp(1:m))
 d2.rho[1:nrow(d.rho), 1:ncol(d.rho)] <- d.rho
 d.rho<- d2.rho
       }
## construction d'un mouvement brownien par la formule de kahnonen-loeve
 nn<-1
 mm<-m2
 Y<-matrix(rnorm(nn*mm), nrow=mm, ncol=nn)
 if (!is.matrix(Y))
```

```
        Y<- as.matrix(Y)
 nn<- nrow(Y)
     cst11 <- ((1:mm) -(1/2)) *( pi/(n*delta) )
 prod1 <- outer(T[1:length(T)], cst11, "*")
 res<- sin(prod1) %*% (Y/cst11)* sqrt(2/(n*delta) )
 res1<- as.fdata(res, dates = 1:ncol(Y))
 WU<- res1[[1]]
 WU1<- matrix(WU, nrow=m, ncol=n)
#### construire un bruit blanc a partir des accroissements
 bruit<-matrix(0, nrow = m, ncol = n)
 bruit[1,1]<- WU1[1,1]
 for(j in 2:n)  bruit[1,j]<- WU1[1,j]-WU1[m,j-1]
 for(j in 1:n) {for (i in (2:m)) bruit[i,j] <-WU1[i,j]- WU1[i-1,j]}
 bruit1<- as.fdata(bruit, dates=1:ncol(bruit))

 Noise<- bruit1[[1]]   #Noise[1]:Noise[n*m]
 noises<- matrix(Noise, nrow=m, ncol=n)
######
f<-function(t) sin(2*pi*t)
#### la saisonalite aux points de discritisations
 a<-matrix(nrow=m, ncol=1)
  a<-f(T) #ajout de parentheses  n*delta

#### construction de la base de shauder par K0
u1<-seq(0, 1/2, by=1/m)
u2<-seq(1/2,1, by=1/m)
u2<- u2[-1]
u3<-c(u1,u2)
v1<-2*u1
```

```
v2<-2*(1-u2)
e<-matrix(0,nrow=m,ncol=K0)
e[,1]<-u3[-(m+1)]
e[,2]<-c(v1,v2)[-(m+1)]
dec<-function(n)
{
 j<-0
 repeat
 {
  s<-2^j
  k<-n-s
  if (k>=s) j<-j+1
  else
      break
 }
# print(j)
 #print(k)
return(c(j,k))
}
for (i in 3:K0)
{
j<-dec(i-1)[1]
l<-dec(i-1)[2]
a1<-l/(2^j)
a2<-(1+2*l)/(2^(j+1))
a3<-(1+l)/(2^j)
u4<-u3[u3<a1]
u4<-rep(0,length(u4))
u5<-u3[u3>=a1]
```

```
u5<-u5[u5<=a2]
u5<-((2^(j+1))*u5)-(2*l)
u6<-u3[u3>a2]
u6<-u6[u6<=a3]
u6<-(2*(1+l))-((2^(j+1))*u6)
u7<-u3[u3>a3]
u7<-rep(0,length(u7))
e[,i]<-c(u4,u5,u6,u7)[-(m+1)]
}
#### la saisonalite dans dim=k0
 alpha<-matrix(nrow=k0, ncol=1)
 for (i in 1:k0)
 {
j<-dec(i-1)[1]
k<-dec(i-1)[2]
alpha[i]<-f((2*k+1)*2*pi/(2^(j+1)))-0.5*(f(k*2*pi/(2^j))
+f((k+1)*2*pi/(2^j)))
}
  funcoeffa<-matrix(0,nrow=m, ncol=k0)
 for (i in 1:k0)  funcoeffa[,i]<-alpha[i,]*e[,i]


 pk0a<-apply(funcoeffa,1,sum)
##### construire le precessus ARC(1)
  x <- matrix(0, nrow = m, ncol = n)
 x[, 1] <- noises[, 1]
 for (i in (2:n)) x[, i] <-d.rho %*%( x[, i - 1]- pk0a)
 + noises[, i - 1]+pk0a
#### calcul de la moyenne emperique
xbar<-rowMeans(x)
```

```
##### calcul de la borne dans l'expression
 de l'estimateur de la dimension
I<-diag(rep(1,m*m),m,m)
H<-solve(I-d.rho)
N<-max(rowSums(abs(H)))
C1<-max(rowSums(abs(e)))
eps<-(1+C1)*N*sqrt(2*log(log(n))/n)*sqrt(max(abs(cov(noises[, 1],
noises[, 1])))))
 # construction de la base de shauder
u1<-seq(0, 1/2, by=1/m)
u2<-seq(1/2,1, by=1/m)
u2<- u2[-1]
u3<-c(u1,u2)
v1<-2*u1
v2<-2*(1-u2)
e0<-matrix(0,nrow=m, ncol=k0)
e0[,1]<-u3[-(m+1)]
e0[,2]<-c(v1,v2)[-(m+1)]
dec<-function(n)
{
 j<-0
 repeat
 {
  s<-2^j
  k<-n-s
  if (k>=s) j<-j+1
  else
     break
 }
```

```
# print(j)
 #print(k)
return(c(j,k))
}
for (i in 3:k0)
{
j<-dec(i-1)[1]
l<-dec(i-1)[2]
a1<-l/(2^j)
a2<-(1+2*l)/(2^(j+1))
a3<-(1+l)/(2^j)
u4<-u3[u3<a1]
u4<-rep(0,length(u4))
u5<-u3[u3>=a1]
u5<-u5[u5<=a2]
u5<-((2^(j+1))*u5)-(2*l)
u6<-u3[u3>a2]
u6<-u6[u6<=a3]
u6<-(2*(1+l))-((2^(j+1))*u6)
u7<-u3[u3>a3]
u7<-rep(0,length(u7))
e0[,i]<-c(u4,u5,u6,u7)[-(m+1)]
}
#### projection des donnees dans des sous espaces engendres
##### par les k0 premiers vecteurs de la base deshauder
Pk0x<-matrix(rep(0,k0*n),k0,n)
for (j in 1:n)
  {
    Pk0x[,j]<-qr.solve(x[,j],e0[,1:k0])
```

```
    }
#### calcul  de la moyenne des projetes des donnes
#### dans la base de shauder
  xbar1<-rowMeans(Pk0x)
coord1<-  e0%*%matrix(xbar1,k0,1)

##### projection des donnes dans des sous espaces engendrs
##### par les k0 premiers vecteurs de la base deshauder
      Pk0xbar<-qr.solve(xbar,e0)
###########coordonee de la projeter ds la base canonique
coord0<-  e0%*%matrix(Pk0xbar,k0,1)
### projection des donnees dans des sous espaces engendres
##### par les K0 premiers vecteurs de la base deshauder
  Pkxbar<-matrix(0,m,K0)
for (i in 1:K0)
Pkxbar[,i]<-e[,1:i]%*%matrix(qr.solve(e[,1:i],coord1),i,1)
 xbar0<- matrix(rep(coord1,K0),m,K0)
d<-abs(Pkxbar-xbar0)
nor<-apply(d,2,max)
kn<-min(which(nor<eps))
return(kn)
}
graf<-function(K0=10,k0=3,n=20,m=80)
{
 graphe<- matrix(0,n,1)
 for (i in 1:2) graphe[i]<- 1
 for (i in 3:n){
  graphe[i]<- proj(K0,k0,i,m)
 plot(graphe, type = "l",ylab = expression(k[n]),
```

```
 xlab = expression("n"),ylim=c(0,K0),bty="l",col="green")
title(main=paste("Dimension Esimator Convergence to the true value",
"\n",sep=""),cex.main=0.9)
title(main=paste("\n","\n","\n","\n",c("m=","n=","K0=","k0="),
c(m,n,K0,k0),sep=""),cex.main=0.9)
# abline(h=k0,col="red",lty=2)
#axis(side=2, at=c(k0),labels=c("k0"),col="red")
text(n,k0+0.3,paste("k0=",k0),col="red")
  }
}
```

We suppose that the mean $m(.)$ lives in a subspace of $C[0,1]$ generated by k_0 first vectors of the Schauder basis $(e_i, i \in \mathbb{N})$ of $C[0,1]$ such that $1 \leq k_0 \leq K_0$ for a fixed integer K_0. The mean $m(t) = \sin(2\pi t)$ and the empirical mean $\bar{X}_n$ is calculated projecting successively on $C[0,1]$'s subspace of dimensions k (generated by k (e_i)) varying from 1 to K_0.

Let $P_k\left(\overline{X}_n\right) \quad k = 1, \cdots, K_0$ denotes this projection and the estimator is given by (cf 3.2) :

$$\begin{aligned}
\widehat{k}_n \quad : \quad &= \min\left\{k \in \{1, \cdots, K_0\}\right/ \\
&\left\|\overline{X}_n - P_k\left(\overline{X}_n\right)\right\|_\infty \leq (1 + C_1)\left\|C_{\varepsilon_0}\right\|^{1/2}\left\|(I-\rho)^{-1}\right\|\sqrt{\frac{2\log\log n}{n}}\right\}
\end{aligned}$$

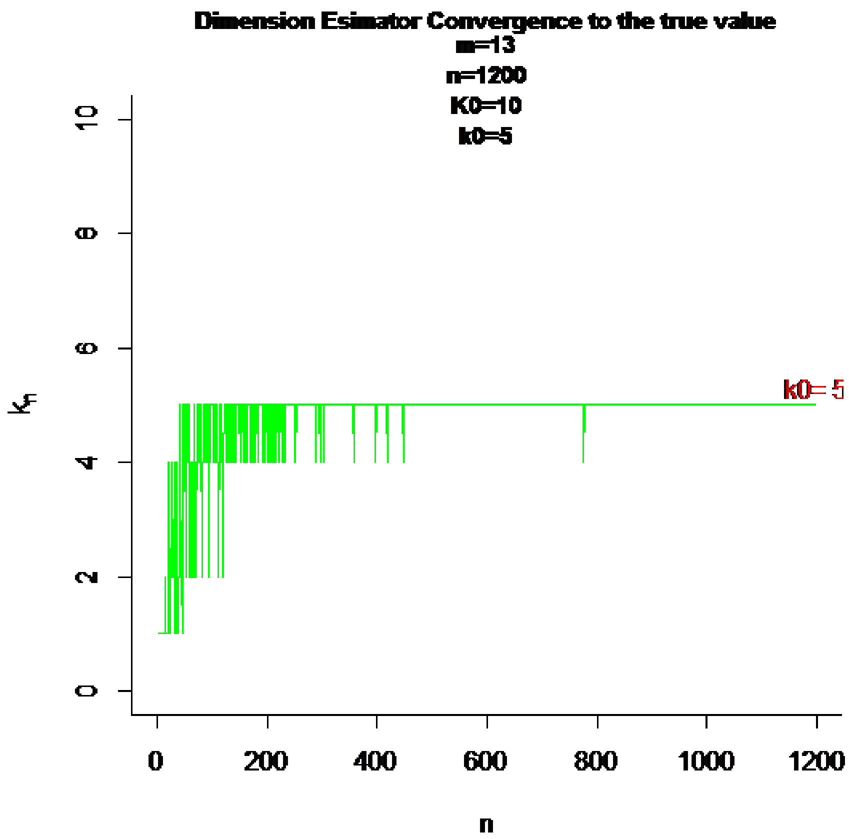

Figure 4.10

This figure shows its convergence to the true dimension $k_0 = 5$ (Recall that $\widehat{k}_n \leq k_0 = 5$).

Conclusion

The final gaol of any statistical study when dealing with time series is focasting.
One of dedicated approches is the autoregression forecasting method which is based on the auto-correlational techniques.
Notice that Autoregression forecasting detects the linear, nonlinear, and seasonal fluctuations in historic data and projects these trends into the future. So Autoregression provides the best forecasting reliability when the driving factors underlying those data are affected by seasonal fluctuations.
An another challenge is The curse of dimensionality which refers to difficulties that arise when dealing with high-dimensional datasets that it can lead to overfitting because when dealing with high-dimensional datasets, it is easy to include irrelevant features that do not contribute to the predictive power of the model.
In this thesis, we have focused on functional autoregressive process with seasonal. We have provide an estimator of a seasonality perturbed by a continuous time process admitting a $C[0,1]$, B-valued autoregressive representation and established its almost sure convergence, asymptotic normality and compact iterated logarithm law. Hence using the compact iterated logarithm law, we have construct a confidence balls for the seasonality in the space $C[0,1]$. Then, we have studied an estimator of seasonality when it belongs to a finite dimensional space and gave its asymptotic properties. The dimension of this space was therefore estimated when has been considered as being unknown. We concluded by giving numerical simulations that illustrate asymptotic results of the estimators.

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الملخص

انصب اهتمامنا في هذه الأطروحة على سلاسل الانحدار التلقائي الوظيفية ذات موسمية. لقد قدمنا مقدّرًا للموسمية مضطربة بسلسلة زمنية متواصلة تقبل تمثيلا انحداريا تلقائيا يأخذ قيما في فضاء الدوال المستمرة على المجال المغلق [0,1] وأثبتنا تقاربه شبه المؤكد، و تقاربه الطبيعي وكذا القانون اللوغاريتمي اللوغاريتمي التكراري المتراص. ومن ثم، باستخدام هذاالقانون، قمنا ببناء كرات ثقة للموسمية في نفس الفضاء درسنا مقدّرًا للموسمية عندما تنتمي إلى فضاء محدود الأبعاد وأعطينا خصائصه التقريبية. وبالتالي قدّرنا بُعد هذا الفضاء عندما تم اعتباره مجهولاً. واختتمنا بإعطاءعمليات محاكاة عددية توضّح النتائج التقاربية للمقدّرات.

الكلمات المفتاحية: تمثيل الانحدار الذاتي، فضاء باناخ، كرة الثقة، التقدير، تقليل الأبعاد، الموسمية.

Abstract

In this thesis, we have focused on functional autoregressive process with seasonality. We have provide an estimator of a seasonality perturbed by a continuous time process admitting a C[0,1]-valued autoregressive representation and established its almost sure convergence, asymptotic normality and compact iterated logarithm law. Hence using the compact iterated logarithm law, we have construct a confidence balls for the seasonality in the space C[0,1]. Then, we have studied an estimator of seasonality when it belongs to a finite dimensional space and gave its asymptotic properties. The dimension of this space was therefore estimated when has been considered as being unknown. We concluded by giving numerical simulations that illustrate asymptotic results of the estimators.

Keywords: Autoregressive representation, Banach Space, Confidence Ball, Estimation, Dimesion reduction, seasonality.

Résumé

Dans cette thèse, nous nous sommes concentrés sur les processus autorégressifs fonctionnels avec saisonnalité. Nous avons fourni un estimateur d'une saisonnalité perturbée par un processus stochastique continu admettant une représentation autorégressive à valeur dans C[0,1] et nous avons établi sa convergence presque sûre, sa normalité asymptotique et sa loi de logarithme itéré compact. Ainsi, en utilisant la loi du logarithme itéré compact, nous avons construit des boules de confiance pour la saisonnalité dans l'espace C[0,1]. Ensuite, nous avons étudié un estimateur de saisonnalité lorsqu'il appartient à un espace de dimension finie et donner ses propriétés asymptotiques. La dimension de cet espace a donc été estimée étant considérée comme inconnue. Nous avons conclu par des simulations numériques qui illustrent les résultats asymptotiques de ces estimateurs.

Mots clés : Représentation autorégressive, espace de Banach, boule de confiance, estimation, réduction de dimension, saisonnalité.