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Presented by

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The global fractional Calderon-Zygmund regularity and applications

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Dedication:

I dedicate this thesis to:

My dearest parents.

My brothers:Salah-Eddin.

My whole family.

All my friends and colleagues without exception.

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Contents

Introduction

The aim of this Master's thesis is to study the existence of solutions to a class of nonlinear fractional elliptic equations involving gradient terms. The analysis relies on regularity results for both the potential and gradient terms. The problems considered are mainly inspired by the results obtained in the thesis [38] and the paper [3, 4].

The study of nonlinear partial differential equations is a very important topic in modern mathematical analysis, with many applications in physics, geometry, and applied sciences. In recent years, a lot of research has focused on nonlocal operators. In particular, the fractional Laplacian $(-\Delta)^s$ for $s \in (0, 1)$ generalizes the classical Laplace operator and is used to model phenomena like long-range interactions, anomalous diffusion, and jump processes [7, 14, 36].

The main objective of this thesis is to prove the **global fractional Calderón-Zygmund regularity** for the weak solution of the fractional Poisson problem with Dirichlet boundary conditions:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $\mathcal{C}^2$ boundary, $f \in L^m(\Omega)$ for some $m \geq 1$, and $s \in (0, 1)$. The fractional Laplacian is defined using the principal value by:

$$(-\Delta)^s u(x) = C_{N,s} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy$$

where $C_{N,s}$ is a normalization constant depending on N and s:

$$C_{N,s} := 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N}{2} + s)}{\Gamma(1 - s)}.$$

This operator is well known in the literature; we refer to [27, 32, 28] for a general introduction to its properties and to [9] for variational methods in this framework.

Understanding the regularity of solutions to these problems is fundamental for both theory and applications. Classical Calderón-Zygmund theory was first developed for second-order elliptic equations [19, 33] to give sharp integrability estimates for solutions based on the data f. In the fractional case, extending this theory is much more difficult because of the nonlocal nature of the operator and the boundary effects. Some related regularity results in the nonlocal setting can be found in [8, 17, 18, 26].

In this work, our approach is based on the **fractional Green's function** $G_s(x, y)$ associated to the problem. This allows us to write the unique weak solution explicitly as:

$$u(x) = \int_{\Omega} G_s(x, y) f(y) \, dy$$

Sharp pointwise estimates for the Green function and its fractional derivatives were established in [11, 16, 15, 20], and they play a central role in our analysis. Using these estimates, we obtain precise regularity results for the nonlocal gradient terms $(-\Delta)^{t/2}u$ and $\nabla^t u$, for $s \leq t < \min\{1, 2s\}$, in Lebesgue spaces $L^p(\mathbb{R}^N)$, Bessel potential spaces $L^{t,p}(\mathbb{R}^N)$, and fractional Sobolev spaces $W^{t,p}(\mathbb{R}^N)$ [6, 34, 35]. These regularity results are obtained globally, over the whole space $\mathbb{R}^N$. This is an important improvement over the local estimates found in [17, 12]. The main results of this thesis are directly connected to the work of [3].

As an application of this regularity theory, we study the **existence and non-existence of weak solutions** for a class of nonlinear problems of Kardar-Parisi-Zhang (KPZ) type [23], which include nonlocal gradient terms:

$$\begin{cases} (-\Delta)^s u = |D_t u|^q + \lambda f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where D_t represents either the half t-Laplacian $(-\Delta)^{t/2}$ or the Riesz t-gradient ∇^t [31, 30, 37], and $q > 1$, $\lambda > 0$ are real parameters. These equations come from models in surface growth theory and stochastic interface dynamics [22, 24, 21]. Their analysis is difficult due to the combination of nonlocal operators and gradient nonlinearities. Similar fractional KPZ problems have been studied in [2, 4, 1, 5, 25].

The thesis is organized as follows. **Chapter 1** introduces the necessary analytical tools, provides key integral estimates, and develops the functional framework, including classical and fractional Sobolev spaces [27, 6, 13], Bessel potential spaces [34, 35], and estimates for the fractional Green's function [11, 16]. **Chapter 2** presents the main regularity theorems, establishing the global fractional Calderón-Zygmund theory and its consequences in fractional spaces [3], as well as compactness results. Finally, **Chapter 3** applies this theory to KPZ problems [4, 1], proving existence results with Schauder's fixed point theorem [13] and non-existence results using energy arguments and sharp boundary estimates [28, 29].

Table of symbols

Symbol	Meaning
$x = (x_1, x_2, \ldots, x_d)$	Element of $\mathbb{R}^d$
$\lvert x\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$	The modulus of $x \in \mathbb{R}^N$
∇u	The gradient of u
Δu	The Laplacian of u
$\nabla^s u$	The Riesz s-gradient of u
$(-\Delta)^s u$	The fractional Laplacian of u
G_s	The Green's function associated to $(-\Delta)^s$
$\mathscr{F} u$	The Fourier transform of u
$\Omega \subset \mathbb{R}^d$	Bounded domain
$\partial\Omega$	Boundary of Ω
D_Ω	$(\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c)$
Ω^c	Complement of Ω in $\mathbb{R}^d$
$\overline{\Omega}$	The closure of $\Omega \subset \mathbb{R}^N$
$B_r(x)$	A ball of center $x \in \mathbb{R}^N$ and radius r
$A \subset B$	A is a subset of B
$A \Subset B$	$A \subset B$ such that $\overline{A} \subset B$
$\lvert\Omega\rvert$	The Lebesgue measure of $\Omega \subset \mathbb{R}^N$
$\mathbb{S}^{N-1}$	The unit sphere in $\mathbb{R}^N$
$\sigma(\mathbb{S}^{N-1})$	The $(N-1)$-dimensional measure of $\mathbb{S}^{N-1}$
$\delta(x)$	The distance from x to $\partial\Omega$
χ_Ω	The indicator function of Ω
$C^0(\Omega)$	The space of continuous functions on Ω
$C_0(\Omega)$	The space of continuous functions on Ω with compact support
$C^{0,\beta}(\Omega)$	The space of Hölder continuous functions on Ω with exponent β
$C^k(\Omega)$	The space of continuous functions of class k differentiability on Ω
$C^\infty(\Omega)$	The space of smooth functions on Ω
$C_0^\infty(\Omega)$	The space of $C^\infty(\Omega)$ functions with compact support
$\mathscr{S}(\mathbb{R}^N)$	Schwartz space
$L^p(\Omega)$	The Lebesgue space on Ω, for $1 \leq p < \infty$
$L^\infty(\Omega)$	The space of essentially bounded functions on Ω
$M^p(\Omega)$	The weak Lebesgue space on Ω (Marcinkiewicz space)
$p' = \frac{p}{p-1}$	The conjugate exponent of p
$W^{k,p}(\Omega)$	The Sobolev space of functions with weak derivatives up to order k in $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$

Symbol	**Meaning**
$H^k(\Omega)$	The Hilbert-Sobolev space $W^{k,2}(\Omega)$
$H_0^k(\Omega)$	The Hilbert-Sobolev space $W_0^{k,2}(\Omega)$
$p^* = \dfrac{Np}{N-kp}$	The Sobolev critical exponent, for $kp < N$
$W^{s,p}(\Omega)$	The fractional Sobolev space of order $s \in (0,1)$ and integrability p
$W_0^{s,p}(\Omega)$	$\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$
$H^s(\Omega)$	The fractional Hilbert-Sobolev space $W^{s,2}(\Omega)$
$H_0^s(\Omega)$	The space $W_0^{s,2}(\Omega)$
$p_s^* = \dfrac{Np}{N-sp}$	The fractional Sobolev critical exponent, for $sp < N$
$L^{s,p}(\mathbb{R}^N)$	The Bessel potential space of order $s \geq 0$ and integrability p
$L_0^{s,p}(\Omega)$	$\{u \in L^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$
J_s	The Bessel potential operator of order s, $J_s(u) := \mathcal{F}^{-1}\left((1+\lvert\cdot\rvert^2)^{-\frac{s}{2}}\mathcal{F}(u)\right)$

Chapter 1

Preliminaries,Useful Tools and functional spaces

In this chapter, we introduce some basic calculus tools and technical results. These tools will help us later to prove our main theorems and estimates. We also give the definitions of the functional spaces used throughout this work. Let us begin with the preliminary tools and integral estimates.

1.1 Useful tools and Preliminaries Integrals estimates

We start with some useful algebraic and analytical lemmas, for the proof see [38].

Lemma 1.1.1 (Young's inequality). *Let take* $a, b > 0$ *and* $p, q \in (0, \infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *then we have :*

$$ab \leq \frac{t^p a^p}{p} + \frac{b^q}{t^q q}$$

for all $t \in (0, \infty)$

Lemma 1.1.2. *Let* $\lambda > 0$ *then:*

$$ln\frac{1}{x} \leq \frac{1}{x^\lambda} + \frac{1}{\lambda}(ln\frac{1}{\lambda} - 1)$$

for all $x \in (0, \infty)$

Proof. We define the function $f : (0, \infty) \to [0, \infty)$ given by $f(x) := x^{-\lambda} + lnx + \frac{1}{\lambda}(ln\frac{1}{\lambda} - 1)$ and we see that $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to \infty} f(x) = \infty$. And moreover , f is derivable and $f'(x) = 0$ if and only if $x = \lambda^{\frac{1}{\lambda}} = x*$. Thus give us that f admit a global minimum on $(0, \infty)$ at $x = x*$ .Since $f(x*) = 0$, we obtain the desired inequality above. $\square$

Lemma 1.1.3. *Let* $p > 0$ *then*

$$(a + b)^p \leq \max\{1, 2^{p-1}\}(a^p + b^p)$$

for all $a, b \geq 0$

Lemma 1.1.4. *Let* $N \geq 1$ *and* $\lambda > 1$ *then:*

$$||u|^\lambda - |w|^\lambda| \leq \lambda \max\{1, 2^{\lambda-2}\}(|u - w|^{\lambda-1} + |u|^{\lambda-1})|u - w|$$

$\forall u, w \in \mathbb{R}^N$

Now, we present some integrals estimates that are useful to prove our main results. We start with:

Lemma 1.1.5. *Let $N \geq 1$,$\rho > 0$ and $\alpha, \beta \in (-\infty, N)$.*
There exists $C := C(N, \alpha, \beta, \rho) > 0$ such that :

- *If $N - \alpha - \beta \neq 0$, then :*

$$\int_{B_\rho(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} \leq C(1 + |x-y|^{N-\alpha-\beta}) \qquad \textit{for all } x, y \in B_\rho(0) \textit{ with } x \neq y$$

- *If $N - \alpha - \beta = 0$, then :*

$$\int_{B_\rho(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} \leq C(1 + |ln|x-y||) \qquad \textit{for all } x, y \in B_\rho(0) \textit{ with } x \neq y$$

Proof. Let fix $x, y \in B_\rho(0)$ with $x \neq y$. Consider the map $T : \mathbb{R}^N \to \mathbb{R}^N$ given by $T(z) = \frac{z-x}{|x-y|}$. Let $A := T(B_\rho(0)) = B_{\frac{\rho}{|x-y|}}(\frac{-x}{|x-y|})$, it follows that :

$$\int_{B_\rho(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} = |x-y|^{N-\alpha-\beta} \int_A \frac{dw}{|w|^\alpha |w+p|^\beta} \qquad \text{with} \qquad p = \frac{x-y}{|x-y|}.$$

Since $|\rho| = 1$,we can split the domain A into three parts:

$$A_1 := A \cap B_{\frac{1}{2}}(0), \qquad A_2 := A \cap (B_2(0) \setminus B_{\frac{1}{2}}(0)) \qquad \text{and} \qquad A_3 := A \cap (\mathbb{R}^N \setminus B_2(0))$$

Thus, we can write:

$$\begin{aligned} \int_{B_\rho(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} &= |x-y|^{N-\alpha-\beta} \int_A \frac{dw}{|w|^\alpha |w+p|^\beta} \\ &= |x-y|^{N-\alpha-\beta} \sum_{i=1}^{3} \int_{A_i} \frac{dw}{|w|^\alpha |w+p|^\beta} \end{aligned} \tag{1.1.5.1}$$

We estimate each integral seperately. First if $w \in A_1$ then $|w+p| \geq \frac{1}{2}$. Since $\alpha < N$, we have :

$$\int_{A_1} \frac{dw}{|w|^\alpha |w+p|^\beta} \leq 2^\beta \int_{B_{\frac{1}{2}}(0)} \frac{dw}{|w|^\alpha} = \frac{2^{\beta+\alpha-N}}{N-\alpha} \sigma(\mathbb{S}^{N-1}). \tag{1.1.5.2}$$

Second, for the integral with A_2, we have $|w| \geq \frac{1}{2}$ and $(B_2(0) \setminus B_{\frac{1}{2}}(0)) \subset B_4(-p)$. Since $\beta < N$, we obtain:

$$\int_{A_2} \frac{dw}{|w|^\alpha |w+p|^\beta} \leq 2^\alpha \int_{B_4(-p)} \frac{dw}{|w+p|^\beta} = \frac{2^{2N-2\beta+\alpha}}{N-\beta} \sigma(\mathbb{S}^{N-1}). \tag{1.1.5.3}$$

Finally, for A_3 we have $|w+p| \geq \frac{|w|}{2}$ and $A_3 \subset \{w \in \mathbb{R}^N : 2 \leq |w| \leq \frac{2\rho}{|x-y|}\}$. This gives

$$\int_{A_3} \frac{dw}{|w|^\alpha |w+p|^\beta} \leq 2^\beta \sigma(\mathbb{S}^{N-1}) \int_2^{\frac{2\rho}{|x-y|}} r^{N-1-\alpha-\beta} dr. \tag{1.1.5.4}$$

If $N - \alpha - \beta = 0$, we obtain:

$$\int_{A_3} \frac{dw}{|w|^\alpha |w+p|^\beta} \leq 2^\beta \sigma(\mathbb{S}^{N-1})(|ln\rho| + |\ln|x-y||). \tag{1.1.5.5}$$

If $N-\alpha-\beta \neq 0$, we obtain :

$$\int_{A_3} \frac{dw}{|w|^\alpha |w+p|^\beta} \leq 2^\beta \sigma(\mathbb{S}^{N-1})(1+\frac{\rho^{N-\alpha-\beta}}{|x-y|^{N-\alpha-\beta}}). \tag{1.1.5.6}$$

Combining estimates (1.1.5.2)-(1.1.5.6) and substituting into (1.1.5.1) we get the final results. □

To study the existence of weak solutions, we will use the Schauder fixed point theorem.

Theorem 1.1.1 (Fixed point of Schawder)**.** *Let E a Banach space, $\mathcal{K}$ un convex closed set of E and $T : \mathcal{K} \longrightarrow \mathcal{K}$, is continuous compact map. Then T admits a fixed point.*

We close this section with the following technical lemma

Lemma 1.1.6. *Let $a, b > 0$, $p > 1$ and $c^* := \frac{p-1}{p}\left(\frac{1}{pa^p b}\right)^{\frac{1}{p-1}}$. Then, the function $g : [0,\infty) \to \mathbb{R}$ given by*

$$g(t) = a^p(bt+c^*)^p - t,$$

has exactly one root $t^* \in (0,\infty)$.

And let $k > 0$ we define the functions, for $t \in \mathbb{R}^+$:

$$T_k(t) = \max\{-k; \min\{k,t\}\} \quad \text{and} \quad G_k(t) = t - T_k(t).$$

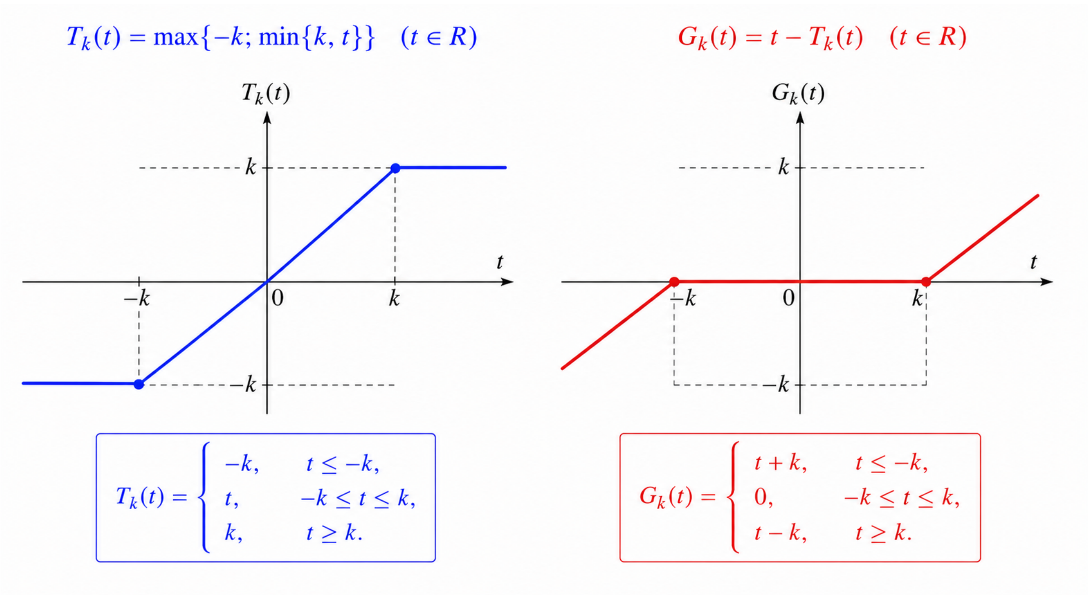

Figure 1.1: The functions T_K and G_k

1.2 Differentiable, Lebesgue and Sobolev Spaces

In this section, we define the main functional spaces used in this work. We start with spaces of continuous and differentiable functions.

Definition 1.2.1. Let $k \in \mathbb{N}$. A function f is called k-differentiable function if $f \in \mathcal{C}^k(\Omega)$ such that:

$$\mathcal{C}^k(\Omega) := \{f \in \mathcal{C}(\Omega) : f^{(j)} \in \mathcal{C}(\Omega), \forall j = (1, ..., k)\}$$

The space of smooth functions is defined by $\mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^{\infty} \mathcal{C}^k(\Omega)$

Proposition 1.2.1. *The space $\mathcal{C}^k(\Omega)$ endowed with the norm :*

$$\|f\|_{\mathcal{C}^k(\Omega)} := \sup_{x\in\Omega} |f(x)| + \sum_{j=1}^{k} \sup_{x\in\Omega} |f^j(x)|$$

is a Banach space. And this holds also for $k = \infty$.

Definition 1.2.2. We denote by $\mathcal{C}_c^\infty(\Omega)$, the space of smooth functions with compact support in Ω. It is equipped with the family of semi-norms :

$$N_{k,\mathcal{K}}(\varphi) := \max_{x\in\mathcal{K}} \max_{|\alpha|\leq k} |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega),$$

for any compact set $\mathcal{K} \subset \Omega$ and $k \in \mathbb{N}$. Its topological dual is the space of distributions denoted by $\mathscr{D}'(\Omega)$

Definition 1.2.3. A smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is called a Schwartz function if :

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \exists C_{\alpha,\beta} > 0, \quad \sup_{x\in\mathbb{R}^d} |x|^\alpha |\partial_x^\beta f(x)| \leqslant C_{\alpha,\beta}.$$

We denote by $\mathscr{S}(\mathbb{R}^d)$ the space of Schwartz functions.
This space by family of semi-norms :

$$N_k(\varphi) := \sup_{x\in\mathbb{R}^d} \max_{|\alpha|\leqslant k, |\beta|\leqslant k} |x^\beta \partial^\alpha \varphi(x)|, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d),$$

for every $k \in \mathbb{N}$.

Proposition 1.2.2. *We have the following strict inclusions*

$$C_0^\infty(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$$

Next, we recall the definitions of Lebesgue spaces.

Definition 1.2.4.

1. Let $1 \leq p < \infty$, and let Ω be an open subset of $\mathbb{R}^N$. For any measurable function $f : \Omega \to \mathbb{R}$, we define :

$$||f||_{L^p(\Omega)} := \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}$$

   We say that $f \in L^p(\Omega)$ if and only if $||f||_{L^p(\Omega)} < \infty$

2. Let $p = \infty$, we define :

$$||f||_{L^\infty(\Omega)} := ess \sup_{x\in\Omega}(|f(x)|) = inf\{C > 0, |f(x)| \leq C, \quad \text{almost every } x \in \Omega\}$$

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable function such that } \|f\|_{L^\infty(\Omega)} < \infty\}$$

Here are some standard properties and inequalities for Lebesgue spaces.

Proposition 1.2.3.

1. *Let* $1 \leq p \leq \infty$. *The space* $L^p(\Omega)$ *endowed with the norm* $||f||_{L^p(\Omega)}$ *is a Banach space.*

2. *For* $p = 2$, *the space* $L^2(\Omega)$ *endowed with the scalar product* $\langle f, g\rangle := \int_\Omega fg\, dx$ *is a Hilbert space.*

We also introduce the weak Lebesgue spaces (or Marcinkiewicz spaces).

Definition 1.2.5. Let $1 \leq p < \infty$, and Ω open set of $\mathbb{R}^N$, if f is measurable function and let define:

$$\|f\|_{M^p(\Omega)} := \sup_{t>0}\{t\mu(|f| > t)^{\frac{1}{p}}\}.$$

We say that f is in the weak lebegue space $M^p(\Omega)$ if and only if $\|f\|_{M^p(\Omega)} < \infty$
In case $p = \infty$ we have $M^\infty(\Omega) = L^\infty(\Omega)$ and with $\|f\|_{M^\infty(\Omega)} := \inf_{t>0}(\mu(|f| > t) = 0)$

Useful inequalities

Theorem 1.2.1 (Hölder's Inequality). *Let* $1 \leq p, q \leq \infty$ *such that,* $\frac{1}{p} + \frac{1}{q} = 1$. $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$, *then* $fg \in L^1\Omega)$, *furthermore*

$$\int_\Omega |fg|\, dx \leq ||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}$$

Theorem 1.2.2 (Minkovski's Inequality). *Let* $1 \leq p, q \leq \infty$, *If* $f, g \in L^p(\Omega)$ *then:*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

Theorem 1.2.3 (Fubini's theorem). *Let* Ω_1, Ω_2 *be an open sets of* $\mathbb{R}^N$, *and* $F : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ *is a measurable function. If one of the statment are true:*

- $$F(x, .) \in L^1(\Omega_2) \quad and \quad \int_{\Omega_2} F(., y)\, dy \in L^1(\Omega_1)$$

  For almost every $x \in \Omega_1$, *or*

- $$F(., y) \in L^1(\Omega_1) \quad and \quad \int_{\Omega_1} F(x, .)\, dx \in L^1(\Omega_2).$$

  For almost every $y \in \Omega_2$

Then, $F \in L^1(\Omega_1 \times \Omega_2)$ *and moreover:*

$$\int_{\Omega_1\times\Omega_2} F(x, y)\, dxdy = \int_{\Omega_1}\left(\int_{\Omega_2} F(x, y)\, dy\right) dx = \int_{\Omega_2}\left(\int_{\Omega_1} F(x, y)\, dx\right) dy$$

Theorem 1.2.4 (Interpolation's inequality). *Let* $1 \leq p < r < q \leq \infty$ *and* $f \in (L^p \cap L^q)(\mathbb{R}^N)$, *then:*
$f \in L^r(\mathbb{R}^N)$, *moreover :*

$$\left(\int_{\mathbb{R}^N} |f|^r dx\right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} |f|^p dx\right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^N} |f|^q dx\right)^{\frac{1-\theta}{q}}$$

with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ *such that* $0 \leq \theta \leq 1$

Theorem 1.2.5. *(Lebesgue's monotone convergence theorem). Let $(f_n)_n$ be a sequence of measurable functions $f_n : \Omega \to \mathbb{R}$ satisfying:*

$$\begin{aligned} &0 \leq f_n(x) \leq f_{n+1}(x), && \forall n \in \mathbb{N}, \text{ almost every } x \in \Omega, \\ &f_n(x) \to f(x), && \text{almost every } x \in \Omega. \end{aligned}$$

Then, f is also measurable and

$$\int_\Omega f_n \, dx \longrightarrow \int_\Omega f \, dx.$$

Theorem 1.2.6. *(Lebesgue's dominated convergence theorem). Let $(f_n)_n$ be a sequence of measurable functions satisfying:*

$$\begin{aligned} &\exists g \in L^1(\Omega) \quad \text{such that } |f_n(x)| \leq g(x), && \forall n \in \mathbb{N}, \text{ almost every} \, x \in \Omega, \\ &f_n(x) \to f(x), && \text{almost every} \, x \in \Omega. \end{aligned}$$

Then, $f \in L^1(\Omega)$ and

$$\int_\Omega |f - f_n| \, dx \longrightarrow 0 \quad \text{and} \quad \int_\Omega f_n \, dx \longrightarrow \int_\Omega f \, dx.$$

Lemma 1.2.1 (Fatou's lemma)**.** *Let Ω be an open set of $\mathbb{R}^N$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of positive measurable functions. Then:*

$$\int_\Omega \liminf_{n\to\infty} f_n \, dx \leqslant \liminf_{n\to\infty} \int_\Omega f_n \, dx$$

Theorem 1.2.7 (Vitali convergence theorem)**.** *Let Ω be an open bounded set of $\mathbb{R}^N$ and that $1 \leqslant p < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(\Omega)$. Then, the following statements are equivalent :*

1. *$(f_n)_{n\in\mathbb{N}}$ converges to f almost every x in Ω and $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega)$.*

2. *$f \in L^p(\Omega)$ and $(f_n)_{\mathbb{N}}$ converges to f in $L^p(\Omega)$.*

Lemma 1.2.2. *Let Ω be an open of $\mathbb{R}^N$ of class $\mathcal{C}^2$, $N \geq 2$ then we have:*

(i) For $\mu \in (0,1)$, $\delta^{-\mu} \in L^p(\Omega)$ if and only if $1 \leq p < \frac{1}{\mu}$

(ii) $\ln \delta \in L^p(\Omega)$ if and only if $1 \leq p < \infty$

Now, we introduce Sobolev spaces, which are based on weak derivatives.

Definition 1.2.6. Let $1 \leq p \leq \infty$ and let Ω be a open subset of $\mathbb{R}^N$ and let $f \in L^p(\Omega)$ for. We say that $f_i \in L^p(\Omega)$ is i-weak derivative for $\forall i = (1, ..., N)$ if:

$$\int_\Omega f_i \varphi \, dx = -\int_\Omega f \partial_{x_i} \varphi \, dx \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)$$

And now we will give defintion of Sobolev space by using definition of weak derivative

Definition 1.2.7. Let $1 \leq p \leq \infty$, and Ω a open set of $\mathbb{R}^N$, the Sobolev space is defined by :

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega), \forall i = (1, ..., N), \exists f_i \in L^p(\Omega),\ f_i \text{ a weak derivative of } f\}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm: For $1 \leq p < \infty$ let :

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

and for $p = \infty$:

$$\|f\|_{W^{1,\infty}(\Omega)} := \left(\|f\|_{L^\infty(\Omega)} + \sum_{i=1}^{N} \|\partial_i f\|_{L^\infty(\Omega)} \right)$$

Proposition 1.2.4.

1. *For $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm $\|.\|_{W^{1,p}(\Omega)}$ is a Banach space.*
2. *For $p = 2$, the Sobolev space $W^{1,2}(\Omega)$ endowed with the scalar product:*

$$\langle f, g \rangle_{W^{1,2}(\Omega)} := \int_\Omega fg\, dx + \sum_{i=1}^{N} \int_\Omega \partial_i f \partial_i g\, dx$$

for all $f, g \in W^{1,2}(\Omega)$ is a Hilbert space.

Now we will give some lemmas that related to those spaces and will help us to proof our main results

Lemma 1.2.3. *Let $f \in W^{1,1}(\mathbb{R}^N)$ then:*

$$f(x) - f(z) = \int_0^1 \langle x - z, \nabla f(\theta x + (1 - \theta)z) \rangle d\theta$$

almost every $x, z \in \mathbb{R}^N$ and for $\theta \in [0, 1]$.

1.3 Fractional Functional spaces

In this section, we introduce fractional Sobolev spaces (see [6], [9] and [27] for more details on these spaces).

Definition 1.3.1. Let $s \in (0, 1)$ and $1 \leq p < +\infty$, let Ω open subset of $\mathbb{R}^N$. The fractional Sobolev space $W^{s,p}(\Omega)$, is defined by:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dxdy < +\infty \right\}.$$

Proposition 1.3.1. *The fractional Sobolev space $W^{s,p}(\Omega)$ endowed with the norm :*

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dxdy \right)^{\frac{1}{p}}.$$

is a Banach space.

Definition 1.3.2. Let $s \in (0,1)$ and $1 \leq p < +\infty$, we can define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ by:

$$W^{s,p}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}}$$

Definition 1.3.3. Let $s \in (0,1)$ and $1 \leq p < +\infty$,let Ω open set of $\mathbb{R}^N$ then:

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Proposition 1.3.2. *If Ω is bounded, then $W_0^{s,p}(\Omega)$ endowed with the norm :*

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}},$$

with,

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c) = (\Omega \times \mathbb{R}^N) \cup (\Omega^c \times \Omega)$$

is a Banach space.

Proposition 1.3.3. *If $s \in (0,1)$ and $p = 2$ then:*

- *We denote $H^s(\Omega) := W^{s,2}(\Omega)$ equipped with the inner product:*

$$\langle u, v \rangle_{H^s(\Omega)} = \int_\Omega u(x)v(x)\, dx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy,$$

  it is a Hilbert space.

- *We denote $H_0^s(\Omega) := W_0^{s,2}(\Omega)$ endowed with scalar product:*

$$\langle u, v \rangle_{H_0^s(\Omega)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy.$$

  it is a Hilbert space.

We now list some embedding theorems for fractional spaces.

Proposition 1.3.4. *Let Ω be an open set of $\mathbb{R}^N$, $0 < s \leq s' < 1$ and $1 \leq p < +\infty$, Then, n, there exists a constant $k > 0$ such that:*

$$\|u\|_{W^{s,p}(\Omega)} \leqslant k \|u\|_{W^{s',p}(\Omega)},$$

for all $u \in W^{s',p}(\Omega)$. Therefore,

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Theorem 1.3.1. *Let $s \in (0,1)$, $1 \leq p < \infty$, and let $\Omega \subset \mathbb{R}^N$be a bounded Lipschitz domain. Then:*

- *If $N > ps$, then there exists a constant $C > 0$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)} \quad \text{for all } p \leq q \leq p_s^*$$

  Moreover,

$$W^{s,p}(\Omega) \subseteq L^q(\Omega) \quad \text{for all } p \leq q \leq p_s^*.$$

  with $p_s^* := \frac{Np}{N-ps}$.
  And if, Ω is bounded, then the embedding $W^{s,p}(\Omega) \subset L^q(\Omega)$ for all $1 \leq q < p_s^*$ is compact.

- *If $N = ps$, then there exists a constant $C > 0$ such that, for any $u \in W^{s,p}(\Omega)$,*
$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)} \quad \textit{for all } p \leq q < \infty.$$
Moreover,
$$W^{s,p}(\Omega) \subseteq L^q(\Omega) \quad \textit{for all } p \leq q < \infty.$$
If also, Ω is a bounded set, then $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \leq q < \infty$.

- *If $N < sp$ and Ω is bounded. Then $W^{s,p}(\Omega)$ is compactly embedded in $C^{0,\beta}(\Omega)$ for all $\beta < \alpha$, where $\alpha = \frac{sp-N}{p}$.*

We also state the fractional version of the Sobolev inequality:

Theorem 1.3.2 (fractional Sobolev's inequality)**.** *Let $s \in (0,1)$ and $N > 2s$. There exists a constant $\mathcal{S}$ such that, for any function $u \in \mathbb{X}^s(\mathbb{R}^N)$ we have:*
$$\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leq \mathcal{S} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$
with $2^*_s = \frac{2N}{N-2s}$ is the so called fractional critical Sobolev exponent. In particular, if $u \in H^s_0(\Omega)$ we have
$$\|u\|^2_{L^{2^*_s}(\Omega)} \leq \mathcal{S} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$
with the same constant $\mathcal{S}$ called the fractional Sobolev constant.

We conclude this part with the fractional interpolation inequality.

Theorem 1.3.3. *Let $0 \leq s_1 \leq s \leq s_2 < 1$ and $1 \leq p_1, p_2, p < \infty$, with:*
$$s = \theta s_1 + (1-\theta)s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \textit{with} \quad 0 < \theta < 1.$$
then, there exists a constant $C > 0$ such that:
$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leqslant C\|u\|^\theta_{W^{s_1,p_1}(\mathbb{R}^N)} \|u\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^N)},$$
for all $u \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N)$.

1.4 Fractional Functional Framework

We will require subsequent theorems related to the Riesz potential, taken from [6, 34], will be needed in the sequel.

Definition 1.4.1. Let $0 < \alpha < N$, the Riesz potential J_α of order α, acting function f on $\mathbb{R}^N$, is defined by :
$$J_\alpha f(x) := a_{N,\alpha} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}\, dy.$$
With $a_{N,\alpha} := 2^{2\alpha-1}\pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N}{2}+\alpha)}{\Gamma(-\alpha)}$

Now, we formulate a fundamental lemma that will serve as a crucial tool later

Lemma 1.4.1. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be an open bounded domain, $0 < \alpha < N$, and $1 \leq p < \ell < \infty$ such that $\frac{1}{\ell} = \frac{1}{p} - \frac{\alpha}{N}$. Moreover, for $g \in L^p(\Omega)$, let*

$$J_\alpha(g)(x) := \int_\Omega \frac{g(y)}{|x-y|^{N-\alpha}}\, dy.$$

Then we have that there exists $C > 0$ such that:

a) J_α is well defined (in the sense that the integral converges absolutely for almost every $x \in \Omega$).

b) $[J_\alpha(g)]_{M^\ell(\Omega)} \leq C\|g\|_{L^1(\Omega)}$. In particular, $\|J_\alpha(g)\|_{L^\sigma(\Omega)} \leq C\|g\|_{L^1(\Omega)}$ for all $1 \leq \sigma < \ell$.

c) If $1 < p < \frac{N}{\alpha}$, then $\|J_\alpha(g)\|_{L^\ell(\Omega)} \leq C\|g\|_{L^p(\Omega)}$.

d) If $p = \frac{N}{\alpha}$, then $\|J_\alpha(g)\|_{L^\sigma(\Omega)} \leq C\|g\|_{L^p(\Omega)}$ for all $1 \leq \sigma < +\infty$.

e) If $p > \frac{N}{\alpha}$, then $\|J_\alpha(g)\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}$.

Theorem 1.4.1. *Assume that Ω is an open bounded domain of $\mathbb{R}^N$ and $0 < \alpha < 2$, then for all $1 \leq p < \frac{N}{N-\alpha}$, the operator defined by:*

$$\begin{aligned} J_\alpha : L^1(\Omega) &\to L^p(\Omega), \\ f \quad &\mapsto \int_{\mathbb{R}^N} \frac{f(y)\chi_\Omega(y)}{|x-y|^{N-\alpha}}\, dy. \end{aligned}$$

is well defined, continuous and compact.

We next introduce the Bessel potential space $L^{s,p}$ along with its structural proper ties. We begin by defining the Fourier transform and the corresponding Bessel potential operator.

Definition 1.4.2. Let $f \in \mathscr{S}(\mathbb{R}^N)$, the Fourier transform of, denoted by $\hat{f}$ is the function defined by:

$$\mathscr{F}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^N} f(x) e^{-2\pi i x.\xi} dx, \quad \xi \in \mathbb{R}^N.$$

Definition 1.4.3. Let $\hat{f} \in \mathscr{S}(\mathbb{R}^N)$ then, we define the Forrier inverse transform by :

$$\mathscr{F}^{-1}(\hat{f})(x) = f(x) := \int_{\mathbb{R}^N} \hat{f}(\xi) e^{2\pi i x.\xi} d\xi, \quad x \in \mathbb{R}^N.$$

Definition 1.4.4. Let $s \geq 0$ and $f \in \mathscr{S}(\mathbb{R}^N)$ The Bessel potential operator of order s of u is denoted by $J_s(u)$

$$J_s(u) := \mathscr{F}^{-1}\left((1+|\cdot|^2)^{\frac{-s}{2}} \mathscr{F}(u)\right)$$

Definition 1.4.5. Let $s \in (0,1)$ and $1 \leq p < \infty$, The Bessel potential space $L^{s,p}$ is defined by:

$$L^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N), \text{ such that } \quad J_s(u) \in L^p(\mathbb{R}^N)\}$$

Proposition 1.4.1. *The Bessel potential space $L^{s,p}(\mathbb{R}^N)$ endowed with the norm :*

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|J_s(u)\|_{L^p(\mathbb{R}^N)} \tag{1.3.5}$$

is a Banach space.

Proposition 1.4.2. *The Bessel potential space can equivalently be defined as the following way :*

$$L^{s,p}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{L^{s,p}(\mathbb{R}^N)}}$$

Equipped with the norm (1.3.5).

Proposition 1.4.3. *For $s \in (0,1)$ and $1 < p < \infty$, the Bessel potential space admits the following equivalents norms:*

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}$$

and

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|\nabla^s u\|_{L^p(\mathbb{R}^N)}$$

where the half s-Laplacian and the Riesz s-gradient are given, respectively, by:

- $$(-\Delta)^{\frac{s}{2}} u(x) = C_{N,\frac{s}{2}}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+s}}\, dy \qquad \textit{Half s Laplacian}$$

- $$(-\nabla)^s u(x) = \mu_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{(x-y)(u(x)-u(y))}{|x-y|^{N+s+1}}\, dy \qquad \textit{Riesz s gradient}$$

with $C_{N,\frac{s}{2}}$ and $\mu_{N,s}$ are the normalization constants.

Let define a subspace of $L^{s,p}$ that will help us in the main results that we want to proof

Definition 1.4.6. Let $s \in (0,1)$ and $1 < p < \infty$ and let Ω an open set of $\mathbb{R}^N$ then let define the space:

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\},$$

Proposition 1.4.4. *If Ω a bounded open set of $\mathbb{R}^N$, then the space $L_0^{s,p}(\Omega)$ endowed with those norms :*

$$\|u\|_{L_0^{s,p}(\Omega)} := \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}$$

and

$$\|u\|_{L_0^{s,p}(\Omega)} := \|\nabla^s u\|_{L^p(\mathbb{R}^N)}.$$

Is a Banach space

After defining the Bessel space $L^{s,p}$, let give some properties of this space

Proposition 1.4.5. *If $0 \leq s \leq t$ then $L^{t,p}(\mathbb{R}^N) \subseteq L^{s,p}(\mathbb{R}^N)$.*
Moreover there exists $k \geq 1$ such that:

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} \leq k \|u\|_{L^{t,p}(\mathbb{R}^N)}$$

for all $u \in L^{t,p}(\mathbb{R}^N)$.

Proposition 1.4.6. *Let $s \in (0,1)$ and $1 < p < \infty$, then for all $\varepsilon > 0$, we have:*

$$L^{s+\varepsilon,p}(\mathbb{R}^N) \subseteq W^{s,p}(\mathbb{R}^N) \subseteq L^{s-\varepsilon,p}(\mathbb{R}^N).$$

Theorem 1.4.2. *Let $s \in (0,1)$ and $1 < p < \infty$. Then:*

- *If $N > ps$,*
$$L^{s,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N) \quad \textit{for all } p \leq q \leq p_s^*$$
Moreover, there exist constants $C_1, C_2 > 0$ such that, for any $u \in L^{s,p}(\mathbb{R}^N)$,
$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_1 \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}.$$
and
$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C_2 \|\nabla^s u\|_{L^p(\mathbb{R}^N)}$$
With $p_s^* := \frac{Np}{N-ps}$.
- *If $N = ps$,*
$$L^{s,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N) \quad \textit{for all } p \leq q < \infty,$$
and there exists a constant $C > 0$ such that, for any $u \in L^{s,p}(\mathbb{R}^N)$,
$$\|u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{L^{s,p}(\mathbb{R}^N)} \quad \textit{for all } p \leq q < \infty.$$
- *If $N < sp$,*
$$L^{s,p}(\mathbb{R}^N) \subseteq C^{0,\alpha}(\mathbb{R}^N),$$
and there exists a constant $C > 0$ such that, for any $u \in L^{s,p}(\Omega)$,
$$\|u\|_{C^{0,\alpha}(\mathbb{R}^N)} := \|u\|_{L^\infty(\mathbb{R}^N)} + \sup_{x,y \in \mathbb{R}^N, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \|u\|_{L^{s,p}(\mathbb{R}^N)},$$
with $\alpha = \frac{sp-N}{p}$

Let give the Bessel potential space like other spaces the Interpolation inequality

Theorem 1.4.3. *Let $0 \leq s_1 \leq s \leq s_2 < \infty$ and $1 < p_1, p_2, p < \infty$, satisfy*
$$s = \theta s_1 + (1-\theta) s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \textit{with} \quad 0 < \theta < 1.$$
Then for all $\theta \in (0,1)$, there exists a constant $C > 0$ such that:
$$\|u\|_{L^{s,p}(\mathbb{R}^N)} \leq C \|u\|^{\theta}_{L^{s_1,p_1}(\mathbb{R}^N)} \|u\|^{1-\theta}_{L^{s_2,p_2}(\mathbb{R}^N)}, .$$
For all $u \in L^{s_1,p_1}(\mathbb{R}^N) \cap L^{s_2,p_2}(\mathbb{R}^N)$

Lemma 1.4.2. *The Bessel potential space $L^{s,p}(\mathbb{R}^N)$ is separable for $1 \leq p < \infty$, as it is isometrically isomorphic to $L^p(\mathbb{R}^N)$ via the Bessel potential operator J_s .*

Remark. For $p = \infty$, the space $L^{t,\infty}(\mathbb{R}^N)$ is not separable, inheriting this property directly from $L^\infty(\mathbb{R}^N)$.

Proposition 1.4.7. *Let $s \geq 0$ and $1 < p < \infty$. The Bessel potential space $L^{s,p}(\mathbb{R}^N)$ is reflexive. Moreover, it is a Hilbert space when $p = 2$.*

Proof. Since the Bessel potential operator $J_s : L^{s,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is an isometric isomorphism [34, 35], and $L^p(\mathbb{R}^N)$ is reflexive for $1 < p < \infty$ [13], it follows immediately that $L^{t,p}(\mathbb{R}^N)$ is reflexive. □

Remark. Since $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ are not reflexives, it follows that $L^{s,1}(\mathbb{R}^N)$ and $L^{s,\infty}(\mathbb{R}^N)$ are not as well as we can see in [13].

And since we talk about the reflexivity of Bessel space, let see its duality.

Proposition 1.4.8. *Let $t \geq 0$ and $1 \leq p < \infty$, and let p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual space of $L^{s,p}(\mathbb{R}^N)$ is given by:*
$$\left(L^{s,p}(\mathbb{R}^N)\right)' = L^{-s,p'}(\mathbb{R}^N)$$
In particular, for $p = 2$, the space $L^{s,2}(\mathbb{R}^N)$ is a Hilbert space and satisfies:
$$\left(L^{s,2}(\mathbb{R}^N)\right)' = L^{-s,2}(\mathbb{R}^N)$$

1.5 Fractional Green function and it estimates

Consider the fractional Poisson problem with homogeneous Dirichlet conditions:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded set with boundary $\partial\Omega$ of class C^2, and $f \in L^m(\Omega)$, for some $m \geq 1$ and $s \in (0,1)$.

Definition 1.5.1. Let $s \in (0,1)$ and Ω a bounded open set of $\mathbb{R}^N$ and also with $\mathbb{R}^{2N}_* := \{(x,y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$, the fractional Green's function is given by:

$$G_s : \mathbb{R}^{2N}_* \to \mathbb{R}.$$

is the unique solution to the distributional problem:

$$\begin{cases} (-\Delta)^s_x G_s(x,y) = d_y(x) & \text{if } x \in \Omega, \\ G_s(x,y) = 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{G}$$

where $y \in \Omega$ is fixed and d_y denotes the Dirac distribution centred at y.

Definition 1.5.2. Let u is a weak solution for the problem (P) it can be represented via the formula:

$$u(x) = \mathbb{G}_s[f](x) := \int_\Omega G_s(x,y) f(y)\, dy, \quad \text{almost every } x \in \Omega.$$

Such that $G_s(x,y)$ Green function is solution for (G).

We now outline sharp kernel and gradient estimates for G_s.

Lemma 1.5.1. *Let $G_s(x,y)$ Green function solution to (G), Then there exist constants $C_1, C_2 > 0$ such that for a.e. $x, y \in \Omega$*

$$\begin{aligned} 1. G_s(x,y) &\leq C_1 \min\{\frac{1}{|x-y|^{N-2s}}, \frac{\delta^s(x)}{|x-y|^{N-s}}, \frac{\delta^s(y)}{|x-y|^{N-s}}\} \quad \textit{almost every}\, x,y \in \Omega \\ 2. |\nabla_x G_s(x,y)| &\leq C_2 G_s(x,y) \max\{\frac{1}{|x-y|}, \frac{1}{\delta(x)}\} \quad \textit{almost every}\, x,y \in \Omega \end{aligned}$$

where $\delta(x) : dist(x, \partial\Omega)$.

Proof. See [38]. □

Lemma 1.5.2. *Let $s \in (0,1)$ and $0 < t < \min\{1, 2s\}$,for any $y \in \Omega$ let : $\alpha_y : \mathbb{R}^N \to \mathbb{R}$ define by :*

$$\alpha_y(x) := |x-y|^{N-(2s-t)} G_s(x,y), \textit{ if } \quad x \neq y \quad \textit{and} \quad \alpha_y(y) = 0.$$

Then : $\alpha_y \in W^{1,p}(\mathbb{R}^N)$ with :

$$\begin{cases} 1 \leq p < \frac{1}{1-s} & \textit{if} \quad s \leq t < min\{1, 2s\} \\ 1 \leq p < \frac{1}{1-t} & \textit{if} \quad 0 < t < s \end{cases}$$

Moreover we have that :

$$\alpha_y(x) - \alpha_y(z) = \int_0^1 \langle x - z, \nabla\alpha_y(\theta x + (1-\theta)z)\rangle d\theta.$$

Almost every $x, y \in \mathbb{R}^N$ and $\theta \in (0,1)$

Lemma 1.5.3. *Let $s \in (0,1)$ and $s \le t < \min\{1, 2s\}$, there exists $C > 0$, such that:*

$$|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| \le \frac{C}{|x-y|^{N-(2s-t)}} \left(1 + |\ln|x-y|| + |\ln\delta(x)| + \frac{|x-y|^{t-s}}{\delta^{t-s}(x)}\right),$$

for almost every $x, y \in \Omega$.

Remark. Let us define the radius R as :

$$R := \frac{1}{3} + \frac{4}{3}(diam(\Omega) + dist(0,\Omega)) \tag{E}$$

such that $\Omega \Subset B_R(0)$.
This choice of reduis R in (E) guarantees the following estimate:

$$|x-y| \ge \frac{1}{4}(1+|x|) \quad \forall y \in \Omega, \quad \forall x \in B_R^c(0) \tag{1.1}$$

We now establish upper bounds for the fractional derivative outside the domain Ω:

Proposition 1.5.1. *Let $s \in (0,1)$, $s \le t < min\{1, 2s\}$ and let R be defined as in (E). There exists $C > 0$ such that :*

(i)

$$|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| \le \frac{C}{|x-y|^{N-2s}} \left(\frac{1}{|x-y|^t} + \frac{1}{\delta(x)^t}\right) \quad \forall y \in \Omega,\ x \in B_R(0) \setminus \Omega$$

(ii)

$$|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| \le \frac{C}{(1+|x|)^{N+t}} \quad \forall y \in \Omega, x \in B_R^c(0)$$

Proof. Since $x \in \mathbb{R}^N \setminus \Omega$ we have by definition of fractional Green function $G_s(x,y) = 0$ so it follow:

$$\begin{aligned} |(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| &= \left| \int_\Omega \frac{G_s(x,y) - G_s(z,y)}{|x-z|^{N+t}} dz + \int_{\mathbb{R}^N\setminus\Omega} \frac{G_s(x,y) - G_s(z,y)}{|x-z|^{N+t}} dz \right| \\ &= \left| \int_\Omega \frac{-G_s(z,y)}{|x-z|^{N+t}} dz \right| = \int_\Omega \frac{G_s(z,y)}{|x-z|^{N+t}} dz \end{aligned}$$

Now according to choice of (E) we have $x \in \mathbb{R}^N \setminus B_R(0)$:

$$|x-z|^{N+t} \ge \frac{1}{4^{N+t}}(1+|x|)^{N+t} \Rightarrow \frac{1}{|x-z|^{N+t}} \le \frac{4^{N+t}}{(1+|x|)^{N+t}}$$

Applying the upper bound from Lemma 1.5.1 :

$$\begin{aligned} |(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| &\le \frac{4^{N+t}}{(1+|x|)^{N+t}} \int_\Omega G_s(z,y) dz \le \frac{4^{N+t} C_1}{(1+|x|)^{N+t}} \int_\Omega \frac{dz}{|z-y|^{N-2s}} \\ &\le \frac{4^{N+t} C_1}{(1+|x|)^{N+t}} \int_{B_{d_\Omega}(y)} \frac{dz}{|z-y|^{N-2s}} \le \frac{C}{(1+|x|)^{N+t}}. \end{aligned}$$

On other hand for $x \in B_R(0) \setminus \Omega$ let split Ω in two parts:
$\Omega_1 := \{z \in \Omega : |z-y| > \frac{1}{2}|x-y|\}$ and $\Omega_2 := \{z \in \Omega : |z-y| \le \frac{1}{2}|x-y|\}$.
Notice that for $z \in \Omega_2$:

$$|x-z| \ge |x-y+y-z| \ge |x-y| - |z-y| \ge \frac{1}{2}|x-y|$$

Let back to $|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)|$ and using Lemma 1.5.1:

$$\begin{aligned}|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| &= \int_\Omega \frac{G_s(z,y)}{|x-z|^{N+t}}dz \leq C_1 \int_\Omega \frac{dz}{|x-z|^{N+t}|z-y|^{N-2s}} \\ &\leq C_1 \left(\int_{\Omega_1} \frac{dz}{|x-z|^{N+t}|z-y|^{N-2s}} + \int_{\Omega_2} \frac{dz}{|x-z|^{N+t}|z-y|^{N-2s}} \right) \\ &\leq \frac{2^{N-2s}C_1}{|x-y|^{N-2s}} \int_{\Omega_1} \frac{dz}{|x-z|^{N+t}} + \frac{2^{N+t}C_1}{|x-y|^{N+t}} \int_{\Omega_2} \frac{dz}{|z-y|^{N-2s}}\end{aligned}$$

In Ω_1:

$$\int_{\Omega_1} \frac{dz}{|x-z|^{N+t}} \leq \sigma(\mathbb{S}^{N-1}) \int_{\delta(x)}^{\infty} \frac{d\rho}{\rho^{1+t}} = \frac{\sigma(\mathbb{S}^{N-1})}{\delta(x)^t}$$

In Ω_2:

$$\int_{\Omega_2} \frac{dz}{|z-y|^{N-2s}} \leq \int_{B_{\frac{1}{2}|x-y|}(y)} \frac{dz}{|z-y|^{N-2s}} = \sigma(\mathbb{S}^{N-1}) \int_0^{\frac{1}{2}|x-y|} \frac{d\rho}{\rho^1} = \sigma(\mathbb{S}^{N-1}) \left(2^{-2s}|x-y|^{2s}\right)$$

Then:

$$\begin{aligned}|(-\Delta)_x^{\frac{t}{2}} G_s(x,y)| &\leq \frac{2^{N-2s}C_1}{|x-y|^{N-2s}} \frac{\sigma(\mathbb{S}^{N-1})}{\delta(x)^t} + \frac{2^{N+t}C_1}{|x-y|^{N+t}} \sigma(\mathbb{S}^{N-1}) \left(2^{-2s}|x-y|^{2s}\right) \\ &\leq \frac{\overline{C_1}}{|x-y|^{N-2s}\delta(x)^t} + \frac{\overline{C_2}}{|x-y|^{N-2s+t}} \\ &\leq \frac{C}{|x-y|^{N-2s}} \left(\frac{1}{\delta(x)^t} + \frac{1}{|x-y|^t} \right)\end{aligned}$$

which completes the proof. □

Now we will give estimates for $\frac{t}{2}$-Laplacian of the weak solution u for the problem (P) based on it fractional Green kernel defined in (G).
Let give this lemma

Lemma 1.5.4. *Let $s \in (0,1)$, $s \leq t < \min\{1, 2s\}$ and let u the unique weak solution of (P) with $f \in L^1(\Omega)$, then there exists $C > 0$ such that:*

$$\left|(-\Delta)^{\frac{t}{2}} u(x)\right| \leq C \left[h_1(x) + |\ln \delta(x)| h_2(x) + \frac{1}{\delta(x)^{t-s}} h_3(x) \right]$$

almost every $x \in \Omega$, such that :

- $$\forall 0 < \lambda < 2s-t, \exists C > 0 \quad \textit{such that} \quad h_1(x) \leq C \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}} dy \quad (1.5.4.1)$$

- $$h_2(x) = \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t)}} dy \quad (1.5.4.2)$$

- $$h_3(x) = (t-s) \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}} dy \quad (1.5.4.3)$$

Proof. Let : $u(x) = \mathbb{G}_s[f](x) := \int_\Omega G_s(x,y)f(y)\,dy$ by using the Fubini's formula:

$$\begin{aligned}
(-\Delta)^{\frac{t}{2}}u(x) &= \int_{\mathbb{R}^N} \frac{\mathbb{G}_s[f](x) - \mathbb{G}_s[f](z)}{|x-z|^{N+t}}dz \\
&= \int_{\mathbb{R}^N} \left(\int_\Omega G_s(x,y)f(y)\,dy - \int_\Omega G_s(z,y)f(y)\,dy \right) \frac{dz}{|x-z|^{N+t}} \\
&= \int_{\mathbb{R}^N} \left(\int_\Omega \frac{(G_s(x,y) - G_s(z,y))}{|x-z|^{N+t}} f(y)dy \right) dz \\
&= \int_\Omega \left(\int_{\mathbb{R}^N} \frac{(G_s(x,y) - G_s(z,y))}{|x-z|^{N+t}} dz \right) f(y)dy \\
&= \int_\Omega (-\Delta)_x^{\frac{t}{2}} G_s(x,y)f(y)\,dy
\end{aligned}$$

This shows that the operator $(-\Delta)^{\frac{t}{2}}$ can be transferred from the solution u to the Green kernel G_s. Using the estimate from Lemma 1.5.3, we obtain:

$$\begin{aligned}
\left|(-\Delta)^{\frac{t}{2}}u(x)\right| &\leq \int_\Omega |(-\Delta)_x^{\frac{t}{2}} G_s(x,y)||f(y)|\,dy \\
&\leq \int_\Omega \frac{C}{|x-y|^{N-(2s-t)}} \left(1 + |\ln|x-y|| + |\ln\delta(x)| + \frac{|x-y|^{t-s}}{\delta^{t-s}(x)} \right) |f(y)|dy \\
&\leq C \left(\int_\Omega \frac{1 + |\ln|x-y||}{|x-y|^{N-(2s-t)}}|f(y)|dy + \int_\Omega \frac{|\ln\delta(x)|}{|x-y|^{N-(2s-t)}}|f(y)|dy + \int_\Omega \frac{\delta^{s-t}(x)}{|x-y|^{N-s}}|f(y)|dy \right)
\end{aligned}$$

We start by an estimate to get h_1, we notice that :

$$|\ln|x-y|| = |-\ln|x-y|| = \left|\ln\frac{1}{|x-y|}\right|.$$

Using Lemma 1.1.2 for $0 < \lambda < 2s - t$

$$\ln\frac{1}{|x-y|} \leq \frac{1}{|x-y|^\lambda} + \frac{1}{\lambda}(ln\frac{1}{\lambda} - 1)$$

So we have:

$$\begin{aligned}
h_1(x) &:= \int_\Omega (1 + |\ln|x-y||) \frac{|f(y)|}{|x-y|^{N-(2s-t)}}dy \\
&\leq \int_\Omega (1 + \frac{1}{\lambda}(ln\frac{1}{\lambda} - 1) + |\frac{1}{|x-y|^\lambda}|) \frac{|f(y)|}{|x-y|^{N-(2s-t)}}dy \\
&\leq C\int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t)}}dy + \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy \\
&\leq C\int_\Omega |x-y|^\lambda \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy + \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy \\
&\leq C\,diam(\Omega)^\lambda \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy + \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy \\
&\leq (C\,diam(\Omega)^\lambda + 1) \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy
\end{aligned}$$

Since $x, y \in \Omega$ so $|x-y| \leq diam(\Omega)$ and let set $C = (C\,diam(\Omega)^\lambda + 1)$ so we get that for $0 < \lambda < 2s - t$ we have:

$$h_1(x) \leq C\int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy$$

And for h_2 we simply get by:

$$\int_\Omega \frac{|\ln \delta(x)|}{|x-y|^{N-(2s-t)}}|f(y)|dy = |\ln \delta(x)| \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t)}}dy = |\ln \delta(x)| h_2(x)$$

we set $h_2(x) = \displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t)}}dy$, and finally for h_3

$$\int_\Omega \frac{\delta^{s-t}(x)}{|x-y|^{N-s}}|f(y)|dy = \frac{1}{\delta^{t-s}(x)} \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy$$

Let set such that:

$$h_3(x) = (t-s)\int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy \leq \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy$$

So we finally obtain our result . □

1.5.1 Local regularity results

Before we got to the fractioal Calderon-Zygmund regularity, let us recall the following first the classical regularity, considering the the Poisson problem with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f \in L^m(\Omega)$ for some $m \geq 1$. Now let give the classical Calderon-Zygmund reglarity theorem for the unique weak solution u of this problem, given by :

Theorem 1.5.1 (Classical Calderon–Zygmund regularity)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, and if $f \in L^p(\Omega)$ for $1 < p < \infty$, then there exists a positive constant $C = C(\Omega, N, p)$ independent of f such that :*

1. *If $1 < p < N$, then*

$$\|\nabla u\|_{L^{p^*}(\Omega)} \leq C\|f\|_{L^p(\Omega)} \text{ where } p^* = \frac{pN}{N-p}.$$

2. *If $p = N$, $|\nabla u| \in L^q(\Omega)$ for all $q \in [1, +\infty)$.*

3. *If $m > N$, $u \in C^{1,\gamma}(\Omega)$ for some $\gamma \in (0,1)$.*

Proof. See [13, 10] □

Remark. The classical Calderon–Zygmund theory shows that the Laplacian $-\Delta$ gains two derivatives in $L^p(\Omega)$ for $1 < p < \infty$. In the fractional setting, the situation is more delicate: the fractional Laplacian $(-\Delta)^s$ gains only $2s$ fractional derivatives in general, and the regularity of $(-\Delta)^{\frac{t}{2}}u$ requires a careful analysis of the boundary behavior of the fractional Green's function, as developed in the next chapter.

Chapter 2

Global Calderon-Zygmund regularity

We consider the fractional Poisson problem with Dirichlet boundary conditions defined by:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $\mathcal{C}^2$ boundary, $f \in L^m(\Omega)$ for some $m \geq 1$, and for $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian.
In this chapter, we present the theorems concerning the global fractional Calderón–Zygmund regularity of $(-\Delta)^{\frac{t}{2}} u$ with $s \leq t < \min\{1, 2s\}$ and u is solution of problem (P). The following theorems are classified according to the value of m: we consider separately the cases $1 \leq m < \frac{N}{2s-t}$ and $m \geq \frac{N}{2s-t}$, and we establish several corollaries concerning the regularity of the solution u in fractional spaces, particularly in Sobolev and Bessel potential spaces.

We conclude this chapter by presenting some compactness results related to the fractional Green kernel, which will be useful in the next chapter for studying the existence and nonexistence of solutions to problem (KPZ).
We now begin by studying the regularity of the solution u in Lebesgue spaces.

2.1 General regularity for the solution

Before going further, we study the regularity of the solution u of problem (P) in Lebesgue spaces, under the assumption that $f \in L^m(\Omega)$ for $m \geq 1$.
From [25], we have the following regularity result for the problem:

2.1.1 Regularity on Lebesgue Spaces:

We start with if $f \in L^m(\Omega)$, for $m > \frac{N}{2s}$ we get the bounded solution in $L^\infty(\Omega)$ given by the theorem.

Theorem 2.1.1. *Let $f \in L^m(\Omega)$ for $m > \frac{N}{2s}$, there exists a constant $C > 0$ such that:*

$$\|u\|_{L^\infty(\Omega)} \leq C$$

To prove this theorem, we use two approaches *the Moser method and the Stampacchia method*. We provide the proof using both techniques.

Before presenting the Moser method, we first state the following proposition :

Proposition 2.1.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\phi(0) = 0$, then if $u \in H_0^s(\Omega)$ we have:*

$$(-\Delta)^s \phi(u) \leq \phi'(u)(-\Delta)^s u \quad \text{weakly in } \Omega.$$

On the other hand, if $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz concave function, then if $u \in H_0^s(\Omega)$ we have:

$$(-\Delta)^s \psi(u) \geq \psi'(u)(-\Delta)^s u \quad \text{weakly in } \Omega.$$

Boundedness of the solution via Moser method.

Proof. For $\beta > 1$ and $T > 0$ large, we define the following convexe function

$$\phi(t) = \begin{cases} t^\beta & \text{if} \quad 0 \leq t < T \\ \beta T^{\beta-1} t - (\beta - 1) T^\beta & \text{if} \quad t > T \end{cases}$$

Since ϕ is Lipschitz and $\phi(0) = 0$, then $\phi(u) \in H_0^s(\Omega)$ and by using the Proposition 2.1.1 we get:

$$(-\Delta)^s \phi(u) \leq \phi'(u)(-\Delta)^s u \tag{2.1.1.1}$$

By Theorem 1.3.2, and since $\phi(u)$ is positive, so we get :

$$\mathcal{S}\|\phi(u)\|^2_{L^{2_s^*}(\Omega)} \leq \|\phi(u)\|^2_{H_0^s(\Omega)} \leq \int_\Omega \phi(u)(-\Delta)^s \phi(u) dx \tag{2.1}$$

In other hand by using (2.1.1.1), we have :

$$\begin{aligned} \int_\Omega \phi(u)(-\Delta)^s \phi(u)\, dx &\leq \int_\Omega \phi(u)\phi'(u)(-\Delta)^s u\, dx = \int_\Omega \phi(u)\phi'(u) f\, dx \\ &\leq \|f\|_{L^m(\Omega)} \|\phi(u)\phi'(u)\|_{L^{\frac{m}{m-1}}(\Omega)} \leq \|f\|_{L^m(\Omega)} \|u^\beta \beta u^{\beta-1}\|_{L^{\frac{m}{m-1}}(\Omega)} \\ &\leq \|f\|_{L^m(\Omega)} \|\beta u^{2\beta-1}\|_{L^{\frac{m}{m-1}}(\Omega)}. \end{aligned}$$

With the fact that $\phi(t) \leq t^\beta$ and $\phi'(t) \leq \beta t^{\beta-1}$.
Setting $m' = \frac{m}{m-1}$, $p = 2_s^*$, $q = 2m'$ and let $T \to \infty$, and using (2.1), we obtain that:

$$\begin{aligned} \|u^\beta\|^2_{L^p(\Omega)} &\leq \mathcal{S}^{-1} \beta \|f\|_{L^m(\Omega)} \left(\int_\Omega u^{(2\beta-1)m'} dx \right)^{\frac{1}{m'}} \\ &\leq \mathcal{S}^{-1} \beta \|f\|_{L^m(\Omega)} \left(\int_\Omega u^{\frac{(2\beta-1)}{2} q} dx \right)^{\frac{2}{q}}. \end{aligned}$$

We obtain that

$$\begin{aligned} \|u\|_{L^{p\beta}(\Omega)} &\leq \left(\mathcal{S}^{-1} \beta \|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}} \left(\int_\Omega u^{\frac{(2\beta-1)}{2} q} dx \right)^{\frac{1}{q\beta}} \\ &\leq \left(\mathcal{S}^{-1} \beta \|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}} \left(1 + \int_\Omega u^{\beta q}\, dx \right)^{\frac{1}{\beta q}}. \end{aligned} \tag{2.1.1.2}$$

Setting $\beta_0 = 1$, we define $\beta_j = (\frac{p}{q})^j$ and it follow that, $\beta_{j+1} q = \beta_j p$.
Let us denote:

$$A_j = \left(\int_\Omega u^{\beta_j p} \right)^{\frac{1}{\beta_j p}}, \quad c_j = \left(\mathcal{S}^{-1} \|f\|_{L^m(\Omega)} \beta_j\right)^{\frac{1}{2\beta_j}}. \tag{2.1.1.3}$$

From (2.1.1.2) and (2.1.1.3) we can set this formula :

$$A_{j+1} \le c_{j+1}\left(1 + A_j^{\beta_j p}\right)^{\frac{1}{\beta_j p}}$$

And by normalizing it so we have $A_0 = 1$ and as consequence $A_j \ge 1$ and after that we apply the log on the formula

$$\log A_{j+1} \le \log c_{j+1} + \frac{1}{\beta_j p}\log\left(1 + A_j^{\beta_j p}\right) \le \log c_{j+1} + \frac{1}{\beta_j p} + \log A_j,$$

By using the fact that $\log(1+x) \le 1 + \log x$ if $x \ge 1$, and by induction we get:

$$\log A_{j+1} \le \sum_{k=1}^{j+1}\log c_k + \sum_{k=0}^{j}\frac{1}{\beta_k p} + \log A_0.$$

We can see that $A_0 \le \|u\|_{H_0^s(\Omega)}$ and also by taking $\beta \to \infty$, we get this two convergent series:

$$\sum_{k=1}^{\infty}\log c_k = \sum_{k=1}^{\infty}\frac{1}{2\beta_k}\log(C\beta_k\|f\|_{L^m(\Omega)}), \qquad \sum_{k=1}^{\infty}\frac{1}{\beta_k p},$$

And by taking the limit $\lim_{j\to\infty} A_j = \|u\|_{L^\infty(\Omega)}$ we get the results . □

Now, we use the second method. Before proceeding, we first state the following lemma, which will be used in the proof.

Boundedness of the solution via Stampacchia method.

Lemma 2.1.1. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that:*

$$\psi(h) \leqslant \frac{M\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0,$$

with $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where $d^\gamma = M\psi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}$.

and we have useful proposition for thus functions

Proposition 2.1.2. *Let w be a function in $H_0^s(\Omega)$:*

i) If ψ is Lipschitz function such that $\psi(0) = 0$, then $\psi(w) \in H_0^s(\Omega)$. In particular, for any $k \ge 0$, $T_k(w), G_k(w) \in H_0^s(\Omega)$.

ii) For any $k \ge 0$

$$\|G_k(w)\|^2_{H_0^s(\Omega)} \le \int_\Omega G_k(w)(-\Delta)^s w\,dx; \tag{1.5.2.1}$$

iii) For any $k \ge 0$

$$\|T_k(w)\|^2_{H_0^s(\Omega)} \le \int_\Omega T_k(w)(-\Delta)^s w\,dx. \tag{1.5.2.2}$$

Proof of Theorem 2.1.1 Let $k > 0$ and use $G_k(u)$ as a test function and using (1.5.2.1) we get that:

$$\|G_k(u)\|^2_{H_0^s(\Omega)} \le \int_{A_k} f\,G_k(u)\,dx.$$

With $A_k = \{x \in \Omega : u \geq k\}$, applying the Sobolev's inequality we get:

$$\mathcal{S}^{-2}\|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leq \|G_k(u)\|^2_{H^s_0(\Omega)}.$$

By using the Holder's inequality, we have:

$$\left|\int_{A_k} f(x)G_k(u(x))\,dx\right| \leqslant \|f\|_{L^m(\Omega)}\|G_k(u)\|_{L^{2^*_s}(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}$$

So finally we have that:

$$\|G_k(u)\|_{L^{2^*_s}(\Omega)} \leq \mathcal{S}^2\|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}.$$

Since for any $h > k$, we have $A_h \subset A_k$ and $G_k(u)\chi_{A_h} \geq (h-k)$ with fact that $A_h \subset \Omega$ we have that :

$$(h-k)|A_h|^{\frac{1}{2^*_s}} \leq \mathcal{S}^2\|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}.$$

Then we deduce that :

$$|A_h| \leqslant \frac{\mathcal{S}^{2\cdot 2^*_s}\|f\|^{2^*_s}_{L^m(\Omega)}|A_k|^{2^*_s(1-\frac{1}{2^*_s}-\frac{1}{m})}}{(h-k)^{2^*_s}}.$$

Using the fact that $m > \frac{N}{2s}$ then

$$2^*_s\left(1-\frac{1}{2^*_s}-\frac{1}{m}\right) > 1$$

Finally by using the lemma 2.1.1 setting $\psi(t) = |A_t|$ so there exists t_0 such that $\psi(t) = 0$ for any $t \geq t_0$ and then $\sup u \leq t_0$, and by this we finish our proof. And for the case $m = \frac{N}{2s}$ the boundedness of the weak solution u is not expected but at least we get the exponential summability given in this theorem.

Theorem 2.1.2. *Let $f \in L^{\frac{N}{2s}}(\Omega)$, then the weak solution u of (P) verifies :*

$$\exists \alpha > 0 \quad \textit{such that} \quad \int_\Omega e^{\alpha u}\,dx < \infty.$$

Or we can say that, $u \in L^q(\Omega)$ for every $q < \infty$.

Proof. For any $T > 0$, let define the convex function by:

$$\phi(t) = \phi_T(t) = \begin{cases} e^{\alpha t} - 1, & \text{if} \quad 0 \leq t < T \\ \alpha e^{\alpha T}(t-T) + e^{\alpha T} - 1, & \text{if} \quad t \geq T \end{cases}$$

with $\alpha > 0$ will fix it later. let $u \in H^s_0(\Omega)$ the unique weak solution of (P) and since $\phi(u)$ is convex and is a Lipschtz function with $\phi(0) = 0$ then we can use it as an admissible test function, also by using Theorem 1.3.2 (the fractional Sobolev's inequality) and by Proposition 2.1.1 we get:

$$\mathcal{S}\|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \leq \int_\Omega \phi'(u)\phi(u)(-\Delta)^s u = \int_\Omega \phi'(u)\phi(u)f.$$

We see that for $u < T$ we have $\phi'(u) = \alpha e^{\alpha u} = \alpha\phi(u) + \alpha$ and so we get that :

$$\mathcal{S}\|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \leq \alpha\int_{\{x\in\Omega\,:\,u<T\}} \big(\phi(u)^2 f + \phi(u)f\big)\,dx \,+\, \alpha e^{\alpha T}\int_{\{x\in\Omega\,:\,u\geq T\}} \phi(u)f\,dx + C.$$

And by using the Holder's inequality we have that:

$$\alpha \int_{\{x\in\Omega\,:\,u<T\}} \left(\phi(u)^2 f + \phi(u) f\right) dx \leq \alpha \|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \|f\|_{L^{\frac{N}{2s}}(\Omega)} \left(1 + |\Omega|^{\frac{N-2s}{2N}}\right),$$

And also since ϕ is increasing, we get:

$$\alpha e^{\alpha T} \int_{\{x\in\Omega\,:\,u\geq T\}} \phi(u) f \, dx \leq \alpha \frac{e^{\alpha T}}{\phi(T)} \int_{\{u\geq T\}} \phi(u)^2 f \leq 2\alpha \|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \|f\|_{L^{\frac{N}{2s}}(\Omega)},$$

given that $\alpha T > 1$. Putting all this estimates together, with fixing α small enough, get the boundedness of $\phi(u)$ in $L^{2^*_s}(\Omega)$, that we give us the results for $T \to \infty$. □

Now we see a Calderon-Zygmund type regularity for the weak solution u on low summability of f,let give this theorem then

Theorem 2.1.3. *Let $f \in L^m(\Omega)$, with $\frac{2N}{N+2s} \leq m < \frac{N}{2s}$ and u the unique weak solution of (P). then, there exists $C := C(N, m, s) > 0$ such that:*

$$\|u\|_{L^{m^{**}_s}(\Omega)} \leq C \|f\|_{L^m(\Omega)}$$

where $m^{**}_s = \frac{mN}{N-2ms}$.

Proof. Let define this convex function by :

$$\phi(t) = \begin{cases} t^\beta & \text{if} \quad 0 \leq t \leq T, \\ \beta T^{\beta-1}(t-T) + T^\beta & \text{if} \quad t > T, \end{cases}$$

Such that it is a differentiable function, with $\beta = \frac{m^{**}_s}{2^*_s} > 1$.
And since that ϕ is convex and $\phi(0) = 0$, we can use $\phi(u)$ as an addmissible test function and by the proposition 2.1.1 we have that:

$$(-\Delta)^s \phi(u) \leq f\, \phi'(u)$$

again we use $\phi(u)$ in the above inequality, and we get :

$$\int_\Omega (-\Delta)^s \phi(u)\, \phi(u)\, dx \leq \int_\Omega f\, \phi(u)\, \phi'(u)\, dx$$

By the Holder's and Sobolev's inequlaties we get :

$$\mathcal{S} \|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \leq \|f\|_{L^m(\Omega)} \left(\int_\Omega |\phi'(u)\phi(u)|^{m'}\right)^{\frac{1}{m'}}.$$

with $m' = \frac{m}{m-1}$.
With our choice of β, there exists a constant $C > 0$ indepent from T, such that:

$$\forall t \geq 0 \quad |\phi'(t)\phi(t)|^{m'} \leq C|\phi(t)|^{2^*_s},$$

So we get

$$\mathcal{S} \left(\int_\Omega |\phi(u)|^{2^*_s}\right)^{\frac{2}{2^*_s} - \frac{1}{m'}} \leq c\|f\|_{L^m(\Omega)}.$$

Finally, with our choose of β we have that

$$\beta 2^*_s = m'(2\beta - 1) = m^{**}_s \quad \text{and} \quad \frac{2}{2^*_s} - \frac{1}{m'} = \frac{1}{m^{**}_s}$$

By taking $T \to \infty$ we get that :

$$\|u\|_{L^{m^{**}_s}(\Omega)} \leq C \|f\|_{L^m(\Omega)}$$

and we got the Calderon-Zygmund regularity. □

Remark. For the case $1 \leq m < \frac{2N}{N+2s}$, see for instance [25], [38].

For a more general regularity result, based on Calderón–Zygmund theory and the estimates provided by the fractional Green representation formula, we establish the following theorem

Theorem 2.1.4. *Let $s \in (0,1)$ with $u := \mathbb{G}_s[f]$ the unique weak solution of (P) with $f \in L^m(\Omega)$ then if :*

$$\begin{cases} r = \infty, & \text{if } m > \dfrac{N}{2s}, \\ r \in [1,\infty), & \text{if } m = \dfrac{N}{2s}, \\ r = \dfrac{mN}{N-2ms}, & \text{if } 1 < m < \dfrac{N}{2s}, \\ r \in \left[1, \dfrac{N}{N-2s}\right), & \text{if } m = 1, \end{cases} \quad \text{and} \quad \begin{cases} q = \infty, & \text{if } m > \dfrac{N}{s}, \\ q \in [1,\infty), & \text{if } m = \dfrac{N}{s}, \\ q = \dfrac{mN}{N-ms}, & \text{if } 1 < m < \dfrac{N}{s}, \\ q \in \left[1, \dfrac{N}{N-s}\right), & \text{if } m = 1, \end{cases} \tag{2.2}$$

there exists $C > 0$ such that:

$$\|u\|_{L^r(\Omega)} + \left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)}. \tag{2.3}$$

Let give a quick proof for Theorem 2.1.4

Proof. Taking into account the representation of the fractional Green kernel, and using Lemma 1.5.1, we obtain the following result, we get:

$$|u(x)| \leq \int_\Omega |G_s(x,y)f(y)|\,dy \leq C_1 \int_\Omega \frac{|f(y)|}{|x-y|^{N-2s}}dy$$

Now, setting $J_\alpha(f) = \displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}}dy$ with $\alpha = 2s$ then Lemma 1.4.1 so we get the following estimates :

$$\|u\|_{L^r(\Omega)} \leq C_1\|J_\alpha(f)\|_{L^r(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

Now for $\frac{u}{\delta^s}$ the regularity in $L^q(\Omega)$ let using Lemma 1.5.1 to get:

$$\left|\frac{u}{\delta^s}\right| \leq \frac{1}{\delta(x)^s}\int_\Omega |G_s(x,y)f(y)|\,dy \leq \frac{C_1}{\delta(x)^s}\int_\Omega \frac{\delta(x)^s|f(y)|}{|x-y|^{N-s}}dy \leq C_1\int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy$$

Then we get our result after using Lemma 1.4.1 by setting $J_\alpha(f) = \displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}}dy$ with $\alpha = s$:

$$\left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C_1\|J_\alpha(f)\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)}$$

where q depends on the values of m. $\square$

2.2 The global fractional Calderon-Zygmund theorems

We now establish a global fractional Calderón–Zygmund regularity result, starting with the case of $m > \frac{N}{2s-t}$.

Theorem 2.2.1. *Let $s, t \in (0,1)$ and u unique weak solution for (P), $f \in L^m(\Omega)$ for $m > \max\{\frac{N}{s}, \frac{N}{2s-t}\}$, then:*

1. *If $0 < t < s$, for all $1 \leq p \leq \infty$, $\exists C > 0$ such that :*
$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$
2. *If $t = s$, for all $1 \leq p < \infty$, $\exists C > 0$ such that:*
$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$
3. *If $s < t < \min\{1, 2s\}$, for all $1 \leq p < \frac{1}{t-s}$, $\exists C > 0$ such that :*
$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

Proof. Since $m > \min\{\frac{N}{s}, \frac{N}{2s-t}\}$, and by [29, Proposition 14,(iii)], there exists $C_s > 0$ such that :
$$\|u\|_{\mathcal{C}^s(\mathbb{R}^N)} \leq C_s\|f\|_{L^m(\Omega)}.$$
For any $x \in \mathbb{R}^N \setminus B_R(0)$, we have from Proposition 1.5.1 and Definition (E), and since $(-\Delta)^{\frac{t}{2}} u(x) = \displaystyle\int_\Omega (-\Delta)_x^{\frac{t}{2}} G_s(x,y) f(y) dy$, there exists $C > 0$ such that:
$$\begin{aligned} |(-\Delta)^{\frac{t}{2}} u(x)| &= \left|\int_\Omega (-\Delta)_x^{\frac{t}{2}} G_s(x,y) f(y) dy\right| \leq \int_\Omega \left|(-\Delta)_x^{\frac{t}{2}} G_s(x,y) f(y)\right| dy \\ &\leq C \int_\Omega \frac{|f(y)|}{(1+|x|)^{N+t}} dy \leq \frac{C}{(1+|x|)^{N+t}} \int_\Omega |f(y)| dy. \end{aligned}$$
By using Holder inequality and setting $C_\Omega = C|\Omega|^{\frac{1}{m'}}$ we have:
$$|(-\Delta)^{\frac{t}{2}} u| \leq \frac{C_\Omega}{(1+|x|)^{N+t}} \|f\|_{L^m(\Omega)}.$$
Since $x \in B_R^c(0)$ that mean $|x| > R$ then :
$$(1+|x|)^{N+t} > (1+R)^{N+t} \Rightarrow \frac{1}{(1+|x|)^{N+t}} < \frac{1}{(1+R)^{N+t}}.$$
Thus
$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^\infty(B_R^c(0))} \leq \frac{C_\Omega}{(1+R)^{N+t}} \|f\|_{L^m(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$
In other hand we have :
$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^1(B_R^c(0))} = \int_{B_R^c(0)} \left|(-\Delta)^{\frac{t}{2}} u(x)\right| dx \leq C_\Omega \|f\|_{L^m(\Omega)} \int_{B_R^c(0)} \frac{dx}{(1+|x|)^{N+t}} \leq C\|f\|_{L^m(\Omega)}.$$
Taking into consideration that $(-\Delta)^{\frac{t}{2}} u \in L^1(B_R^c(0))$ and $(-\Delta)^{\frac{t}{2}} u \in L^\infty(B_R^c(0))$ then by interpolation inequality we get:
$$(-\Delta)^{\frac{t}{2}} u \in L^p(B_R^c(0)) \quad \text{and} \quad \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(B_R^c(0))} \leq \|f\|_{L^m(\Omega)} \tag{I_1}$$

for $1 \leq p \leq \infty$.
Now, we split in three cases:
First case : 0<t<s
Let start with $0 < t < s$ we will use the estimation given in [29, Proposition 14], and by the fact that $u \in \mathcal{C}^s(\Omega)$, we have

$$\begin{aligned} \left|(-\Delta)^{\frac{t}{2}} u(x)\right| &\leq \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+t}} dy \\ &= \int_{B_R(0)} \frac{|u(x)-u(y)|}{|x-y|^{N+t}} dy + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|u(x)-u(y)|}{|x-y|^{N+t}} dy \end{aligned}$$

We analyze each term separately, since $u \in \mathcal{C}^s(\Omega)$, we have :

$$\begin{aligned} \int_{B_R(0)} \frac{|u(x)-u(y)|}{|x-y|^{N+t}} dy &\leq \sup_{x,y \in B_R(0)} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{s}} \right\} \int_{B_R(0)} \frac{dy}{|x-y|^{N-(s-t)}} \\ &\leq \|u\|_{\mathcal{C}^s(\mathbb{R}^N)} \int_{B_R(0)} \frac{dy}{|x-y|^{N-(s-t)}} \leq C_s \|f\|_{L^m(\Omega)} \int_{B_R(0)} \frac{dy}{|x-y|^{N-(s-t)}}. \end{aligned}$$

In other hand, with the condition that $m > \frac{N}{2s}$ by using Theorem 2.1.1, u is in $L^\infty(\mathbb{R}^N)$, we have :

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|u(x)-u(y)|}{|x-y|^{N+t}} dy &\leq \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|u(x)|+|u(y)|}{|x-y|^{N+t}} dy \leq 2 \sup_x \{u(x)\} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{dy}{|x-y|^{N+t}} \\ &\leq 2\|u\|_{\mathcal{C}^s(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{dy}{|x-y|^{N+t}} \\ &\leq 2C_s \|f\|_{L^m(\Omega)} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{dy}{|x-y|^{N+t}}. \end{aligned}$$

By combining the above integrals and by applying the change of variables $\rho = \frac{|x-y|}{R}$, we get:

$$\begin{aligned} \left|(-\Delta)^{\frac{t}{2}} u(x)\right| &\leq C_s \|f\|_{L^m(\Omega)} \left(\int_{B_R(0)} \frac{dy}{|x-y|^{N-(s-t)}} + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{dy}{|x-y|^{N+t}} \right) \\ &\leq C_s \|f\|_{L^m(\Omega)} \sigma\left(\mathbb{S}^{N-1}\right) \left(\int_0^R \frac{d\rho}{\rho^{-(s-t)}} + 2\int_R^\infty \frac{d\rho}{\rho^t} \right) \\ &\leq C \|f\|_{L^m(\Omega)}. \end{aligned}$$

Almost every $x \in \mathbb{R}^N$, such that $\sigma\left(\mathbb{S}^{N-1}\right)$ is the Lebesgue measure of the unit sphere.
In one hand we have :

$$ess \sup_{x \in B_R(0)} \left|(-\Delta)^{\frac{t}{2}} u(x)\right| = \|(-\Delta)^{\frac{t}{2}} u\|_{L^\infty(B_R(0))} \leq C \|f\|_{L^m(\Omega)}.$$

And in other hand :

$$\int_{B_R(0)} \left|(-\Delta)^{\frac{t}{2}} u(x)\right| dx \leq C \|f\|_{L^m(\Omega)} \int_{B_R(0)} dx \leq C_1 \|f\|_{L^m(\Omega)}.$$

So we get that $\|(-\Delta)^{\frac{t}{2}} u\|_{L^1(B_R(0))}$, then we have that $(-\Delta)^{\frac{t}{2}} u \in L^1(B_R(0)) \cap L^\infty(B_R(0))$, as consequence we use the interpolation inequality and according to (I_1), we obtain :

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C \|f\|_{L^m(\Omega)}$$

for $1 \leq p \leq \infty$.

Second case : t=s

For $x, y \in \Omega$ and $\displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-s}} dy < \infty$, we have:

$$h_3(x) = (t-s) \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}} dy = 0.$$

We also have $m > \frac{N}{s}$, choosing $0 < \lambda < s - \frac{N}{m}$, there exists $C > 0$ such that:

$$h_1(x) = J_\alpha(f)(x) \leq C \int_\Omega \frac{|f(y)|}{|x-y|^{N-(s-\lambda)}} dy$$

With $\alpha = s - \lambda$ and by condition of λ we have $m > \frac{N}{s-\lambda}$, by Lemma 1.4.1

$$h_1(x) \leq \|J_\alpha(f)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

and by set $h_2(x) = J_\alpha(f)(x) = \displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}} dy$, with $\alpha = s$ then using Lemma 1.4.1, we have:

$$h_2(x) = \|J_\alpha(f)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

By using Theorem 1.5.4 and by the fact that $h_3 = 0$, we obtain:

$$\left|(-\Delta)^{\frac{t}{2}} u(x)\right| \leq C\left[h_1(x) + |\ln \delta(x)|\, h_2(x)\right] \leq C\|f\|_{L^m(\Omega)} \left(1 + |\ln \delta(x)|\right).$$

Integrating both sides over Ω, and by the fact $\ln \delta(x) \in L^p(\Omega)$, using Lemma 1.2.2, we obtain:

$$\begin{aligned}
\int_\Omega \left|(-\Delta)^{\frac{t}{2}} u(x)\right|^p dx &\leq C^p \|f\|^p_{L^m(\Omega)} \int_\Omega \left(1 + |\ln \delta(x)|\right)^p dx \\
&\leq C_p \|f\|^p_{L^m(\Omega)} \left(|\Omega| + \int_\Omega |\ln \delta(x)|^p dx\right) \leq C\|f\|^p_{L^m(\Omega)}.
\end{aligned}$$

then we have :

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)}. \qquad (I_2)$$

Now, we have $|x-y| \geq \min\{\delta(x), \delta(y)\}$ for $x \in B_R(0) \setminus \Omega$, and $y \in \Omega$, it is result that:

$$|x-y| \geq \delta(y) \Rightarrow |x-y|^s \geq \delta(y)^s \Rightarrow \frac{1}{|x-y|^s} \leq \frac{1}{\delta(y)^s}$$

Let back to $\left|(-\Delta)^{\frac{t}{2}} u(x)\right|$ by using Holder's inequality and Proposition 2.1.4, we obtain :

$$\begin{aligned}
\left|(-\Delta)^{\frac{t}{2}} u(x)\right| &\leq \int_\Omega \frac{|u(y)|}{|x-y|^{N+s}} dy = \int_\Omega \frac{|u(y)|}{|x-y|^N |x-y|^s} dy \\
&\leq \int_\Omega \frac{|u(y)|}{\delta(y)^s} \frac{1}{|x-y|^N} dy \leq \left\|\frac{u}{\delta^s}\right\|_{L^\infty(\Omega)} \int_\Omega \frac{dy}{|x-y|^N} \\
&\leq C\|f\|_{L^m(\Omega)} \sigma(\mathbb{S}^{N-1}) \int_{\delta(x)}^{2R} \frac{d\rho}{\rho} \leq C\|f\|_{L^m(\Omega)} \sigma(\mathbb{S}^{N-1}) \left[\ln 2R - \ln \delta(x)\right] \\
&\leq C\|f\|_{L^m(\Omega)} \sigma(\mathbb{S}^{N-1}) \left[\ln 2R + |\ln \delta(x)|\right] \leq C\|f\|_{L^m(\Omega)} \left(1 + \ln \delta(x)\right).
\end{aligned}$$

Such that $\sigma(\mathbb{S}^{N-1})$ is the Lebesgue measure of the unit sphere, now let estimate $(-\Delta)^{\frac{t}{2}}u$ in $L^p(B_R(0)\setminus\Omega)$. We have:

$$\int_{B_R(0)\setminus\Omega}\left|(-\Delta)^{\frac{t}{2}}u(x)\right|^p dx \leq C^p\|f\|^p_{L^m(\Omega)}\int_{B_R(0)}(1+|\ln\delta(x)|)^p\,dx$$
$$\leq C_p\|f\|^p_{L^m(\Omega)}\left(|B_R(0)|+\int_{B_R(0)}|\ln\delta(x)|^p\,dx\right)\leq C_p\|f\|^p_{L^m(\Omega)}$$

By the same argument as before, since $\ln\delta\in L^p(\Omega)$ for all $1\leq p<\infty$, we obtain:

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(B_R(0)\setminus\Omega)}\leq C\|f\|_{L^m(\Omega)}. \tag{I_3}$$

Thus by(I_1), (I_2) and (I_3) we deduce that

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\mathbb{R}^N)}\leq C\|f\|_{L^m(\Omega)} \qquad \text{for all } 1\leq p<\infty$$

.

Third case : $s<t<min\{1,2s\}$
For $x\in\Omega$, let start by estimate of h_1, by choosing $0<\lambda<2s-t-\frac{N}{m}$, there exists $C>0$ such that:

$$h_1(x)\leq C\int_\Omega\frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}}dy.$$

By Lemma1.4.1 and setting $J_\alpha(f)(x)=\displaystyle\int_\Omega\frac{|f(y)|}{|x-y|^{N-\alpha}}dy$, with $\alpha=2s-t-\lambda$ and by the choice of λ we have $m>\frac{N}{2s-t-\lambda}$. So:

$$h_1(x)\leq\|J_\alpha(f)\|_{L^\infty(\Omega)}\leq C\|f\|_{L^m(\Omega)}.$$

By using again Lemma 1.4.1, such that :

$$h_2(x)=J_\alpha(f)(x)=\int_\Omega\frac{|f(y)|}{|x-y|^{N-\alpha}}dy$$

with $\alpha=2s-t$ and since $s<t<\min\{1,2s\}$ we have $m>\frac{N}{2s-t}$ so :

$$h_2(x)\leq\|J_\alpha(f)\|_{L^\infty(\Omega)}\leq C\|f\|_{L^m(\Omega)}.$$

And for h_3 since $t-s<1$:

$$h_3(x)=(t-s)\int_\Omega\frac{|f(y)|}{|x-y|^{N-s}}dy\leq\int_\Omega\frac{|f(y)|}{|x-y|^{N-s}}dy$$

By Lemma 1.4.1 $J_\alpha(f)(x)=\displaystyle\int_\Omega\frac{|f(y)|}{|x-y|^{N-\alpha}}dy$ with $\alpha=s$ so $m>\frac{N}{2s-t}>\frac{N}{s}$ we get:

$$h_3(x)\leq\|J_\alpha(f)\|_{L^\infty(\Omega)}\leq C\|f\|_{L^m(\Omega)}.$$

So finally we get:

$$\left|(-\Delta)^{\frac{t}{2}}u(x)\right|\leq C\left(h_1(x)+|\ln\delta(x)|h_2(x)+\frac{1}{\delta(x)^{t-s}}h_3(x)\right)$$
$$\leq C\|f\|_{L^m(\Omega)}\left(1+|\ln\delta(x)|+\frac{1}{\delta(x)^{t-s}}\right)$$

We now establish the L^p regularity of $(-\Delta)^{\frac{t}{2}}u$:

$$\int_\Omega \left|(-\Delta)^{\frac{t}{2}}u(x)\right|^p dx \leq C^p \|f\|^p_{L^m(\Omega)} \int_\Omega \left(1 + |\ln \delta(x)| + \frac{1}{\delta(x)^{t-s}}\right)^p dx$$
$$\leq C_p \|f\|^p_{L^m(\Omega)} \left(|\Omega| + \int_\Omega |\ln \delta(x)|^p dx + \int_\Omega \delta(x)^{-p(t-s)} dx\right)$$

By the Lemma 1.2.2 we have $\ln \delta \in L^p(\Omega)$ for $1 \leq p < \infty$ and $\delta(x)^{-(t-s)} \in L^p(\Omega)$ for $1 \leq p < \frac{1}{t-s}$, thus:

$$\|(-\Delta)^{\frac{t}{2}}u(x)\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)} \tag{I_4}$$

Now for $x \in B_R(0)\backslash\Omega$ and $y \in \Omega$, using the fact that $|x-y| \geq \delta(y)$, and applying Holder's inequality together with Proposition 2.1.4 with $q = \infty$, we obtain:

$$\left|(-\Delta)^{\frac{t}{2}}u(x)\right| \leq \int_\Omega \frac{|u(y)|}{|x-y|^{N+t}} dy \leq \int_\Omega \frac{|u(y)|}{|x-y|^{N+t-s}} \frac{1}{|x-y|^s} dy$$
$$\leq \int_\Omega \frac{|u(y)|}{\delta(y)^s} \frac{dy}{|x-y|^{N+t-s}} \leq \|\frac{u}{\delta^s}\|_{L^\infty(\Omega)} \int_\Omega \frac{dy}{|x-y|^{N+t-s}}$$
$$\leq C\|f\|_{L^m(\Omega)} \int_{\delta(x)}^{2R} \frac{d\rho}{\rho^{t-s+1}} \leq C\|f\|_{L^m(\Omega)} \left((2R)^{-(t-s)} - \delta(x)^{-(t-s)}\right)$$
$$\leq C\|f\|_{L^m(\Omega)} \left(1 + \delta(x)^{-(t-s)}\right)$$

It follows that the L^p in $B_R(0) \setminus \Omega$ regularity estimate holds:

$$\int_{B_R(0)\setminus\Omega} \left|(-\Delta)^{\frac{t}{2}}u(x)\right|^p dx \leq C_p\|f\|_{L^m(\Omega)} \left(|B_R(0) \setminus \Omega| + \int_{B_R(0)} \left|\delta(x)^{-(t-s)}\right|^p dx\right)$$

By Proposition 2.1.4, $\delta(x)^{-(t-s)} \in L^p(B_R(0))$ for $1 \leq p < \frac{1}{t-s}$ we obtain :

$$\|(-\Delta)^{\frac{t}{2}}u(x)\|_{L^p(B_R(0)\setminus\Omega)} \leq C\|f\|_{L^m(\Omega)} \tag{I_5}$$

Finally, using (I_1),(I_4) and (I_5) thus, for $s < t < \min\{1, 2s\}$:

$$\|(-\Delta)^{\frac{t}{2}}u(x)\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for all } 1 \leq p < \frac{1}{t-s}.$$

$\square$

We next consider the Calderon-Zygmund regularity when $1 \leq m < \frac{N}{2s-t}$.

Theorem 2.2.2. *Let $s \in (0,1)$, $s \leq t < \min\{1, 2s\}$ and u unique weak solution of (P), with $f \in L^m(\Omega)$ for $1 \leq m < \frac{N}{2s-t}$. Then for all $1 \leq p < p^*(m,s,t)$, there exists $C > 0$ such that:*

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

With:

$$p^*(m,s,t) := \begin{cases} \dfrac{mN}{N-ms}, & \text{for } t = s, \\ \min\left\{\dfrac{mN}{N-ms+mN(t-s)}, \dfrac{1}{t-s}\right\}, & \text{for } t > s. \end{cases}$$

Proof. First let see for $x \in B^c_R(0)$ by using Holder's inequality and Proposition 1.5.1 we get:

$$\left|(-\Delta)^{\frac{t}{2}}u(x)\right| \leq \int_\Omega \left|(-\Delta)^{\frac{t}{2}}_x G_s(x,y)f(y)\right| dy \leq \|f\|_{L^m(\Omega)} \left(\int_\Omega \left|(-\Delta)^{\frac{t}{2}}_x G_s(x,y)\right|^{m'} dy\right)^{\frac{1}{m'}}$$
$$\leq C\|f\|_{L^m(\Omega)} \left(\int_\Omega \frac{dy}{(1+|x|)^{(N+t)m'}}\right)^{\frac{1}{m'}} \leq \frac{C}{(1+|x|)^{(N+t)}}\|f\|_{L^m(\Omega)}$$

We start by :

$$\int_{B^c_R(0)} \left|(-\Delta)^{\frac{t}{2}}u(x)\right| dx \leq \int_{B^c_R(0)} \frac{C\|f\|_{L^m(\Omega)}}{(1+|x|)^{(N+t)}} dx \leq C\|f\|_{L^m(\Omega)} \int_{B^c_R(0)} \frac{dx}{|x|^{(N+t)}}$$
$$\leq C\|f\|_{L^m(\Omega)}\sigma(\mathbb{S}^{N-1}) \int_R^\infty \frac{d\rho}{\rho^{1+t}} \leq C\|f\|_{L^m(\Omega)}.$$

Where $\sigma(\mathbb{S}^{N-1})$ is the Lebesgue measure of the unit sphere. Then $(-\Delta)^{\frac{t}{2}}u \in L^1(B^c_R(0))$. And since $|x| > R$, we have :

$$\left|(-\Delta)^{\frac{t}{2}}u(x)\right| \leq \frac{C}{(1+|x|)^{(N+t)}}\|f\|_{L^m(\Omega)} \leq \frac{C}{(1+R)^{(N+t)}}\|f\|_{L^m(\Omega)}$$

So we get:

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^\infty(B^c_R(0))} \leq \frac{C}{(1+R)^{(N+t)}}\|f\|_{L^m(\Omega)}$$

Then $(-\Delta)^{\frac{t}{2}}u \in L^\infty(B^c_R(0))$ and we have $\|(-\Delta)^{\frac{t}{2}}u\|_{L^\infty(B^c_R(0))} \leq C\|f\|_{L^m(\Omega)}$.
Since $(-\Delta)^{\frac{t}{2}}u \in L^1(B^c_R(0)) \cap L^\infty(B^c_R(0))$ and by Interpolation inequality, we have

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(B^c_R(0))} \leq C\|f\|_{L^m(\Omega)} \qquad \forall 1 \leq p \leq \infty \tag{I_1}$$

Now, for $x \in B_R(0)$ and $1 \leq m < \frac{N}{s}$**:**
Since $x \in \Omega$ we have :

$$\left|(-\Delta)^{\frac{t}{2}}u(x)\right| \leq C\left[h_1(x) + |\ln\delta(x)|\, h_2(x) + \frac{1}{\delta(x)^{t-s}}h_3(x)\right]$$

Notice that $0 < \lambda < 2s - t$, we set :

$$h_1(x) \leq J_\alpha(f)(x) = \int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}}dy.$$

With $\alpha = 2s - t - \lambda$ and by using Lemma 1.4.1 (b) and (c) together, we obtain for $1 \leq m < \frac{N}{2s-t-\lambda}$

$$\|h_1\|_{L^p(\Omega)} \leq \|J_\alpha(f)\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)}$$

for all $1 \leq p \leq \ell$ with $\frac{1}{\ell} = \frac{1}{m} - \frac{\alpha}{N}$, where

$$\frac{1}{\ell} = \frac{N - m\alpha}{mN} = \frac{N - m(2s-t-\lambda)}{mN} > \frac{N - m(2s-t)}{mN}$$

Which applies that $\ell < \frac{mN}{N-m(2s-t)}$ thus:

$$\|h_1\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for all } 1 \leq p < \frac{mN}{N-m(2s-t)}. \tag{2.4}$$

Now, for h_2 we use the Holder's inequality with $1<\theta<\infty$ such that $\frac{1}{\theta}+\frac{1}{\theta'}=1$, so:

$$\left(\int_\Omega ||\ln\delta(x)|h_2(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_\Omega |\ln\delta(x)|^{p\theta'}\,dx\right)^{\frac{1}{p\theta'}}\left(\int_\Omega |h_2(x)|^{p\theta}\,dx\right)^{\frac{1}{p\theta}}$$

By Lemma 1.2.2, we have $\ln\delta\in L^{p\theta'}(\Omega)$ for $1\le p\theta'<\infty$.
For

$$h_2(x)=J_\alpha(f)(x)=\int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}}dy \qquad \text{with} \qquad \alpha=2s-t.$$

Using (b) and (c) in Lemma 1.4.1, we get if $1\le m<\frac{N}{2s-t}$

$$\|h_2\|_{L^{p\theta}(\Omega)}=\|J_\alpha(f)\|_{L^{p\theta}(\Omega)}\le C\|f\|_{L^m(\Omega)}.$$

For all $1\le p\theta\le\ell$ with $:\dfrac{1}{\ell}=\dfrac{1}{m}-\dfrac{\alpha}{N}=\dfrac{N-m\alpha}{mN}=\dfrac{N-m(2s-t)}{mN}\Rightarrow \ell=\dfrac{mN}{N-m(2s-t)}$.

We deduce that:

$$\|\,|\ln\delta|\,h_2\|_{L^p(\Omega)}\le C\|f\|_{L^m(\Omega)} \qquad \text{for all} \qquad 1\le p<\frac{mN}{N-m(2s-t)}. \tag{2.5}$$

Now if $t>s$ then $h_3\neq 0$.
By Holder's inequality for $1<\theta<\infty$ such that $\frac{1}{\theta}+\frac{1}{\theta'}=1$, so:

$$\left\|\frac{h_3}{|\delta|^{t-s}}\right\|_{L^p(\Omega)}\le\left(\int_\Omega \left|\delta(x)^{-(t-s)}\right|^{p\theta'}dx\right)^{\frac{1}{p\theta'}}\left(\int_\Omega |h_3(x)|^{p\theta}\,dx\right)^{\frac{1}{p\theta}}$$

Using Lemma 1.2.2, we have $\delta(x)^{-(t-s)}\in L^{p\theta'}(\Omega)$ for $1\le p\theta'<\frac{1}{t-s}$. In other hand we have :

$$h_3(x)=(t-s)\int_\Omega\frac{|f(y)|}{|x-y|^{N-s}}dy\le\int_\Omega\frac{|f(y)|}{|x-y|^{N-s}}dy.$$

Let set

$$J_\alpha(f)(x)=\int_\Omega\frac{|f(y)|}{|x-y|^{N-\alpha}}dy \qquad \text{for } \alpha=s.$$

Applying Lemma 1.4.1 (b) and (c) again to get:

$$\|h_3\|_{L^{p\theta}(\Omega)}=\|J_\alpha(f)\|_{L^{p\theta}(\Omega)}\le C\|f\|_{L^m(\Omega)} \qquad \text{For } 1\le p\theta\le\ell. \tag{2.6}$$

With

$$\frac{1}{\ell}=\frac{1}{m}-\frac{\alpha}{N}=\frac{N-m\alpha}{mN}=\frac{(N-ms)}{mN}$$

We now determine values for p:

$$\begin{cases}1\le p\theta'<\dfrac{1}{t-s}\\[2mm] 1\le p\theta<\dfrac{mN}{N-ms}\end{cases}\Rightarrow\begin{cases}0<t-s<\dfrac{1}{p\theta'}\le 1\\[2mm] 0<\dfrac{N-ms}{mN}\le\dfrac{1}{p\theta}\le 1\end{cases}\Rightarrow t-s+\frac{N-ms}{mN}<\frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right)\le 2$$

$$\Rightarrow\frac{N-ms+(t-s)Nm}{mN}<\frac{1}{p}\le 2\Rightarrow\frac{1}{2}\le p\le\frac{mN}{N-ms+(t-s)Nm}$$

$$\Rightarrow 1\le p\le\frac{mN}{N-ms+(t-s)Nm}.$$

So finally :

$$\left\|\frac{h_3}{|\delta|^{t-s}}\right\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p \leq \frac{mN}{N-ms+(t-s)Nm}.$$

By combining (2.4), (2.5) and (2.6) we get :

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p < p^* \tag{I_2}$$

Such that:

$$p^* := \begin{cases} \dfrac{mN}{N-ms}, & \text{for } t = s, \\ \dfrac{mN}{N-ms+mN(t-s)}, & \text{for } t > s. \end{cases}$$

For $x \in B_R(0) \setminus \Omega$, we have that $|x-y| \geq \min\{\delta(x), \delta(y)\}$, $\forall y \in \Omega$ and $\forall x \in B_R(0) \setminus \Omega$, and we have $\forall \varepsilon > 0$:

$$\begin{aligned} \left|(-\Delta)^{\frac{t}{2}}u(x)\right| &\leq \int_\Omega \frac{|u(y)|}{|x-y|^{N+t}}dy = \int_\Omega \frac{|u(y)|}{|x-y|^{N-\varepsilon}|x-y|^{t-s+\varepsilon}|x-y|^s}dy \\ &\leq \int_\Omega \frac{|u(y)|}{|x-y|^{N-\varepsilon}}\frac{1}{\delta(x)^{t-s+\varepsilon}}\frac{1}{\delta(y)^s}dy = \frac{1}{\delta(x)^{t-s+\varepsilon}}\int_\Omega \frac{|u(y)|}{\delta(y)^s}\frac{1}{|x-y|^{N-\varepsilon}}dy \end{aligned}$$

Let setting

$$g(x) = \int_\Omega \frac{|u(y)|}{\delta(y)^s}\frac{1}{|x-y|^{N-\varepsilon}}dy.$$

By using Holder's inequality we get:

$$\begin{aligned} \|(-\Delta)^{\frac{t}{2}}u\|_{L^p(B_R(0)\setminus\Omega)} &\leq \left(\int_{B_R(0)\setminus\Omega}\left|\frac{1}{\delta(x)^{t-s+\varepsilon}}\right|^p |g(x)|^p dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_{B_R(0)\setminus\Omega}\left|\frac{1}{\delta(x)^{t-s+\varepsilon}}\right|^{p\theta'} dx\right)^{\frac{1}{p\theta'}}\left(\int_{B_R(0)\setminus\Omega}|g(x)|^{p\theta}dx\right)^{\frac{1}{p\theta}} \end{aligned}$$

By Lemma 1.2.2, we have $\delta(x)^{-(t-s+\varepsilon)} \in L^{p\theta'}(B_R(0) \setminus \Omega)$, if and only if $1 \leq p\theta' < \frac{1}{t-s+\varepsilon}$.
Now, for $g(x) = \displaystyle\int_\Omega \frac{|u(y)|}{\delta(y)^s}\frac{1}{|x-y|^{N-\varepsilon}}dy$, let set :

$$J_\alpha(h) = \int_\Omega \frac{h(y)}{|x-y|^{N-\varepsilon}}, \quad \text{with, } h(x) = \frac{u(y)}{\delta(y)^s},\ \alpha = \varepsilon.$$

We have from Proposition 2.1.4, since $1 \leq m < \frac{N}{s}$ so $1 \leq q < \frac{mN}{N-ms}$ and :

$$\left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)}$$

Since $1 \leq q < \frac{mN}{N-ms} < \frac{N}{\varepsilon}$, we choose ε such that $\varepsilon < \frac{N}{m} - s$, by Lemma 1.4.1, we have

$$\|g\|_{L^{p\theta}(B_R(0)\setminus\Omega)} \leq \|J_\alpha(h)\|_{L^{p\theta}(B_R(0))} \leq C\left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p\theta \leq \ell.$$

With

$$\frac{1}{\ell} = \frac{1}{q} - \frac{\varepsilon}{N} = \frac{qN}{N-q\varepsilon}, \text{ such that } : \ell = \frac{qN}{N-q\varepsilon} < \frac{Nm}{N-ms-m\varepsilon}$$

Thus $1 \leq p\theta < \frac{Nm}{N-ms-m\varepsilon}$.
We now determine the values of p:

$$\begin{cases} 1 \leq p\theta' < \dfrac{1}{t-s+\varepsilon} \\ 1 \leq p\theta < \dfrac{mN}{N-ms-m\varepsilon} \end{cases} \Rightarrow \begin{cases} 0 < t-s+\varepsilon < \dfrac{1}{p\theta'} \leq 1 \\ 0 < \dfrac{N-ms-m\varepsilon}{mN} \leq \dfrac{1}{p\theta} \leq 1 \end{cases}$$

$$\Rightarrow t-s+\varepsilon+\frac{N-ms-m\varepsilon}{mN} < \frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right) \leq 2$$
$$\Rightarrow \frac{1}{2} \leq p < \frac{Nm}{N-ms+(t-s)Nm+\varepsilon m(N-1)}$$
$$\Rightarrow 1 \leq p < \frac{Nm}{N-ms+(t-s)Nm+\varepsilon m(N-1)}$$

For $1 \leq m < \dfrac{N}{s}$ and $1 \leq p < p^*$, we fix:

$$\varepsilon \in (0,\varepsilon^*) \quad \text{with} \quad \varepsilon^* = \min\left\{\frac{N}{m}-s, \frac{1}{m(N-1)\left(\frac{mN}{p}-(N-ms+mN(t-s))\right)}\right\}$$

Therefore, we conclude that:

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(B_R(0)\setminus\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p < p^*. \tag{I_3}$$

Thus if $1 \leq m < \frac{N}{s}$, we have :

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)} \quad \text{for} \quad 1 \leq p < p^*.$$

Case : $\frac{N}{s} \leq m < \frac{N}{2s-t}$**:**
In this case, since $s < t < \min\{1, 2s\}$.
For $x \in \Omega$, arguing as in the first case, we obtain:

$$\|h_1\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)}, \quad \forall\, 1 \leq p < \frac{mN}{N-m(2s-t)}$$

and

$$\||\ln\delta|h_2\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)}, \quad \forall\, 1 \leq p < \frac{mN}{N-m(2s-t)}$$

Now for $\delta^{-(t-s)}h_3$ with $h_3(x) \leq \displaystyle\int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy$, by Holder's inequality we have :

$$\left\|\frac{h_3}{|\delta|^{t-s}}\right\|_{L^p(\Omega)} \leq \left(\int_\Omega \left|\delta(x)^{-(t-s)}\right|^{p\theta'} dx\right)^{\frac{1}{p\theta'}} \left(\int_\Omega |h_3(x)|^{p\theta} dx\right)^{\frac{1}{p\theta}}$$

Using Lemma 1.2.2 we get $\delta(x)^{-(t-s)} \in L^{p\theta'}(\Omega)$ for $1 \leq p\theta' < \frac{1}{t-s}$.
For h_3, we set

$$J_\alpha(f)(x) = \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}}dy, \qquad \text{with} \quad \alpha = s.$$

Applying Lemma 1.4.1, with $m = \frac{N}{s}$ then:

$$\|h_3\|_{L^{p\theta}(\Omega)} \leq \|J_\alpha(f)\|_{L^{p\theta}(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for all } 1 \leq p\theta < \infty$$

We now determine the values of p for which the estimate holds:

$$\begin{cases} 1 \leq p\theta' < \dfrac{1}{t-s} \\ 1 \leq p\theta < \infty \end{cases} \Rightarrow \begin{cases} 0 < t-s < \dfrac{1}{p\theta'} \leq 1 \\ 0 < \dfrac{1}{p\theta} \leq 1 \end{cases} \Rightarrow t-s < \frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right) \leq 2$$

$$\Rightarrow \frac{1}{2} \leq p < \frac{1}{t-s} \Rightarrow 1 \leq p < \frac{1}{t-s}$$

Recalling that $\frac{N}{s} \leq m$, we deduce that $\frac{mN}{N-m(2s-t)} > \frac{1}{t-s} = p^*$, and hence:

$$\left\|\frac{h_3}{\delta^{t-s}}\right\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p < \frac{1}{t-s}.$$

Finally we have for $1 \leq p < p^*$:

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\Omega)} \leq C\|f\|_{L^m(\Omega)}. \tag{I_4}$$

Now for $x \in B_R(0) \setminus \Omega$ and $y \in \Omega$, since $|x-y| \geq \min\{\delta(x), \delta(y)\}$, and arguing as above, for $\varepsilon > 0$, we have:

$$|(-\Delta)^{\frac{t}{2}}u(x)| \leq \frac{1}{\delta(x)^{t-s+\varepsilon}} \int_\Omega \frac{|u(y)|}{\delta(y)^s} \frac{1}{|x-y|^{N-\varepsilon}} dy.$$

By setting $g(x) = \displaystyle\int_\Omega \frac{|u(y)|}{\delta(y)^s} \frac{1}{|x-y|^{N-\varepsilon}} dy$, and using Holder's inequality:

$$\|(-\Delta)^{\frac{t}{2}}u\|_{L^p(B_R(0)\setminus\Omega)} \leq \left(\int_{B_R(0)\setminus\Omega} \left|\frac{1}{\delta(x)^{t-s+\varepsilon}}\right|^{p\theta'} dx\right)^{\frac{1}{p\theta'}} \left(\int_{B_R(0)\setminus\Omega} |g(x)|^{p\theta} dx\right)^{\frac{1}{p\theta}}$$

Using Lemma 1.2.2, we have $\delta(x)^{-(t-s+\varepsilon)} \in L^{p\theta'}(B_R(0) \setminus \Omega)$, for all $1 \leq p\theta' < \frac{1}{t-s+\varepsilon}$. For

$$g(x) = J_\alpha(h)(x) = \int_\Omega \frac{|h(y)|}{|x-y|^{N-\alpha}} dy, \quad \text{with} \quad h(y) = \frac{|u(y)|}{\delta(y)^s} \text{ and } \alpha = \varepsilon.$$

First, by using Proposition 2.1.4, since $m = \frac{N}{s}$, we have:

$$\|h\|_{L^q(\Omega)} = \left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for all } 1 \leq q < \infty.$$

Applying Lemma 1.4.1, which require $1 \leq q < \dfrac{N}{\varepsilon}$, we choose $\varepsilon \in \left(0, \dfrac{N}{q}\right)$, and we get:

$$\|J_\alpha(h)\|_{L^{p\theta}(B_R(0)\setminus\Omega)} \leq C\left\|\frac{u}{\delta^s}\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)}$$

For all $1 \leq p\theta \leq \ell$ with $\ell = \dfrac{qN}{N-q\varepsilon}$. As above, we now determine the values of p:

$$\begin{cases} 1 \leq p\theta' < \dfrac{1}{t-s+\varepsilon} \\ 1 \leq p\theta < \dfrac{qN}{N-q\varepsilon} \end{cases} \Rightarrow \begin{cases} 0 < t-s+\varepsilon < \dfrac{1}{p\theta'} \leq 1 \\ 0 < \dfrac{N-q\varepsilon}{qN} \leq \dfrac{1}{p\theta} \leq 1 \end{cases} \Rightarrow t-s+\varepsilon+\frac{N-q\varepsilon}{qN} < \frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right) \leq 2$$

$$\Rightarrow \frac{1}{2} \leq p < \frac{qN}{N + qN(t-s) + \varepsilon q(N-1)} \Rightarrow 1 \leq p < \frac{qN}{N + qN(t-s) + \varepsilon q(N-1)}$$

Now considering $\frac{N}{2} \leq m < \infty$ and $1 \leq p < \frac{1}{t-s}$, we choose:
$q \in \left(\frac{p}{1-p(t-s)}, \infty\right)$ and $\varepsilon \in (0, \varepsilon^*)$ with $\varepsilon^* = \left\{\frac{N}{q}, \frac{1}{q(N-1)}\left(\frac{Nq}{p} - (N + Nq(t-s))\right)\right\}$.
Thus we get that :

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(B_R(0)\setminus\Omega)} \leq C\|f\|_{L^m(\Omega)} \qquad \text{for } 1 \leq p <= \frac{1}{t-s}. \qquad (I_5)$$

So gathering (I_1), (I_4) and (I_5), we conclude that for $\dfrac{N}{s} \leq m < \dfrac{N}{2s-t}$ and $1 \leq p < p^* = \dfrac{1}{t-s}$:

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$$

Finally if $1 \leq m < \frac{N}{2s-t}$, we have :

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}, \quad \text{for} \quad 1 \leq p < p^*.$$

With:

$$p^*(m,s,t) := \begin{cases} \dfrac{mN}{N-ms}, & \text{for } t = s, \\ \min\left\{\dfrac{mN}{N - ms + mN(t-s)}, \dfrac{1}{t-s}\right\}, & \text{for } t > s. \end{cases}$$

$\square$

Now and since we give the proofs of Theorems 2.2.1 and 2.2.2 we get as consequence the precise rate of explosion of the half t-Laplacian of the solution u of (P) at the boundary of Ω

Theorem 2.2.3. *Let $s \in (0,1)$, $s \leq t < \min\{1, 2s\}$ and let u be the unique weak solution to (P) with $f \in L^m(\Omega)$ for $m \geq 1$.*

i) If $m > \frac{N}{2s-t}$, there exists $C > 0$ such that:

$$\left\||\ln\delta|^{-1}(-\Delta)^{\frac{s}{2}} u\right\|_{L^\infty(\Omega)} + (t-s)\left\|\delta^{t-s}(-\Delta)^{\frac{t}{2}} u\right\|_{L^\infty(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

ii) If $1 \leq m < \frac{N}{2s-t}$, for all $1 \leq p < \frac{mN}{N-ms}$ and $1 \leq q < \frac{mN}{N-m(2s-t)}$, there exists $C > 0$ such that:

$$\left\||\ln\delta|^{-1}(-\Delta)^{\frac{s}{2}} u\right\|_{L^p(\Omega)} + (t-s)\left\|\delta^{t-s}(-\Delta)^{\frac{t}{2}} u\right\|_{L^q(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

Remark. A direct consequence of the above results is the local Calderon-Zygmund regularity. For any $\omega \subset \Omega$, we have :

- If $m \leq \frac{N}{2s-t}$, there exists $C_\omega > 0$ such that:

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^\infty(\omega)} + (t-s)\|(-\Delta)^{\frac{t}{2}} u\|_{L^\infty(\omega)} \leq C_\omega\|f\|_{L^m(\Omega)}$$

- If $1 \leq m < \frac{N}{2s-t}$, for all $1 \leq p < \frac{mN}{N-ms}$ and $1 \leq q < \frac{mN}{N-m(2s-t)}$, there exists $C_\omega > 0$ such that:

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\omega)} + (t-s)\|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\omega)} \leq C_\omega\|f\|_{L^m(\Omega)}$$

Proof. See [38]. $\square$

Remark. In the same way as for the half t-Laplacian $(-\Delta)^{\frac{t}{2}}$, the Calderon-Zygmund regularity also holds for another fractional differential operator, the Riesz fractional gradient ∇^t.

2.3 Regularity of the solution in the fractional spaces

Since we have the regularity of $(-\Delta)^{\frac{t}{2}}$ and ∇^t we can go further and see the regularity of the solution in fractional spaces, we start in the Bessel potential space if $m > \max\left\{\frac{N}{s}, \frac{N}{2s-t}\right\}$ with this Corollary

Corollary 2.3.1. *Let $s,t \in (0,1)$ and u the unique weak solution given by $u := \mathbb{G}_s[f]$ with $f \in L^m(\Omega)$ for $m > \max\left\{\frac{N}{s}, \frac{N}{2s-t}\right\}$:*

i) If $0 < t \leq s$, then $u \in L^{t,p}(\mathbb{R}^N)$ for all $1 < p < \infty$ and $\exists C > 0$ such that:

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

ii) If $s < t < \min\{1, 2s\}$, then $u \in L^{t,p}(\mathbb{R}^N)$ for all $1 < p < \frac{1}{t-s}$ and $\exists C > 0$:

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

Let give a proof for this corollary

Proof. Since u is solution of (P) then $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, so we just need to see if $u \in L_0^{t,p}(\Omega)$ and since we have an equivalent norm for this space, given by $\|u\|_{L_0^{t,p}(\Omega)} := \|(-\Delta)^{\frac{t}{2}}u\|_{L^p(\mathbb{R}^N)}$ by proposition 1.4.4, since we have already the results of regularity for $(-\Delta)^{\frac{t}{2}}u$ in $L^p(\mathbb{R}^N)$ in Theorem 2.2.1, so we have that :

1. If $0 < t \leq s$, for all $1 < p < \infty$, $u \in L_0^{t,p}(\Omega)$ and $\exists C > 0$ such that :

$$\|u\|_{L_0^{t,p}(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

2. If $s < t < \min\{1, 2s\}$, for all $1 < p < \frac{1}{t-s}$, $u \in L_0^{t,p}(\Omega)$ and $\exists C > 0$ such that :

$$\|u\|_{L_0^{t,p}(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

And since $L_0^{t,p}(\Omega)$ is a subset of $L^{t,p}(\mathbb{R}^N)$ we have same results for the last space. □

Now let see the regularity of solution u in the fractional Sobolev space $W^{\gamma,p}$ for $m > \max\left\{\frac{N}{s}, \frac{N}{2s-t}\right\}$,

Corollary 2.3.2. *Let $s, \gamma \in (0,1)$ and u the unique weak solution of (P) given by $u := \mathbb{G}_s[f]$ with $f \in L^m(\Omega)$ for $m > \max\left\{\frac{N}{s}, \frac{N}{2s-\gamma}\right\}$:*

i) If $0 < \gamma \leq s$, then $u \in W^{\gamma,p}(\mathbb{R}^N)$ for all $1 < p < \infty$ and $\exists C > 0$ such that:

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

ii) If $s < \gamma < \min\{1, 2s\}$, then $u \in W^{\gamma,p}(\mathbb{R}^N)$ for all $1 < p < \frac{1}{\gamma-s}$ and $\exists C > 0$:

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

Proof. For $0 < \gamma < s$, we get the result immediately by using Corollary 2.3.1 with Proposition 1.4.6 the embadding

$$L^{\gamma+\varepsilon,p}(\mathbb{R}^N) \subset W^{\gamma,p}(\mathbb{R}^N)$$

For all $0 < \varepsilon < \gamma < 1$ and $1 < p < \infty$.

Now for $s \leq \gamma < \min\{1, 2s\}$ and $\overline{p} := \begin{cases} \infty, & \text{for } \gamma = s, \\ \dfrac{1}{\gamma - s}, & \text{for } \gamma > s. \end{cases}$

And since $m > \frac{N}{2s-\gamma}$ and $1 < p < \overline{p}$, we can choose:
$\varepsilon \in (0, \varepsilon^*)$ with $\varepsilon^* = \min\left\{\min\{1, 2s\} - \gamma, 2s - \gamma - \frac{N}{m}, \frac{1}{p} - (\gamma - s)\right\}$.
By the corollary 2.3.1 *ii)* we set $t = \gamma + \varepsilon$, we get:

$$\|u\|_{L^{\gamma+\varepsilon,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$$

$$t = \gamma + \varepsilon \Rightarrow \gamma = t - \varepsilon \Rightarrow s - \varepsilon < \gamma < \min\{1, 2s\} - \varepsilon < \min\{1, 2s\}.$$

And by the choice made for ε we get $s \leq \gamma < \min\{1, 2s\}$.
Finally by the embedding $L^{\gamma+\varepsilon,p}(\mathbb{R}^N) \subset W^{\gamma,p}(\mathbb{R}^N)$ and corollary 2.3.1 we have $u \in W^{\gamma,p}(\mathbb{R}^N)$,
for $1 < p < \overline{p}$ and:

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

□

Now let see if $1 \leq m < \frac{N}{2s-t}$, then what is the regularity of u in fractional spaces, we start with Bessel potential space.

Corollary 2.3.3. *Let $s \in (0, 1)$, $s \leq t < \min\{1, 2s\}$ and u is unique weak solution of* (P) *given by $u := \mathbb{G}_s[f]$ with $f \in L^m(\Omega)$ for $1 \leq m < \frac{N}{2s-t}$. Then, for all $1 < p < p^*(m, s, t)$, there exists $C > 0$ such that:*

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}. \tag{2.7}$$

Proof. According to Theorem 2.2.2 $\quad \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)} \quad$ for $1 \leq p < p^*$,
and since $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.
So by the definition of the space $L_0^{t,p}(\Omega)$, we have:

$$\|u\|_{L_0^{t,p}(\Omega)} = \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$$

And since $L_0^{t,p}(\Omega) \subset L^{t,p}(\mathbb{R}^N)$, then $u \in L^{t,p}(\mathbb{R}^N)$, and

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$$

□

and let see the regularity in the fractional Sobolev space if $1 \leq m < \frac{N}{2s-t}$.

Corollary 2.3.4. *Let $s \in (0, 1)$, $s \leq \gamma < \min\{1, 2s\}$ and u is unique weak solution of* (P) *given by $u := \mathbb{G}_s[f]$ with $f \in L^m(\Omega)$ for $1 \leq m < \frac{N}{2s-\gamma}$. Then, for all $1 < p < p^*(m, s, \gamma)$, there exists $C > 0$ such that:*

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}. \tag{2.8}$$

Proof. Let $1 \leq m < \frac{N}{2s-t}$ and $1 < p < p^*$, let choose $\varepsilon \in (0, \varepsilon^*)$ with .

$$\varepsilon^* = \min\left\{s, \min\{1, 2s\} - \gamma, \frac{1}{p} + \frac{1}{p^*}\right\}$$

We have from proposition 1.4.6 that $L^{\gamma+\varepsilon,p}(\mathbb{R}^N) \subset W^{\gamma,p}(\mathbb{R}^N)$ for $0 < \varepsilon < \gamma < 1$ and $1 < p < \infty$.
We have from corollary 2.3.3 that $\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$.
Let set $t = \gamma + \varepsilon$, so we have :

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq \|u\|_{L^{\gamma+\varepsilon,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}$$

So we get for $1 < p < p^*$:

$$\|u\|_{W^{\gamma,p}(\mathbb{R}^N)} \leq C\|f\|_{L^m(\Omega)}.$$

□

We can resume all corollaries 2.3.1-2.3.4 in this proposition, let define, for $0 < s \leq t < \min\{1, s(1 + N^{-1})\}$, let us set:

$$\widetilde{p}(m,s,t) := \begin{cases} \dfrac{1}{(t-s)^+}, & \text{if } m > \frac{N}{2s-t}, \\ \min\left\{\dfrac{mN}{N - ms + mN(t-s)}, \dfrac{1}{(t-s)^+}\right\}, & \text{if } 1 \leq m < \frac{N}{2s-t}. \end{cases}$$

Proposition 2.3.1. *Let $0 < s \leq t < \min\{1, s(1 + N^{-1})\}$ and let u be the unique weak solution to (P) with $f \in L^m(\Omega)$ for $m \geq 1$. Then, for all $1 < p < \widetilde{p}$, there exists $\widetilde{C} > 0$ such that:*

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \leq \widetilde{C}\|f\|_{L^m(\Omega)} \quad \textit{and} \quad \|u\|_{W^{t,p}(\mathbb{R}^N)} \leq \widetilde{C}\|f\|_{L^m(\Omega)}. \tag{2.9}$$

2.4 Compactness results

In this subsection we give some compactness results for the fractional poisson problem P that will help us in the next section.

Proposition 2.4.1. *Let $0 < t < \min\{1, s(1 + N^{-1})\}$ and $1 < p < N/(N(1 + t - s) - s)$. The solution map:*

$$\begin{aligned} \mathbb{G}_s : L^1(\Omega) &\to L_0^{t,p}(\Omega) \\ f \quad &\mapsto \mathbb{G}_s[f] := \int_\Omega G_s(x,y)f(y)\,dy, \end{aligned}$$

is well-defined, continuous and compact.

Proof. Since $\mathbb{G}_s$ is well defined and continuous (it linear also) by the Corollary 2.3.3, for all $1 \leq p < p^*(1, t, s = \frac{N}{N-s+N(t-s)})$.
Now, we need show that $\mathbb{G}_s$ is compact:
Let $(f_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ a Cauchy's sequence such that $\|f\|_{L^1(\Omega)}$ for all $n \in N$, and let define $u_n = \mathbb{G}_s(f_n)$ for all $n \in N$, then from corollaries 2.3.3 and 2.3.4 we deduce that u_n is bounded since we have:

$$\|u\|_{L^{s,\gamma}(\Omega)\cap W^{s,\gamma}(\Omega)} \leq C\|f\|_{L^1(\Omega)} \leq C \qquad \text{for } 1 \leq \gamma \frac{N}{N - s + N(t-s)}.$$

By Theorem 1.4.1 Proposition 2.1.4, there exists a sub-sequence $(u_n)_{n\in\mathbb{N}}$ and $u \in W^{s,\gamma}(\Omega)$

$$u_n \xrightarrow[n\to\infty]{} u \quad \text{strongly in } L^q(\mathbb{R}^N), \quad \text{for all } 1 \leq q < \frac{N}{N-2s}, \tag{$*$}$$

and also, up to a subsequence, L^p convergence implies almost everywhere convergence (see [13]), so that:

$$u_n \xrightarrow[n\to\infty]{} u \quad \text{almost every } x \in \mathbb{R}^N$$

By using Vitali's convergence Theorem 1.2.7 and Proposition 2.1.4 we obtain :

$$\frac{u_n}{\delta^s} \xrightarrow[n\to\infty]{} \frac{u}{\delta^s} \quad \text{strongly in } L^q(\mathbb{R}^N), \quad \text{for all } 1 \leq q < \frac{N}{N-s}, \tag{$**$}$$

since we have now that $(u_n)_{n\in\mathbb{N}}$ converges strongly in $L^q(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-s}$. Now we show that$((-\Delta)^{\frac{t}{2}} u_n)_{n\in\mathbb{N}}$ strongly converges in $L^p(\mathbb{R}^N)$, we have for $i, j \in \mathbb{N}$:

$$\begin{cases} (-\Delta)^s (u_i - u_j) = f_i - f_j & \text{in } \Omega \\ \qquad u_i - u_j = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

We have 2 cases:
case 1:$s \leq t < \min\{1, s(1 + \frac{1}{N})\}$ First let start if $x \in \mathbb{R}^N \setminus B_R(0)$, by choice of (E) we have that $|x - y| \geq \frac{1}{4}(1 + |x|)$, $\forall x \in \mathbb{R}^N \setminus B_R(0)$ and $y \in \Omega$, then for all $i, j \in \mathbb{N}$ we have:

$$\left|(-\Delta)^{\frac{t}{2}}(u_i - u_j)\right| \leq \int_\Omega \frac{|u_i(y) - u_j(y)|}{|x-y|^{N+t}} dy \leq \frac{C}{(1+|x|)^{N+t}} \|u_i - u_j\|_{L^1(\Omega)}.$$

By the same argument used in the proof of Theorem 2.2.1, it follows that $(-\Delta)^{\frac{t}{2}}(u_i - u_j) \in L^1(\mathbb{R}^N \setminus B_R(0)) \cap L^\infty(\mathbb{R}^N \setminus B_R(0))$, and :

$$\|(-\Delta)^{\frac{t}{2}}(u_i - u_j)\|_{L^1(\mathbb{R}^N\setminus B_R(0))} + \|(-\Delta)^{\frac{t}{2}}(u_i - u_j)\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} \leq C\|u_i - u_j\|_{L^1(\Omega)} \quad \forall i, j \in \mathbb{N}$$

Then using the interpolation's inequality we get for all $1 \leq p \leq \infty$, $(-\Delta)^{\frac{t}{2}}(u_i - u_j) \in L^p(\mathbb{R}^N \setminus B_R(0))$, and :

$$\|(-\Delta)^{\frac{t}{2}}(u_i - u_j)\|_{L^p(\mathbb{R}^N\setminus B_R(0))} \leq C\|u_i - u_j\|_{L^1(\Omega)} \quad \forall i, j \in \mathbb{N}$$

and since we have ($*$) then:

$$(-\Delta)^{\frac{t}{2}}(u_i - u_j) \xrightarrow[i,j\to\infty]{} 0 \quad \text{strongly in } L^p(\mathbb{R}^N \setminus B_R(0)), \forall 1 \leq p < \infty. \tag{2.3.1.1}$$

Now if $x \in \Omega$ and $y \in \Omega$: by the theorem 1.5.4 we have:

$$\left|(-\Delta)^{\frac{t}{2}}(u_i - u_j)\right| \leq C\left[J_{2s-t-\lambda}(f_i - f_j) + |\ln\delta| J_{2s-t}(f_i - f_j) + \frac{1}{\delta^{t-s}} J_s(f_i - f_j)\right]$$

such that $J_\alpha(f)(x) := \int_\Omega \frac{|f(y)|}{|x-y|^{N-\alpha}} dy$. For $J_{2s-t-\lambda}(f_i - f_j)$ we use the theorem 1.4.1 so we have :

$$J_{2s-t-\lambda}(f_i - f_j) : L^1(\Omega) \to L^p(\Omega)$$

is well defined, continuous and compact for all $1 \leq p < \frac{N}{N-(2s-t-\lambda)}$, let choose $0 < \lambda < 2s - t$ so we have:

$$J_{2s-t-\lambda}(f_i - f_j) \xrightarrow[i,j\to\infty]{} 0 \quad \text{strongly in } L^p(\Omega), \quad \forall 1 \leq p < \frac{N}{N-2s+t} \tag{$L1$}$$

For $|\ln\delta|J_{2s-t}(f_i-f_j)$, by using Holder's inequality and lemma 1.2.2 (ii) we get:

$$\begin{aligned}\||\ln\delta|J_{2s-t}(f_i-f_j)\|_{L^p(\Omega)} &\leq \|\ln\delta\|_{L^{p\theta'}(\Omega)}\|J_{2s-t}(f_i-f_j)\|_{L^{p\theta}(\Omega)}\\ &\leq C\|J_{2s-t}(f_i-f_j)\|_{L^{p\theta}(\Omega)}\end{aligned}$$

for $1\leq p\theta'<\infty$ and $1\leq p\theta<\frac{N}{N-2s+t}$, then using theorem 1.4.1 :

$$|\ln\delta|J_{2s-t}(f_i-f_j)\xrightarrow[i,j\to\infty]{}0 \quad \text{stongly in } L^p(\Omega),\quad \forall 1\leq p<\frac{N}{N-2s+t}. \tag{L2}$$

Finally $\frac{1}{\delta^{t-s}}J_s(f_i-f_j)$ by using Holder's inequality and lemma 1.2.2 (i)we get:

$$\begin{aligned}\left\|\frac{1}{\delta^{t-s}}J_s(f_i-f_j)\right\|_{L^p(\Omega)} &\leq \|\delta^{-(t-s)}\|_{L^{p\theta'}(\Omega)}\|J_s(f_i-f_j)\|_{L^{p\theta}(\Omega)}\\ &\leq C\|J_s(f_i-f_j)\|_{L^{p\theta}(\Omega)}\end{aligned}$$

for:

$$\begin{cases}1\leq p\theta'<\dfrac{1}{t-s}\\ 1\leq p\theta<\dfrac{N}{N-s}\end{cases}\Rightarrow\begin{cases}0<t-s<\dfrac{1}{p\theta'}\leq 1\\ 0<\dfrac{N-s}{N}\leq\dfrac{1}{p\theta}\leq 1\end{cases}\Rightarrow t-s+\frac{N-s}{N}<\frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right)\leq 2$$

$$\Rightarrow\frac{N(t-s)+N-s}{N}<\frac{1}{p}\leq 2\Rightarrow 1\leq p<\frac{N}{N(1+t-s)-s}$$

Then using Theorem 1.4.1, we get:

$$\delta^{-(t-s)}J_{2s-t}(f_i-f_j)\xrightarrow[i,j\to\infty]{}0 \quad \text{strongly in } L^p(\Omega),\quad \forall 1\leq p<\frac{N}{N(1+t-s)-s}. \tag{L3}$$

Combining $(L1)$,$(L2)$ and $(L3)$, we get:

$$(-\Delta)^{\frac{t}{2}}(u_i-u_j)\xrightarrow[i,j\to\infty]{}0 \quad \text{strongly in } L^p(\Omega),\quad \forall 1\leq p<\frac{N}{N(1+t-s)-s}. \tag{2.3.1.2}$$

Finally we see the convergence in $B_R(0)\setminus\Omega$,we have $|x-y|\geq\min\{\delta(x),\delta(y)\}$, for all $y\in\Omega$ and $x\in B_R(0)\setminus\Omega$.
For almost every $x\in B_R(0)\setminus\Omega$ and $i,j\in\mathbb{N}$, for $\varepsilon>0$:

$$\begin{aligned}\left|(-\Delta)^{\frac{t}{2}}(u_i-u_j)(x)\right| &\leq\int_\Omega\frac{|(u_i-u_j)(y)|}{|x-y|^{N+t}}dy\\ &\leq\frac{1}{\delta(x)^{t-s+\varepsilon}}\int_\Omega\frac{|(u_i-u_j)(y)|}{\delta(y)^s}\frac{dy}{|x-y|^{N-\varepsilon}}\end{aligned}$$

setting $g(x)=\int_\Omega\frac{|(u_i-u_j)(y)|}{\delta(y)^s}\frac{dy}{|x-y|^{N-\varepsilon}}$,and using Holder's inequality and lemma 1.2.2 (i):

$$\left\|(-\Delta)^{\frac{t}{2}}(u_i-u_j)\right\|_{L^p(B_R(0)\setminus\Omega)}\leq\left\|\delta^{-(t-s+\varepsilon)}\right\|_{L^{p\theta'}(B_R(0)\setminus\Omega)}\|g\|_{L^{p\theta}(B_R(0)\setminus\Omega)}\leq C\,\|g\|_{L^{p\theta}(B_R(0)\setminus\Omega)}$$

for $1\leq p\theta'<\frac{1}{t-s+\varepsilon}$.
Now for g using lemma 1.4.1 (b), (c) we have $\|g\|_{L^{p\theta}(B_R(0)\setminus\Omega)}\leq\left\|\frac{(u_i-u_j)}{\delta^s}\right\|_{L^q(B_R(0)\setminus\Omega)}$,
for $1\leq p\theta\leq\frac{qN}{N-q\varepsilon}$.

$$\begin{cases}1\leq p\theta'<\dfrac{1}{t-s+\varepsilon}\\ 1\leq p\theta<\dfrac{qN}{N-q\varepsilon}\end{cases}\Rightarrow\begin{cases}t-s+\varepsilon<\dfrac{1}{p\theta'}\leq 1\\ 0<\dfrac{N-q\varepsilon}{qN}\leq\dfrac{1}{p\theta}\leq 1\end{cases}$$

$$\Rightarrow \frac{qN(t-s+\varepsilon)+N-q\varepsilon}{qN} < \frac{1}{p}\left(\frac{1}{\theta}+\frac{1}{\theta'}\right) \leq 2$$
$$\Rightarrow \frac{qN(t-s)-q\varepsilon(N-1)+N}{qN} < \frac{1}{p} \leq 2 \Rightarrow 1 \leq p < \frac{qN}{qN(t-s)-q\varepsilon(N-1)+N}$$

So by the choice of $\varepsilon \in (0,\varepsilon^*)$ and $q \in \left(\frac{p}{1-p(t-s)}, \frac{N}{N-s}\right)$,
with $\varepsilon^* = \min\left\{\frac{N}{q}, \frac{1}{q(N-1)(\frac{Nq}{p}-(N+Nq(t-s)))}\right\}$ we have:

$$\left\|(-\Delta)^{\frac{t}{2}}(u_i-u_j)\right\|_{L^p(B_R(0)\setminus\Omega)} \leq C\left\|\frac{(u_i-u_j)}{\delta^s}\right\|_{L^q(B_R(0)\setminus\Omega)}$$

for $1 \leq p < \frac{N}{N(1+t-s)-s}$ and $\forall i,j \in \mathbb{N}$, and by using ($**$) we get:

$$(-\Delta)^{\frac{t}{2}}(u_i-u_j) \xrightarrow[i,j\to\infty]{} 0 \quad \text{stongly in } L^p(B_R(0)\setminus\Omega), \quad \forall 1 \leq p < \frac{N}{N(1+t-s)-s}. \tag{2.3.1.3}$$

Gathering (2.3.1.1), (2.3.1.2) and (2.3.1.3) we get:

$$(-\Delta)^{\frac{t}{2}}(u_i-u_j) \xrightarrow[i,j\to\infty]{} 0 \quad \text{strongly in } L^p(\mathbb{R}^N), \quad \forall 1 \leq p < \frac{N}{N(1+t-s)-s}.$$

Since $\left((-\Delta)^{\frac{t}{2}}u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^p(\mathbb{R}^N)$, so is strongly convergent in $L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{N(1+t-s)-s}$.
case 2:$0 < t < s$ Since we have convergence for $s \leq t < \min\{1, s(1+N^{-1})\}$, we can get the convergence for $0 < t < s$ by using Proposition 1.4.5. □

Corollary 2.4.1. *Let* $0 < t < \min\{1, s(1+N^{-1})\}$ *and* $1 < p < N/(N(1+t-s)-s)$. *The solution map:*

$$\begin{aligned} \mathbb{G}_s : L^1(\Omega) &\to W_0^{t,p}(\Omega) \\ f \quad &\mapsto \mathbb{G}_s[f] := \int_\Omega G_s(x,y)f(y)\,dy, \end{aligned}$$

is well-defined, continuous and compact.

Proof. The proof of this corollary is the same as the one of the proposition since $L_0^{t,p}(\Omega)$ and $W_0^{t,p}(\Omega)$ have equivalent norms. □

Chapter 3

Existence and non-existence of solutions for KPZ problems

We now present the Kardar-Parisi-Zhang type equations, known as KPZ problems, involving the nonlocal gradient term (the half t-Laplacian). Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class C^2. For $s, t \in (0, 1)$, we consider problems of the form:

$$\begin{cases} (-\Delta)^s u = |(-\Delta)^{\frac{t}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{KPZ}$$

where $q > 1$ and $\lambda > 0$ are real parameters, f belongs to a suitable Lebesgue space, and $(-\Delta)^{\frac{t}{2}} u$ denotes the nonlocal gradient term, defined as follows:

$$(-\Delta)^{\frac{t}{2}} u(x) := a_{N,\frac{t}{2}} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+t}} \, dy \tag{KPZ_1}$$

For any function $u \in C_c^\infty(\mathbb{R}^N)$, and the normalization constants :

$$C_{N,\alpha} := -2^{2\alpha} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N}{2} + \alpha)}{\Gamma(-\alpha)}$$

We now state the definition of a weak solution to problem (KPZ).

Definition 3.0.1. We say that a function u is a weak solution to KPZ if u and $|(-\Delta)^{\frac{t}{2}} u|^q$ belong to $L^1(\Omega)$, $u \equiv 0$ almost everywhere in $\mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega u (-\Delta)^s \varphi \, dx = \int_\Omega (|(-\Delta)^{\frac{t}{2}} u|^q + \lambda f(x)) \varphi \, dx,$$

for all φ belonging to

$$\mathbb{X}^s(\Omega) := \left\{ \varphi \in C^s(\mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s \varphi \in L^\infty(\Omega) \right\}.$$

In this chapter, we analyze the existence and non-existence of weak solutions to problem (KPZ) for suitable values of λ and q. We begin with the existence results.

3.1 The existence results for KPZ problems

For $q \in (1, +\infty)$ we now analyze the existence of weak solutions to the problem involving our non-local gradient term $(-\Delta)^{\frac{t}{2}}$, we now analyze the existence of weak solutions to the problem:

$$\begin{cases} (-\Delta)^s u = |(-\Delta)^{\frac{t}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{KPZ_1}$$

Let set $0 < t < \min\{1, s(1+N^{-1})\}$ and define:

$$\overline{q}(m,s,t) := \begin{cases} +\infty, & \text{if } t \leq s \text{ and } m \geq \frac{N}{s}, \\ \dfrac{s}{N(t-s)}, & \text{if } t > s \text{ and } m > \frac{N}{s}, \\ \dfrac{N}{N-ms}, & \text{if } t \leq s \text{ and } 1 \leq m < \frac{N}{s}, \\ \dfrac{N}{(N-sm+mN(t-s))}, & \text{if } t > s \text{ and } 1 \leq m \leq \frac{N}{s}. \end{cases} \tag{$\overline{q}$}$$

Then we have the main results for the existence of the solution of (KPZ):

Theorem 3.1.1. *Assume that $0 < t < \min\{1, s(1+N^{-1})\}$, $f \in L^m(\Omega)$ for some $m \geq 1$. Then, for all $1 < q < \overline{q}$, there exists $\lambda^* > 0$ such that, for all $0 < \lambda \leq \lambda^*$, ($KPZ_1$) has a weak solution u. Moreover:*

- *If $m \geq N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*
- *If $1 \leq m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < \frac{mN}{(N-ms)}$.*

Proof. The proof is divided into two cases:
Cas 1:$m > \frac{N}{s}$
First of all, we notice without loss of generality that $\frac{N}{s} < m < \dfrac{1}{(q(t-s))^+}$.
Then let set $r = r(m,s,t,q) > 0$, verifies $1 < qm < r < \frac{1}{(t-s)^+}$, we define:

$$\lambda^* := \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(q(\widetilde{C}k)^q \| \|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \right)^{\frac{-1}{q-1}}$$

By setting $C = (\widetilde{C}k)^q \| \|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}}$, the maximum of the function

$$\eta(t) = t^{\frac{1}{q}} - Ct + C(qC)^{\frac{q}{1-q}} - \frac{(qC)^{\frac{1}{1-q}}}{q}$$

is attained at $t = t_*$, where t_* satisfies:

$$\frac{1}{q} t_*^{\frac{1}{q}-1} - C = 0$$

and the maximum value is given by:

$$\lambda^* \|f\|_{L^m(\Omega)} = t_*^{\frac{1}{q}} - Ct_* + C(qC)^{\frac{q}{1-q}} - \frac{(qC)^{\frac{1}{1-q}}}{q}$$

This determines the definition of λ^*.
Since $q > 1$, using Lemma 1.1.6, setting $a = \widetilde{C}$, $b = \| \|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} k^q$ and $C^* = \lambda^* \|f\|_{L^m(\Omega)}$ so the function can have unique root ℓ such that:

$$\widetilde{C}^q \left(|\Omega|^{\frac{r-qm}{mr}} k^q \ell + \lambda^* \|f\|_{L^m(\Omega)} \right)^q - \ell = 0$$

equivalent to:

$$\widetilde{C} \left(|\Omega|^{\frac{r-qm}{mr}} k^q \ell + \lambda^* \|f\|_{L^m(\Omega)} \right) = \ell^{\frac{1}{q}} \tag{3.1.1.1}$$

Since we have λ^* and ℓ let define:

$$E_\eta := \left\{ v \in L_0^{\gamma,1+\eta}(\Omega) : \|v\|_{L_0^{\gamma,r}(\Omega)} \leq \ell^{\frac{1}{q}} \right\}$$

with $\gamma := \max\{t,s\}$ and $0<\eta<\min\left\{q-1, \frac{s-N(\gamma-s)}{N(1+(\gamma-s))-s}\right\}$
E_η is closed and convex set of $L_0^{\gamma,1+\eta}(\Omega)$, we can prove those by using Fatou's Theorem and trigonometry inequation.
To get the existence of a weak solution for (KPZ_1) in E_η, we will use the Schawder's fixed point Theorem 1.1.1, let define the operator:

$$\begin{aligned} T : E_\eta &\to L_0^{\gamma,1+\eta}(\Omega)\\ \varphi &\mapsto u \end{aligned}$$

such that u is the unique weak solution for the problem:

$$\begin{cases} (-\Delta)^s u = |(-\Delta)^{\frac{t}{2}}\varphi|^q + \lambda f(x), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

we can also see that the operator T as the Green representation given by:

$$T(\varphi)(x) := \int_\Omega G(x,y)\left(|(-\Delta)^{\frac{t}{2}}\varphi(y)|^q + \lambda f(y)\right)dy = u(x)$$

The existence of the weak solution in $L_0^{\gamma,1+\eta}(\Omega)$ immediately by using the fixed point of Schawder theorem to T. First of all we have by Proposition 2.4.1 that T is well defined in $L_0^{\gamma,1+\eta}(\Omega)$. we split the proof in three steps
Step-1: $T(E_\eta)\subset E_\eta$:
Let $\varphi\in E_\eta$ and $u=T(\varphi)$, by using Propositions 2.3.1 and (3.1.1.1) we get:

$$\begin{aligned} \|u\|_{L_0^{\gamma,r}(\Omega)} &\le \widetilde{C}\||(-\Delta)^{\frac{t}{2}}\varphi|^q + \lambda f\|_{L^m(\Omega)}\\ &\le \widetilde{C}\left(\||(-\Delta)^{\frac{t}{2}}\varphi|^q\|_{L^m(\Omega)} + \lambda\|f\|_{L^m(\Omega)}\right)\\ &\le \widetilde{C}\left(\||(-\Delta)^{\frac{t}{2}}\varphi|^q\|_{L^m(\Omega)} + \lambda^*\|f\|_{L^m(\Omega)}\right)\\ &\le \widetilde{C}\left(\|(-\Delta)^{\frac{t}{2}}\varphi\|^q_{L^r(\Omega)}|\Omega|^{\frac{r-qm}{mr}} + \lambda^*\|f\|_{L^m(\Omega)}\right)\\ &\le \widetilde{C}\left(\|\varphi\|^q_{L_0^{t,r}(\Omega)}|\Omega|^{\frac{r-qm}{mr}} + \lambda^*\|f\|_{L^m(\Omega)}\right)\\ &\le \widetilde{C}\left(k^q\|\varphi\|^q_{L_0^{\gamma,r}(\Omega)}|\Omega|^{\frac{r-qm}{mr}} + \lambda^*\|f\|_{L^m(\Omega)}\right)\\ &\le \widetilde{C}\left(k^q\ell|\Omega|^{\frac{r-qm}{mr}} + \lambda^*\|f\|_{L^m(\Omega)}\right) = \ell^{\frac{1}{q}}. \end{aligned}$$

Thus $\|u\|_{L_0^{\gamma,r}(\Omega)}\le \ell^{\frac{1}{q}}$ and by definition of T we have that $u\in L_0^{\gamma,1+\eta}(\Omega)$ so obtained that $u\in E_\eta$. Which is equivalent to say $T(E_\eta)\subset E_\eta$.
Step-2: T is compact:
Let $(\varphi_n)_n\subset E_\eta$ such that $\|\varphi_n\|_{L_0^{\gamma,1+\eta}(\Omega)}\le 1$ and $(h_n)_n$ such that:

$$h_n = |(-\Delta)^{\frac{t}{2}}\varphi_n|^q + \lambda f$$

" Now, we prove that $(h_n)_n$ is bounded in $L^1(\Omega)$. Since $1+\eta\le q\le r$, it follows from Hölder's and interpolation inequalities that:

$$\begin{aligned} \|h_n\|_{L^1(\Omega)} &\le \int_\Omega |(-\Delta)^{\frac{t}{2}}\varphi_n(x)|^q dx + \lambda\int_\Omega |f(x)|dx\\ &\le \int_\Omega |(-\Delta)^{\frac{t}{2}}\varphi_n(x)|^q dx + \lambda^*|\Omega|^{\frac{m-1}{m}}\|f\|_{L^m(\Omega)}\\ &\le k^q\|(-\Delta)^{\frac{\gamma}{2}}\varphi_n\|^q_{L^q(\Omega)} + \lambda^*|\Omega|^{\frac{m-1}{m}}\|f\|_{L^m(\Omega)}\\ &\le k^q\left(\|(-\Delta)^{\frac{\gamma}{2}}\varphi_n\|^{1-\theta}_{L^{1+\eta}(\Omega)}\|(-\Delta)^{\frac{\gamma}{2}}\varphi_n\|^{\theta}_{L^r(\Omega)}\right)^q + \lambda^*|\Omega|^{\frac{m-1}{m}}\|f\|_{L^m(\Omega)}\\ &\le k^q\ell^\theta + \lambda^*|\Omega|^{\frac{m-1}{m}}\|f\|_{L^m(\Omega)} \end{aligned}$$

So $(h_n)_n$ is bounded in $L^1(\Omega)$ and since T can be given by the Green representation, then we use Proposition 2.4.1 to get that T is compact.

Thus, $(h_n)_n$ is bounded in $L^1(\Omega)$ is given by the Green representation, Proposition 2.4.1 ensures the compactness of T.

Step-3: T is continuous:

Since T is not a linear operator, we need to verify its continuity, we consider $(\varphi_n)_n \subset E_\eta$ be a sequence converging to φ in $L_0^{\gamma,1+\eta}(\Omega)$, and let $(u_n)_n \subset E_\eta$ such that $u_n = T(\varphi_n)$ for all $n \in \mathbb{N}$ and $u = T(\varphi)$ such that:

$$\begin{cases} (-\Delta)^s(u_n - u) = \left(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\right), & \text{in } \Omega, \\ \qquad (u_n - u) = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Therefore, it suffices to show that $\left\|\left(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\right)\right\|_{L^1(\Omega)} \xrightarrow[n\to\infty]{} 0.$

By applying Hölder's inequality and the Mean Value Theorem, we obtain

$$\begin{aligned} \left\|\left(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\right)\right\|_{L^1(\Omega)} &\le \left\|\left(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\right)\right\|_{L^1(\Omega)} \\ &\le C\|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)} \end{aligned}$$

with $C = |\Omega|^{\frac{q-1}{q}}$, On the other hand, by Proposition 1.4.5 and the interpolation inequality, it follows that:

$$\begin{aligned} \|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)} &\le \|\varphi_n - \varphi\||_{L_0^{t,q}(\Omega)} \le k\|\varphi_n - \varphi\||_{L_0^{\gamma,q}(\Omega)} = k\|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|_{L^q(\mathbb{R}^N)} \\ &\le k\|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|^{\theta}_{L^{1+\eta}(\mathbb{R}^N)}\|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|^{1-\theta}_{L^r(\mathbb{R}^N)} \\ &= k\|\varphi_n - \varphi\||^{\theta}_{L_0^{\gamma,1+\eta}(\Omega)}\|\varphi_n - \varphi\||^{1-\theta}_{L_0^{\gamma,r}(\Omega)} \le (2\ell^{\frac{1}{q}})^{1-\theta}k\|\varphi_n - \varphi\||^{\theta}_{L_0^{\gamma,1+\eta}(\Omega)} \end{aligned}$$

with $\frac{1}{q} = \frac{\theta}{1+\eta} + \frac{1-\theta}{r}$, and since $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ in $L_0^{\gamma,1+\eta}(\Omega)$, we obtain that:

$$\left\|\left(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\right)\right\|_{L^1(\Omega)} \le C(2\ell^{\frac{1}{q}})^{1-\theta}k\|\varphi_n - \varphi\||^{\theta}_{L_0^{\gamma,1+\eta}(\Omega)} \xrightarrow[n\to\infty]{} 0$$

Therefore, by Proposition 2.4.1, we conclude that T is continuous.

ince T is a continuous, compact operator mapping E_η into itself $(T(E_\eta) \subset E_\eta)$, the Schauder fixed-point theorem ensures the existence of a fixed point $u \in E_\eta$ such that $T(u) = u$.

Since T is a continuous compact operator and $T(E_\eta) \subset E_\eta$ and by the Schawder fixed point theorem we get that T admit unique u such that $T(u) = u$, E_η into itself $(T(E_\eta) \subset E_\eta)$, Under these conditions, we establish the existence (and uniqueness, if applicable) of the solution $u \in E_\eta$. Next, by applying Proposition 1.4.6 and Theorem 1.3.1 for $\frac{N}{s} \le m$, we obtain:

$$u \in W^{s,p}(\mathbb{R}^N) \cap \mathcal{C}^{0,s}(\mathbb{R}^N) \quad \text{for all } 1 < p < \infty$$

Case 2: $1 \le m \le \frac{N}{s}$

First, let us choose $r = r(m,t,s,q) > 0$ such that $1 < qm < r < \frac{mN}{(N-ms+mN(\gamma-s))^+}$. Using the same λ^* and ℓ from Case 1, we define:

$$\widetilde{E}_\eta := \left\{v \in L_0^{\gamma,1+\eta}(\Omega) : \|v\|_{L_0^{\gamma,1+\eta}(\Omega)} \le M \text{ and } \|v\|_{L_0^{\gamma,r}(\Omega)} \le \ell^{\frac{1}{q}}\right\}$$

such that $\gamma := \max\{t,s\}$, $0 < \eta < \min\left\{q-1, \frac{s-N(\gamma-s)}{N(1+(\gamma-s))-s}\right\}$, and :

$$M := \widetilde{C}_1(|\Omega|^{\frac{r-q}{r}}k^q\ell + \lambda^*\|f\|_{L^1(\Omega)})$$

where $\widetilde{C}_1 > 0$ is the same constant as in Proposition 2.3.1 for $m = 1$. It is clear that $\widetilde{E}_\eta$ is a closed, bounded, and convex subset of $L_0^{\gamma,1+\eta}(\Omega)$. Similarly to Case 1, let us define:

$$\begin{aligned} \widetilde{T} : \widetilde{E}_\eta &\to L_0^{\gamma,1+\eta}(\Omega) \\ \varphi &\mapsto u \end{aligned}$$

This can be obtained via the Green representation formula. Following the same approach as in Case 1, one can prove that $\widetilde{T}$ is a continuous and compact operator satisfying $\widetilde{T}(\widetilde{E}_\eta) \subset \widetilde{E}_\eta$. Thus, applying Schauder's fixed-point theorem (Theorem 1.1.1) ensures the existence of a solution u in $\widetilde{E}_\eta$. Furthermore, by Lemma 1.4.1, we obtain

$$u \in W^{s,p}(\mathbb{R}^N)` \quad \text{for all } 1 < p < \frac{mN}{N - ms}$$

□

Remark. Similar results can be obtained for the case where the nonlocal gradient term is replaced by the Riesz t-gradient ∇^t, which is given by:

$$\nabla^t u(x) := \mu_{N,t} \int_{\mathbb{R}^N} \frac{(x-y)(u(x)-u(y))}{|x-y|^{N+t+1}}\, dy \quad \text{such that} \quad \mu_{N,\alpha} := 2^\alpha \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+\alpha+1}{2})}{\Gamma(\frac{1-\alpha}{2})}.$$

Specifically, regarding the existence of a weak solution to the problem:

$$\begin{cases} (-\Delta)^s u = |\nabla^t u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{KPZ$_2$}$$

The main result for (KPZ$_2$) is given by the next theorem.

Theorem 3.1.2. *Assume that $0 < t < \min\{1, s(1 + N^{-1})\}$, $f \in L^m(\Omega)$ for some $m \geq 1$. Then, for all $1 < q < \overline{q}$, there exists $\lambda^* > 0$ such that, for all $0 < \lambda \leq \lambda^*$, (KPZ$_2$) has a weak solution u. Moreover:*

- *If $m \geq N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*
- *If $1 \leq m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < \frac{mN}{(N-ms)}$.*

Proof. See [38] □

3.2 Non-existence results for KPZ problems

After proving the existence theorems, we now state a nonexistence theorem for problem (KPZ_1).

Theorem 3.2.1. *Let $0 < t < \min\{1, 2s\}$, $f \in L^1(\Omega)$ with $f \gneqq 0$ and $q > \frac{2(s+1)}{t+2}$. Then, there exists $\lambda^{**} > 0$ such that, for all $\lambda > \lambda^{**}$, (KPZ_1) has no weak solution in $W_0^{s,2}(\Omega)$.*

Proof. We suppose that $u \in W_0^{s,2}(\Omega)$, and let $\phi \in W^{s,2}(\Omega) \cap \mathcal{C}^{0,s}(\mathbb{R}^N)$ be unique energy solution of :

$$\begin{cases} (-\Delta)^s \phi = 1, & \text{in } \Omega \\ \phi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We denote that $\phi \in \mathbb{X}^s(\Omega)$, so we will use it as test function in (KPZ_1), and we get:

$$\begin{aligned}\int_\Omega (-\Delta)^s u\phi\,dx = \int_\Omega u(-\Delta)^s\phi\,dx = \int_\Omega u\,dx = \int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + \lambda\int_\Omega f\phi\,dx \\ \geq \int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + \lambda\int_\Omega f\phi\,dx.\end{aligned} \tag{3.2.1.1}$$

In other hand, let $\psi \in W_0^{\frac{t}{2},2}(\Omega)\cap\mathcal{C}^{0,s}(\mathbb{R}^N)$, the unique energy solution of:

$$\begin{cases}(-\Delta)^{\frac{t}{2}}\psi = 1, & \text{in } \Omega\\ \psi = 0, & \text{in } \mathbb{R}^N\setminus\Omega.\end{cases} \tag{3.2.1.2}$$

Since $W_0^{s,2}(\Omega)\subset W_0^{\frac{t}{2},2}(\Omega)$ we can use u as test function in (3.2.1.2), by integration by part and (3.2.1.1), we get:

$$\int_\Omega (-\Delta)^{\frac{t}{2}}\psi u\,dx = \int_\Omega \psi(-\Delta)^{\frac{t}{2}}u\,dx = \int_\Omega u\,dx \geq \int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + \lambda\int_\Omega f\phi\,dx \tag{3.2.1.3}$$

Moreover by Young's inequality:

$$\begin{aligned}\int_\Omega \psi(-\Delta)^{\frac{t}{2}}u\,dx &= \int_\Omega \left((-\Delta)^{\frac{t}{2}}u(\phi)^{\frac{1}{q}}\right)\left(\frac{\psi}{(\phi)^{\frac{1}{q}}}\right)dx\\ &\leq \int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + C_q\int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}}dx\end{aligned} \tag{3.2.1.4}$$

from (3.2.1.3) and (3.2.1.4) we get:

$$\int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + \lambda\int_\Omega f\phi\,dx \leq \int_\Omega |(-\Delta)^{\frac{t}{2}}u|^q\phi\,dx + C_q\int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}}dx.$$

Which is equivalent to:

$$\lambda\int_\Omega f\phi\,dx \leq C_q\int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}}dx \tag{3.2.1.5}$$

Finally we have from [18], there exists $C_0 > 1$ such that:

$$C_0^{-1}\delta^s(x) \leq \phi \leq C_0\delta^s(x) \quad \text{and} \quad C_0^{-1}\delta^s(x) \leq \psi \leq C_0\delta^s(x) \quad \text{in } \overline{\Omega} \tag{3.2.1.6}$$

Since $q > \frac{2(s+1)}{t+2}$, we get the right side in (3.2.1.5) is bounded and :

$$\lambda \leq \frac{C_p\displaystyle\int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}}dx}{\displaystyle\int_\Omega f\phi\,dx} =: \lambda^{**}$$

□

Remark. The regularity assumption on f can be slightly relaxed. In fact, for the nonexistence result in Theorem 3.2.1 to hold, we only need $f \in L^1(\Omega, \delta^s(x)dx)$. Furthermore, note that we only used the fact that u is a weak super-solution to (KPZ_1). Conversely, the restriction on the power q seems to be of a technical nature.

Conclusion

This Master's thesis, based on the works [3, 4, 1, 38], is devoted to the study of the global fractional Calderón–Zygmund regularity theory for the fractional Poisson problem with homogeneous Dirichlet boundary conditions, and to its application to the existence and non-existence of weak solutions for a class of nonlocal Kardar-Parisi-Zhang (KPZ) type equations involving nonlocal gradient term.

We prove the global fractional Calderón–Zygmund estimates for the nonlocal gradient term $(-\Delta)^{t/2}u$, with $s \leq t < \min\{1, 2s\}$, in the Lebesgue space $L^p(\mathbb{R}^N)$, distinguishing the regimes $m > \frac{N}{2s-t}$ and $1 \leq m < \frac{N}{2s-t}$. These estimates were extended to obtain global regularity for the solution u itself in the Bessel potential spaces $L^{t,p}(\mathbb{R}^N)$ and the fractional Sobolev spaces $W^{t,p}(\mathbb{R}^N)$. We further derived precise rates of blow-up of the fractional derivatives of u near the boundary $\partial\Omega$, and we obtained compactness results for the Green operator G_s, which played an essential role in the subsequent existence theory.

This regularity theory was then applied to the nonlocal KPZ-type problem involving either the half t-Laplacian $(-\Delta)^{t/2}u$ as the gradient nonlinearity. Using a Schauder fixed-point argument built upon the compactness of the solution map G_s, we proved the existence of a weak solution for small enough values of the parameter λ and for exponents q below an explicit threshold $\overline{q}(m, s, t)$, with the solution shown to belong to appropriate fractional Sobolev or Hölder spaces depending on the summability of f. Conversely, by a suitable testing procedure combined with sharp boundary estimates for both the fractional Green's function and an auxiliary function associated to the half t-Laplacian, we proved a non-existence result for large values of λ, when the exponent q exceeds the critical value $\frac{2(s+1)}{t+2}$.
Altogether, the results of this thesis extend, in a global setting over the whole space $\mathbb{R}^N$, regularity estimates that were previously available only locally, and they provide a fairly complete existence and non-existence theory for a family of nonlocal KPZ-type problems driven by fractional gradient terms.

Bibliography

[1] B. Abdellaoui, K. Biroud, A. Primo, F. Soria, and A. Younes. Fractional KPZ equations with fractional gradient term and Hardy potential. *Mathematics in Engineering*, 5(2):1–36, 2023.

[2] B. Abdellaoui and A. J. Fernández. Nonlinear fractional Laplacian problems with nonlocal 'gradient terms'. *Proceedings of the Royal Society of Edinburgh Section A*, 150(5):2682–2718, 2020.

[3] B. Abdellaoui, A. J. Fernández, T. Leonori, and A. Younes. Global fractional Calderón-Zygmund regularity. *Communications in Contemporary Mathematics, 2025, v. 27, n. 8, p. 1, doi. 10.1142/S021919972550004X.*

[4] B. Abdellaoui, A. J. Fernández, T. Leonori, and A. Younes. Deterministic KPZ-type equations with nonlocal "gradient terms". *Annali di Matematica Pura ed Applicata (1923-)*, pages 1–18, 2022.

[5] B. Abdellaoui and I. Peral. Towards a deterministic KPZ equation with fractional diffusion: the stationary problem. *Nonlinearity*, 31(4):1260–1298, 2018.

[6] R. A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[7] D. Applebaum. *Lévy processes and stochastic calculus.* Cambridge University Press, 2009.

[8] U. Biccari, M. Warma, and E. Zuazua. Local elliptic regularity for the Dirichlet fractional Laplacian. *Advanced Nonlinear Studies*, 17(2):387–409, 2017.

[9] G. M. Bisci, V. D. Rădulescu, and R. Servadei. *Variational methods for nonlocal fractional problems*, volume 162. Cambridge University Press, 2016.

[10] L. Boccardo and G. Croce Elliptic partial differential equations: existence and regularity of distributional solutions. *Stud Math.*, 2014;55. De Gruyter

[11] K. Bogdan and T. Jakubowski. Estimates of the Green function for the fractional Laplacian perturbed by gradient. *Potential Analysis*, 36(3):455–481, 2012.

[12] J. P. Borthagaray, R. H. Nochetto, and A. J. Salgado. Weighted Sobolev regularity and rate of approximation of the obstacle problem for the integral fractional Laplacian. *Mathematical Models and Methods in Applied Sciences*, 29(14):2679–2717, 2019.

[13] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[14] L. Caffarelli. Non-local diffusions, drifts and games. In *Nonlinear Partial Differential Equations: The Abel Symposium 2010*, pages 37–52. Springer, 2012.

[15] Z.-Q. Chen, P. Kim, and R. Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *Journal of the European Mathematical Society*, 12(5):1307–1329, 2010.

[16] Z.-Q. Chen and R. Song. Estimates on Green functions and Poisson kernels for symmetric stable processes. *Mathematische Annalen*, 312(3):465–501, 1998.

[17] M. Cozzi. Interior regularity of solutions of non-local equations in Sobolev and Nikol'skii spaces. *Annali di Matematica Pura ed Applicata*, 196(2):555–578, 2017.

[18] M. M. Fall. Regularity results for nonlocal equations and applications. *Calculus of Variations and Partial Differential Equations*, 59(5):Paper No. 181, 53, 2020.

[19] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, 2001.

[20] T. Grzywny and M. Ryznar. Estimates of Green functions for some perturbations of fractional Laplacian. *Illinois Journal of Mathematics*, 51(4):1409–1438, 2007.

[21] J. A. Mann Jr and W. A. Woyczynski. Growing fractal interfaces in the presence of self-similar hopping surface diffusion. *Physica A: Statistical Mechanics and its Applications*, 291(1-4):159–183, 2001.

[22] G. Karch and W. Woyczyński. Fractal Hamilton-Jacobi-KPZ equations. *Transactions of the American Mathematical Society*, 360(5):2423–2442, 2008.

[23] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889, 1986.

[24] E. Katzav. Growing surfaces with anomalous diffusion: Results for the fractal Kardar-Parisi-Zhang equation. *Physical Review E*, 68(3):031607, 2003.

[25] T. Leonori, I. Peral, A. Primo, and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete and Continuous Dynamical Systems*, 35(12):6031–6068, 2015.

[26] T. Mengesha, A. Schikorra, and S. Yeepo. Calderón-Zygmund type estimates for nonlocal PDE with Hölder continuous kernel. *Advances in Mathematics*, 383:107692, 2021.

[27] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bulletin des sciences mathématiques*, 136(5):521–573, 2012.

[28] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014.

[29] X. Ros-Oton and J. Serra. The extremal solution for the fractional Laplacian. *Calculus of Variations and Partial Differential Equations*, 50(3-4):723–750, 2014.

[30] T. Shieh and D. Spector. On a new class of fractional partial differential equations II. *Advances in Calculus of Variations*, 11(3):289–307, 2018.

[31] T.-T. Shieh and D. E. Spector. On a new class of fractional partial differential equations. *Advances in Calculus of Variations*, 8(4):321–336, 2015.

[32] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana University Mathematics Journal*, 55:1155–1174, 2006.

[33] G. Stampacchia. Équations elliptiques du second ordre à coefficients discontinus. *Séminaire Jean Leray*, 3:1–77, 1963.

[34] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, NJ, 1970.

[35] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 1983.

[36] E. Valdinoci. From the long jump random walk to the fractional Laplacian. *arXiv preprint arXiv:0901.3261*, 2009.

[37] M. Šilhavý. Fractional vector analysis based on invariance requirements (critique of coordinate approaches). *Continuum Mechanics and Thermodynamics*, 32(1):207–228, 2020.

[38] A. Younes Nonlinear elliptic problems in relation with the fractional Laplacian : Existence and regularity. *Thesis.Univ-Tlemcen,2023*
https://dspace.univ-tlemcen.dz/items/5638678c-bf4b-431d-bcb9-8bd2aa173a96.2013