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DYNAMICS OF A DIFFUSIVE AGE-STRUCTURED HBV MODEL WITH SATURATING INCIDENCE

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Dedication

This thesis is dedicated to every person who helped me, even if with a word,
to every person who motivated me from the beginning of my studies until the day of my graduation,
to every person who contributed to raising my morale,
and to every person who told me, "You can do it."

To my incredible parents, my wonderful sister ***Esma****,*
my supportive brother ***Mohammed Nor****, my kind sister-in-law* ***Ikram****,*
my dearest best friend ***Khadidja****, my charming nephew* ***Riyad****,*
my lovely niece ***Alaa****,*
and my uncles, aunts, and cousins, both paternal and maternal.

I dedicate this thesis to the past, the future, and the present day,
to every person who opens it and reads it.

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Abstract

In this thesis, we studied the dynamics of a diffusive age-structured HBV model with saturating incidence. We demonstrated the existence of solutions and aimed to prove the global stability of the steady state, which is the essential part of this work. Numerical simulations were provided to validate the theoretical results obtained.
Theoretical epidemiology of communicable diseases has become a distinct discipline, offering a vast field of application for mathematics and statistics. Beyond theoretical results, this discipline aims to provide quantitative foundations for public health considerations regarding infectious diseases.

Résumé

Dans ce mémoire, nous avons étudié la dynamique d'un modèle HBV structuré par âge avec diffusion et incidence saturante. Nous avons démontré l'existence des solutions et visé à prouver la stabilité globale de l'état d'équilibre, qui constitue la partie essentielle de ce travail. Des simulations numériques ont été fournies pour valider les résultats théoriques obtenus.
L'épidémiologie théorique des maladies transmissibles est devenue une discipline distincte, offrant un vaste champ d'application pour les mathématiques et les statistiques. Au-delà des résultats théoriques, cette discipline vise à fournir des bases quantitatives pour les considérations de santé publique concernant les maladies infectieuses.

ملخص

في هذا البحث، قمنا بدراسة ديناميكيات نموذج التهاب الكبد الفيروسي حط المهيكل حسب العمر مع الانتشار والوقوع المشبع. لقد أثبتنا وجود الحلول وهدفنا إثبات الاستقرار العالمي لحالة التوازن، والتي تشكل الجزء الأساسي من هذا العمل. تم تقديم محاكاة رقمية للتحقق من صحة النتائج النظرية التي تم الحصول عليها.
أصبحت الأوبئة النظرية للأمراض المعدية تخصصًا متميزًا، حيث تقدم مجالًا واسعًا لتطبيقات الرياضيات والإحصاءات. وبالإضافة إلى النتائج النظرية، تهدف هذه التخصص إلى توفير أسس كمية للنظر في الصحة العامة المتعلقة بالأمراض المعدية.

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Notation

Ω : Open set in $\mathbb{R}^n$

$\partial\Omega$: Boundary of Ω

$\overline{\Omega}$: Closure of Ω

∇f : Gradient of f

Δf : Laplacian of f

$L(X)$: Space of linear operators in X

$C(\Omega)$: Space of continuous functions defined on Ω

$L^p(\Omega)$: Lebesgue space

$L^\infty(\Omega)$: Space of bounded functions almost everywhere

$C^\infty(\Omega)$: Space of infinitely differentiable functions on Ω

$C^k(\Omega)$: Space of C^k functions on Ω

$C(\Omega; X)$: Space of continuous functions from Ω to X

$C^1(\Omega; X)$: Space of continuously differentiable functions from Ω to X

$\mathcal{C} := \mathrm{BUC}(\overline{\Omega}, \mathbb{R})$: the set of all bounded and uniformly continuous functions from $\overline{\Omega}$ to $\mathbb{R}$.

$\mathcal{C}^2(\overline{\Omega})$: the set of all functions in $\mathcal{C}$ whose derivatives up to order 2 .

$\mathcal{C}^2_+(\overline{\Omega})$: the set of all functions in $\mathcal{C}_+$ whose derivatives up to order 2 .

A : Linear operator

$D(A)$: Domain of operator A

$\dfrac{\partial}{\partial n}$: Normal derivative

$W^{k,p}$: Sobolev space, with derivative up to order k in $L^p(\Omega)$

Introduction

Epidemiology studies how diseases evolve within human populations, aiming to understand dynamics. Its goal is to enhance disease treatment and prevention. Infectious diseases are caused by microorganisms such as viruses, bacteria, parasites, or fungi. Some well-known examples include tuberculosis, HIV/AIDS, and hepatitis B.

Hepatitis B, often abbreviated as Hep B, is a liver infection caused by the hepatitis B virus (HBV). Imagine it as an unwelcome intruder that sneaks into your liver and causes serious trouble. This virus can persist in the body for long periods, sometimes even a lifetime, leading to severe complications such as liver damage, cirrhosis, or even liver cancer.

What makes Hep B particularly daunting is that it can be transmitted through infected blood, from mother to child during childbirth...
This disease affects millions of people worldwide and is a major public health concern.

Chronic HBV infection often starts early in life, persisting due to weakened immune responses [2]. In highly affected areas, prevalence rates range from 8% to 15% [17]. China has the highest number of cases at 120 million, followed by India and Indonesia [17]. Annually, approximately 600,000 deaths are attributed to hepatitis B [26].

In this study, we will delve into the mysteries of Hep B and explore mathematical models to better understand its spread and dynamics. Among these models, Nowak et al. [15] introduced a fundamental model for HBV infection, followed by subsequent research exploring its dynamics [25, 23, 1]:

$$\begin{cases} \dot{x} = \lambda - dx - \beta xv, \\ \dot{y} = \beta xv - ay, \\ \dot{v} = ky - uv, \end{cases} \tag{1}$$

where x, y, v represent uninfected cells, infected cells and free virus particles, respectively; x are produced at a constant rate λ, and die at the rate d.
v infects x to produce y at rate β. Infected cells die at rate a. New virus is produced from infected cells at rate k and dies at rate u.
After that, Wang and Wang [13] proposed a spatial model to simulate HBV infection and es-

tablished the existence of traveling waves, considering the movement of viruses within the liver.

They proposed the following model with diffusion in the free virus:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \lambda - au(x,t) - \beta u(x,t)v(x,t), \\ \dfrac{\partial w}{\partial t} = \beta u(x,t)v(x,t) - bw(x,t), \\ \dfrac{\partial v}{\partial t} = D\Delta v + kw(x,t) - dv(x,t). \end{cases} \tag{2}$$

Where $u(x,t)$, $w(x,t)$, and $v(x,t)$ represent the densities of uninfected cells, infected cells, and free virus at location x and time t, respectively; susceptible cells are produced at rate λ, die at rate a; infected cells are produced at rate β and die at rate b; free viruses are produced from infected cells at rate k and are removed at rate d.
The parameters λ, a, β, b, k, d are all positive constants.
The Laplacian operator is defined by:

$$\Delta v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} \qquad , n = 1, 2, 3...$$

With D is the diffusion coefficient.

They showed the stability of HBV infection in different environments, whether homogeneous or not.

In most virus infection models, it's thought that the speed of infection relies on the amount of virus and healthy cells. However, experiments suggest [4] that with microparasitic infections—tiny parasites—the more parasites there are, the faster the infection spreads. This indicates that hepatitis B infection might follow a similar pattern.

In this memory, we delve into the study of model with an age-since-infection and saturating incidence into model (2). Here are the equations of the model[27]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = S - \mu u(x,t) - \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial \theta} = -\delta(\theta)w(x,t), & x \in \Omega, \quad \theta > 0, \quad t > 0, \\ \dfrac{\partial v}{\partial t} = D\Delta v + \displaystyle\int_0^\infty p(\theta)w(x,\theta,t)d\theta - dv(x,t), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

With the initials and boundaries conditions:

$$\begin{cases} w(x,0,t) = \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, & x \in \Omega, \\ u(x,0) = \vartheta_{10}(x), & x \in \Omega, \\ w(x,\theta,0) = \vartheta_{20}(x,\theta), & x \in \Omega, \quad \theta > 0, \\ v(x,0) = \vartheta_{30}(x), & x \in \Omega, \end{cases} \tag{4}$$

In **the first chapter**, we begin by introducing certain preliminary results that will serve as the foundation for the subsequent chapter. In **the second chapter**, we discuss both local and global stability, and conduct numerical simulations.
These mathematical models will aid us in gaining deeper insights into the mechanisms of Hepatitis B spread and in devising novel strategies to control this disease.

Chapter 1

Preliminaries

1.1 Some notions about stability

Consider the following non-autonomous system:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$ with $\Omega \subset \mathbb{R}^n$ is a locally Lipschitz continuous function.
One primary approach in the study of dynamical systems is to look for equilibrium points.

Definition 1.1.1 ***"The equilibrium point"*** *An equilibrium point x^* for the differential equation $\dot{x}(t) = f(x(t))$ is a constant solution with respect to time, that is $\dfrac{dx^*}{dt} = 0$, thus*

$$f(x^*(t)) = f(x^*) = 0.$$

Definition 1.1.2 ***"Attractivity"***

— *An equilibrium point x^* is said to be attractive if there exists a neighborhood $U(x^*)$ such that:*

$$\forall x_0 \in U(x^*), \quad \lim_{t \to +\infty} x(t) = x^*.$$

— *An equilibrium point x^* is said to be globally attractive if for all ,*

$$\forall x_0 \in \Omega, \quad \lim_{t \to +\infty} x(t) = x^*.$$

Definition 1.1.3 ***"Stability of an Equilibrium Point"***

— *x^* is **stable** if $\forall \epsilon > 0$, there $\exists \delta > 0$ such that if $|x_0 - x^*| < \delta$, then $|x(t) - x^*| < \epsilon$ for all $t \geq 0$.*

— *x^* is said to be **locally asymptotically stable** if it is stable and attractive.*

— *x^* is said to be **globally asymptotically stable** if it is locally asymptotically stable and globally attractive.*

Definition 1.1.4 ***"A positively invariant set"***
A set Ω is positively invariant if every solution $y(t)$ such that $y(0) \in \Omega$ remains in Ω for $t \geqslant 0$.

1.2 Some definitions in topology

Definition 1.2.1 *[16]* ***"Sobolev space"***
Let p be an element of $[1, +\infty]$, Ω an open subset of $\mathbb{R}^n$. We define the first-order Sobolev space, denoted $W^{1,p}(\Omega)$ (or $H^1(\Omega)$ if $p = 2$) , as the set:

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) | \forall i, 1 \leq i \leq n, \frac{\partial u}{\partial x_i} \in L^p(\Omega)\}$$

Definition 1.2.2 ***" Topological space"***
Let $\mathcal{O}$ be a family of subsets of a set X. We suppose that:

1. *The empty set and X are elements of $\mathcal{O}$.*
2. *The union of any arbitrary family of elements of $\mathcal{O}$ is still an element of $\mathcal{O}$.*
3. *The intersection of a finite family of elements of $\mathcal{O}$ is still an element of $\mathcal{O}$.*

Then we say that $\mathcal{O}$ is ***a topology for*** *X. In this case, the set X equipped with this family is called* ***a topological space****.*

Definition 1.2.3 ***"Semi-flow [7]"***
A semi-flow on a topological space X is a family of continuous maps $\phi_t : X \to X$, $t \in \mathbb{R}^+$, satisfying the following properties :

1. *ϕ_0 is the identity on X.*
2. *For all $s, t \in \mathbb{R}^+$, $\phi_{s+t} = \phi_s \circ \phi_t$.*

Definition 1.2.4 ***"Asymptotically compact[8]"***
Let J be a time-set and $\Phi : J \times X \to X$ a map, with $M \subset X$.

Φ is called asymptotically compact on M, if, for any sequences (t_i) in J, $t_i \to \infty$ as $i \to \infty$, and (x_i) in M, $(\Phi(t_i, x_i))$ has a convergent subsequence.

Definition 1.2.5 ***"The Kuratowski measure"***

The Kuratowski measure of noncompactness (see [12]), K is a givenbounded subset of a Banach space, which is defined by

$$K(B) := \inf\{r : B \text{ has a finite cover of diameter} \leq r\},$$

1.3 Semi-groups

Definition 1.3.1 ***"C_0 Semi group[20]"***
Let X be a Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if:

— *$T(0) = I$ (where I is the identity operator on X).*

— *$T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.*

— *For all $u \in X$, $\lim_{t\to 0} T(t)u = u$.*
A semigroup of bounded linear operators that is strongly continuous is called a C_0 semigroup.

Definition 1.3.2 ***"The infinitesimal generator[20]"***
Let $\{T(t)\}_{t\geq 0}$ be a linear C_0-semigroup on a Banach space X. The infinitesimal generator of $\{T(t)\}_{t\geq 0}$ is a linear operator $A : D(A) \subset X \to X$ satisfying the following properties:

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ exists \right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \forall x \in D(A).$$

Definition 1.3.3 ***"Integrated semigroup[20]"***
Let $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on a Banach space $(X, \|\cdot\|)$. We say that $\{S(t)\}_{t\geq 0}$ is an integrated semigroup on X if the following properties are satisfied:

(i) $S(0) = 0$;

(ii) The map $t \mapsto S(t)x$ is continuous from $[0, +\infty)$ into X for each $x \in X$;

(iii) $S(t)$ satisfies

$$S(t)S(s) = \int_0^t [S(r + s) - S(r)]\, dr, \quad \forall t, s \in [0, +\infty).$$

1.4 Compact attractors of individual sets

Definition 1.4.1 *[8]*
Let J be a time-set and $\Phi : J \times X \to X$ a semiflow.

— *A set $K \subset X$ is said to attract a set $M \subset X$, if $K \neq \emptyset$ and*

$$d(\Phi_t(M), K) \to 0, \quad as \quad t \to \infty.$$

We also say that M is attracted by K.

— *K is called an attractor of M, if K is invariant and attracts M. In this situation, we also say that M has the attractor K.*
K is called a compact attractor of M if K is compact in addition.

Lemma 1.4.1 *[8] Let K and M be subsets of X. Then the following statements are equivalent*

1. *K attracts M.*
2. *$d(\Phi_t(x), K) \to 0$, $t \to \infty$, uniformly for $x \in M$.*
3. *All sequences (t_j) in J, $t_j \to \infty$ as $j \to \infty$, and (x_j) in M satisfy $d(\Phi(t_j, x_j), K) \to 0$, $j \to \infty$.*

Definition 1.4.2 *[8]*
The ω-limit set of a subset M of X is defined as:

$$\omega(M) = \bigcap_{t\in J} \overline{\Phi(J_t \times M)},$$

where $J_t = J \cap [t, \infty)$. Obviously, $\omega(M)$ is a closed set, which may be empty. Certain special cases require specific attention. If M is forward invariant, then $\Phi(J_t \times M) = \Phi(t, M)$, so

$$\omega(M) = \bigcap_{t \in J} \overline{\Phi(t, M)} \subset \overline{\Phi(s, M)} \subset \overline{M}, \quad s \in J.$$

If M is invariant, then $\Phi(t, M) = M$ and $\omega(M) = M$.

The following alternative characterization will prove to be very useful.

Lemma 1.4.2 *[8]*
An element x in X belongs to $\omega(M)$ if and only if there exist sequences (t_j) in J, $t_j \to \infty$ as $j \to \infty$, and (x_j) in M such that $\Phi(t_j, x_j) \to x$ as $j \to \infty$.

Definition 1.4.3 *[8]*
Let $\Phi : J \times X \to X$ be a state-continuous semiflow.

- *Φ is called **point-dissipative** (or ultimately bounded) if there exists a bounded subset B of X which attracts all points in X.*
- *Φ is called **asymptotically smooth** if Φ is asymptotically compact on every forward invariant bounded closed set.*
- *Φ is called **eventually bounded** on a set $M \subset X$ if $\Phi(J_r \times M)$, $J_r = J \cap [r, \infty)$, is bounded for some $r \in J$.*

Proposition [8] Let J be a time-set and $\Phi : J \times X \to X$ a map, and let M be the empty set in X. Then the following statements are equivalent:

1. Φ is asymptotically compact on M.
2. M is attracted by a nonempty compact set $K \subset X$.
3. $\omega(M)$ is nonempty, compact, and attracts M.

Theorem 1.4.1 *[8]*
If one, and then all, of these three statements hold, $\omega(M) \subset K$ for every compact set $K \subset X$ that attracts M. The following are equivalent for a state-continuous semiflow $\Phi : J \times X \to X$:

(a) Φ is point-dissipative, asymptotically smooth, and eventually bounded on every compact subset K of X.

(b) There exists a compact attractor A of neighborhoods of compact sets in X; further, A attracts every subset of X on which Φ is eventually bounded.

1.4.1 Cauchy Problem with non-dense domain

Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Consider the inhomogeneous Cauchy problem:[?]

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t), & 0 < t < T, \\ u(0) = x \in \overline{D(A)}, \end{cases} \tag{1.2}$$

Where $f \in L^1_{loc}((0,\tau),X)$ for some $\tau > 0$. Recall that $u \in C([0,\tau],X)$ is an integrated solution of (1.2) if u satisfies :

$$\begin{cases} \int_0^t u(s)ds \in D(A), \\ u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds. \forall t \in [0,\tau). \end{cases} \tag{1.3}$$

1.5 The Parabolic Comparison Principle

Theorem 1.5.1 ***"The Parabolic Comparison Principle[9]"***

Let $T > 0$ (which may possibly be infinite). Consider a function $f = f(t,x,u)$ such that f and $\frac{\partial f}{\partial u}$ are in $C([0,T]\times\overline{\Omega}\times\mathbb{R})$. Let $\overline{u}$ and $\underline{u}$ be in $C^2_1((0,T]\times\Omega))\cap C([0,T]\times\overline{\Omega})$ (where f is C^2 class with respect to x and C^1 with respect to t), satisfying

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} \geq D\Delta\overline{u} + f(t,x,\overline{u}), & t \in (0,T], x \in \Omega, \\ \dfrac{\partial \underline{u}}{\partial t} \leq D\Delta\underline{u} + f(t,x,\underline{u}), & t \in (0,T], x \in \Omega, \\ \underline{u}(0,x) \leq \overline{u}(0,x), & x \in \Omega, \end{cases}$$

we also assume either:

$$\frac{\partial \underline{u}}{\partial \nu} \leq 0 \leq \frac{\partial \overline{u}}{\partial \nu}, \quad x \in \partial\Omega,$$

or:

$$\underline{u}(t,x) \leq 0 \leq \overline{u}(t,x), \quad t \in (0,T], x \in \partial\Omega.$$

Then $\underline{u} \leq \overline{u}$ in $t \in (0,T]\times\Omega$. Moreover, either there exists $(t_0,x_0) \in (0,T]\times\Omega$ such that $\overline{u}(t_0,x_0) = u(t_0,x_0)$, and in this case $\overline{u} \equiv \underline{u}$ in $[0,t_0]\times\overline{\Omega}$, or $\overline{u} > \underline{u}$ in $(0,T]\times\Omega$.

Theorem 1.5.2 ***"Maximum principle"[9]***

Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in L^2((0,T);W^{1,2}(\Omega))\cap C([0,T],L^2(\Omega))$. If

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \geq 0, & (x,t) \in \Omega\times(0,T), \\ u(x,0) \geq 0, & (x,t) \in \Omega\times\{0\}, \\ u(x,t) \geq 0, & (x,t) \in \partial\Omega\times(0,T), \end{cases} \tag{1.2}$$

then $u \geq 0$ in $\overline{\Omega}\times(0,T)$. If $u \geq 0$, then $u > 0$ in $\Omega\times(0,T)$.

Chapter 2

Model with an age-since-infection and saturating incidence

Understanding how viruses spread is very important for controlling them. One key factor is how long an infection lasts, which affects how a disease moves through a population. Our study looks at a model that includes the time since infection and how the infection rate changes as more people get infected. By using this model, we aim to understand how viruses spread and what drives this process.

Our model shown in equation (2.1), builds on the basic idea in equation (2) and gives a clearer view of the infection process. This helps us learn more about how diseases spread. Therefore, we can learn more about how the disease spread by studying the following model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = S - \mu u(x,t) - \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial \theta} = -\delta(\theta) w(x,\theta,t), & x \in \Omega, \quad \theta > 0, \quad t > 0, \\ \dfrac{\partial v}{\partial t} = D\Delta v + \displaystyle\int_0^\infty p(\theta) w(x,\theta,t) d\theta - dv(x,t), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial \eta} = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.1}$$

$$\begin{cases} w(x,0,t) = \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, & x \in \Omega, \\ u(x,0) = \vartheta_{10}(x), & x \in \Omega, \\ w(x,\theta,0) = \vartheta_{20}(x,\theta), & x \in \Omega, \quad \theta > 0, \\ v(x,0) = \vartheta_{30}(x), & x \in \Omega, \end{cases} \tag{2.2}$$

$w(x,\theta,t)$ is the density of infected cells of infection age θ at location x and at time t. The term $\dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}$ is the saturating incidence of HBV infection. The functions $p(\theta)$ and $\delta(\theta)$ denote respectively the rate of the production virus depend of the infection age θ and death rate of production infected cells.

Assumption: The maps $\theta \to \delta(\theta)$, $\theta \to p(\theta)$ are almost everywhere bounded, and belong to $L^{\infty}_{+}((0,+\infty),\mathbb{R}) \setminus \{0_{L^{\infty}}\}$. Moreover, $\delta(\theta) > \delta_{\min}$ for some positive number $\delta_{\min}$.

Since $\delta(\theta)$ denotes the infection age-specific death rate of productively infected cells, it is reasonable to assume that $\delta_{\min} \geq \mu$. Then we study the well-posedness for the initial-boundary value problem of system (2.1)-(2.2).

$$\underline{\delta} = \inf_{\theta \in (0,+\infty)} \delta(\theta),$$

Let's look for **the steady states** of model , therefore the variation with respect to time is zero[27],After calculations, we obtain:

— A uninfected steady state :

$$E_0 = (\frac{S}{\mu}, 0, 0).$$

— A unique infected steady state:

$$E^* = (u^*, w^*(\theta), v^*).$$

Where

$$u^* = \frac{SK\alpha + d}{K(\beta + \alpha\mu)},$$

with $\prod(\theta) = p(\theta)e^{-\int_0^\theta \delta(l)dl}d\theta$ and $K = \int_0^\infty \prod(\theta)d\theta$

$$v^* = \frac{(\frac{\beta SK}{\mu d} - 1)}{\alpha\mu + \beta}.$$

such that v^* exists if: $\frac{\beta SK}{\mu d} > 1$

$$w^*(\theta) = \frac{\beta u^* v^*}{1 + \alpha v^*} e^{-\int_0^\theta \delta(l)dl}.$$

Now, we consider a subsystem of the model :

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial \theta} = -\delta(\theta) w(x,t), \\ \\ w(x,0,t) = \dfrac{\beta u(x,t) v(x,t)}{1+\alpha v(x,t)}, \\ \\ w(x,\theta,0) = \vartheta_{20}(x,\theta). \end{cases} \tag{2.3}$$

We solve the equation along **the characteristics**:

$$\frac{dw}{dh} = \frac{dw}{d\theta}\frac{d\theta}{dh} + \frac{dw}{dt}\frac{dt}{dh} = -\delta(\theta) w(x,t),$$

By identification:

$$\begin{cases} \dfrac{d\theta}{dh} = 1, \\ \\ \dfrac{dt}{dh} = 1, \end{cases} \Rightarrow \begin{cases} \theta = h + \theta_0, & \theta_0 > 0, \\ \\ t = h + t_0, & t_0 > 0, \end{cases}$$

Let's put:

$$\begin{cases} \overline{w}(h) = w(\theta,t) = w(h+\theta_0, h+t_0), \\ \\ \overline{\delta}(h) = \delta(\theta) = \delta(h+\theta_0), \end{cases}$$

$\dfrac{d\overline{w}}{dh} = -\overline{\delta}(h)\overline{w} \Rightarrow \overline{w} = \overline{w}(0) e^{-\int_0^h \delta(l)dl} \Rightarrow w(\theta,t) = w(\theta_0,t_0) e^{-\int_0^h \delta(l+\theta_0)dl}$.

— **If** $\theta > t$

$$\begin{cases} \theta = h + \theta_0, \\ \\ t = h, \end{cases} \Rightarrow \begin{cases} \theta_0 = \theta - t, \\ \\ t = h, \end{cases}$$

$\overline{w}(0) = w(x,\theta_0,t_0) = w(x,\theta-t,0) = \vartheta_{20}(x,\theta-t)$.

— **If** $\theta < t$

$$\begin{cases} \theta = h, \\ \\ t = h + t_0, \end{cases} \Rightarrow \begin{cases} t_0 = t - \theta, \\ \\ \theta = h, \end{cases}$$

$$\overline{w}(0) = w(\theta_0, t_0) = w(0, t-\theta) = \frac{\beta u(t-\theta)v(t-\theta)}{1+\alpha v(t-\theta)},$$

$$\overline{w}(h) = w(x,\theta,t) = \begin{cases} \dfrac{\beta u(x,t-\theta)v(x,t-\theta)}{1+\alpha v(x,t-\theta)} e^{-\int_0^\theta \delta(l)dl}, & \theta \leq t, \\ \vartheta_{20}(x,\theta-t)e^{-\int_0^\theta \delta(l+\theta-t)dl}, & \theta > t. \end{cases} \tag{2.4}$$

2.1 Existence of the solution

We reformulate the system (2.1)-(2.2) as a Cauchy problem. Furthermore, we define the following space:

$$\overline{X} = \mathcal{C} \times \mathcal{C} \times L^1((0,+\infty),\mathcal{C}) \times \mathcal{C}^2(\overline{\Omega}).$$

Then, let's set:

$$\overline{X}_0 = \mathcal{C} \times \{0_{\mathcal{C}}\} \times L^1((0,+\infty),\mathcal{C}) \times \mathcal{C}^2(\overline{\Omega}),$$

$$\overline{X}_+ = \mathcal{C}_+ \times \mathcal{C}_+ \times L^1_+((0,+\infty),\mathcal{C}) \times \mathcal{C}^2_+(\overline{\Omega}),$$

$$\overline{X}_{0+} = \overline{X}_0 \cap \overline{X}_+,$$

where, $\mathcal{C} = BUC(\overline{\Omega},\mathbb{R})$, the set of all bounded and uniformly continuous functions from $\overline{\Omega}$ to $\mathbb{R}$, $\mathcal{C}_+ = B \cup C(\overline{\Omega},\mathbb{R}_+)$,
$C^2(\overline{\Omega})$, the set of all functions in $\mathcal{C}_+$ whose derivatives up to order 2

$$A\begin{pmatrix} u \\ \begin{pmatrix} 0 \\ w \end{pmatrix} \\ v \end{pmatrix} = \begin{pmatrix} -\mu u \\ \begin{pmatrix} -w(0) \\ \dfrac{-\partial w}{\partial \theta} - \delta(\theta)w(x) \end{pmatrix} \\ D\Delta v - dv \end{pmatrix},$$

with the Dom(A) is written as follows:

$$Dom(A) = \mathcal{C} \times \{0_{\mathcal{C}}\} \times W^{1,1}((0,+\infty),\mathcal{C}) \times W^{2,2}(\overline{\Omega}) \subset \bar{X},$$

where $W^{1,1}$ is a Sobolev space and $W^{2,2}$ denotes the Sobolev space of functions in $\mathcal{C}_+$ whose distributional derivatives up to order 2. By theorem 6.2 in [6], we have the linear operator A

generates a nondegenerate integrated semigroup on $\overline{X}$. We define a map $F : [0, +\infty) \longrightarrow \overline{X}$ by

$$F(t) = \begin{pmatrix} S - \dfrac{\beta u(.,t)v(.,t)}{1+\alpha v(.,t)} \\ \begin{pmatrix} \dfrac{\beta u(.,t)v(.,t)}{1+\alpha v(.,t)} \\ 0_{L^1} \end{pmatrix} \\ \int_0^\infty p(\theta)w(\theta,.,t)d\theta \end{pmatrix},$$

we obtain a Cauchy problem:

$$\begin{cases} \dfrac{dU}{dt} = AU(t) + F(t), \\ U(0) = U_0 \in \overline{X}_{0+}, \end{cases} \tag{2.5}$$

with

$$U(t) = \begin{pmatrix} u(.,t) \\ \begin{pmatrix} 0 \\ w(.,.,t) \end{pmatrix} \\ v(.,t) \end{pmatrix}$$

From Theorem 6.5 in [6]. We say that there exists an unique continuous solution.

Lemma 2.1.1 *[Theorem 2.2.1, [28]] Let $\mu(x)$ be a continuous, strictly positive, and uniformly bounded function on $\overline{\Omega}$. The following equation*

$$\frac{\partial \omega(t,x)}{\partial t} = \lambda(x) - \mu(x)\omega(t,x), \quad x \in \overline{\Omega}, \quad t > 0$$

has a unique, strictly positive, and globally asymptotically stable steady state $\frac{\lambda(x)}{\mu(x)}$ in $C^1(\overline{\Omega}, \mathbb{R})$.

Proposition 2.1.1 *[11] $\mathcal{D}$ is defined:*

$$\mathcal{D} := \{U \in \overline{X}_{0+} : u(x,t) \leq \frac{S}{\mu}, u(x,t) + \int_0^\infty w(x,\theta,t) \leq \frac{S}{\underline{a}},$$

$$v(t,x) \leq \frac{\bar{p}}{d}\frac{S}{\underline{a}}\}.$$

$\mathcal{D}$ is positively invariant

Proof: Let $U_0 = (u_0(x), w_0(x,\theta), v_0(x)) \in D$. Following the positivity of the solution, we have:

$$\frac{\partial u}{\partial t} \leq S - \mu u(x,t), \qquad x \in \Omega, \qquad t > 0,$$

using Lemma(2.1.1), we easily obtain:

$$u(t,x) \leq \frac{S}{\mu},$$

Then summing up the first two equations to the model(2.1):

$$\frac{\partial(u(t,x) + \int_0^\infty w(t,\theta,x)\,d\theta)}{\partial t} = S - \mu u(t,x) - \int_0^\infty \delta(\theta) w(t,\theta,x)\,d\theta$$

.

$$\leq S - \underline{a}\left(u(t,x) + \int_0^\infty w(t,\theta,x)\right)\,d\theta.$$

where $\underline{a} = \inf\{\underline{\delta}, \mu\}$, hence, for all:

$$x \in \overline{\Omega}, \qquad u(t,x) + \int_0^\infty w(t,\theta,x)\,d\theta \leq \frac{S}{\underline{a}}.$$

Finally, by considering the third equation of system (2.1), we obtain the following inequality:

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} \leq D\Delta v(t,x) + \bar{p}\dfrac{S}{\underline{a}} - dv(t,x), & x \in \Omega,\ t \in (0,T_0), \\ \dfrac{\partial v(t,x)}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

where $\bar{p} = \sup_{\theta\in\mathbb{R}^+}\ p(\theta)$, From the comparison principle, with

$$v_0(x) \leq \frac{\bar{p}}{d}\frac{S}{\underline{a}},$$

we conclude that:

$$v(t,x) \leq \frac{\bar{p}}{d}\frac{S}{\underline{a}},$$

Therefore, $\mathcal{D}$ is a positively invariant set.
To conclude, we deduce that U is bounded if we choose the initial values from $\mathcal{D}$.□

Therefore, Therefore, we define the solution semiflow $\psi(t) : \overline{X}_+ \to \overline{X}_+$ of system (2.1),(2.2) by

$$\psi(t)\phi = (u(\cdot,t,\phi), w(\cdot,\cdot,t,\phi), v(\cdot,t,\phi)),\ \forall t \geq 0,$$

where $\phi = (\vartheta_{10}(x), \vartheta_{20}(x,\theta), \vartheta_{30}(x)) \in \overline{X}_+$, the initial conditions of model (2.1). For $\psi(t)$ in the sense that:

$$\psi(t)(\phi) \in \mathcal{D},$$

$\forall t \geq 0$, $\varphi \in \mathcal{D}$. The following lemma then follows from Theorem 3.4 developed by Redlinger [22].

Lemma 2.1.2 *[27] For any given initial values $(\vartheta_{10}(x), \vartheta_{20}(x,\theta), \vartheta_{30}(x)) \in \mathcal{D}$, system (3)-(5) has exactly one regular solution $U(x,t) = (u(x,t), w(x,\cdot,t), v(x,t))$ satisfying $U(x,t) \in \mathcal{D}$ on $[0,+\infty)$.*
Since the first two equations in system (2.1)-(2.2) have no diffusion terms, its solution map $\psi(t)$ is not compact. To handle this situation, we employ the Kuratowski measure of noncompactness, for any bounded set B. We set $K(B) = \infty$ whenever B is unbounded. It is evident that B is precompact (i.e., $\overline{B}$ is compact) if and only if $K(B) = 0$. We then derive the following results.

Lemma 2.1.3 *[27] The solution semiflow $\psi(t)$ is K-contracting in the sense that*

$$\lim_{t\to+\infty} K(\psi(t)(B)) = 0,$$

for any bounded set $B \subset X^+$.

Proof. See [10]

Theorem 2.1.1 *[27] The solution semiflow $\psi(t)$ admits a global attractor on X^+.*

Proof. By Lemma (2.1.3), it follows that $\psi(t)$ is K-contracting on X^+. Additionally, Lemma (2.1.2) implies that $\psi(t)$ is point dissipative on X^+, and forward orbits of bounded subsets of X^+ for $\psi(t)$ are bounded. According to Theorem 2.6 in [19], $\psi(t)$ has a global attractor that attracts each bounded set in X^+. □

2.2 Local stability analysis of steady states

Using the following Volterra formulation, we have:

$$w(x,\theta,t) = \begin{cases} \dfrac{\beta u(x,t-\theta)v(x,t-\theta)}{1+\alpha v(x,t-\theta)} e^{-\int_0^\theta \delta(l)dl}, & \theta \le t, \\ \vartheta_{20}(x,\theta-t)e^{-\int_0^\theta \delta(l+\theta-t)dl}, & \theta > t. \end{cases} \tag{2.6}$$

Then, system (2.1)-(2.2) can be modified as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = S - \mu u(x,t) - \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, \\ \dfrac{\partial v}{\partial t} = D\Delta v + \displaystyle\int_0^\infty p(\theta)w(x,\theta,t)d\theta - dv(x,t), \end{cases} \tag{2.7}$$

for $t > 0$, $x \in \Omega$, with homogeneous Neumann boundary conditions. Substituting the expression (2.6) into the second equation of system (2.7), we obtain:

$$\begin{cases} \dfrac{\partial u}{\partial t} = S - \mu u(x,t) - \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, \\ \dfrac{\partial v}{\partial t} = D\Delta v + \displaystyle\int_0^t \prod(\theta) \frac{\beta u(x,t-\theta)v(x,t-\theta)}{1+\alpha v(x,t-\theta)} d\theta - dv(x,t) + F_\omega(x,t). \end{cases} \tag{2.8}$$

Where

$$\prod(\theta) = p(\theta)e^{-\int_0^\theta \delta(\tau)d\tau}d\theta,$$

for $t > 0$, $x \in \Omega$, with:

$$F_\omega(x,t) = \int_t^\infty p(\theta)w(x,\theta-t,0)e^{-\int_0^t \delta(\tau+\theta-t)}d\theta.$$

By **Assumption**, if $p(\theta)$ is in $L^\infty((0,+\infty),\mathbb{R}) \setminus \{0_{L^\infty}\}$, then $F_w(x,t)$ approaches zero when t becomes sufficiently large. Thus, system (2.8) can be expressed as following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = S - \mu u(x,t) - \dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}, \\ \dfrac{\partial v}{\partial t} = D\Delta v + \displaystyle\int_0^\infty \prod(\theta)\frac{\beta u(x,t-\theta)v(x,t-\theta)}{1+\alpha v(x,t-\theta)}d\theta - dv(x,t), \end{cases} \tag{2.9}$$

for $t > 0$, $x \in \Omega$, system (2.9) has an uninfected steady state $E_0 = (\frac{S}{\mu}, 0)$. If $R_0 > 1$, the system (2.9) has a unique infected steady state $E^* = (u^*, v^*)$. Let $E(u_0, v_0)$ represent any feasible steady state . Then, the linearization of system (2.9) at $E(u_0, v_0)$ is given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(\mu + A)u(x,t) - Bv(x,t), \\ \dfrac{\partial v}{\partial t} = D\Delta v + A\displaystyle\int_0^{+\infty} \prod(\theta)u(x,t-\theta)d\theta + B\int_0^{+\infty} \prod(\theta)v(x,t-\theta)d\theta - dv(x,t), \end{cases} \tag{2.10}$$

where

$$A = \frac{\beta v_0}{1+\alpha v_0},$$

and

$$B = \frac{\beta u_0}{(1+\alpha v_0)^2}.$$

Let $0 = \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary conditions, and $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\Omega)$. Let

$$X = [C^1(\Omega)]^2, \qquad \{\phi_{ij}; j = 1, 2, \ldots, \dim E(\mu_i)\},$$

be an orthogonal basis of

$$E(\mu_i),$$

and

$$X_{ij} = \{c\phi_{ij} \quad | \quad c \in \mathbb{R}^2\}.$$

Then

$$X = \bigoplus_{i=0}^{\infty} X_i,$$

and

$$X_i = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}.$$

Substituting

$$u(x,t) = e^{\lambda t}\phi(x),$$

and

$$v(x,t) = e^{\lambda t}\psi(x),$$

into system (2.10), we have the associated eigenvalue problem:

$$\begin{cases} \lambda\phi(x) = -(\mu + A)\phi(x) - B\psi(x), \\ \\ \lambda\psi(x) = D\psi''(x) + A\bar{\prod}(\lambda)\phi(x) + B\bar{\prod}(\lambda)\psi(x) - d\psi(x), \end{cases} \tag{2.11}$$

where

$$\bar{\prod}(\lambda) = \int_0^\infty e^{-\lambda\theta}\prod(\theta)d\theta,$$

represents the Laplace transform of the function $\prod(\theta)$. Following the proof of Lemma 2.2 in [24], we find that the problem (2.11) has a principal eigenvalue, denoted by λ^*, which satisfies the equation:

$$(\lambda + d + \mu_i D)(\lambda + \mu + A) - (\lambda + \mu)B\bar{\prod}(\lambda) = 0. \tag{2.12}$$

Case 01:

For $E_0 = (\frac{S}{\mu}, 0)$, we obtain:

$$A = 0,$$

and

$$B = \frac{\beta S}{\mu},$$

the principal eigenvalue λ^* satisfies the equation:

$$(\lambda + \mu)\left(\lambda + d + \mu_i D - \frac{\beta S}{\mu}\bar{\prod}(\lambda)\right) = 0. \tag{2.13}$$

Since $\lambda_1 = -\mu$ is a root of this Eq, we only consider the roots of the following equation:

$$\lambda + d + \mu_i D - \frac{\beta S}{\mu}\bar{\prod}(\lambda) = 0. \tag{2.14}$$

In fact, if λ is a root of Eq. (2.14). By contradiction, we suppose $\mathrm{Re}(\lambda) \geq 0$, then we have

$$|\bar{\prod}(\lambda)| = \left|\int_0^\infty p(\theta)e^{-\lambda\theta}e^{-\int_0^\theta \delta(\tau)d\tau}\,d\theta\right| \leq K,$$

where

$$K = \int_0^\infty \prod(\theta)d\theta,$$

and also,

$$\frac{\mu}{\beta S}|\lambda + d + \mu_i D| \geq \frac{\mu(d + \mu_i D)}{\beta S}.$$

It follows from the above inequalities and Eq. (2.14) that

$$\mu d \leq \mu(d + \mu_i D) \leq \beta SK,$$

$$\Rightarrow \frac{\beta SK}{\mu d} \geq 1$$

i.e.,

$$R_0 = \frac{\beta SK}{\mu d}.$$

So we have a **contradiction**. Then, if $R_0 < 1$, all the roots of Eq. (2.14), and therefore Eq. (2.13), have negative real parts $\mathrm{Re}(\lambda) < 0$.
Accordingly, the principal eigenvalue λ^* has a negative real part, and the uninfected steady state $E_0\left(\frac{S}{\mu}, 0\right)$ of system (2.8) is **locally asymptotically stable** if $R_0 < 1$.

Case 02:

For $E^* = (u^*, v^*)$, we obtain:

$$A = \frac{\beta v^*}{1 + \alpha v^*},$$

and

$$B = \frac{\beta u^*}{(1 + \alpha v^*)^2}.$$

The principal eigenvalue λ^* satisfies the following equation

$$(\lambda + d + \mu_i D)\left(\lambda + \mu + \frac{\beta v^*}{1 + \alpha v^*}\right) - (\lambda + \mu)\frac{\beta u^*}{(1 + \alpha v^*)^2}\bar{\prod}(\lambda) = 0. \tag{2.15}$$

Whether the value of μ equals that of $d + \mu_i D$ for some i can determine the types of the roots to Eq. (2.15).
Let consider two cases:

(i) If $\mu \neq d + \mu_i D$:

For all $i \in \mathbb{N}^+$, then $\lambda_1 = -\mu$ is not the root of Eq. (2.15). So this Eq can be written as:

$$\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*}Z - \frac{\beta u^*}{(1 + \alpha v^*)^2}\bar{\prod}(\lambda) = 0. \tag{2.16}$$

where

$$Z = \frac{\lambda + d + \mu_i D}{\lambda + \mu}.$$

If λ is a root of Eq (2.16). By absurdity, we suppose $\mathrm{Re}(\lambda) \geq 0$, then we have $\mathrm{Re}(Z) \geq 0$,

$$\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*} Z \geq |d + \mu_i D| \geq d,$$

and

$$\frac{\beta u^*}{(1 + \alpha v^*)^2} \bar{\prod} (\lambda) \leq \frac{K \beta u^*}{(1 + \alpha v^*)^2} = \frac{d}{1 + \alpha v^*}.$$

Combining the above inequalities and Eq. (2.16), we have that:

$$d \leq \frac{d}{1 + \alpha v^*},$$

$$d(1 + \alpha v^*) \leqslant d,$$

$$\Rightarrow v^* < 0,$$

which contradicts:

$$v^* > 0.$$

(ii) If $\underline{\mu = d + \mu_i D}$:

Then $\lambda_1 = -\mu$ is a root of Eq (2.16). So, we need only to consider the following

$$\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*} - \frac{\beta u^*}{(1 + \alpha v^*)^2} \bar{\prod} (\lambda) = 0. \tag{2.17}$$

If λ is a root of this Eq with $\mathrm{Re}(\lambda) \geq 0$, then it leads to the contradiction:

$$d \leq |\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*}| = |\frac{\beta u^*}{(1 + \alpha v^*)^2} \bar{\prod} (\lambda)| \leq |\frac{K \beta u^*}{(1 + \alpha v^*)^2}| = \frac{d}{1 + \alpha v^*}. \tag{2.18}$$

$$d \leq \frac{d}{1 + \alpha v^*},$$

which contradicts:

$$v^* > 0.$$

To summarize, if $R_0 > 1$.
Then all roots of Eq(2.15) have negative real parts.
Therefore, the principal eigenvalues λ^* have negative real parts, and the infected steady state $E^* = (u^*, v^*)$ of the system (2.8) is **locally asymptotically stable**.

Summarizing the above discussion, with the following conclusions:

Theorem 2.2.1 *If $R_0 < 1$, the uninfected steady state E_0 of system (2.1) is **locally asymptotically stable**; otherwise, it is unstable.*
*If $R_0 > 1$, the infected steady state E^* of system (2.1) is **locally asymptotically stable**.*

2.3 Global stability analysis of steady states:

Let's examine the global stability of the uninfected steady state E_0 and the infected steady state E^*.

Lemma 2.3.1 *There exists $\varepsilon_u > 0$ such that*

$$u(x,t) > \varepsilon_u,$$

for all $x \in \Omega$, $t > 0$.

Proof: By examining the first equation of system (2.1), we have:

$$\frac{\partial u}{\partial t} \geq S - \left(\mu + \frac{\beta}{\alpha}\right) u(x,t),$$

for each $x \in \Omega$, $t > 0$. Hence, $u(x,\cdot)$ is increasing if it is less than $\dfrac{\alpha S}{\alpha\mu + \beta}$. Moreover, solving this differential inequality for each x using the integrating factor:

$$\frac{\partial}{\partial t}(u \quad e^{(\mu + \frac{\beta}{\alpha})t}) = S \quad e^{(\mu + \frac{\beta}{\alpha})t}.$$

After the calculations, we obtain:

$$u(x,t) = \frac{\alpha S}{\mu\alpha + \beta}(1 - e^{-(\mu + \frac{\beta}{\alpha})t}) + u(x,0)e^{-(\mu + \frac{\beta}{\alpha})t},$$

$$u(x,t) \geqslant \frac{\alpha S}{\mu\alpha + \beta}(1 - e^{-(\mu + \frac{\beta}{\alpha})t}),$$

Thus, for any positive t_1, there exists $\varepsilon_u = \varepsilon_u(t_1) > 0$ such that

$$u(x,t) > \varepsilon_u,$$

for all $x \in \Omega$, $t \geq t_1$.

Theorem 2.3.1 *[27] If $R_0 \leq 1$, we have:*

$$\lim_{t \to +\infty} (u(x,t), w(x,\theta,t), v(x,t)) = (u_0, 0_{L^1}, 0),$$

*uniformely for $x \in \overline{\Omega}$, ie: the E_0 of the model (2.1-2.2) is **globally asymptotically stable**.*

Proof:
We need only to prove that the omega limit set contains only the uninfected steady state E_0. We set the function

$$H(z) = z - 1 - \ln z, \quad z \in \mathbb{R}^+$$

and consider the following Lyapunov functional

$$V(t) = \int_\Omega \left[u^0 H\left(\frac{u(x,t)}{u^0}\right) + \frac{1}{K}\int_0^\infty f(\theta) w(x,\theta,t) d\theta + \frac{1}{K} v(x,t) \right] dx,$$

where

$$f(\theta) = \int_\theta^\infty p(\tau) e^{-\int_\theta^\tau \delta(\sigma) d\sigma} d\tau.$$

Obviously, the function $f(\theta)$ is bounded and satisfies $f(0) = K$, $f(\theta) > 0$ for $0 < \theta < +\infty$, and $f'(\theta) = \delta(\theta) f(\theta) - p(\theta)$.

Next, we calculate the derivative of $V(t)$ along the solution of system (2.1-2.2), following the same approach as in the proof of lemma 9.18 in [8]:

$$\frac{dV(t)}{dt} = \int_\Omega \left[\left(1 - \frac{u^0}{u(x,t)}\right) \frac{\partial u(x,t)}{\partial t} + \frac{1}{K}\int_0^\infty f(\theta) \frac{\partial w(x,\theta,t)}{\partial t} d\theta + \frac{1}{K}\frac{\partial v(t)}{\partial t} \right] dx,$$

$$\frac{dV(t)}{dt} = \int_\Omega [\left(1 - \frac{u^0}{u(x,t)}\right)(S - \mu u(x,t) - \frac{\beta u(x,t) v(x,t)}{1 + \alpha v(x,t)}) + \frac{1}{K}\int_0^\infty f(\theta)\frac{\partial w(x,\theta,t)}{\partial t} d\theta$$
$$+ \frac{1}{K}(D\Delta v + \int_0^\infty p(\theta) w(x,\theta,t) d\theta - d v(x,t))] dx,$$

$$\frac{dV(t)}{dt} = \int_\Omega [(-\frac{\mu}{u}[(u^0 - u) + \frac{\beta u^0(x,t) v(x,t)}{\mu(1 + \alpha v(x,t))}) + \frac{1}{K}\int_0^\infty f(\theta)\frac{\partial w(x,\theta,t)}{\partial \theta} d\theta$$
$$+ \frac{1}{K}(D\Delta v + \int_0^\infty p(\theta) w(x,\theta,t) d\theta - d v(x,t))] dx,$$

$$\frac{dV(t)}{dt} = \int_\Omega [-\frac{\mu}{u}[(u^0 - u)^2 + \frac{\beta u^0(x,t) v(x,t)}{1 + \alpha v(x,t)} - \frac{\beta u(x,t) v(x,t)}{1 + \alpha v(x,t)} + \frac{1}{K}(D\Delta v + \int_0^\infty p(\theta) w(x,\theta,t) d\theta -$$
$$d v(x,t))] dx,$$

$$\frac{dV(t)}{dt} = \int_\Omega [-\frac{\mu}{u}[(u^0 - u)^2 + \frac{\beta u^0(x,t) v(x,t)}{1 + \alpha v(x,t)} - \frac{1}{K} D\Delta v - \frac{1}{K} d v(x,t))] dx - \int_\Omega E(x,t) dx.$$

Where,

$$E(x,t) = \frac{\beta u(x,t) v(x,t)}{1 + \alpha v(x,t)} + \frac{1}{K}\int_0^\infty f(\theta)(\frac{\partial w(x,\theta,t)}{\partial \theta} + \delta(\theta) w(x,\theta,t)) d\theta - \int_0^\infty p(\theta)$$
$$w(x,\theta,t) d\theta,$$

$$E(x,t) = \frac{\beta u(x,t) v(x,t)}{1 + \alpha v(x,t)} + \frac{1}{K}\int_0^\infty f(\theta)\frac{\partial w(x,\theta,t)}{\partial \theta} d\theta + \frac{1}{K}\int_0^\infty (\delta(\theta) f(\theta) - p(\theta))$$
$$w(x,\theta,t) d\theta.$$

Then, we perform integration by parts of:

$$\int_0^\infty f(\theta)\frac{\partial w(x,\theta,t)}{\partial \theta} d\theta = f(\theta) w(x,\theta,t)\Big|_{\theta=0}^{\theta=\infty} - \int_0^\infty w(x,\theta,t) f'(\theta)\, d\theta$$

$$= f(\theta)w(x,\theta,t)\Big|_{\theta=\infty} - K\frac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)} - \int_0^\infty w(x,\theta,t)f'(\theta)\,d\theta,$$

with $f(0) = K,\ f'(\theta) = \delta(\theta)f(\theta) - p(\theta)$,
We deduce:

$$E(x,t) = f(\theta)w(x,\theta,t)\Big|_{\theta=\infty},$$

which is bounded due to (2.1.2) and the boundedness of the function $f(\theta)$.
Since,

$$\frac{dV(t)}{dt} = \int_\Omega [-\frac{\mu}{u}[(u^0-u)^2 + \frac{\beta u^0(x,t)v(x,t)}{1+\alpha v(x,t)} - \frac{1}{K}D\Delta v - \frac{1}{K}dv(x,t))]dx$$

$$- \int_\Omega f(\theta)w(x,\theta,t)\Big|_{\theta=\infty} dx,$$

We have that:

$$R_0 = \frac{\beta SK}{\mu d} \Rightarrow R_0 = \frac{\beta u^0 K}{d} \Rightarrow u^0 = \frac{dR_0}{\beta K},$$

and

$$\int_\Omega \Delta v dx = 0.$$

It then follows that:

$$\frac{dV(t)}{dt} = \int_\Omega [-\frac{\mu}{u}[(u^0-u)^2 + \frac{\beta(\frac{dR_0}{\beta K})v(x,t)}{1+\alpha v(x,t)} - \frac{1}{K}dv(x,t))]dx$$

$$- \int_\Omega f(\theta)w(x,\theta,t)\Big|_{\theta=\infty} dx,$$

$$\frac{dV(t)}{dt} = -\int_\Omega \frac{\mu}{u}[(u^0-u(x,t))^2 dx + \int_\Omega \frac{d(R_0-1)-\alpha d v(x,t)}{K(1+\alpha v(x,t))}v(x,t)dx$$

$$- \int_\Omega f(\theta)w(x,\theta,t)\Big|_{\theta=\infty} dx,$$

If $R_0 \leq 1$, it's apparent that $\frac{dV(t)}{dt} \leq 0$, and equality occurs only at E_0.
We conclude that the largest invariant set

$$X = \{(u,w,v) \in \mathbb{R}^+ \times L^1_+((0,\infty),\mathbb{R}) \times \mathbb{R}^+ : \frac{dV(t)}{dt} = 0\}$$

in X_0^+ is just $\{E_0\}$. Since the attractor is compact we can use **the LaSalle's invariant principle**, E_0 exhibits **global asymptotic stability** when $R_0 \leq 1$.□

To achieve this, it's important to ensure that solutions remain positive, thus guaranteeing the finiteness of the Lyapunov functional. The following definition outlines the initial conditions necessary for our future discussion.

Definition 2.3.1 *We define the disease to be initially present if:*

$$\int_{\Omega} v(x,0)\,dx > 0.$$

Lemma 2.3.2 *Suppose the disease is initially present. If $R_0 > 1$, there exists $\varepsilon > 0$ such that*

$$w(x,\theta,t), v(x,t) > \varepsilon,$$

for all $x \in \Omega$, $t > 0$, $\theta > 0$.

Proof: Define

$$\overline{v}(t) = \int_{\Omega} v(x,t)\,dx,$$

we note that $\overline{v}(t)$ is non-negative. From Definition (2.3.1), it follows that $\overline{v}(t) > 0$. Next, we demonstrate that using $\int_{\Omega} D\Delta v\ dx = 0$, we deduce that

$$\frac{d\overline{v}(t)}{dt} = \frac{d}{dt}\int_{\Omega} v(x,t)\,dx = \int_{\Omega} \frac{\partial v}{\partial t}\,dx$$

$$= \int_{\Omega} \left(D\Delta v(x,t) + \int_0^{\infty} p(\theta) w(x,\theta,t)\,d\theta - dv(x,t) \right) dx,$$

$$\frac{d\overline{v}(t)}{dt} = \int_{\Omega} dx \int_0^{\infty} p(\theta) w(x,\theta,t) d\theta - d\overline{v}(t),$$

It follows from $v(x,t) \leq \dfrac{R_0}{\delta_{min}}$ and equation (2.9) that:

$$\frac{d\overline{v}(t)}{dt} = \int_{\Omega} dx \int_0^{\infty} \prod(\theta) \frac{\beta u(x,t-\theta) v(x,t-\theta)}{1+\alpha v(x,t-\theta)}\,d\theta - d\overline{v}(t),$$

we know that $u(x,t) > \varepsilon_u$(2.3.1), we obtain:

$$\frac{d\overline{v}(t)}{dt} \geq \int_{\Omega} dx \int_0^{\infty} \prod(\theta) \frac{\beta \varepsilon_u v(x,t-\theta)}{1+\alpha v(x,t-\theta)}\,d\theta - d\overline{v}(t),$$

$$\geq \int_{\Omega} dx \int_0^{\infty} \prod(\theta) \frac{\beta \varepsilon_u v(x,t-\theta)}{1+\alpha(\dfrac{R_0}{\delta_{min}})}\,d\theta - d\overline{v}(t),$$

$$\geq \int_0^{\infty} \prod(\theta) \frac{\beta \varepsilon_u \delta_{\min}}{\delta_{\min} + \alpha R_0} \int_{\Omega} v(x,t-\theta) dx d\theta - d\overline{v}(t),$$

$$\geq \int_0^{\infty} \prod(\theta) \frac{\beta \varepsilon_u \delta_{\min}}{\delta_{\min} + \alpha R_0} \overline{v}(t-\theta) d\theta - d\overline{v}(t),$$

since $R_0 > 1$, we have that

$$\int_0^{\infty} \prod(\theta) \frac{\beta \varepsilon_u \delta_{\min}}{\delta_{\min} + R_0}\,d\theta \geq \frac{\mu \delta_{\min} \varepsilon_u}{s(\delta_{\min} + R_0)}.$$

According to Lemma 5.1 in [14], $\overline{v}(t)$ is unbounded. Since Ω is connected, the diffusion term causes the support of $v(\cdot,t)$ to spread instantaneously to all of the interior of Ω. Thus, $v(x,t)>0$ for all $x\in \text{int}(\Omega)$, $t>0$.

We now demonstrate that v is positive on $\partial\Omega$. Suppose $v(x_0,t)=0$ for some $x_0\in\partial\Omega$, $t>0$. By Proposition 13.3 in [18] (a version of the Maximum Principle used for the boundary), it follows that $\left.\dfrac{\partial v}{\partial \nu}\right|_{x=x_0}<0$, contradicting the boundary condition. Therefore, $v(x,t)>0$ for all $x\in\partial\Omega$, $t>0$, and consequently for all $x\in\overline{\Omega}$, $t>0$.
Through the equation of $w(x,\theta,t)$ which depends on $u(x,t)$ and $v(x,t)$, we easily show that

$$w(x,\theta,t)>\varepsilon_u,$$

Lemma 2.3.3 *[3] Let q be a non-negative, bounded Lebesgue measurable function. Let z_1 and z_2 be non-zero solutions of*

$$\frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}=-q(a)z(t,a),$$

for $t\in\mathbb{R}$ and $a>0$, with $z_j(t,0)=Z_j(t)>0$ for all $t\in\mathbb{R}$, for $j=1,2$. Let

$$U(t)=\int_0^\infty \alpha(a)G\left(\frac{z_1(t,a)}{z_2(t,a)}\right)da,$$

where G is continuous and $\alpha(a)=\int_a^\infty \xi(\sigma)d\sigma$, with $\xi,\alpha\in L^1_+$. Then

$$\frac{dU}{dt}=\int_0^\infty \xi(a)\left[G\left(\frac{z_1(t,0)}{z_2(t,0)}\right)-G\left(\frac{z_1(t,a)}{z_2(t,a)}\right)\right]da.$$

Theorem 2.3.2 *Suppose the disease is initially present. If $R_0>1$, then we have that:*

$$\lim_{t\to+\infty}(u(x,t),w(x,\theta,t),v(x,t))=(u^*,w^*(\theta),v^*)$$

uniformly for $x\in\overline{\Omega}$, namely, the infected steady state E^* of model (2.1)-(2.2) is ***globally asymptotically stable****.*

Proof

We know that the infected steady state E^* of system (2.1) is locally asymptotically stable if $R_0>1$. Hereafter, employing $f(\theta)$ as defined in equation:

$$f(\theta)=\int_\theta^\infty p(\tau)e^{-\int_\theta^\tau \delta(\sigma)d\sigma}d\tau.$$

and

$$H(z)=z-1-\ln z,$$

we formulate the subsequent Lyapunov functional:

$$V(t)=\int_\Omega\left(u^*H\left(\frac{u(x,t)}{u^*}\right)+\frac{1}{K}\int_0^\infty f(\theta)w^*(\theta)H\left(\frac{w(x,\theta,t)}{w^*(\theta)}\right)d\theta\right)dx+$$

$$\frac{1}{K}\int_\Omega v^* H\left(\frac{v(x,t)}{v^*}\right)dx.$$

We now demonstrate that $\frac{dV(t)}{dt} \leq 0$ along the solution of system (2.1)-(2.2). Through direct computation, we derive:

$$\begin{aligned}\frac{dV(t)}{dt} &= \int_\Omega \left(1-\frac{u^*}{u(x,t)}\right)\left(S-\mu u(x,t)-\frac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)}\right)dx\\ &+\frac{1}{K}\int_\Omega\int_0^\infty f(\theta)\left(1-\frac{w^*(\theta)}{w(x,\theta,t)}\right)\frac{\partial}{\partial t}w(x,\theta,t)d\theta dx\\ &+\int_\Omega\left(\frac{D}{K}\left(1-\frac{v^*}{v(x,t)}\right)\Delta v-\frac{d}{K}v(x,t)+\frac{d}{K}v^*\right)dx\\ &+\frac{1}{K}\int_\Omega\left(1-\frac{v^*}{v(x,t)}\right)\int_0^\infty p(\theta)w(x,\theta,t)d\theta dx.\end{aligned}$$

Then, we use the lemma (2.3.3) , we get:

$$\begin{aligned}&\int_0^\infty f(\theta)\left(1-\frac{w^*(\theta)}{w(x,\theta,t)}\right)\frac{\partial}{\partial t}w(x,\theta,t)d\theta\\ &=-\int_0^\infty f(\theta)w^*(\theta)\left(\frac{w(x,\theta,t)}{w^*(\theta)}-1-\ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)d\theta,\\ &=-f(\theta)w^*(\theta)\left(\frac{w(\theta,t)}{w^*(\theta)}-1-\ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)\Big|_{\theta=0}^\infty+\\ &\int_0^\infty\left(\frac{w(x,\theta,t)}{w^*(\theta)}-1-\ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)(f'(\theta)w^*(\theta)+f(\theta)\frac{dw^*(\theta)}{d\theta}d\theta),\end{aligned}$$

it then follows from

$$f'(\theta)w^*(\theta)+f(\theta)\left(\frac{dw^*(\theta)}{d\theta}\right)=-p(\theta)w^*(\theta),$$

and

$$\begin{cases} f(0)=K,\\ w^*(0)=\dfrac{\beta u^*v^*}{1+\alpha v^*},\\ w(x,0,t)=\dfrac{\beta u(x,t)v(x,t)}{1+\alpha v(x,t)},\\ S=\mu u^*+\dfrac{\beta u^*v^*}{1+\alpha v^*},\\ \dfrac{\beta u^*}{1+\alpha v^*}=\dfrac{d}{K},\\ w^*(0)=\dfrac{1}{K}\displaystyle\int_0^\infty p(\theta)w^*(\theta)d\theta. \end{cases} \tag{2.19}$$

We have that:

$$\frac{dV}{dt} = -\int_\Omega \frac{\mu(u(x,t)-u^*)^2}{u(x,t)}dx - \frac{1}{K}\int_\Omega f(\theta)w^*(\theta)\left(\frac{w(\theta,t)}{w^*(\theta)} - 1 - \ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)\Big|_{\theta=\infty} dx$$
$$+\int_\Omega \left(\frac{D}{K}\left(1-\frac{v^*}{v(x,t)}\right)\Delta v + C_1(x,t)\right)dx,$$

with

$$C_1(x,t) = -\frac{u^*}{u(x,t)}\left(w^*(0) - w(x,0,t)\right) - w^*(0)\ln\frac{w(x,0,t)}{w^*(0)}$$
$$-\frac{1}{K}\int_0^\infty p(\theta)w^*(\theta)\left(\frac{w(x,\theta,t)}{w^*(\theta)} - 1 - \ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)d\theta$$
$$+\frac{1}{K}\left(1-\frac{v^*(x,t)}{v(x,t)}\right)\int_0^\infty p(\theta)w(x,\theta,t)d\theta + w^*(0) - w^*(0)\frac{v(x,t)}{v^*(x,t)}.$$

Using the equalities in (2.19) and simple calculations we have:

$$C_1(x,t) = -\frac{1}{K}\int_0^\infty p(\theta)w^*(\theta)\left(\frac{v^*}{v(x,t)}\frac{w(x,\theta,t)}{w^*(\theta)} - 1 - \ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)d\theta$$

$$+w^*(0)\left(1 - \ln\frac{w(x,0,t)}{w^*(0)} - \frac{v(x,t)}{v^*} + \frac{w(x,0,t)}{w^*(0)}\frac{u^*}{u(x,t)} - \frac{u^*}{u(x,t)}\right),$$

$$= -\frac{1}{K}\int_0^\infty p(\theta)w^*(\theta)\left(H\left(\frac{u^*}{u(x,t)}\right) + H\left(\frac{w(x,\theta,t)}{w^*(\theta)}\frac{v^*}{v(x,t)}\right)\right)d\theta$$

$$-\frac{1}{K}\int_0^\infty p(\theta)w^*(\theta)\left(H\left(\frac{v(x,t)}{v^*}\right) - H\left(\frac{w(x,0,t)}{w^*(0)}\frac{u^*}{u(x,t)}\right)\right)d\theta.$$

Due to the monotonicity of function:

$$g(z) = \frac{z}{1+\alpha z},$$

in $(0,+\infty)$, combined with Equation (2.19), we conclude that

$$\begin{cases} \dfrac{v(x,t)}{v^*} \le \dfrac{v(x,t)}{v^*}\dfrac{1+\alpha v^*}{1+\alpha v(x,t)} = \dfrac{w(x,0,t)}{w^*(0)}\dfrac{u^*}{u(x,t)} \le 1, & \text{if } v(x,t)\le v^*, \\ \dfrac{v(x,t)}{v^*} \ge \dfrac{v(x,t)}{v^*}\dfrac{1+\alpha v^*}{1+\alpha v(x,t)} = \dfrac{w(x,0,t)}{w^*(0)}\dfrac{u^*}{u(x,t)} \ge 1, & \text{if } v(x,t)\ge v^*. \end{cases} \tag{2.20}$$

Since the function $H(z)$ is monotonically decreasing in $(0,1]$ and increasing in $[1,+\infty)$, it then follows from Equation (2.20) that

$$H\left(\frac{v(x,t)}{v^*}\right) \ge H\left(\frac{w(x,0,t)}{w^*(0)}\frac{u^*}{u(x,t)}\right),$$

and therefore $C_1(x,t) \leq 0$ for all $u, w, v > 0$.
Recalling that

$$\int_\Omega \frac{\Delta v}{v}\, dx = \int_\Omega \frac{||\nabla v||^2}{v^2}\, dx,$$

we obtain:

$$\frac{dV(t)}{dt} = -\int_\Omega \frac{\mu(u(x,t)-u^*)^2}{u(x,t)} dx - \frac{Dv^*}{K}\int_\Omega \frac{||\nabla v||^2}{v^2} dx + \int_\Omega C_1(x,t) dx$$

$$-\frac{1}{K}\int_\Omega f(\theta) w^*(\theta) \left(\frac{w(\theta,t)}{w^*(\theta)} - 1 - \ln\frac{w(x,\theta,t)}{w^*(\theta)}\right)\Bigg|_{\theta=\infty} dx.$$

Given that $u^*, w^*(\theta), v^* > 0$, if $R_0 > 1$, then we observe that $\dfrac{dV(t)}{dt} \leq 0$ for all $u, w, v > 0$, with equality holding if and only if at E^*.

We conclude that the largest invariant set

$$Q = \left\{(u,w,v) \in \mathbb{R}^2_+ \times L^1_+((0,+\infty),\mathbb{R}) : \frac{dV(t)}{dt} = 0\right\},$$

in X^0_+ is the singleton $\{E^*\}$.

Since the attractor is compact we can use **LaSalle's invariant principle** , E^* is **globally asymptotically stable when** $R_0 > 1$.

2.4 Numerical simulations

We perform some numerical simulations to illustrate the theoretical results, by taking the explicit function forms of $p(\theta)$ and $\delta(\theta)$ used in [21, 5]
The infection age-specific viral production rate can be expressed as:

$$p(\theta) = \begin{cases} P_{\max}(1 - \exp^{-b(\theta - d_1)}), & \theta \geq d_1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.21}$$

where b determines how quickly the saturation level, $P_{\max}$, is achieved. The parameter d_1 signifies the delay in viral production; meaning it takes d_1 days after the initial infection for the first viral particles to be produced.
The death rate of infected cells is given by

$$\delta(\theta) = \begin{cases} \delta_0, & \theta < d_2, \\ \delta_0 + \delta_m(1 - \exp^{-c(\theta - d_2)}), & \theta \geq d_2, \end{cases} \tag{2.22}$$

where δ_0 represents the baseline death rate, $\delta_0 + \delta_m$ denotes the maximum death rate, c controls how quickly the maximum is reached, and d_2 is the delay between infection and the onset of cell-mediated killing.

We take the parameter values in (2.21) and (2.22) as follows:

$$b = 1, \quad d_1 = 0, \quad c = 0.5, \quad \delta_0 = 0.1, \quad \delta_m = 1, \quad d_2 = 0.$$

We use numerical methods to simulate the dynamics of our system. Specifically, we employ finite difference methods for discretizing the Laplacian operator and the Euler method for time integration. These methods enable us to accurately approximate spatial derivatives and efficiently propagate the system in time, infection rate, and space.

We consider the parameters in system (2.1) as follows:

$$S = 10^6, \quad \beta = 5 \times 10^{-10}, \quad \mu = 0.1, \quad \alpha = 0.002, \quad d = 5, D = 1$$

and we use a day as the unit of time.

Case $R_0 \leq 1$:

If $P_{\max} = 50, R_0 = 0.5785 < 1$, the disease-free equilibrium $E_0 = (\frac{S}{\mu}, 0, 0)$of model (2.1) is globally asymptotically stable. By Theorem (2.3.1), numerical simulation illustrates our result.

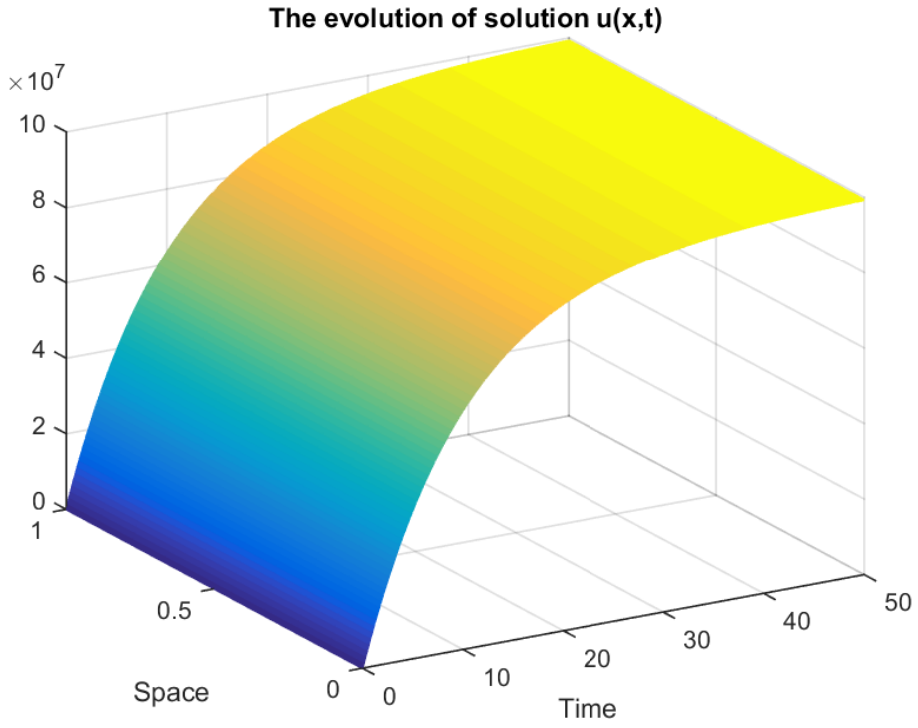

Figure 2.1 – Evolution of Uninfected Cells $u(x,t)$ with Respect to Time t and Space x for $R_0 \leq 1$

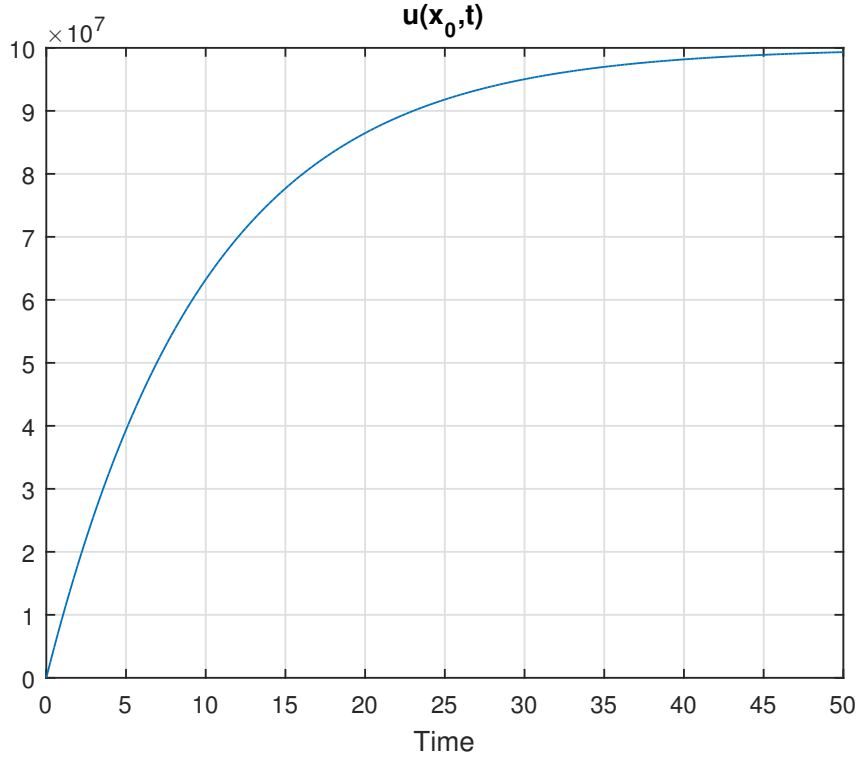

Figure 2.2 – Uninfected cells $u(x_0,t)$ at fixed space x_0 with respect to time t for $R_0 \leq 1$

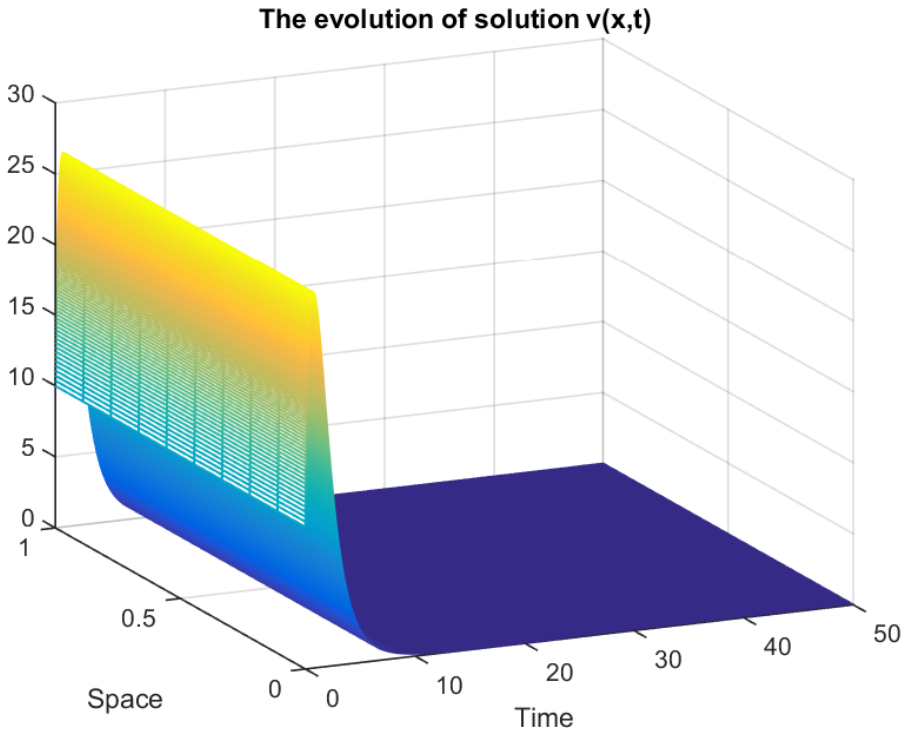

Figure 2.3 – Evolution of free virus $v(x,t)$) with respect to time t and space x for $R_0 \leq 1$

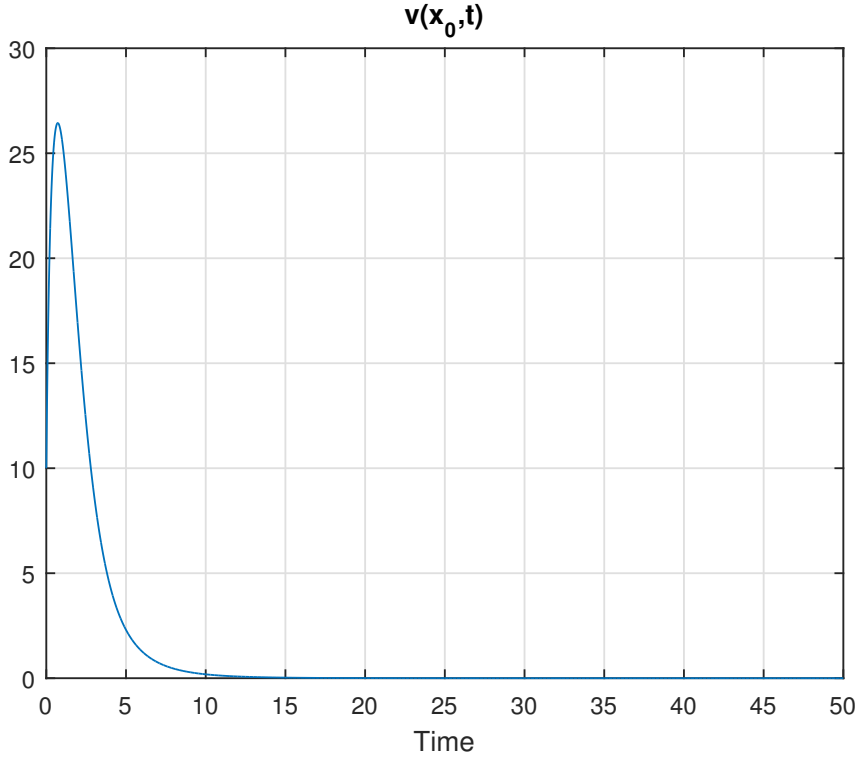

Figure 2.4 – Free virus $v(x_0,t)$ at fixed space x_0 with respect to time t for $R_0 \leq 1$.

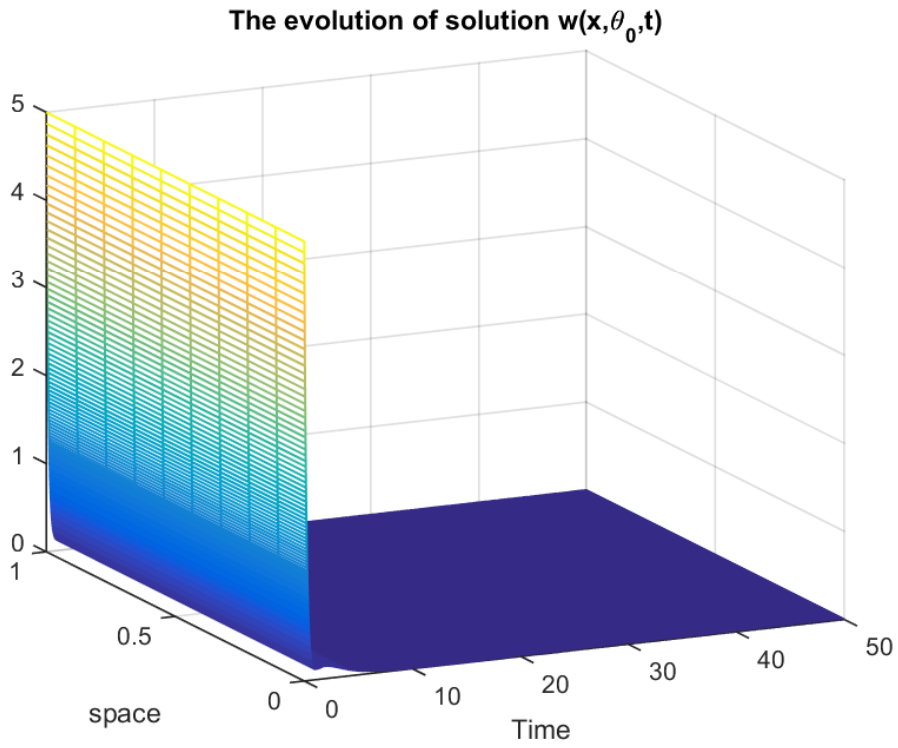

Figure 2.5 – Evolution of infected cells $w(x, \theta_0, t)$ at fixed space θ_0 with respect to time t and space x for $R_0 \leq 1$.

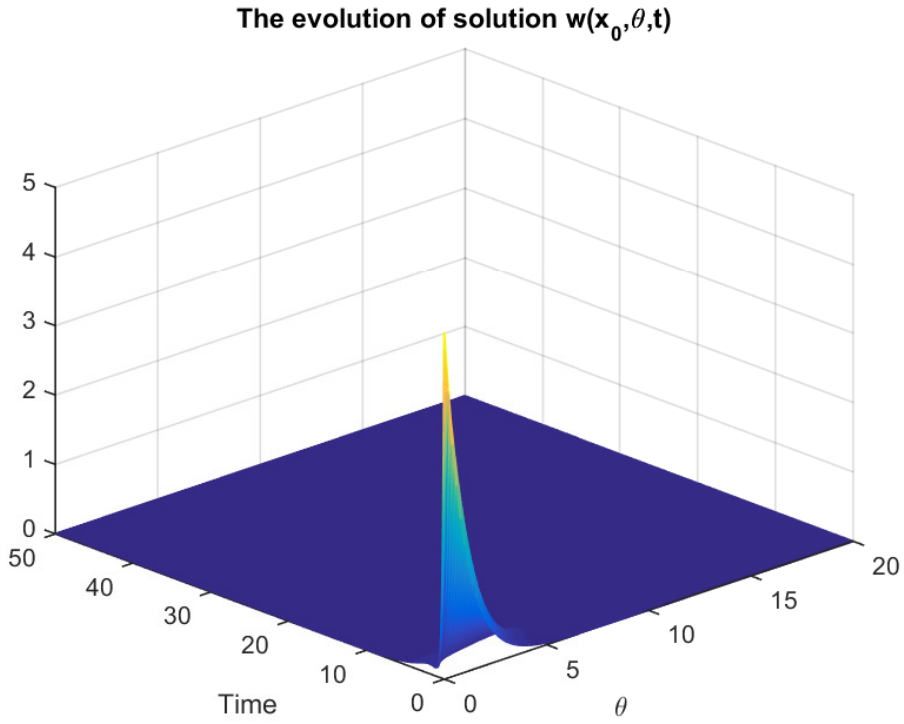

Figure 2.6 – Evolution of infected cells $w(x0, \theta, t))$ at fixed Space x_0 with Respect to Time t and age θ for $R_0 \leq 1$

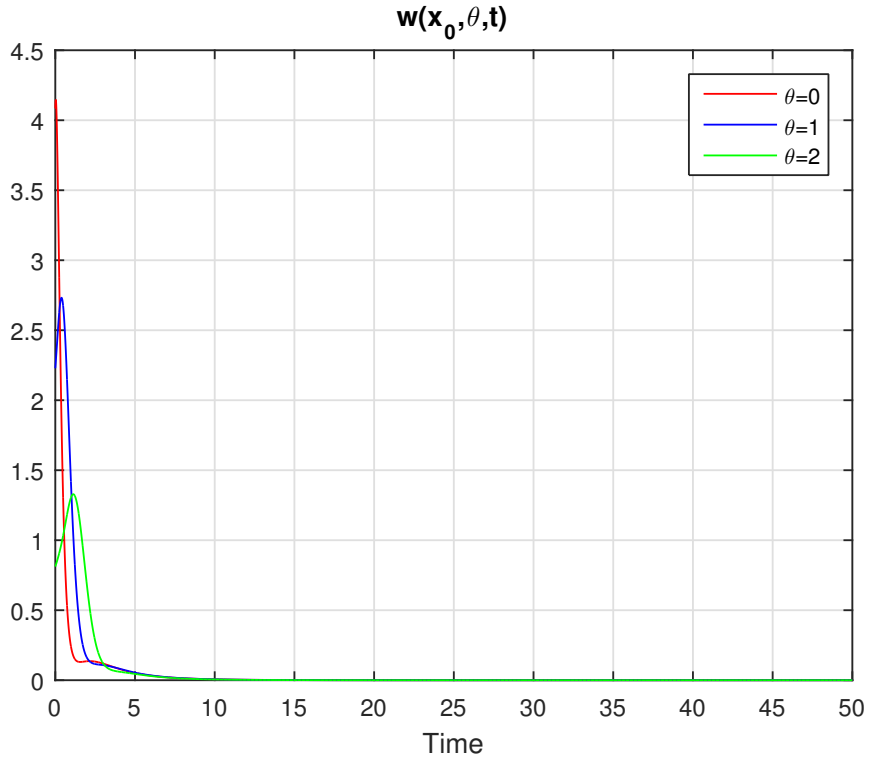

Figure 2.7 – Evolution of infected cells $w(x_0, \theta, t))$ at fixed space x_0 with respect to time t for $R_0 \leq 1$ for $\theta = 0, 1, 2$.

Case $R_0 > 1$:

If $P_{\max} = 350, R_0 = 4.0493 > 1$, the uninfected steady state $E^* = (u^*, w^*(\theta), v^*)$ of model (2.1) is globally stable.By Theorem (2.3.1), numerical simulation illustrates our result.

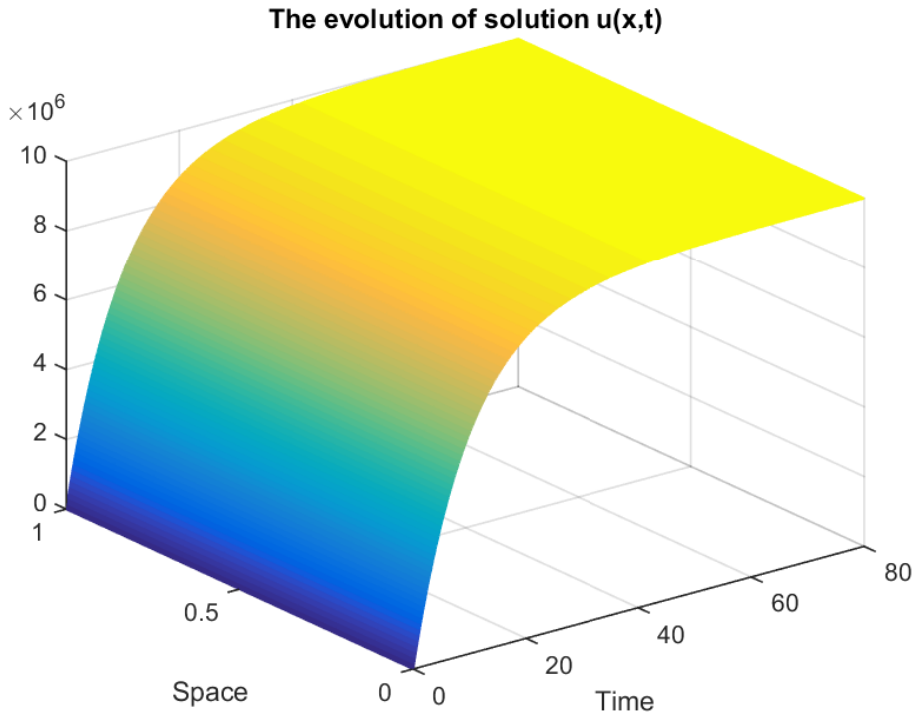

Figure 2.8 – Evolution of uninfected cells $u(x,t)$ with respect to time t and space x for $R_0 > 1$

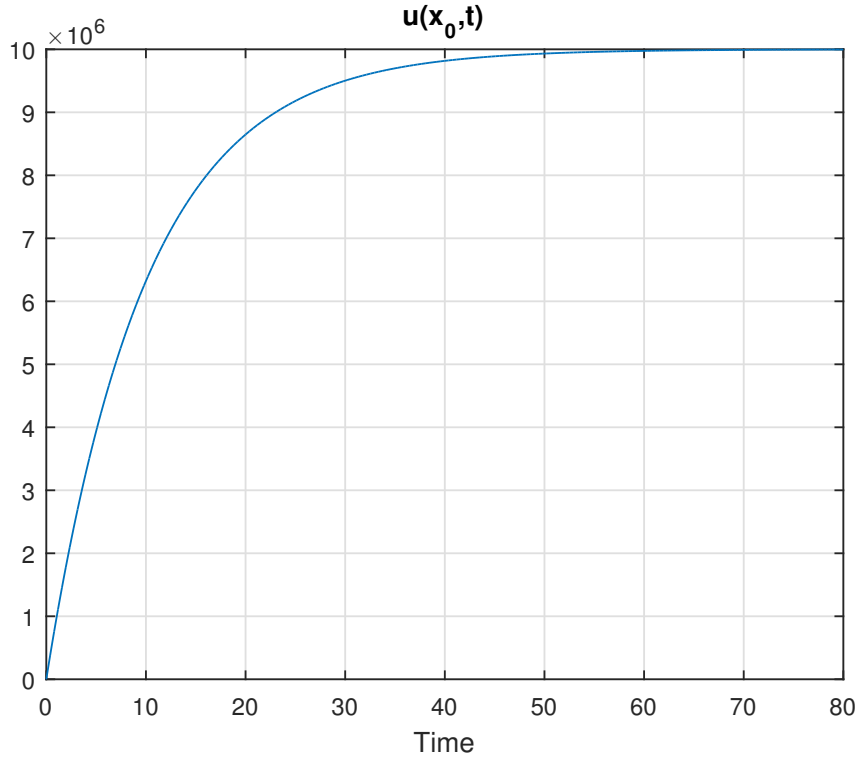

Figure 2.9 – Uninfected cells $u(x_0,t)$ at fixed space x_0 with respect to time t for $R_0 > 1$

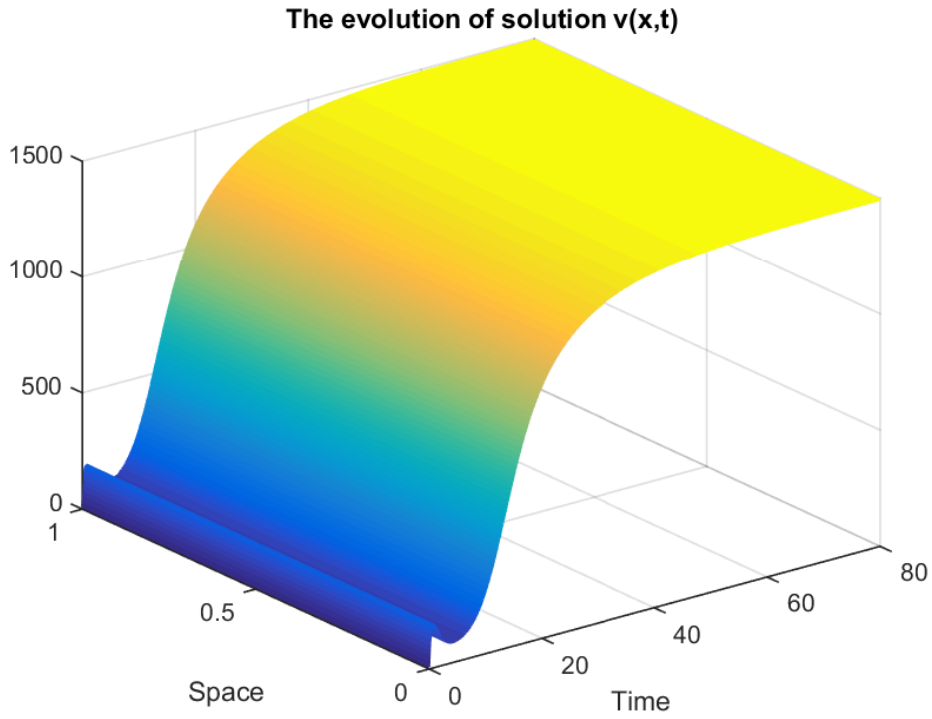

Figure 2.10 – Evolution of free virus $v(x,t)$ with respect to time t and space x for $R_0 > 1$.

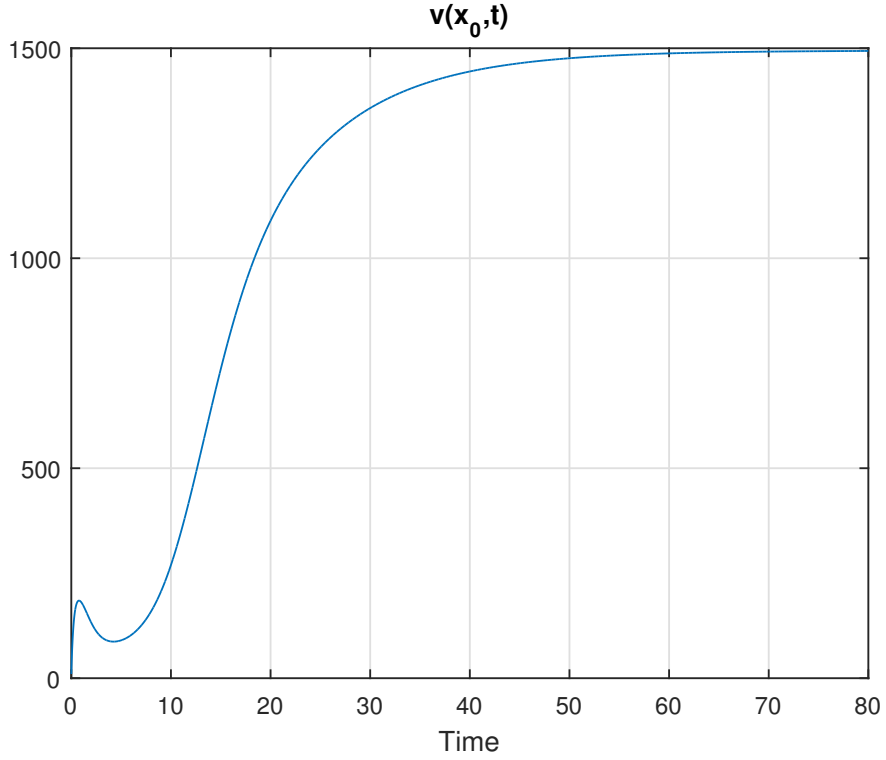

Figure 2.11 – Free virus $v(x_0,t)$ at fixed space x_0 with respect to time t for $R_0 > 1$.

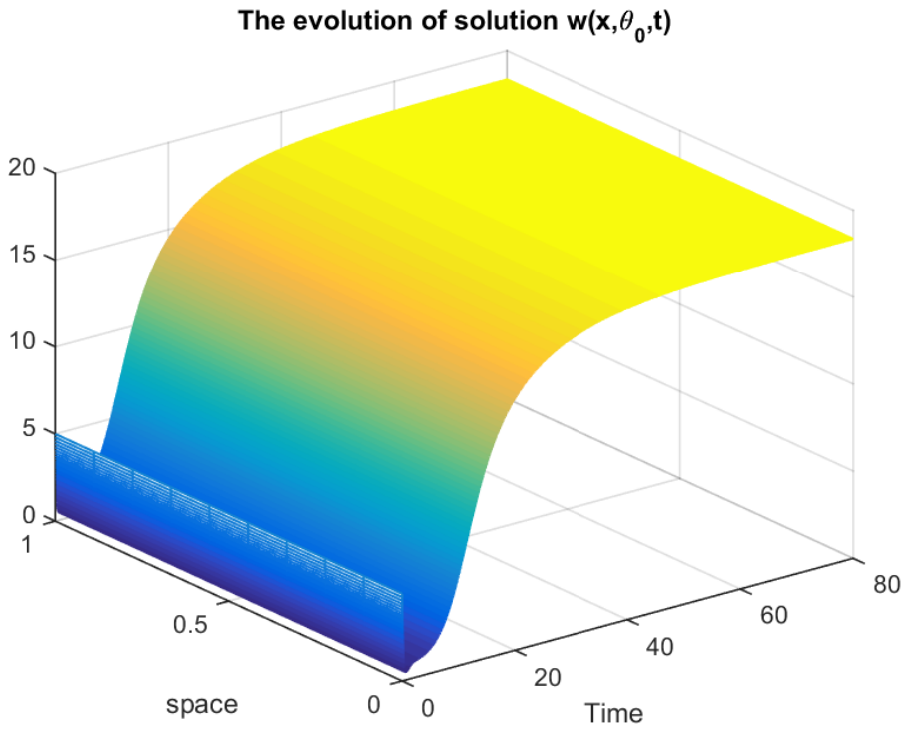

Figure 2.12 – Evolution of infected cells $w(x, \theta_0, t)$ at fixed age θ_0 with respect to time t and space x for $R_0 > 1$

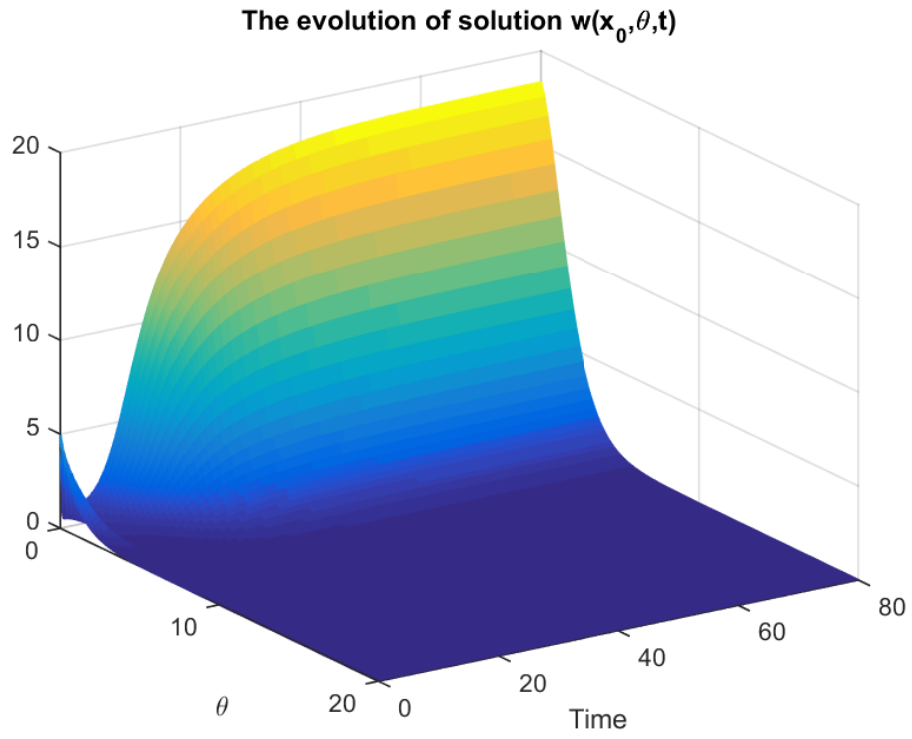

Figure 2.13 – Evolution of infected cells $w(x_0, \theta, t)$ at fixed Space x_0 with respect to time t and age θ for $R_0 > 1$

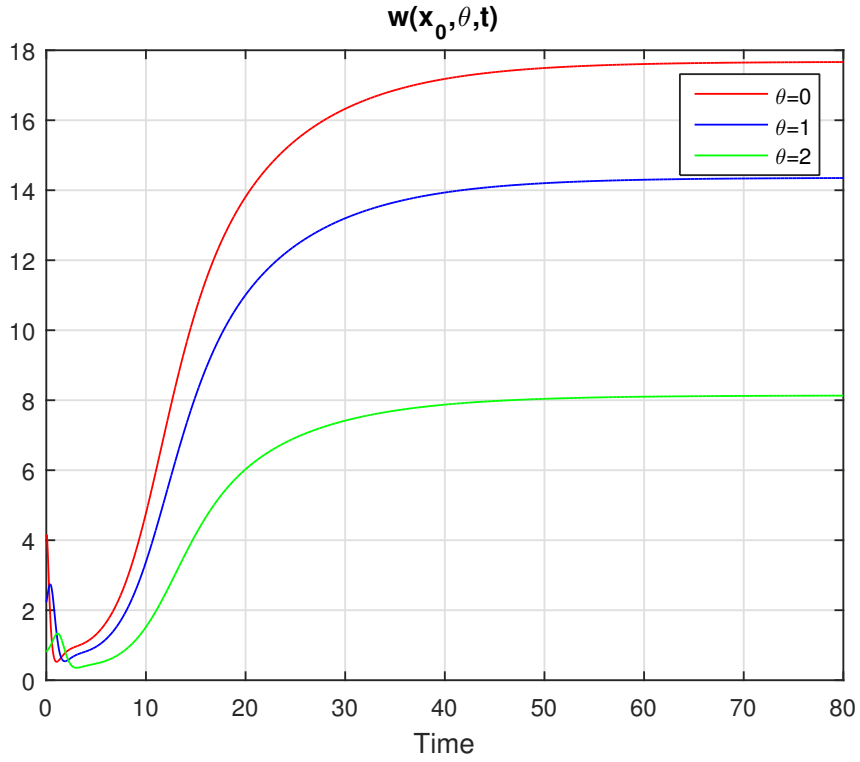

Figure 2.14 – Evolution of infected cells $w(x_0, \theta, t))$ at fixed space x_0 with respect to time t for $R_0 > 1$ for $\theta = 0, 1, 2$

Conclusion

In this thesis, we have studied the dynamics of a diffusive age-structured HBV model with saturating incidence. Firstly, we have showed the existence of solutions and aimed to prove the global stability of the steady state, which is the essential part of this work. Further, numerical simulation validate our theoretical results obtained

The theoretical epidemiology of communicable diseases has become a distinct discipline, separate from theoretical demography and mathematical ecology, and offers a vast field of application for mathematics and statistics. Beyond the theoretical results, this discipline aims to provide quantitative foundations for public health considerations regarding infectious diseases. In this direction, it is noted that the use of more realistic (and mathematically more complex) models would be desirable.

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