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University of Tlemcen
Faculty of Sciences
Department of Mathematics

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# Nonlinear elliptic equations defined by a class of monotone operators with singular nonlinearity having variable exponent

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Dissertation submitted to the Department of Mathematics as a partial fulfillment of the requirements for the degree of Master in partial differential equations

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# Dedication

I dedicate this thesis to :

My dear grandmother, who may no longer be with us, but will always remain in our hearts.

My dear parents, both of you have taught me respect, determination, courage, and so many other important values.

My dear sisters, for their constant encouragement and moral support.

To all my family and to all the people who have known how to be present when I needed them.

# Contents

# Notation

| **Symbols** | **Definition** |
|---|---|
| $x = (x_1, x_2, x_3, .., x_N)$ | Element of $\mathbb{R}^N$ |
| $\lvert x\rvert= \sqrt{x_1^2 + x_2^2 + x_3^2 + .. + x_N^2}$ | Norm of $x$ |
| $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$ | The gradient of $u$ |
| $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ | The laplacian of $u$ |
| q or $p^{'}$ | Conjugate exponent of $p$, $\frac{1}{p} + \frac{1}{q} = 1$ |
| $mes(A) = \lvert A\rvert$ | The lebesgue measure of a measurable set $A \subset \mathbb{R}^N$ |
| $\lvert\lvert u\rvert\rvert_s$ | Norm of $u$ in $L^s(\Omega)$ |
| $\lvert\lvert u\rvert\rvert_X$ | Norm of $u$ in the space $X$ |
| $X^{'}$ | The dual space of $X$ |
| $s^* = \frac{Ns}{N-s}$ | Critical sobolev exponent |
| $\partial\Omega$ | Boundary of $\Omega$ |
| $f^+$ | The positive part of $f$ |

| **Symbols** | **Definition** |
|---|---|
| $\langle\cdot,\cdot\rangle$ *or* $(\cdot,\cdot)$ | The scalar product in $\mathbb{R}^N$ |
| $C(\Omega)$ *or* $C^0(\Omega)$ | The space of continuous function on $\Omega$ |
| $C_0(\Omega)$ | The space of continuous function on $\Omega$ with compact support in $\Omega$ |
| $C^k(\Omega)$ | set of function on $\Omega$, for which the $k$-th partial derivatives are continuous |
| $C_0^k(\Omega)$ | Space of $C^k(\Omega)$ with compact support in $\Omega$ |
| $C^\infty(\Omega)$ | Space of infinitely differentiable functions on $\Omega$ |
| $C_0^\infty(\Omega)$ | Space of $C^\infty(\Omega)$ with compact support in $\Omega$ |
| $L^p(\Omega)$ | $\left\{u:\Omega\to\mathbb{R}^N \mid u \textit{ measurable }, \int_\Omega \lvert u\rvert^p<\infty\right\}$; $1\le p<\infty$ |
| $L^\infty(\Omega)$ | $\{u:\Omega\to\mathbb{R}^N \mid u \textit{ measurable and} \exists C$, such that $\lvert u\rvert\le C, a.e. x\in\Omega\}$ ; $1\le p<\infty$ |
| $L^q(\Omega)$ | The dual space of $L^p(\Omega)$ |
| $W^{k,p}(\Omega)$ | Sobolev space , with weak derivatives up to order $k$ in $L^p(\Omega)$ |
| $W_0^{k,p}(\Omega)$ | Sobolev space , with zero trace |
| $W^{-k,q}(\Omega)$ | Dual space of $W_0^{k,p}(\Omega)$ |
| supp$(\phi)$ | Support of $\phi$ defined by supp$(\phi)=\overline{\{\mathbf{x}\in\mathbb{R}^n \mid \phi(\mathbf{x})\neq 0\}}$. |

# Intoduction

This master's thesis focuses on the study of Nonlinear Elliptic equations defined by a class of monotone operators with singular nonlinearity having variable exponent, where the function $f$ is in $L^m(\Omega)$.
To solve this type of problem, we proceed by approximation, reducing it to suitable variational framework, where we can sometimes demonstrate the existence and regularity of solutions for these approximation problems, which are preserved when passing to limits.

In the first chapter, we provide a review of $L^p$ space and Sobolev space, which will be of great utility in this work, along with some theorems on existence and uniqueness , such as the Minty-Browder theorem and the Schauder fixed point theorem for nonlinear case.

In the second chapter, we study the existence of the solution to Nonlinear problem with a class of monotone operator of the form :

$$\begin{cases} -div(a(x, \nabla u)) + \lambda |u|^{p-2}\ u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega$ is an open bounded set in $\mathbb{R}^N$ and $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a function, $\lambda > 0$, and $f \in W^{-1,q}(\Omega)$, we assume that there exist two constants $b_1, b_2 \geq 0$, and $\alpha, \beta$ with $0 \leq \alpha \leq \min\{1, p-1\}$ and $\max\{p, 2\} \leq \beta < \infty$. $a$ fulfills the standards for both continuity and monotonicity following :

$$a(y, 0) = 0$$

$$|a(y, \xi_1) - a(y, \xi_2)| \leq b_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha$$

$$b_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \leq (a(y, \xi_1) - a(y, \xi_2)).(\xi_1 - \xi_2)$$

for a.e y $\in \mathbb{R}^N$ and $\xi_1, \xi_2 \in \mathbb{R}^N$.

In the third chapter, we study the existence and regularity of solutions to a nonlinear problem involving monotone operators and a singular nonlinearity having variable exponent

$$\begin{cases} -Au = \frac{f}{u^{\gamma(x)}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where

$$Au = div(a(x, \nabla u)) - \mu |u|^{p-2}\ u$$

where $\Omega$ is a bounded open set in $\mathbb{R}^N$ , $1 < p < N$ ,$\gamma(x) > 0$ is assumed to be a regular function, say for example $\gamma(x) \in C(\overline{\Omega})$, f is a non-negative function belonging to a suitable Lebesgue space $L^m(\Omega)$.

By approximation methods , we obtain the existence and regularity of positive solution to the considered problem .

The research conducted in [5] , delved into the case with specific parameters $p = 2$, $\mu = 0$ and $\gamma(x) = \gamma > 0$ . The authors proved various existence and regularity outcomes , particularly focusing on different values of $\gamma$(specifically , $\gamma = 1$, $\gamma < 1$ , $\gamma < 1$ , and on the summability characteristics of $f$ . Moreover [4] contributes a result on $G$-convergence , which was subsequently extended to nonlinear setting in [9] , and to case involving the anisotropic operator $L$ in [12] .

In [8] , the authors examine a semilinear problem where $p = 2$ and $\mu = 0$ featuring a singular nonlinearity characterized by a variable exponent. They establish the existence and regularity of solution contingent upon certain conditions regarding the behavior of the function $\gamma(x)$ in the vicinity of the boundary of $\Omega$ .

In [17] , Miri achieved the existence and regularity of solution to a problem featuring singular nonlinearity with a variable exponent , where the differential operator is presumed to be anisotropic and $\mu = 0$ . Here $\gamma(x) > 0$ represents a smooth function exhibiting favorable in properties in the vicinity of $\partial\Omega$.

For insights into the interplay between a singular nonlinearity and the fractional Laplacian , we refer readers to woks [3] and [7] . Additionally , for examinations regarding the interaction between the Hardy potential and the singular term $u^{-\gamma}$ , we direct readers to [1].

# Chapter 1

# Background Material

In this chapter, we recall some Theorems and Definition that will be useful to us in the following chapters , as echoed by several seminal works such as [6] , [2] , [10]

## 1.1 Weak Convergence

**Definition 1.1.1** (**weak convergence** ). *Let $E$ be a Banach space , and $E^*$ is the dual space and $< .,. >$ the duality Bracket on $E \times E^*$ .*
*We say that the sequence $(x_h)_h$ in $E$ weakly convergence to $x \in E$ if :*

$$< x^*, x_h > \to < x^*, x > \quad \forall x^* \in E^* \quad \forall h \in \mathbb{N}$$

*and we write :*

$$x_h \underset{h \to +\infty}{\rightharpoonup} x \text{ weakly in } E$$

**Theorem 1.1.** *Let $E$ be a banach space , $E^*$ is the dual space of $E$, let $(x_h)_h$, and $(x_h^*)_h$ be two sequence in $E$ and $E^*$ respectively .*

- *if $x_h \underset{h \to +\infty}{\rightharpoonup} x$ (weakly in $E$ ), we have :*

$$\begin{cases} \exists k > 0, \forall h \in \mathbb{N} : ||x_h||_E \leq K \\ \quad ||x||_E \leq \lim_{h \to +\infty} inf \, ||x_h||_E \end{cases}$$

- *if $x_h \underset{h \to +\infty}{\to} x$ (Strongly in $E$ ) , so we have : $x_h \underset{h \to +\infty}{\rightharpoonup} x$ weakly in $E$*

**Definition 1.1.2** (**Reflexive space** ). *Let $E$ be a Banach space , and $E^*$ its dual space with the norm*

$$||f||_{E^*} = \sup_{\substack{x \in E \\ ||x||_E \leq 1}} |\langle f, x \rangle|$$

*The bidual $E^{**}$ is the dual of $E^*$ , $g \in E^{**}$ with the norm :*

$$||g||_{E^{**}} = \sup_{\substack{f \in E^* \\ ||f||_{E^*} \leq 1}} |\langle g, f \rangle|$$

*We define a canonical injection $J : E \to E^{**}$ as follows : given $x \in E$ fixed , $f \mapsto \langle f, x \rangle$ from $E^*$ to $\mathbb{R}$ constitutes a continuous linear form on $E^*$ , i.e an element of $E^{**}$ denoted by $Jx$.Thus*

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E} \qquad \forall x \in E, \forall f \in E^*$$

*J is an isometry , that's means* $||Jx||_{E^{**}} = ||x||_E \quad \forall x \in E$
*and we have that J is linear , in fact*

$$||Jx||_{E^{**}} = \sup_{\substack{f\in E^* \\ ||f||_{E^*}\leq 1}} |\langle Jx, f\rangle| = \sup_{\substack{f\in E^* \\ ||f||^*_E\leq 1}} |\langle f, x\rangle| = ||x||_E$$

*When J is surjective , we say that E is a reflexive space .*

**Definition 1.1.3 (Separable space ).** *The Banach space E is separable if there exists a subset D of E that is countable and dense E.*

**Theorem 1.2.** *Let E be a reflexive Banach space , and let* $(x_h)_h$ *be a bounded sequences in E so there exists a subsequence* $(x_{\sigma(h)})$ *of* $(x_h)_h$ *, and* $x \in E$ *, such that*

$$x_{\sigma(h)} \underset{h\to+\infty}{\rightharpoonup} x \text{ weakly in } E$$

*If every subsequence converges weakly to the same limit x, then :*

$$x_h \underset{h\to+\infty}{\rightharpoonup} x \text{ weakly in } E.$$

## 1.2 The $L^p$ spaces

**Definition 1.2.1.** *Let* $\Omega$ *be a bounded open set in* $\mathbb{R}^N$ *, and* $p \in \mathbb{R}$ *with* $1 \leq p < +\infty$ *, The space* $L^p(\Omega)$ *consists of equivalence classes of measurable functions* $f : \Omega \to \mathbb{R}^N$ *, such that :*

- *if* $1 \leq p < \infty$ *:*
$$\int_\Omega |f(x)|^p\, dx < +\infty$$
*we note the p-norm :* $\|\mathbf{f}\|_{L^p(\Omega;\mathbb{R}^N)} = (\int_\Omega |f(x)|^p\, dx)^{1/p} < +\infty$
- *if* $p = \infty$ *we have*
$$||f||_{L^\infty(\Omega;\mathbb{R}^N)} = inf\{M \geq 0 : |f(x)| \leq M, almost\ everywhere\ on\ \Omega\}$$

**Definition 1.2.2.** *let* $1 \leq p \leq +\infty$ *, we define the conjugate* $p'$ *of* $p$ *by :*

$$\frac{1}{p'} + \frac{1}{p} = 1 \qquad if\ 1 < p < \infty$$

- *if* $p = 1$ *we have* $p' = \infty$
- *if* $p = \infty$ *we have* $p' = 1$

**Remark 1.1.** *Let* $1 \leq p < \infty$ *, The dual space of* $L^p(\Omega;\mathbb{R}^N)$ *is* $L^q(\Omega;\mathbb{R}^N)$ *, such that* $\frac{1}{q} + \frac{1}{p} = 1$

**Theorem 1.3 (Dominated Convergence).** *Let* $(f_n)_{n\in N}$ *be a sequence of measurable functions on a measured space* $(E, A, \mu)$*, with values in the set of real or complex numbers when :*

- *the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to a function f*

- *There exists an integrable function g such that:*

$$\forall n \in N \ \ \text{almost everywhere } x \in E \ \ |f(x)| \leq g(x)$$

*Then f is integrable and*

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0$$

**Theorem 1.4** (**Green′s Formula**). *Consider $\Omega$ a bounded regular open set in $R^N$ of boundary $\Gamma$ , Let u and v be functions mapping from $\Omega$ to $\mathbb{R}$ , such that $\forall u \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$ then :*

$$\int_\Omega (\Delta u) v \, dx = \int_\Gamma v(\nabla u) \cdot \nu \, d\Gamma - \int_\Omega \nabla u \cdot \nabla v \, dx \tag{1.1}$$

*where $\nu$ represents the unit outward normal vector for $\Gamma$, and $d\Gamma$ represents the surface measure on $\Gamma$.*

**Theorem 1.5** ( **Holder′s inequality**). *If $\Omega$ is an open subset of $\mathbb{R}^N$ , f,g : $\Omega \to \mathbb{R}$ such that $f \in L^p(\Omega; R^N)$ and $g \in L^q(\Omega; R^N)$ with $1 \leq p < +\infty$ , then we have :*

$$\int_\Omega |f(x)g(x)| \, dx \leq ||f||_{L^p(\Omega;R^N)} ||g||_{L^q(\Omega;R^N)}$$

**Theorem 1.6** (**Holder Inequality for Three Functions** ). *Let $f, g, h : X \to \mathbb{R}$ (or $\mathbb{C}$), where X is a measurable set equipped with a measure $\mu$. For three exponents $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, the Hölder inequality states:*

$$\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r,$$

**Theorem 1.7** (**The reverse Holder inequality**). *let $f$ , $g : \mathbb{R} \to \mathbb{R}$ , and $a$ , $b \in \mathbb{R}$ with $a < b$ , we suppose that $f.g \in L^1([a,b])$ , such that $0 < p < 1$ with $q = \frac{p}{p-1}$*

$$\left(\int_a^b |f(x)|^p \, dx\right)^p \left(\int_a^b |g(x)|^q \, dx\right)^q \leq \int_a^b |f(x)g(x)| \, dx \tag{1.2}$$

*Proof.* see [10] □

**Theorem 1.8** (**Young′s inequality**). *If a and b are nonnegative real numbers , p and q are real numbers greater than 1 , with $\frac{1}{q} + \frac{1}{p} = 1$ , then we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Theorem 1.9** (**Minkowski inequality**). *Let $1 \leq p < \infty$ , and let f and g be element of $L^p(\Omega)$ , then $f + g \in L^p(\Omega)$ , and we have :*

$$\left(\int |f+g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int |g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

### 1.2.1 Weak convergence in $L^p(\Omega)$ space

**Theorem 1.10.** *Let $(f_n)$ be a bounded sequence in $L^p(\Omega)$. Then, it admits a subsequence that converges weakly. The concept of weak convergence in $L^p(\Omega; \mathbb{R}^N)$ is defined as follows:*

- *If* $1 < p < +\infty$, *then* $f_h \underset{h\to+\infty}{\rightharpoonup} f$ *weakly in* $L^p(\Omega;\mathbb{R}^N)$ *so:*

$$\int_\Omega \langle f_h(x), g(x)\rangle\, dx \underset{h\to+\infty}{\to} \int_\Omega \langle f(x), g(x)\rangle\, dx \quad \forall g \in L^q(\Omega;\mathbb{R}^N)$$

- *If* $p = +\infty$, *then* $f_h \underset{h\to+\infty}{\rightharpoonup} f$ *weakl * in* $L^\infty(\Omega;\mathbb{R}^N)$ *so:*

$$\int_\Omega \langle f_h(x), g(x)\rangle\, dx \underset{h\to+\infty}{\to} \int_\Omega \langle f(x), g(x)\rangle\, dx \quad \forall g \in L^1(\Omega;\mathbb{R}^N)$$

**Theorem 1.11.** *The space* $L^p(\Omega;\mathbb{R}^N)$ *is reflexive for* $1 < p < +\infty$. *Furthermore,* $L^2(\Omega;\mathbb{R}^N)$ *is a Hilbert space with the scalar product defined by:*

$$\langle f, g\rangle_{L^2(\Omega;\mathbb{R}^N)} = \int_\Omega \langle f(x), g(x)\rangle\, dx.$$

## 1.3 Sobolev space

**Definition 1.3.1.** *Let's consider* $\Omega$ *as an open set in* $\mathbb{R}^N$ *and* $1 \leq p \leq +\infty$ . *The Sobolev space* $W^{1,p}(\Omega)$ *is defined as:*

$$W^{1,p}(\Omega) = u \in L^p(\Omega) : \nabla u \in L^p(\Omega;\mathbb{R}^N),$$

*where* $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$ *represents the first derivative in the sense of distributions of the real-valued function* $u$.

*In this space, we define the following norm :*

$$\|u\|_{W^{1,p}(\Omega)} = \{\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}\}$$

*or sometimes an equivalent norm:*

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)}\right)^{1/p} \quad if\ ( \quad 1 \leq p < +\infty).$$

**Definition 1.3.2.** *Let* $1 \leq p < +\infty$. *The space* $W_0^{1,p}(\Omega)$ *is defined as the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p}(\Omega)$. *The dual space of* $W_0^{1,p}(\Omega)$ *is denoted by* $W^{-1,q}(\Omega)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

**Remark 1.2.** *If* $p = 2$, *the space* $W^{1,2}(\Omega)$ *is denoted by* $H^{1,2}(\Omega)$ *or simply* $H^1(\Omega)$. *Similarly,* $W_0^{1,2}(\Omega)$ *is denoted by* $H_0^{1,2}(\Omega)$ *or simply* $H_0^1(\Omega)$.

**proposition 1.1.** *[2]*

1. *The space* $W^{1,p}(\Omega)$ *is a Banach space for* $1 \leq p \leq +\infty$.
2. *The space* $W^{1,p}(\Omega)$ *is a reflexive space for* $1 < p < +\infty$.
3. *The space* $W^{1,p}(\Omega)$ *is a separable space for* $1 \leq p < +\infty$.
4. *The space* $W_0^{1,p}(\Omega)$ *is a separable Banach space and is also reflexive for* $1 < p < +\infty$.
5. *The spaces* $H^1(\Omega)$ *and* $H_0^1(\Omega)$ *are Hilbert spaces equipped with the following inner product:*

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2(\Omega)}.$$

**Theorem 1.12 (Poincaré inequality).** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Then there exists a constant $K(\Omega)$, depending only on $\Omega$ and $p$ , then $\forall u \in W_0^{1,p}$ we have*

$$||u||_{L^p(\Omega)} \leq K(\Omega)||\nabla u||_{L^p(\Omega;\mathbb{R}^N)} \tag{1.3}$$

**Theorem 1.13 (Sobolev inequality).** *let $\Omega$ be a regular open of $\mathbb{R}^N$ and $1 \leq p < \infty$ , so there exists a constant $K$ , depending only on $N$ and $p$ , then $\forall u \in W_0^{1,p}$*

$$||u||_{L^{p^*}(\Omega)} \leq K(p,N)||\nabla u||_{L^p(\Omega;\mathbb{R}^N)}$$

**Remark 1.3.** *[2]*
*we have $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}$ is a norm on $W_0^{1,p}(\Omega)$, denoted by $\|u\|_{W_0^{1,p}(\Omega)}$, which is equivalent to the norm $\|u\|_{W^{1,p}(\Omega)}$.*

**Definition 1.3.3 (Compact Operator).** *Let $E$ and $F$ be two Banach spaces, and let $A : E \to F$ be a continuous operator (not necessarily linear). We say that $A$ is a compact operator if the image of every bounded set in $E$ under $A$ is relatively compact in $F$. In other words, if $(u_n)_n \subset E$ is a bounded sequence, then the sequence $(v_n = A(u_n))_n \subset F$ has a convergent subsequence in $F$.*

## 1.4 Embeddings Theorem

**Theorem 1.14 (Compact Embedding).** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$.*

- *if $1 \leq p < +\infty$ , then $W^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall q \in [1, \frac{Np}{N-p}]$ with compact embedding for $q \in [1, \frac{Np}{N-p}[$.*
- *if $p = N$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall q \in [1, +\infty[$ with compact embedding.*
- *if $p > N$, then $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ with compact embedding.*

**Remark 1.4.** *Compact embedding allows us to pass from weak convergence to strong convergence as follows: Let $u_h \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

- *if $1 \leq p < +\infty$ , then $u_{\sigma(h)} \to u$ strongly in $L^q(\Omega)$, $1 \leq p < \frac{Np}{N-p}$*
- *if $p = N$, then $u_{\sigma(h)} \to u$ strongly in $L^q(\Omega)$, $1 \leq p < +\infty$*
- *if $p > N$, then $u_{\sigma(h)} \to u$ strongly in $C(\overline{\Omega})$.*

**proposition 1.2.** *Let $T : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega)$ be an operator.*

- *The operator $T$ is called monotone if*

$$\langle Tu - Tv, u - v\rangle \ \geq 0 \qquad \forall u, v \in W_0^{1,p}(\Omega) \tag{1.4}$$

- *The operator $T$ is called strictly monotone if*

$$\langle Tu - Tv, u - v\rangle \ > 0 \qquad u \neq v \tag{1.5}$$

- *The operator $T$ is hemicontinuous*

$$\lim_{t\to 0} < T(u + tv), w > = < Tu, w > \qquad \forall u, v, w \in W_0^{1,p}(\Omega) \tag{1.6}$$

- *The operator $T$ is coercive*

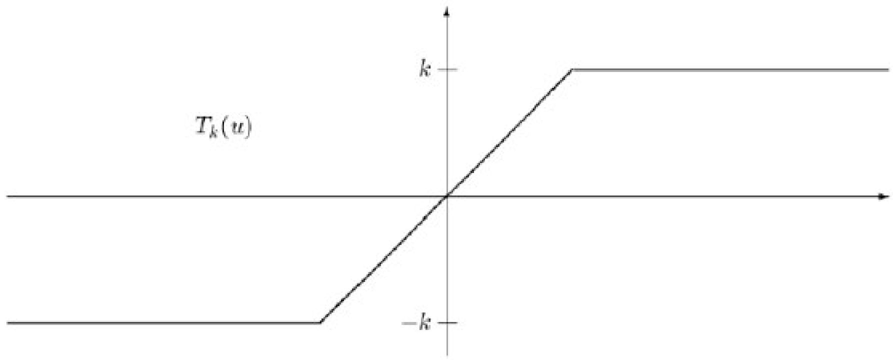
$$\lim_{||u||\to\infty} \frac{\langle Tu, u\rangle}{||u||} = +\infty \tag{1.7}$$

## 1.5 Concept of Truncation

Truncation , within the realm of mathematical PDEs ( Partial Differential Equations ) , typically involves approximating a function or an infinite series by keeping only a finite number of its terms , it often relies on the utilization of two functions $T_k(s)$ and $G_k(s)$ , where $k > 0$ , defined as follows :

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ k\frac{s}{|s|} & \text{if } |s| \ge k \end{cases}$$

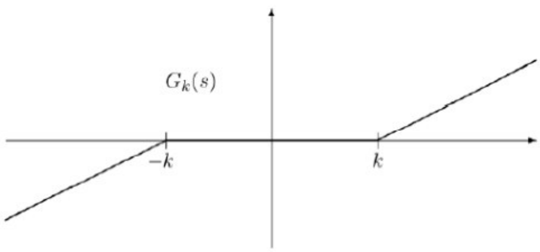
Figure 1.1: Truncature




and

$$G_k(s) \;=\; s - T_k(s)$$

Figure 1.2: function $G_k(s)$




**Lemma 1.1.** • *if* $1 \le p < 2$ *, There exists a constant* $c > 0$ *such that for real numbers a and b , we have :*

$$\left||a|^{p-2}\, a - |b|^{p-2}\, b\right| \le c|a-b|^{p-1} \tag{1.8}$$

• *if* $2 \le p < +\infty$ *, There exists a constant* $c > 0$ *such that for real numbers a and b, we have :*

$$\left||a|^{p-2}\, a - |b|^{p-2}\, b\right| \le c\,[|a| - |b|]^{p-2}\,|a-b| \tag{1.9}$$

*Proof.* see [6] □

**Lemma 1.2.** • *if* $1 \leq p < 2$ , *There exists a constant* $c \geqslant 0$ *such that for real numbers a and b , we have :*

$$c_0(p)\,(|a|+|b|)^{p-2}\,|a-b|^2 \leq (a-b)\left[|a|^{p-2}\,a-|b|^{p-2}\,b\right] \tag{1.10}$$

- *if* $2 \leq p < +\infty$ , *There exists a constant* $c \geqslant 0$ *such that for real numbers a and b , we have :*

$$c_0(p)\,\,|a-b|^p \leq (a-b)\left[|a|^{p-2}\,a-|b|^{p-2}\,b\right] \tag{1.11}$$

*Proof.* see [6] □

## 1.6 Existence theorem

**Definition 1.6.1** (**bilinear form**). *E is a vector space . Consider the function* $f : E \times E \to \mathbb{R}$. *We define a as a bilinear form on E if , For any fixed* $u \in E$ , *the following mappings are both linear :*

1. $f(u,.) : v \in E \to f(u,v) \in \mathbb{R}$
2. $f(.,u) : v \in E \to f(v,u) \in \mathbb{R}$

**Definition 1.6.2** (**Coercive bilinear form**). *[6] Let V be a Hilbert space, and let f be a bilinear form on V. We say that f is coercive on V if there exists a constant* $\alpha > 0$ *such that*

$$f(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in V.$$

**Theorem 1.15** (**Minty – Browder**). *Let X be a Banach space and let* $T : X \to X^*$, *be everywhere defined* $(i.e, D(T) = X)$ , *monotone and hemicontinuous . Then T is maximal monotone . In addition , if X is reflexive and T is coercive , i.e ,*

$$\lim_{||x||_X \to +\infty} \frac{\langle Tx, x\rangle_{X,X^*}}{||x||_X} = +\infty,$$

*then* $Img(T) = X^*$

*Proof.* see [14] □

**Theorem 1.16** (**Schauder fixed point**). *[15] Assume that K is a closed convex set of a Banach space X. Let S be a continuous and compact mapping from K into itself. Then S has a fixed point in K.*

# Chapter 2

# Nonlinear elliptic equations defined by a class of monotone operators

## 2.1 Introduction

In this chapter we show the existence and uniqueness of the solution of the following problem :

$$\begin{cases} -div(a(x, \nabla u)) + \lambda |u|^{p-2}\ u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

Let $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a function , $\lambda > 0$ , and $f \in W^{-1,q}(\Omega)$ , we suppose that there exists two positives constants $b_1, b_2$,$1 < p < \infty$, and another two constants $\alpha \quad and \quad \beta$ with :

- $0 \leq \alpha \leq min\{1, p-1\}$
- $max\{p, 2\} \leq \beta < +\infty$

such that for every y $\in \mathbb{R}^N$ and $\xi_1, \xi_2 \in \mathbb{R}^N$, $a(.,.)$ fulfills the standards for both continuity and monotonicity following : for a.e y $\in \mathbb{R}^N$ and $\xi_1, \xi_2 \in \mathbb{R}^N$

$$a(y, 0) = 0 \tag{2.2}$$

$$|a(y, \xi_1) - a(y, \xi_2)| \leq b_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha \tag{2.3}$$

$$b_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \leq (a(y, \xi_1) - a(y, \xi_2)).(\xi_1 - \xi_2) \tag{2.4}$$

for the function $a(.,.)$ we invite the reader to see the work [13]

**Definition 2.1.1.** *We say that $u \in W_0^{1,p}(\Omega)$ is an energy solution of 2.1 if and only if :*

$$\int_\Omega a(x, \nabla u) \times \nabla\phi\, dx + \int_\Omega \lambda |u|^{p-2}\ u \times \phi\, dx = \int_\Omega f\phi\, dx, \forall \phi \in C_0^1(\Omega)$$

**Theorem 2.1.** *Let $T$ be the following operator*

$$T : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$$
$$u \longmapsto T(u) = -div(a(x, \nabla u)) + \lambda |u|^{p-2}\ u.$$

*we get the equation 2.1 is equivalent to the operator equation*

$$T(u) = \ f, \quad u \in W_0^{1,p}(\Omega).$$

*Proof.* see [13] □

## 2.2 Existence Result

**Theorem 2.2.** *The problem* (2.1) *has a unique solution* $u \in W_0^{1,p}(\Omega)$.

*Proof.* We will show that T is strictly monotone and hemicontinuous , we have two cases :

- **if** $1 \leq p < 2$

  1. **hemicontinuous**
     By (1.8) we get :

$$\lambda \int_\Omega \big||u+tv|^{p-2}\ (u+tv) - |u|^{p-2}\ u\big|\,|w|\,dx \leq \lambda c \int_\Omega |tv|^{p-1}|w|\,dx$$

and in the other hand side , by (2.3) we have :

$$|a(x,\nabla u + t\nabla v) - a(x,\nabla u)| \leq b_1 \int_\Omega (1+|\nabla u + t\nabla v| + |\nabla u|)^{p-1-\alpha}|t\nabla v|^\alpha\,dx$$

we obtain :

$$\begin{aligned}|< T(u+tv) - T(u), w >| &\leq \int_\Omega |a(x,\nabla u + t\nabla v) - a(x,\nabla u)||\nabla w|\,dx \\ &+ \lambda \int_\Omega ||u+tv|^{p-2}\ (u+tv) - |u|^{p-2}\ u||w|\,dx \\ &\leq |t|^\alpha b_1 \int_\Omega (1+|\nabla u + t\nabla v| + |\nabla u|)^{p-1-\alpha}|\nabla v|^\alpha |\nabla w|\,dx \\ &+ \lambda c \int_\Omega |tv|^{p-1}|w|\,dx\end{aligned}$$

Utilizing the Holder inequality leads to :

$$\begin{aligned}|< T(u+tv) - T(u), w >| &\leq b_1|t|^\alpha \left(\int_\Omega (1+|\nabla u + t\nabla v| + |\nabla u|)^{(p-1-\alpha)q}|\nabla v|^{\alpha q}\,dx\right)^{\frac{1}{q}} \\ &\times \left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{1}{p}} + \lambda c|t|^{p-1}\left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{q}}\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

$$\begin{aligned}|< T(u+tv) - T(u), w >| &\leq b_1|t|^\alpha \left(\int_\Omega (1+|\nabla u + t\nabla v| + |\nabla u|)^p\,dx\right)^{1-\frac{\alpha}{p-1}}\left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{\alpha}{p-1}} \\ &\times \left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{1}{p}} + \lambda c|t|^{p-1}\left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{q}}\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

in the other hand side by using Minkowski inequality we get

$$\left(\int_\Omega (1+|\nabla u + t\nabla v| + |\nabla u|)^p\,dx\right) \leq \left(2\left(\int_\Omega |\nabla u|^p\ dx\right)^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}} + |t|^p\left(\int_\Omega |\nabla v|^p\ dx\right)^{\frac{1}{p}}\right)^p$$

so we have

$$\left(\int_\Omega (1+|\nabla u+t\nabla v|+|\nabla u|)^p\,dx\right)^{1-\frac{\alpha}{p-1}}$$
$$\leq \left(2\left(\int_\Omega |\nabla u|^p\ dx\right)^{\frac{1}{p}}+|\Omega|^{\frac{1}{p}}+|t|^p\left(\int_\Omega |\nabla v|^p\ dx\right)^{\frac{1}{p}}\right)^{p-\alpha q}$$

finally we obtain

$$|<T(u+tv)-T(u),w>|\leq |t|^\alpha b_1\left(2\left(\int_\Omega |\nabla u|^p\ dx\right)^{\frac{1}{p}}+|\Omega|^{\frac{1}{p}}+|t|^p\left(\int_\Omega |\nabla v|^p\ dx\right)^{\frac{1}{p}}\right)^{p-\alpha q}$$
$$\times\left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{\alpha}{p}}\left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{1}{p}}$$
$$+\lambda c|t|^{p-1}\left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{q}}\times\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}$$

Given that all integrals are bounded , we conclude that

$$\lim_{t\to 0}|<T(u+tv)-T(u),w>|=0$$

That means for $1\leq p<2$ , T is hemicontinuous

2. **strictly monotone** we have

$$<T(u)-T(v),u-v>=\int_\Omega (a(x,\nabla u)-a(x,\nabla v)).(\nabla u-\nabla v)\,dx$$
$$+\lambda\int_\Omega \left[|u|^{p-2}\ u-|v|^{p-2}\ v|\right](u-v)\,dx$$

By 2.4 we get :

$$b_2\int_\Omega (1+|\nabla u|+|\nabla v|)^{p-\beta}|\nabla u-\nabla v|^\beta\,dx+\lambda\int_\Omega \left(|u|^{p-2}\ u-|v|^{p-2}\ v|\right)(u-v)\,dx$$
$$\leq\ <T(u)-T(v),u-v>$$

using the reverse Hölder inequality (1.7)( with the dual exponents $0\leq\frac{p}{\beta}\leq 1$ **and** $\frac{p}{p-\beta}$ ) and (1.10) leads to :

$$b_2\left(\int_\Omega (1+|\nabla u|+|\nabla v|)^p\,dx\right)^{\frac{p-\beta}{p}}\times\left(\int_\Omega |\nabla u-\nabla v|^p\,dx\right)^{\frac{\beta}{p}}+\lambda c\int_\Omega (|u|+|v|)^{p-2}|u-v|^2\,dx$$
$$\leq\ <T(u)-T(v),u-v>$$

we use the reverse Hölder inequality ( with the dual exponents $0\leq\frac{p}{2}\leq 1$ **and** $\frac{p}{p-2}$ ) on the second term :

$$b_2\left(\int_\Omega (1+|\nabla u|+|\nabla v|)^p\,dx\right)^{\frac{p-\beta}{p}}\left(\int_\Omega |\nabla u-\nabla v|^p\,dx\right)^{\frac{\beta}{p}}+\lambda c\left(\int_\Omega (|u|+|v|)^p\,dx\right)^{\frac{p-2}{p}}$$
$$\times\left(\int_\Omega |u-v|^p\,dx\right)^{\frac{2}{p}}\,dx\ \leq\ \ <T(u)-T(v),u-v>$$

This demonstrates that T is strictly monotonic

- **if** $2 \leq p < \infty$

1. **hemicontinuous** By 1.9 we get :

$$\begin{aligned}|< T(u+tv)-T(u),w>| &\leq \int_\Omega |a(x,\nabla u+t\nabla v)-a(x,\nabla u)||\nabla w|\,dx+\\ &\lambda\int_\Omega \left||u+tv|^{p-2}\,(u+tv)-|u|^{p-2}\,u\right|\,|w|\,dx\\ &\leq |t|^\alpha b_1\int_\Omega (1+|\nabla u+t\nabla v|+|\nabla u|)^{p-1-\alpha}|\nabla v|^\alpha|\nabla w|\,dx\\ &+\lambda c|t|\int_\Omega [|u+tv|-|u|]^{p-2}\,|v||w|\,dx\end{aligned}$$

By using the Hölder inequality we get :

$$\begin{aligned}|< T(u+tv)-T(u),w>| &\leq |t|^\alpha b_1\int_\Omega (1+|\nabla u+t\nabla v|+|\nabla u|)^{p-1-\alpha}|\nabla v|^\alpha|\nabla w|\,dx\\ &+\lambda\,c|t|\left(\int_\Omega (|u+tv|-|u|)^p\,dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}}\\ &\times\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

Repeating the previous steps , we obtain

$$\begin{aligned}|< T(u+tv)-T(u),w>| &\leq |t|^\alpha b_1\left(2\left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}}+|\Omega|^{\frac{1}{p}}+|t|^p\left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{1}{p}}\right)^{p-\alpha q}\\ &\times\left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{\alpha}{p}}\left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{1}{p}}\\ &+\lambda\,c|t|\left(\int_\Omega (|u+tv|-|u|)^p\,dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}}\\ &\times\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

Given that all integrals are bounded , we conclude that

$$\lim_{t\to 0}|< T(u+tv)-T(u),w>| = 0$$

That means for $2 \leq p < \infty$ , T is Hemicontinuous.

2. **strictly monotone** we have

$$\begin{aligned}< T(u)-T(v),u-v> &= \int_\Omega (a(x,\nabla u)-a(x,\nabla v)).(\nabla u-\nabla v)\,dx\\ &+\lambda\int_\Omega \left[|u|^{p-2}\,u-|v|^{p-2}\,v|\right](u-v)\,dx\end{aligned}$$

By (2.4) we get :

$$b_2\int_\Omega (1+|\nabla u|+|\nabla v|)^{p-\beta}|\nabla u-\nabla v|^\beta\,dx+\lambda\int_\Omega \left[|u|^{p-2}\,u-|v|^{p-2}\,v|\right](u-v)\,dx$$

$$\leq \quad < T(u) - T(v), u - v >$$

using the reverse Hölder inequality ( with the dual exponents $0 \leq \frac{p}{\beta} \leq 1$ and $\frac{p}{p-\beta}$ ) leads to

$$< T(u) - T(v), u - v > \geq b_2 \left( \int_\Omega (1 + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{p-\beta}{p}} \times \left( \int_\Omega |\nabla u - \nabla v|^p \, dx \right)^{\frac{\beta}{p}}$$
$$+\lambda \int_\Omega \left[ |u|^{p-2} \; u - |v|^{p-2} \; v| \right] (u - v) \, dx$$

By 1.11 we get :

$$b_2 \left( \int_\Omega (1 + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{p-\beta}{p}} \times \left( \int_\Omega |\nabla u - \nabla v|^p \, dx \right)^{\frac{\beta}{p}} + \lambda \int_\Omega c \; |u - v|^p \, dx$$

$$\leq \quad < T(u) - T(v), u - v >$$

That means for $2 \leq p < \infty$ , T is strictly monotone .

Now we will show that T is coercive for $p \in [1, \infty[$ . By 2.2 and 2.4 we get that

$$b_2 \int_\Omega (1 + |\nabla u|)^{p-\beta} |\nabla u|^\beta \, dx + \lambda \int_\Omega |u|^p \; dx \quad \leq \quad < T(u), u >$$

using the reverse Hölder inequality ( with the dual exponents $0 \leq \frac{p}{\beta} \leq 1$ **and** $\frac{p}{p-\beta}$ ) leads to :

$$b_2 \left( \int_\Omega (1 + |\nabla u|)^p \, dx \right)^{\frac{p-\beta}{p}} \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{\beta}{p}} + \lambda \; ||u||^p_{L^p(\Omega)} \quad \leq \quad < T(u), u >$$

so we have

$$b_2 ||1 + \nabla u||^{p-\beta}_{L^p(\Omega)} ||\nabla u||^\beta_{L^p(\Omega)} + \lambda ||u||^p_{L^p(\Omega)} \quad \leq \quad < T(u), u >$$
$$b_2 \left( ||1||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} \right)^{p-\beta} ||\nabla u||^\beta_{L^p(\Omega)} + \lambda \; ||u||^p_{L^p(\Omega)} \quad \leq \quad < T(u), u >$$

Let $r = min\{b_2, \lambda\}$ and $s = ||1||_{L^p(\Omega)}$ then

$$r \frac{(s + ||\nabla u||)^p}{||\nabla u||} \left[ \frac{||\nabla u||}{s + ||\nabla u||} \right]^\beta + r \frac{||u||^p}{||\nabla u||} \; \leq \; \frac{< T(u), u >}{||\nabla u||}$$

if $||\nabla u|| \to \infty$ , The left hand side converges to $\infty$ that's mean T is coercive.

Given that the operator T exhibits properties of hemicontinuous, strict monotonicity, and coerciveness, the Minty-Browder theorem guarantees the existence and uniqueness of a solution to problem (2.1) such that $u \in W^{1,p}_0(\Omega)$ . □

# Chapter 3

# Nonlinear elliptic equations defined by a class of monotone operators with a singular nonlinearity having variable exponent

The results of this chapter are obtained in the paper [15]

## 3.1 Introduction

In this chapter, we are going to prove the existence and regularity of solutions to nonlinear elliptic equations with singular nonlinearities expressed as :

$$\begin{cases} -Au = \frac{f}{u^{\gamma(x)}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ , \end{cases} \tag{3.1}$$

where

$$Au = div(a(x, \nabla u)) - \mu |u|^{p-2}\ u$$

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ , and $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ a function such that $a(., \xi)$ is mesurable for every $\xi \in \mathbb{R}^N$ and $a(., .)$ fulfills the conditions specified in 2.2, 2.3 and 2.4, with $1 < p < N$, we assume $\gamma(x) > 0$ to be a regular function such that $\gamma(x) \in C(\overline{\Omega})$ and $f$ is non-negative function belonging to a suitable Lebesgue space $L^m(\Omega)$, and $\mu > 0$ is a real number.

We get existence and regularity of solutions to the considered problem by approximation methods .

## 3.2 Approximation problems

Let $n \in \mathbb{N}^*$ and $f$ be a positive function measurable that is not identically zero , we consider this approximation problems :

$$\begin{cases} -Au_n = \frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega \\ \quad u_n > 0 & \text{in } \Omega \\ \quad u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

such that $Au_n = div(a(x, \nabla u_n)) - \mu|u_n|^{p-2}\, u_n$.
Let's define $f_n(x) = T_n(f)$, and $\overline{\gamma} = \sup_{x\in\overline{\Omega}} \gamma(x)$.

Fix $n \in \mathbb{N}$ , and Let $w = S(v) \in W_0^{1,p}(\Omega)$ with $v \in L^p(\Omega)$ be the unique solution of

$$\begin{cases} -Aw = \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

We're going to prove that $w = S(v) \in E$ such that $v \in E$ , i.e $S(E) \subset E$
Let $w$ be a test function in (3.3), we get :

$$\int_\Omega a(x, \nabla w)\nabla w + \mu|w|^p\, dx = \int_\Omega \frac{f_n w}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\, dx \le n^{\overline{\gamma}+1} \int_\Omega w\, dx.$$

We apply the Hölder inequality on the right-hand side , we obtain :

$$n^{\overline{\gamma}+1} \int_\Omega w\, dx \le n^{\overline{\gamma}+1} (mes\ (\Omega))^{\frac{1}{q}} \left(\int_\Omega |w|^p\, dx\right)^{\frac{1}{p}}$$

such that $\frac{1}{p} + \frac{1}{q} = 1$ .
By (2.2) and monotonicity condition (2.4), we have :

$$\begin{aligned} \mu \int_\Omega |w|^p\, dx &\le \int_\Omega b_2(1 + |\nabla w|)^{p-\beta} |\nabla w|^\beta + \mu|w|^p\, dx \\ &\le \int_\Omega a(x, \nabla w)\nabla w + \mu|w|^p\, dx \end{aligned}$$

According to the preceding inequalities , we have :

$$\mu \int_\Omega |w|^p\, dx \le n^{\overline{\gamma}+1} (mes\ (\Omega))^{\frac{1}{q}} \left(\int_\Omega |w|^p\, dx\right)^{\frac{1}{p}}$$

so we obtain :

$$||w||_{L^p(\Omega)}^{p-1} \le C n^{\overline{\gamma}+1}$$

and then we get :

$$||S(v)||_{L^p(\Omega)} = ||w||_{L^p(\Omega)} \le \left(C n^{\overline{\gamma}+1}\right)^{\frac{1}{p-1}}$$

This implies that for $n$ fixed, the ball of radius $R = (Cn^{\overline{\gamma}+1})^{\frac{1}{p-1}}$ in $L^p(\Omega)$ remains invariant under S ,which means $w \in E$ and $S(E) \subset E$ .

Now we will show that S is continuous and compact .

**Lemma 3.1.** *S is continuous operator .*

*Proof.* Let's define $w_k = S(v_k)$ and $w = S(v)$ , we need to demonstrate if :

$$\lim_{k\to\infty} ||v_k - v||_{W_0^{1,p}} = 0 \Longrightarrow \lim_{k\to\infty} ||w_k - w||_{W_0^{1,p}} = 0$$

such that $w_k$ is the solution of :

$$\begin{cases} -div(a(x,\nabla w_k)) + \mu|w_k|^{p-2}\ w_k = \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega \\ \quad w_k = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

and $w$ is the solution of :

$$\begin{cases} -div(a(x,\nabla w)) + \mu|w|^{p-2}\ w = \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

we take $w_k$ as test function in (3.4) , and then we get :

$$b_2 \int_\Omega (1 + |\nabla w_k|)^{p-\beta} |\nabla w_k|^\beta\, dx \leq \int_\Omega \frac{f_n w_k}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}}\, dx \tag{3.6}$$

We apply the Hölder inequality on the right-hand side, followed by the Poincare inequality in the precedent inequality , we obtain :

$$\begin{aligned} \int_\Omega \frac{f_n w_k}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}}\, dx &\leq n^{\overline{\gamma}+1} \int_\Omega |w_k|\, dx \\ &\leq n^{\overline{\gamma}+1} (mes\ (\Omega))^{\frac{1}{p'}} \left( \int_\Omega |w_k|^p\, dx \right)^{\frac{1}{p}} \\ &\leq k(p)\ n^{\overline{\gamma}+1} (mes\ (\Omega))^{\frac{1}{p'}} ||\nabla w_k||_{L^p(\Omega)} \end{aligned} \tag{3.7}$$

such that $p' = \frac{p}{p-1}$
Using the reverse Hölder inequality in (3.6) on the left-hand side ( with the dual exponents $0 \leq \frac{p}{\beta} \leq 1$ and $\frac{p}{p-\beta}$ ) leads to :

$$b_2 ||1 + |\nabla w_k|||^{p-\beta}_{L^p(\Omega)} ||\nabla w_k||^\beta_{L^p(\Omega)} \leq \int_\Omega \frac{f_n w_k}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}}\, dx \tag{3.8}$$

By (3.7) , we get :

$$||1 + |\nabla w_k|||^{p-\beta}_{L^p(\Omega)} ||\nabla w_k||^\beta_{L^p(\Omega)} \leq b_3\ n^{\overline{\gamma}+1} (mes\ \Omega)^{\frac{1}{p'}} ||\nabla w_k||_{L^p(\Omega)}$$

That's means

$$||1 + |\nabla w_k|||^{p-\beta}_{L^p(\Omega)} ||\nabla w_k||^{\beta-1}_{L^p(\Omega)} \leq b_3\ n^{\overline{\gamma}+1} (mes\ \Omega)^{\frac{1}{p'}}.$$

In the following , we will demonstrate that $||\nabla w_k||_{L^p(\Omega)}$ is uniformly bounded with respect to $k$ .
Let's $\sigma = (mes\ \Omega)^{\frac{1}{p}}$ , we have :

$$\left(\sigma + ||\nabla w_k||_{L^p(\Omega)}\right)^{p-\beta} ||\nabla w_k||^{\beta-1}_{L^p(\Omega)} \leq ||1 + |\nabla w_k|\ ||^{p-\beta}_{L^p(\Omega)} ||\nabla w_k||^{\beta-1}_{L^p(\Omega)}$$

If $||\nabla w_k||_{L^p(\Omega)}$ is less than or equal to $\sigma$ , then our demonstration is complete , Let us now consider the case while $||\nabla w_k||_{L^p(\Omega)}$ is greater than or equal to $\sigma$ , then we get :

$$\big(2||\nabla w_k||_{L^p(\Omega)}\big)^{p-\beta}\,||\nabla w_k||^{\beta-1}_{L^p(\Omega)} \leq \big(\sigma + ||\nabla w_k||_{L^p(\Omega)}\big)^{p-\beta}\,||\nabla w_k||^{\beta-1}_{L^p(\Omega)}$$

which means

$$||\nabla w_k||^{p-1}_{L^p(\Omega)} \leq (2)^{\beta-p} b_3\; n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}} \tag{3.9}$$

$$\Longrightarrow ||\nabla w_k||_{L^p(\Omega)} \leq \max\{\sigma, 3.9\} \quad \forall w_k \in W^{1,p}_0(\Omega).$$

Now , we take $w$ as test function in (3.5) , and then we get :

$$b_2\int_\Omega (1+|\nabla w|)^{p-\beta}|\nabla w|^\beta\,dx \leq \int_\Omega \frac{f_n w}{\big(|v|+\frac{1}{n}\big)^{\gamma(x)}}\,dx \tag{3.10}$$

we apply the Hölder inequality on the right-hand side, Followed by the Poincare inequality in the precedent inequality , we achieve :

$$\begin{aligned}\int_\Omega \frac{f_n w}{\big(|v|+\frac{1}{n}\big)^{\gamma(x)}}\,dx &\leq n^{\overline{\gamma}+1}\int_\Omega |w|\,dx\\ &\leq n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}}\left(\int_\Omega |w|^p\,dx\right)^{\frac{1}{p}}\\ &\leq k(p)\; n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}}||\nabla w||_{L^p(\Omega)}\end{aligned} \tag{3.11}$$

such that $p' = \frac{p}{p-1}$
using the reverse Hölder inequality in (3.10) on the left-hand side ( with the dual exponents $0 \leq \frac{p}{\beta} \leq 1$ and $\frac{p}{p-\beta}$ ) leads to :

$$b_2||1+|\nabla w|||^{p-\beta}_{L^p(\Omega)}||\nabla w||^{\beta}_{L^p(\Omega)} \leq \int_\Omega \frac{f_n w}{\big(|v|+\frac{1}{n}\big)^{\gamma(x)}}\,dx$$

By using (3.11) , we get :

$$||1+|\nabla w|||^{p-\beta}_{L^p(\Omega)}||\nabla w||^{\beta}_{L^p(\Omega)} \leq b_4\; n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}}||\nabla w||_{L^p(\Omega)}$$

That's means

$$||1+|\nabla w|||^{p-\beta}_{L^p(\Omega)}||\nabla w||^{\beta-1}_{L^p(\Omega)} \leq b_4\; n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}}$$

In the following , we will demonstrate that $||\nabla w||_{L^p(\Omega)}$ is uniformly bounded.
we have :

$$\big(\sigma + ||\nabla w||_{L^p(\Omega)}\big)^{p-\beta}\,||\nabla w||^{\beta-1}_{L^p(\Omega)} \leq ||1+|\nabla w|\;||^{p-\beta}_{L^p(\Omega)}||\nabla w||^{\beta-1}_{L^p(\Omega)}$$

If $||\nabla w||_{L^p(\Omega)}$ is less than or equal to $\sigma$ , then our demonstration is complete , Let us now consider the case while $||\nabla w||_{L^p(\Omega)}$ is greater than or equal to $\sigma$ , then we get :

$$\big(2||\nabla w||_{L^p(\Omega)}\big)^{p-\beta}\,||\nabla w||^{\beta-1}_{L^p(\Omega)} \leq \big(\sigma + ||\nabla w||_{L^p(\Omega)}\big)^{p-\beta}\,||\nabla w||^{\beta-1}_{L^p(\Omega)}$$

which means

$$||\nabla w||^{p-1}_{L^p(\Omega)} \leq (2)^{\beta-p} b_4\; n^{\overline{\gamma}+1}(mes\;\Omega)^{\frac{1}{p'}}. \tag{3.12}$$

$$\Longrightarrow ||\nabla w||_{L^p(\Omega)} \leq \max\{\sigma, 3.12\} \quad \forall w \in W^{1,p}_0(\Omega).$$

Through the subtraction operation between (3.4) and (3.5), and we take $(w_k - w)$ as a test function , we get :

$$\int_\Omega (w_k - w)\left(\frac{f_n}{\left(|v_n| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right) dx = \int_\Omega \left(a(x,\nabla w_k) - a(x,\nabla w)\right)\nabla(w_k - w)\,dx$$
$$+\mu\int_\Omega \left(|w_k|^{p-2}\, w_k - |w|^{p-2}\, w\right)(w_k - w)\,dx$$

We have the following remarks :
- For $1 < p < 2$ we have :

$$0 \le c_0(p)\int_\Omega (1 + |w_k| + |w|)^{p-2}|w_k - w|^2\,dx \le \int_\Omega \left(|w_k|^{p-2}\, w_k - |w|^{p-2}\, w\right)(w_k - w)\,dx.$$

- For $2 \le p < \infty$ we have :

$$0 \le c_0(p)\int_\Omega |w_k - w|^p\,dx \le \int_\Omega \left(|w_k|^{p-2}\, w_k - |w|^{p-2}\, w\right)(w_k - w)\,dx.$$

In the both cases, we have :

$$\int_\Omega \left(a(x,\nabla w_k) - a(x,\nabla w)\right)\nabla(w_k - w)\,dx \le \int_\Omega \left(a(x,\nabla w_k) - a(x,\nabla w)\right)\nabla(w_k - w)\,dx$$
$$+\,\mu\int_\Omega \left(|w_k|^{p-2}\, w_k - |w|^{p-2}\, w\right)(w_k - w)\,dx$$

By using (2.4) we get :

$$b_2\int_\Omega (1 + |\nabla w_k| + |\nabla w|)^{p-\beta}|\nabla w_k - \nabla w|^\beta\,dx$$
$$\le \int_\Omega (w_k - w)\left(\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right) dx$$

using the reverse Hölder inequality ( with the dual exponents $0 \le \frac{p}{\beta} \le 1$ and $\frac{p}{p-\beta}$ ) leads to :

$$b_2||1 + |\nabla w_k| + |\nabla w|\ ||^{p-\beta}_{L^p(\Omega)}||\nabla w_k - \nabla w||^\beta_{L^p(\Omega)}$$
$$\le \int_\Omega (w_k - w)\left(\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right) dx$$

then

$$||\nabla w_k - \nabla w||^\beta_{L^p(\Omega)} \le \frac{1}{b_2}||1 + |\nabla w_k| + |\nabla w|\ ||^{\beta-p}_{L^p(\Omega)}\left[\int_\Omega (w_k - w)\left(\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right) dx\right]$$

Hence

$$||w_k - w||^\beta_{W_0^{1,p}(\Omega)} \le K_1(\overline{\gamma}, n, p, p', \Omega, b_3, b_4, \beta, \sigma)\int_\Omega (w_k - w)\left(\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right) dx$$

And then , we use Hölder inequality, we got :

$$||w_k - w||^\beta_{W_0^{1,p}(\Omega)} \le K_1(\overline{\gamma}, n, p, p', \Omega, b_3, b_4, \beta, \sigma)||(w_k - w)||_{L^{p^*}(\Omega)}$$
$$\times\left(\int_\Omega \left|\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}\right|^{(p^*)'} dx\right)^{\frac{1}{(p^*)'}}$$

such that $(p^*)' = \frac{p^*}{p^*-1}$.
We apply The sobolev inequality, we obtain :

$$||w_k - w||^{\beta}_{W_0^{1,p}(\Omega)} \leq K_2(\overline{\gamma}, n, p, p', \Omega, b_3, b_4, \beta, \sigma, N)||(w_k - w)||_{W_0^{1,p}(\Omega)} \times \left( \int_\Omega \left| \frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} \right|^{(p^*)'} dx \right)^{\frac{1}{(p^*)'}}$$

that's means

$$||w_k - w||^{\beta-1}_{W_0^{1,p}(\Omega)} \leq K_2(\overline{\gamma}, n, p, p', \Omega, b_3, b_4, \beta, \sigma, N) \left( \int_\Omega \left| \frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} \right|^{(p^*)'} dx \right)^{\frac{1}{(p^*)'}}$$

Considering that :

$$\left| \frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} \right| \leq 2n^{\gamma(x)+1} \leq 2n^{\overline{\gamma}+1}$$

given the dominated convergence theorem and acknowledging that :

$$v_k(x) \to v(x) \quad in \quad \Omega$$

we conclude that :

$$\lim_{k \to +\infty} ||w_k - w||_{W_0^{1,p}(\Omega)} = 0$$

□

**Lemma 3.2.** *S is compact operator .*

*Proof.* Let $\{v_k\}_k \subset W_0^{1,p}(\Omega)$ be a bounded sequence , denoting $w_k = S(v_k)$ , and by (3.9) we get :$||w_k||_{W_0^{1,p}(\Omega)} = ||S(v_k)||_{W_0^{1,p}(\Omega)} \leq c(n)$ .
such that $W_0^{1,p}(\Omega)$ is a reflexive space, so we have :

- $S(v_k) \rightharpoonup w$ in $W_0^{1,p}(\Omega)$.

and we have :

- $S(v_k) \to w$ in $L^m(\Omega)$ such that $1 \leq m < p^* = \frac{Np}{N-p}$.

Now , we will show that the sequence $\{\nabla w_k\}$ converge strongly to $\{\nabla w\}$ in $L^p(\Omega)$ , let's $(w_k - w)$ be a test function in (3.4), we get :

$$\int_\Omega (a(x, \nabla w_k), \nabla(w_k - w)) \; dx + \mu \int_\Omega |w_k|^{p-2} \; w_k(w_k - w) \, dx = \int_\Omega \frac{f_n(w_k - w)}{\left(|v_k| + \frac{1}{n}\right)^{\gamma(x)}} \, dx$$

we add

$$-\int_\Omega (a(x, \nabla w), \nabla(w_k - w)) \; dx - \mu \int_\Omega |w|^{p-2} \; w(w_k - w) \, dx$$

we obtain :

$$\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\ dx+\mu\int_\Omega \left(|w_k|^{p-2}\ w_k-|w|^{p-2}\ w\right)(w_k-w)\,dx$$
$$=-\int_\Omega (a(x,\nabla w),\nabla(w_k-w))\ dx-\mu\int_\Omega |w|^{p-2}\ w(w_k-w)\,dx+\int_\Omega \frac{f_n(w_k-w)}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}}\,dx$$

By Lemma 1.2 , we have :

$$\int_\Omega \left(|w_k|^{p-2}\ w_k-|w|^{p-2}\ w\right)(w_k-w)\,dx\geq 0$$

so we have :

$$\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\ dx\leq\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\ dx$$
$$+\mu\int_\Omega \left(|w_k|^{p-2}\ w_k-|w|^{p-2}\ w\right)(w_k-w)\,dx$$

That means that

$$\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\ dx\leq\int_\Omega \frac{f_n(w_k-w)}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}}\,dx-\int_\Omega (a(x,\nabla w),\nabla(w_k-w))\ dx$$
$$-\mu\int_\Omega |w|^{p-2}\ w(w_k-w)\,dx \tag{3.13}$$

By (2.2) and (2.3) we get :

$$|a(x,\nabla w_k)|\leq b_1(1+|\nabla w_k|)^{p-1-\alpha}|\nabla w_k|^\alpha\leq b_1(1+|\nabla w_k|)^{p-1}$$
$$\Longrightarrow\int_\Omega |a(x,\nabla w_k)|^{\frac{p}{p-1}}\,dx\leq b_1\int_\Omega (1+|\nabla w_k|)^p\,dx$$

we conclude that $a(x,\nabla w_k)$ is bounded in $L^q(\Omega)$ whenever $\nabla w_k$ is bounded in $L^p(\Omega)$, such that $q=\frac{p}{p-1}$ .
We have :

$$\left|\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}}\right|\leq n^{\overline{\gamma}+1}$$

Passing to the limit in (3.13) we get :

$$\lim_{k\to+\infty}\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\ dx=0. \tag{3.14}$$

Hence

$$\int_\Omega |\nabla w_k-\nabla w|^p\,dx=\int_\Omega \frac{|\nabla w_k-\nabla w|^p}{(1+|\nabla w_k|+|\nabla w|)^{\frac{p(\beta-p)}{\beta}}}(1+|\nabla w_k|+|\nabla w|)^{\frac{p(\beta-p)}{\beta}}\,dx$$

By Hölder inequality with exponents $(\frac{\beta}{p},\frac{\beta}{\beta-p}\ )$ , we get:

$$\int_\Omega |\nabla w_k-\nabla w|^p\,dx\leq\left(\int_\Omega \frac{|\nabla w_k-\nabla w|^\beta}{(1+|\nabla w_k|+|\nabla w|)^{\beta-p}}\,dx\right)^{\frac{p}{\beta}}\left(\int_\Omega (1+|\nabla w_k|+|\nabla w|)^p\,dx\right)^{\frac{\beta-p}{\beta}}$$
$$\leq c\left(\int_\Omega \frac{|\nabla w_k-\nabla w|^\beta}{(1+|\nabla w_k|+|\nabla w|)^{\beta-p}}\,dx\right)^{\frac{p}{\beta}}.$$

By (2.4) , we obtain :

$$\int_\Omega (a(x,\nabla w_k)-a(x,\nabla w),\nabla(w_k-w))\,dx \geq \int_\Omega \frac{|\nabla w_k-\nabla w|^\beta}{(1+|\nabla w_k|+|\nabla w|)^{\beta-p}}\,dx$$
$$\geq \left(\int_\Omega |\nabla w_k-\nabla w|^p\,dx\right)^{\frac{\beta}{p}}.$$

Using (3.14) , we conclude that

$$\lim_{k\to+\infty}\left(\int_\Omega |\nabla w_k-\nabla w|^p\,dx\right)^{\frac{\beta}{p}}=0.$$

So the sequences $\{\nabla w_k\}$ converge strongly to $\{\nabla w\}$ in $L^p(\Omega)$
Let $\varphi\in W_0^{1,p}(\Omega)$ , We conclude that :

$$\lim_{k\to+\infty}\int_\Omega a(x,\nabla w_k)\nabla\varphi\,dx+\mu\int_\Omega |w_k|^{p-2}\ w_k\varphi\,dx=\int_\Omega a(x,\nabla w)\nabla\varphi\,dx+\mu\int_\Omega |w|^{p-2}\ w\varphi\,dx$$

where

$$\int_\Omega a(x,\nabla w_k)\nabla\varphi\,dx+\mu\int_\Omega |w_k|^{p-2}\ w_k\varphi\,dx=\int_\Omega \frac{f_n\varphi}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}}\,dx$$

and

$$\int_\Omega a(x,\nabla w)\nabla\varphi\,dx+\mu\int_\Omega |w|^{p-2}\ w\varphi\,dx=\int_\Omega \frac{f_n\varphi}{\left(|v|+\frac{1}{n}\right)^{\gamma(x)}}\,dx$$

□

**Lemma 3.3.** *for any fixed n , the approximate problem has solution* $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ .

*Proof.* By Lemma 3.1, and Lemma 3.2 , given that all conditions of Schauder's fixed point theorem 1.16 are fulfilled , we conclude that for any fixed n , the approximate problem has solution $u_n\in W_0^{1,p}(\Omega)$ , such that $u_n=S(u_n)$ , and $u_n$ is a weak solution of :

$$\begin{cases} -div(a(x,\nabla u_n))+\mu|u_n|^{p-2}\ u_n=\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega\\ \quad u_n=0 & \text{on } \partial\Omega \end{cases} \tag{3.15}$$

According to the result in [18], Theorem 4.2 , since $\left(\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma(x)}}\right)\in L^\infty(\Omega)$ , then $u_n\in L^\infty(\Omega)$. Using the maximum principle [11] we deduce that $u_n\geq 0$ ,since $\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma(x)}}\geq 0$ □

**Lemma 3.4.** *The sequence* $\{u_n\}_n$ *is increasing with respect to n.*

*Proof.* We observe that $0\leq f_n\leq f_{n+1}$

$$-Au_n=\frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\leq\frac{f_{n+1}}{\left(u_n+\frac{1}{n+1}\right)^{\gamma(x)}},$$

we have :

$$-Au_{n+1} = \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}.$$

Then we get :

$$\begin{aligned}-Au_n + Au_{n+1} &\leq \left[\frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}} - \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}\right] \\ &= f_{n+1}\left[\frac{1}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}} - \frac{1}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}\right]\end{aligned}$$

Selecting $(u_n - u_{n+1})^+$ as a test function , then we get :

$$\int_\Omega f_{n+1}\left[\frac{1}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}} - \frac{1}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}\right](u_n - u_{n+1})^+ \leq 0$$

That means that

$$\int_\Omega (-Au_n + Au_{n+1})(u_n - u_{n+1})^+ \leq 0,$$

and we have :

$$\begin{aligned}\int_\Omega (-Au_n + Au_{n+1})(u_n - u_{n+1})^+ = &\int_\Omega \left(a(x,\nabla u_n) - a(x,\nabla u_{n+1}), \nabla(u_n - u_{n+1})^+\right)\,dx \\ &+ \mu\int_\Omega (|u_n|^{p-2}\ u_n - |u_{n+1}|^{p-2}\ u_{n+1})(u_n - u_{n+1})^+\,dx.\end{aligned}$$

The hypotheses 2.4 helps us to get :

$$\begin{aligned}&b_2\int_\Omega (1 + |\nabla u_n| + |\nabla u_{n+1}|)^{p-\beta}|\nabla(u_n - u_{n+1})^+|^\beta\,dx \\ &\leq \int_\Omega (a(x,\nabla u_n) - a(x,\nabla u_{n+1})), \nabla(u_n - u_{n+1})^+)\,dx\end{aligned}$$

using the reverse Hölder inequality ( with the dual exponents $0 \leq \frac{p}{\beta} \leq 1$ and $\frac{p}{p-\beta}$ ) leads to :

$$\begin{aligned}&b_2\left(\int_\Omega (1 + |\nabla u_n| + |\nabla u_{n+1}|)^p\,dx\right)^{\frac{p-\beta}{p}} \times \left(\int_\Omega |\nabla(u_n - u_{n+1})^+|^p\,dx\right)^{\frac{\beta}{p}} \\ &\leq b_2\int_\Omega (1 + |\nabla u_n| + |\nabla u_{n+1}|)^{p-\beta}|\nabla(u_n - u_{n+1})^+|^\beta\,dx\end{aligned} \tag{3.16}$$

and then we have two cases :

1. for $1 \leq p < 2$ :
   We have :

$$\begin{aligned}c_0(p)\int_\Omega (1 + |u_n| + |u_{n+1}|)^{p-2}|(u_n - u_{n+1})^+|^2\,dx \leq &\int_\Omega \left(|u_n|^{p-2}\ u_n - |u_{n+1}|^{p-2}\ u_{n+1}\right) \\ &\times (u_n - u_{n+1})^+\,dx\end{aligned}$$

using the reverse Hölder inequality ( with the dual exponents $\frac{p}{2} \leq 1$ and $\frac{p}{p-2}$ ) leads to :

$$\begin{aligned} c_0(p)\left(\int_\Omega (1+|u_n|+|u_{n+1}|)^p\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{2}{p}} \\ \leq c_0(p)\int_\Omega (1+|u_n|+|u_{n+1}|)^{p-2}|(u_n-u_{n+1})^+|^2\,dx \end{aligned} \tag{3.17}$$

According to the inequalities precedents 3.16 and 3.17 , we obtain :

$$\begin{aligned} 0 \geq \mu\ c_0(p)\left(\int_\Omega (1+|u_n|+|u_{n+1}|)^p\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{2}{p}} \\ + b_2\left(\int_\Omega (1+|\nabla u_n|+|\nabla u_{n+1}|)^p\,dx\right)^{\frac{p-\beta}{p}} \left(\int_\Omega |\nabla(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{\beta}{p}} \end{aligned}$$

Consequently $(u_n - u_{n+1})^+ = 0$ almost everywhere in $\Omega$ , indicating that:

$$u_n \ \leq \ u_{n+1}$$

2. for $2 \leq p < \infty$:
We follow the same previous steps in the 1st case , we have :

$$c_0(p)\int_\Omega |(u_n-u_{n+1})^+|^2\,dx \leq \int_\Omega \left(|u_n|^{p-2}\ u_n - |u_{n+1}|^{p-2}\ u_{n+1}\right)(u_n-u_{n+1})^+\,dx$$

using the reverse Hölder inequality ( with the dual exponents $\frac{p}{2} \leq 1$ and $\frac{p}{p-2}$ ) leads to :

$$\begin{aligned} c_0(p)(mes\ \Omega)^{\frac{p-2}{p}} \left(\int_\Omega |(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{2}{p}} \\ \leq c_0(p)\int_\Omega (1+|u_n|+|u_{n+1}|)^{p-2}|(u_n-u_{n+1})^+|^2\,dx \end{aligned} \tag{3.18}$$

According to the inequalities precedents 3.16 and 3.18 , we obtain :

$$\begin{aligned} b_2\left(\int_\Omega (1+|\nabla u_n|+|\nabla u_{n+1}|)^p\,dx\right)^{\frac{p-\beta}{p}} \left(\int_\Omega |\nabla(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{\beta}{p}} \\ +\mu\ c_0(p)(mes\ \Omega)^{\frac{p-2}{p}} \left(\int_\Omega |(u_n-u_{n+1})^+|^p\,dx\right)^{\frac{2}{p}} \leq 0 \end{aligned}$$

Consequently $(u_n - u_{n+1})^+ = 0$ almost everywhere in $\Omega$ , indicating that:

$$u_n \ \leq \ u_{n+1}$$

□

**Remark 3.1.** *Let's assume that the problem 3.2 has two solutions , denoted as $u_n$ and $v_n$ , employing the same technique as demonstrated in the proof of lemma 3.4 implies that the solution of 3.2 is unique .*

**Lemma 3.5.** *Consider a sequence $\{u_n\}_n$ of nonnegative functions , where $\{u_n\}_n$ is uniformly bounded in the space $W_0^{1,p}(\Omega)$ , and $u_n$ converge weakly to $u$ in $W_0^{1,p}(\Omega)$ with $u_n \leq u \quad \forall n \in \mathrm{N}$ . Assuming that $-div(a(x, \nabla u_n)) \geq 0$ then $u_n$ converges strongly to $u$ in $W_0^{1,p}(\Omega)$.*

*Proof.* Given that $-div(a(x,\nabla u_n)) \geq 0$ and $u_n \leq u$ , then

$$\int_\Omega -div(a(x,\nabla u_n))(u_n - u)\, dx \leq 0$$

Hence

$$\int_\Omega \left(-div(a(x,\nabla u_n)) + div(a(x,\nabla u))\right)(u_n - u)\, dx + \int_\Omega -div(a(x,\nabla u))(u_n - u)\, dx \leq 0$$

Thus

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u)\,,\nabla u_n - \nabla u)\, dx + \int_\Omega (a(x,\nabla u)\,,\nabla u_n - \nabla u)\, dx \leq 0.$$

Given that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ then

$$\int_\Omega (a(x,\nabla u)\,,\nabla u_n - \nabla u)\, dx \to 0.$$

Hence

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u)\,,\nabla u_n - \nabla u)\, dx \leq o(1).$$

We already have

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u)\,,\nabla u_n - \nabla u)\, dx \geq b_2 \int_\Omega (1 + |\nabla u_n| + |\nabla u|)^{p-\beta} |\nabla u_n - \nabla u|^\beta\, dx$$

Using the Hölder inequality , we get :

$$\int_\Omega |\nabla u_n - \nabla u|^p\, dx \leq \left(\int_\Omega \frac{|\nabla u_n - \nabla u|^\beta}{(1 + |\nabla u_n| + |\nabla u|)^{\beta-p}}\, dx\right)^{\frac{p}{\beta}} \left(\int_\Omega (1 + |\nabla u_n| + |\nabla u|)^p\, dx\right)^{\frac{\beta-p}{\beta}}$$

$$\Longrightarrow \int_\Omega |\nabla u_n - \nabla u|^p\, dx \leq o(1)$$

□

**Lemma 3.6.** *For all $n \in \mathbb{N}$ , the solution of problem , is such that For all $E \subset\subset \Omega, 0 < C_E \leq u_n$*

*Proof.* Given that $\forall n \in \mathbb{N}$ fixed , we have $u_n \in L^\infty(\Omega)$. Thus , for $n = 1$ we get:

$$-div(a(x,\nabla u_1)) + \mu |u_1|^{p-2}\, u_1 = \frac{f_n}{(|u_1| + 1)^{\gamma(x)}} \geq \frac{f_n}{(||u_1||_\infty + 1)^{\gamma(x)}} \geq 0$$

The right-hand side is not identically zero , applying the strong maximum principle implies $u_1 > 0$ in $\Omega$ . Hence there exists a constant $C_E > 0$ such that $0 < C_E \leq u_n$ since $u_1 \leq u_n$ for every $n \in \mathbb{N}$ □

## 3.3 Passage to the limit in $n$

Noting that $\Omega_\epsilon = \{x \in \Omega, dist(x, \partial\Omega) < \epsilon\}$ for fixed $\epsilon > 0$ , and putting $\omega_\epsilon = \Omega \setminus \overline{\Omega_\epsilon}$.

**Theorem 3.1.** *Suppose $s = \frac{Np}{N(p-1)+p}$ , $f \in L^s(\Omega)$ and there exists $\epsilon > 0$ , such that $\gamma(x) \leq 1$ in $\Omega_\epsilon$ , then the solution $u_n$ of 3.2 is bounded in $W_0^{1,p}(\Omega)$ .*

*Proof.* Given the outcomes mentioned earlier , we have $0 < C_{\Omega_\epsilon} \leq u_n$ . Now Let $u_n$ be a test function in 3.2

$$b_2 \int_\Omega (1+|\nabla u_n|)^{p-\beta} |\nabla u_n|^\beta \, dx \leq \int_\Omega \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \, dx.$$

By the reverse Hölder inequality with the dual exponent $\frac{p}{p-\beta}$ and $0 < \frac{p}{\beta} < 1$, and Let's $\sigma = (mes\ \Omega)^{\frac{1}{p}}$ , we have :

$$\begin{aligned} b_2 \left(\sigma + ||\nabla u_n||_{L^p(\Omega)}\right)^{p-\beta} ||\nabla u_n||^\beta_{L^p(\Omega)} &\leq b_2 ||1+|\nabla u_n|\ ||^{p-\beta}_{L^p(\Omega)} ||\nabla u_n||^\beta_{L^p(\Omega)} \\ &\leq b_2 \int_\Omega (1+|\nabla u_n|)^{p-\beta} |\nabla u_n|^\beta \, dx \end{aligned}$$

If $||\nabla u_n||_{L^p(\Omega)}$ is less than or equal to $\sigma$ , then our demonstration is complete , Let us now consider the case while $||\nabla u_n||_{L^p(\Omega)}$ is greater than or equal to $\sigma$ , then we get :

$$b_2 \left(2||\nabla u_n||_{L^p(\Omega)}\right)^{p-\beta} ||\nabla u_n||^\beta_{L^p(\Omega)} \leq b_2 \left(\sigma + ||\nabla u_n||_{L^p(\Omega)}\right)^{p-\beta} ||\nabla u_n||^\beta_{L^p(\Omega)}.$$

Hence , using Hölder inequality , Young inequality and then Sobolev inequality , we get :

$$\begin{aligned} b_2 (2)^{p-\beta} ||\nabla u_n||^p_{L^p(\Omega)} &\leq \int_\Omega \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \, dx \\ &= \int_{\overline{\Omega_\epsilon}} \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \, dx + \int_{\omega_\epsilon} \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \, dx \\ &\leq \int_{\overline{\Omega_\epsilon}} f(x) u_n^{1-\gamma(x)} \, dx + \int_{\omega_\epsilon} \frac{f(x)}{C^{\gamma(x)}_{\omega_\epsilon}} u_n \, dx \\ &\leq \int_{\overline{\Omega_\epsilon} \cap \{u_n \leq 1\}} f(x) \, dx + \int_{\overline{\Omega_\epsilon} \cap \{u_n \geq 1\}} f(x) u_n \, dx + \int_{\omega_\epsilon} \frac{f(x)}{C^{\gamma(x)}_{\omega_\epsilon}} u_n \, dx \\ &\leq ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \int_\Omega f(x) u_n \, dx \\ &\leq ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \left(\int_\Omega |f(x)|^s \, dx\right)^{\frac{1}{s}} \left(\int_\Omega |u_n|^{s'} \, dx\right)^{\frac{1}{s'}} \\ &= ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \left(\int_\Omega |f(x)|^s \, dx\right)^{\frac{1}{s}} \left(\int_\Omega |u_n|^{p^*} \, dx\right)^{\frac{1}{p^*}} \\ &\leq ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \left[\frac{1}{q}\left(\frac{1}{\epsilon}||f||_{L^s(\Omega)}\right)^q + \frac{1}{p}\left(\epsilon ||u_n||_{L^{p^*}(\Omega)}\right)^p\right] \\ &\leq ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \frac{1}{q}\left(\frac{1}{\epsilon}||f||_{L^s(\Omega)}\right)^q + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \\ &\times \frac{K(N,P)\ \epsilon^p}{p} \left(||\nabla u_n||_{L^p(\Omega)}\right)^p \end{aligned}$$

then

$$\begin{aligned} b_2 (2)^{p-\beta} ||\nabla u_n||^p_{L^p(\Omega)} - \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \frac{K(N,P)\ \epsilon^p}{p} \left(||\nabla u_n||_{L^p(\Omega)}\right)^p \\ \leq ||f||_{L^1(\Omega)} + \left(1 + ||C^{-\gamma(x)}_{\omega_\epsilon}||_{L^\infty(\Omega)}\right) \frac{1}{q}\left(\frac{1}{\epsilon}||f||_{L^s(\Omega)}\right)^q \end{aligned}$$

that means that

$$||u_n||^p_{W_0^{1,p}(\Omega)}\left(b_2\,(2)^{p-\beta}-\left(1+||C_{\omega_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}\right)\frac{K(N,P)\;\epsilon^p}{p}\right)$$
$$\leq ||f||_{L^1(\Omega)}+\left(1+||C_{\omega_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}\right)\frac{1}{q}\left(\frac{1}{\epsilon}||f||_{L^s(\Omega)}\right)^q$$

whenever

$$\epsilon\leq\left(\frac{b_2\,(2)^{p-\beta}\,p}{K(N,p)\;\left(1+||C_{\omega_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}\right)}\right)^{\frac{1}{p}}$$

$$\Longrightarrow ||u_n||_{W_0^{1,p}(\Omega)}\leq C$$

such that $C$ is a constant independent of n and $s^{'}=\frac{Np}{N-p}$ , with $s^{'}=p^*$ . □

**Theorem 3.2.** *Let $s=\frac{Np}{N(p-1)+p}$ and $f$ is a positives function in $L^s(\Omega)$ and supposing there exists a $\epsilon>0$ , such that $\gamma(x)\leq 1$ in $\Omega_\epsilon$ , it follows that there exists a solution $u$ in $W_0^{1,p}(\Omega)$ to 3.1 , in the sense that :*
*$\forall\phi\in C_0^1(\Omega)$ , we have*

$$\int_\Omega a(x,\nabla u)\nabla\phi\,dx+\mu\int_\Omega|u|^{p-2}\;u\phi\,dx=\int_\Omega\frac{f}{u^{\gamma(x)}}\phi\,dx.$$

*Proof.* According to the preceding theorem 3.1 , we have $\{u_n\}_n$is bounded in $W_0^{1,p}(\Omega)$ , Thus up to sequence , we obtain the existence of $u\in W_0^{1,p}(\Omega)$ , since $W_0^{1,p}(\Omega)$ is a reflexive space , we get:

- $u_n\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$
- $u_n\longrightarrow u$ Strongly in $L^m(\Omega)$ such that $m<p^*=\frac{Np}{N-p}$

Hence

$$0\leq\left|\frac{f_n\phi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\right|\leq||\phi C_{w_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}f(x)$$

such that $\phi\in C_0^1(\Omega)$ , and $\omega$ is the support of $\phi$ using the dominated convergence theorem , we get :

$$\lim_{n\to+\infty}\int_\Omega\frac{f_n\phi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\,dx=\int_\Omega\frac{f}{u^{\gamma(x)}}\phi\,dx$$

we suppose $\phi\in W_0^{1,p}(\Omega)$ , then :

$$\int_\Omega\left(a(x,\nabla u_n),\nabla\phi\right)\,dx+\mu\int_\Omega|u_n|^{p-2}\;u_n\phi\,dx=\int_\Omega\frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\phi\,dx \tag{3.19}$$

First of all we must show that the sequence $\{\nabla u_n\}$ converge strongly to $\nabla u$ in $L^p(\Omega)$ , since the weak convergence is not enough to pass to the limit in (3.19)
Let's $(u_n-u)\leq 0$ be a test function in (3.4)

$$\int_\Omega\left(a(x,\nabla u_n),\nabla(u_n-u)\right)\,dx+\mu\int_\Omega|u_n|^{p-2}\;u_n(u_n-u)\,dx=\int_\Omega\frac{f_n(u_n-u)}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\,dx\leq 0$$

Then

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u), \nabla(u_n - u))\, dx + \mu \int_{\Omega} \left( |u_n|^{p-2}\, u_n - |u|^{p-2}\, u \right) (u_n - u)\, dx$$
$$\leq - \int_{\Omega} (a(x, \nabla u), \nabla(u_n - u))\, dx - \mu \int_{\Omega} |u|^{p-2}\, u(u_n - u)\, dx.$$

We know that :

$$\int_{\Omega} \left( |u_n|^{p-2}\, u_n - |u|^{p-2}\, u \right) (u_n - u)\, dx \geq 0$$

so , we get :

$$0 \leq \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u), \nabla(u_n - u))\, dx$$
$$\leq - \int_{\Omega} (a(x, \nabla u), \nabla(u_n - u))\, dx - \mu \int_{\Omega} |u|^{p-2}\, u(u_n - u)\, dx.$$

Passing to the limit , we obtain :

$$\lim_{n \to \infty} \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u), \nabla(u_n - u))\, dx = 0$$

Hence

$$\int_{\Omega} |\nabla u_n - \nabla u|^p\, dx = \int_{\Omega} \frac{|\nabla u_n - \nabla u|^p}{(1 + |\nabla u_n| + |\nabla u|)^{\frac{p(\beta-p)}{\beta}}} (1 + |\nabla u_n| + |\nabla u|)^{\frac{p(\beta-p)}{\beta}}\, dx$$

By Hölder inequality with exponents $(\frac{\beta}{p}, \frac{\beta}{\beta-p}\ )$ , we get:

$$\int_{\Omega} |\nabla u_n - \nabla u|^p\, dx \leq \left( \int_{\Omega} \frac{|\nabla u_n - \nabla u|^\beta}{(1 + |\nabla u_n| + |\nabla u|)^{\beta-p}}\, dx \right)^{\frac{p}{\beta}} \left( \int_{\Omega} (1 + |\nabla u_n| + |\nabla u|)^p\, dx \right)^{\frac{\beta-p}{\beta}}$$
$$\leq c \left( \int_{\Omega} \frac{|\nabla u_n - \nabla u|^\beta}{(1 + |\nabla u_n| + |\nabla u|)^{\beta-p}}\, dx \right)^{\frac{p}{\beta}}.$$

By (2.4) , we obtain :

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u), \nabla(u_n - u))\, dx \geq \int_{\Omega} \frac{|\nabla u_n - \nabla u|^\beta}{(1 + |\nabla u_n| + |\nabla u|)^{\beta-p}}\, dx$$
$$\geq \left( \int_{\Omega} |\nabla u_n - \nabla u|^p\, dx \right)^{\frac{\beta}{p}}.$$

Using (3.14) , we conclude that

$$\lim_{k \to +\infty} \left( \int_{\Omega} |\nabla u_n - \nabla u|^p\, dx \right)^{\frac{\beta}{p}} = 0.$$

so the sequences $\{\nabla u_n\}$ converge strongly to $\nabla u$ in $L^p(\Omega)$ .
Let $\varphi \in W_0^{1,p}(\Omega)$ , We conclude that :

$$\lim_{k \to +\infty} \int_{\Omega} a(x, \nabla u_n) \nabla \varphi\, dx + \mu \int_{\Omega} |u_n|^{p-2}\, u_n \varphi\, dx = \int_{\Omega} a(x, \nabla u) \nabla \varphi\, dx + \mu \int_{\Omega} |u|^{p-2}\, u \varphi\, dx$$

such that

$$\int_\Omega a(x,\nabla u_n)\nabla\varphi\,dx+\mu\int_\Omega |u_n|^{p-2}\ u_n\varphi\,dx=\int_\Omega \frac{f_n\varphi}{\left(|v_k|+\frac{1}{n}\right)^{\gamma(x)}}\,dx$$

and

$$\int_\Omega a(x,\nabla u)\nabla\varphi\,dx+\mu\int_\Omega |u|^{p-2}\ u\varphi\,dx=\int_\Omega \frac{f_n\varphi}{\left(|v|+\frac{1}{n}\right)^{\gamma(x)}}\,dx.$$

□

**Remark 3.2.** *if $\mu=0$, we can apply Lemma (3.5) to prove theorem (3.2).*

**Theorem 3.3.** *Let's suppose that there exists a constant $\gamma^*>1$ and another constant $\epsilon$ such that $||\gamma||_{L^\infty(\Omega)}\leq\gamma^*$ .*
*Given that $f\in L^s(\Omega)$ with $s=\frac{N(p+\gamma^*-1)}{N(p-1)+p\gamma^*}$ , problem 3.1 has a solution $u$ in $L^r(\Omega)$ with $r=\frac{N(p+\gamma^*-1)}{N-p}$ , and $u^{\frac{p+\gamma^*-1}{p}}\in W_0^{1,p}(\Omega)$.*

*Proof.* Let $u_n^{\gamma^*}$ be a test function in 3.2, we obtain

$$\begin{aligned}
b_2\int_\Omega (1+|\nabla u_n|)^{p-\beta}|\nabla u_n|^\beta\ u_n^{\gamma^*-1}\,dx &\leq \int_{\overline{\Omega_\epsilon}} f(x)u_n^{\gamma^*-\gamma(x)}\,dx+\int_{\omega_\epsilon}\frac{f(x)}{C_{\omega_\epsilon}^{\gamma(x)}}u_n^{\gamma^*}\,dx\\
&\leq \int_{\overline{\Omega_\epsilon}\cap\{u_n\leq 1\}} f(x)\,dx+\int_{\overline{\Omega_\epsilon}\cap\{u_n\geq 1\}} f(x)u_n^{\gamma^*}\,dx+\int_{w_\epsilon}\frac{f(x)}{C_{w_\epsilon}^{\gamma(x)}}u_n^{\gamma^*}\,dx\\
&\leq ||f||_{L^1(\Omega)}+\left(1+||C_{\omega_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}\right)\int_\Omega f(x)u_n^{\gamma^*}\,dx\\
&\leq ||f||_{L^1(\Omega)}+\left(1+||C_{\omega_\epsilon}^{-\gamma(x)}||_{L^\infty(\Omega)}\right)\left(\int_\Omega |f(x)|^s\,dx\right)^{\frac{1}{s}}\left(\int_\Omega u_n^{\gamma^*z}\,dx\right)^{\frac{1}{z}}
\end{aligned}$$

such that $z=\frac{N(p+\gamma^*-1)}{(N-p)\gamma^*}$ and then , we get :

$$b_2\int_\Omega (1+|\nabla u_n|)^{p-\beta}|\nabla u_n|^\beta\ u_n^{\gamma^*-1}\,dx\leq C_1+C_2\left(\int_\Omega u_n^{\gamma^*z}\,dx\right)^{\frac{1}{z}}.$$

Hence

$$\int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx=\int_\Omega\frac{|\nabla u_n|^p}{(1+|\nabla u_n|)^{\frac{p(\beta-p)}{\beta}}}(1+|\nabla u_n|)^{\frac{p(\beta-p)}{\beta}}u_n^{\gamma^*-1}\,dx,$$

by Hölder inequality with exponents $(\frac{\beta}{p},\frac{\beta}{\beta-p}$ ) , we get:

$$\begin{aligned}
\int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx &\leq \left(\int_\Omega\frac{|\nabla u_n|^\beta u_n^{\gamma^*-1}}{(1+|\nabla u_n|)^{\beta-p}}\,dx\right)^{\frac{p}{\beta}}\left(\int_\Omega (1+|\nabla u_n|)^p u_n^{\gamma^*-1}\,dx\right)^{\frac{\beta-p}{\beta}}\\
&\leq \left(C_1+C_2\left(\int_\Omega u_n^{\gamma^*z}\,dx\right)^{\frac{1}{z}}\right)^{\frac{p}{\beta}}\left(\int_\Omega (1+|\nabla u_n|)^p u_n^{\gamma^*-1}\,dx\right)^{\frac{\beta-p}{\beta}}\\
&\leq \left(C_1+C_2\left(\int_\Omega u_n^{\gamma^*z}\,dx\right)^{\frac{1}{z}}\right)^{\frac{p}{\beta}}\left((2)^{p-1}\int_\Omega u_n^{\gamma^*-1}\,dx+(2)^{p-1}\int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx\right)^{\frac{\beta-p}{\beta}}
\end{aligned}$$

by employing the Young inequality , we achieve :

$$\int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx \le \left(\frac{1}{\epsilon_1}\right)^{\frac{\beta}{p}} \frac{p}{\beta}\left(C_1 + C_2\left(\int_\Omega u_n^{\gamma^* z}\,dx\right)^{\frac{1}{z}}\right)$$
$$+\,\epsilon_1^{\frac{\beta}{\beta-p}}\frac{\beta-p}{\beta}\left((2)^{p-1}\int_\Omega u_n^{\gamma^*-1}\,dx + (2)^{p-1}\int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx\right)$$

Noting that

$$\begin{aligned}\int_\Omega u_n^{\gamma^*-1}\,dx &\le \int_{\overline{\Omega_\epsilon}} u_n^{\gamma^*-1}\,dx + \int_{\omega_\epsilon}\frac{u_n^{\gamma^*}}{C_{w_\epsilon}}\,dx\\ &\le \int_{\overline{\Omega_\epsilon}\cap\{u_n\le 1\}} dx + \int_{\overline{\Omega_\epsilon}\cap\{u_n\ge 1\}} u_n^{\gamma^*}\,dx + \int_{\omega_\epsilon}\frac{u_n^{\gamma^*}}{C_{\omega_\epsilon}}\\ &\le ||1||_{L^1(\Omega)}+(1+C_{\omega_\epsilon}^{-1})\int_\Omega u_n^{\gamma^*}\,dx\\ &\le ||1||_{L^1(\Omega)}+(1+C_{\omega_\epsilon}^{-1})(mes\Omega)^{\frac{1}{s}}\left(\int_\Omega u_n^{\gamma^* z}\,dx\right)^{\frac{1}{z}}\end{aligned}$$

such that $\frac{1}{s}+\frac{1}{z}=1$
By selecting $\epsilon_1$ to be sufficiently small , we achieve :

$$\int_\Omega |\nabla\left(u_n^{\frac{p+\gamma^*-1}{p}}\right)|^p\,dx = \int_\Omega |\nabla u_n|^p u_n^{\gamma^*-1}\,dx \le C_1(\epsilon_1,p,\beta) + C_2(\epsilon_1,p,\beta)\left(\int_\Omega u_n^{\gamma^* z}\,dx\right)^{\frac{1}{z}}$$

Using Sobolev inequality , we get :

$$\frac{1}{(c(N,p))^p}\left(\int_\Omega u_n^r\,dx\right)^{\frac{p}{p^*}} \le C_1(\epsilon_1,p,\beta) + C_2(\epsilon_1,p,\beta)\left(\int_\Omega u_n^{\gamma^* z}\,dx\right)^{\frac{1}{z}}$$

and by the fact that

$$\frac{p}{p^*} > \frac{1}{z}$$

we conclude that $\{u_n\}_n$ is bounded in $L^r(\Omega)$ such that $r = \frac{N(p+\gamma^*-1)}{N-p}$ , and by the monotone convergence theorem $\{u_n\}_n$ converges strongly to $u \in L^r(\Omega)$

$\square$

# Conclusion

In this work , we provided a proof of the existence and regularity of solutions to nonlinear elliptic equations , which represent a wide of monotone operators . These equations are distinguished by a singular nonlinearity characterized by a variable exponent .
Due to the nonlinearity of function $a$ , in theorem 3.2 of chapter 3 , it is necessary to demonstrate that the sequence $\{\nabla u_{n+1}\}$ converges strongly to $\{\nabla u\}$ in order to take the limit . However , in work [17] , [12] and [5], weak convergence is sufficient .

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## Résumé

Cette thèse de master se concentre sur l'étude des équations elliptiques non linéaires définies par une classe d'opérateurs monotones avec une non-linéarité singulière ayant un exposant variable, où la fonction $f$ appartient à $L^m(\Omega)$. Pour résoudre ce type de problème, l'approche utilisée est l'approximation, en le réduisant à un cadre variationnel approprié. Cela permet parfois de démontrer l'existence et la régularité des solutions pour ces problèmes d'approximation, régularités qui sont préservées lors du passage à la limite.

## abstract

This master's thesis focuses on the study of nonlinear elliptic equations defined by a class of monotone operators with singular nonlinearity having variable exponent, where the function $f$ is in $L^m(\Omega)$. To solve this type of problem, the approach used is approximation, reducing it to a suitable variational framework. This sometimes allows for the demonstration of the existence and regularity of solutions for these approximation problems, which are preserved when passing to limits.