People's democratic republic of Algeria

Ministry of Higher Education and Scientific Research

Aboubekr BELKAÏD University - TLEMCEN

Faculty of Sciences

Mathematic's Department

MATHS2 - ST

COURSES AND CORRECTED EXERCISES

Addressed to students of level: Licence (L 1)

Domain : Sciences.

Sector: Technical Sciences.

Specialty: Mathematics.

Established by:

RAHMOUN Amel

Year : 2024.

FACULTÉ DES SCIENCES

Faculty of Sciences

كلية العلوم

Faculté des Sciences – Tidjani HADDAM

Tél.: 043 21 63 70 / Tél &Fax: 043 21 63 68 / 043 21 63 71

Site Web: www.fs.univ-tlemcen.dz

Email : vdrpg.facscience@gmail.com

بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

Aknowledgments

Praises be to God (hallowed be his name in heaven and on earth), through whom all things are made, this typescript is no exception...

I express all my gratitude to:

Mr Houbad Mekki
Mme Hakem Amel
Mr Mssirdi Bachir
Mr Touaoula Tarik

My sincere thanks are also conveyed to ALL my teachers throughout my years of learning who have deepened my knowledge and shaped my character.

Dedication

I dedicate this course to my two children:

Moulay Réda

And

Fatiha Malek.

Introduction

This course is intended for first-year university students in the technical sciences (ST) sector. It covers the mathematics module "Maths2" with the harmonized program of the LMD system.

It is the result of long years of teaching courses and tutorials in the Sciences Faculty of Tlemcen University.

Inspired by my own educational experience, I wrote it in simple language, introducing theoretical notions through practical examples and gave ultimate priority to pedagogy, even if it meant making my impending mathematics teachers scold!

A minimum of sentences explains the procedure to follow to solve an exercise and multiple examples illustrate most possible cases.

My challenges were first of all to simplify the theoretical notions in order to bring the student in this sector – generally not very interested in pure mathematics – closer to everything that can encourage them to acquire logical thinking. Then secondly, to give - as many times as possible - the solutions to the proposed exercises! Because oh how many times I was disappointed not to find the solution to an interesting problem in the book I held in my hands.

This typescript follows the official harmonized program and is divided into chapters, each chapter is made up of several lessons, followed by a series of exercises to definitively fix the ideas, because a *wise man once said: "methods are the economies of memory and the habits of mind".*

I hope from the bottom of my heart that this modest work can help anyone in difficulty in the chapters presented.

Like anything produced by a human, this book is far from perfect, but its author remains open mind and heart to any comments or suggestions that could raise his level and/or correct his errors.

Contents

1 Chapter 1 Matrices and Determinants

1.1 Lesson N°1 Matrices: definitions and basic operations

- **Definitions**

A matrix is -for the moment- a representation of any table of numbers on which we can perform some algebraic operations like addition, subtraction, multiplication...

Example

1^{rst} representation: by a table

	1↓	2↓	3↓	4↓
1→	0	-1	$\frac{1}{2}$	1
2→	1	0	2	3
3→	π	0	0	0

2^{nd} representation: by a matrix

$$A = \begin{pmatrix} 0 & -1 & \frac{1}{2} & 1 \\ 1 & 0 & 2 & 3 \\ \pi & 0 & 0 & 0 \end{pmatrix}$$

Like the table, the matrix has rows and columns.

If n is the number of rows of a matrix M and m the number of its columns, then the dimension of M, (or the size of M) is (n, m). We also note $M_{(n,m)}$.

Remark

Size = (number of rows, number of columns) and never the opposite!

Example

1*) The size of the matrix* A *is* $(3, 4)$.

2) The size of $B = \begin{pmatrix} 2 & 1 \\ \pi & e \\ 0 & 3 \\ -1 & -1 \end{pmatrix}$ is $(4, 2)$.

3) The size of $C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is $(3, 1)$. It is a column vector, (matrix with only one column).

4) The size of $D = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$ is $(1, 3)$. It is a row vector, (matrix with only one row).

Question

Give an example of an (1,1) matrix. What can you conclude?

Answer
$M = (\pi)_{(1,1)}$, we conclude that the numbers can be considered as matrices of dimension (1,1).

- **Finding the item** m_{ij}

We denote by m_{ij} the item of the matrix M that is at the i^{eth} line and the j^{eth} column, following the order.

Example
a_{32}**=0,** b_{11}**=2,** c_{21}**=2,** c_{12} does not exist**.** d_{13}**=1**...etc.

- **Particular case**

When $n = m$, the matrix is said to be square of order n.

Example

$$M_1 = \begin{pmatrix} 2 & 3 \\ -4 & 0 \end{pmatrix}_{(2,2)}, M_2 = \begin{pmatrix} \pi & 4 & \sqrt{2} \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{pmatrix}_{(3,3)}.$$

- **Definitions**
  - The **diagonal** of a square matrix $M_{(n,n)}$ is formed by all the elements m_{ii}**,** (i.e when $i = j$), we write:

$$diago(M) = (m_{11}, m_{22}, ..., m_{nn}).$$

- - The **trace** of a square matrix (n, n) is the sum of its diagonal elements, we write:

$$\begin{aligned} tr(M) &= \sum_{i=1}^{n} m_{ii} \\ &= m_{11} + m_{22} +, ... + m_{nn} \end{aligned}$$

Examples

$$\begin{aligned} diago(M_1) &= (2,0),\ diago(M_2) = (\pi, -\frac{1}{2}, 0). \\ tr(M_1) &= 2+0 = 2; tr(M_2) = \left(\pi - \frac{1}{2}\right). \end{aligned}$$

1.1.1 Some particular matrices

- **The null matrix**

  All of its elements are zero. **Example** $O_{(2,3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(2,3)}$.

- **The diagonal square matrix**

  All its off-diagonal elements are zero, i.e.$\forall i \neq j, a_{ij} = 0$.

  Example $N = \begin{pmatrix} \pi & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(3,3)}$.

  Attention: do not confuse the diagonal of a square matrix with the diagonal square matrix!

- **The identity matrix** (very important):

  Denoted Id, it is a diagonal square matrix, where all of its diagonal elements are equal to 1.

  Example

$$Id_{(3,3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(3,3)}, Id_{(2,2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(2,2)}$$

Question

Find $Id_{(1,1)}$.

Answer

$$Id_{(1,1)} = (1).$$

- **Upper triangular matrix**

  It is a square matrix all of which sub-diagonal elements are zero, i.e. $\forall i, j$ if $i > j$ then $a_{ij} = 0$.

  Example

$$M = \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Lawer triangular matrix**

  It is a square matrix of which all the over-diagonal elements are zero, i.e. $\forall i, j$ if $i < j$ then $a_{ij} = 0$.

  Example

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

1.1.2 Matrix operations

Equality

We say that the two matrices A and B are equal ($A = B$) if and only if A and B have the same dimension and all the elements of A are equal to the corresponding elements of B, (i.e. which are in the same position (ij)), i.e.

$$\forall i, j, a_{ij} = b_{ji}.$$

We then say that the elements of A and B are equal term by term.

Example

One has : $E = \begin{pmatrix} 2x+3 & 5 \\ 3 & -2y-4 \end{pmatrix}_{(2,2)}$ and $F = \begin{pmatrix} -1 & 5 \\ 3 & 5 \end{pmatrix}_{(2,2)}$.

Find x and y to get $E = F$?

Solution

1) E and F have the same size $(2, 2)$.

2) So that $E = F$ it is now necessary that :

$$\begin{cases} 2x+3=-1 \\ -2y-4=5 \end{cases}.$$

We must therefore take :

$$\begin{cases} x=-2 \\ y=\frac{-9}{2} \end{cases}.$$

Transposition

To transpose a matrix $A_{(n,m)}$ is to change its rows to columns and vice versa, i.e.

$$\forall i, j, a_{ij}^{'} = a_{ji}.$$

The new matrix obtained after this operation is noted A^t, (A^T or A'). Its size is (m,n).

Attention : in the general case $A \neq A^t$.

Examples

1) $A = \begin{pmatrix} 0 & -1 & \frac{1}{2} & 1 \\ 1 & 0 & 2 & 3 \\ \pi & 0 & 0 & 0 \end{pmatrix}_{(3,4)}, \quad A^t = \begin{pmatrix} 0 & 1 & \pi \\ -1 & 0 & 0 \\ \frac{1}{2} & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}_{(4,3)}$

2) $M_1 = \begin{pmatrix} 2 & 3 \\ -4 & 0 \end{pmatrix}_{(2,2)}, (M_1)^t = \begin{pmatrix} 2 & -4 \\ 3 & 0 \end{pmatrix}_{(2,2)}$

3) $C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{(3,1)}, C^t = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}_{(1,3)},$

4) $D = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}_{(1,3)}, D^t = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}_{(3,1)}.$

Addition and subtraction

To add or subtract two or more matrices, they must have the same dimension (size). The addition of matrices is done term by term, and the result is a matrix of the same dimension as the matrices added (or subtracted), i.e.

$$\forall i,j,(a+b)_{ij}=(a_{ij})+(b_{ij})\,.$$

Examples

1)

$$E=\begin{pmatrix}1&2&3\\0&-1&-5\end{pmatrix}_{(2,3)},F=\begin{pmatrix}1&2&3\\-5&0&-1\end{pmatrix}_{(2,3)};$$

$$(E+F)=\begin{pmatrix}1+1&2+2&3+3\\0+(-5)&-1+0&-5+(-1)\end{pmatrix}_{(2,3)}=\begin{pmatrix}2&4&6\\-5&-1&-6\end{pmatrix}_{(2,3)}.$$

2)

$$M_1=\begin{pmatrix}2&3\\-4&0\end{pmatrix}_{(2,2)},Id_{(2,2)}=\begin{pmatrix}1&0\\0&1\end{pmatrix}_{(2,2)};$$

$$\left(M_1-Id_{(2,2)}\right)=\begin{pmatrix}2-1&3-0\\-4-0&0-1\end{pmatrix}_{(2,2)}=\begin{pmatrix}1&3\\-4&-1\end{pmatrix}_{(2,2)}.$$

3) We cannot add or subtract $O_{(2,3)}$ and $N_{(3,3)}$ because they do not have the same dimensions.

Multiply a matrix by a scalar We multiply a scalar (a number) $\lambda\in\mathbb{R}$ and a matrix by multiplying it by all the elements of this matrix.

$$\forall i,j,(\lambda a)_{ij}=\lambda\,(a_{ij})\,.$$

Example

$$A=\begin{pmatrix}1&3\\-4&-1\end{pmatrix}_{(2,2)}$$

So that :

$$5A=\begin{pmatrix}5\times1&5\times3\\5\times(-4)&5\times(-1)\end{pmatrix}_{(2,2)}=\begin{pmatrix}5&15\\-20&-5\end{pmatrix}_{(2,2)}.$$

In the same way, to factorize a number of a matrix, we factorize it of all the elements of this matrix, for example:

$$A = \frac{1}{2} \begin{pmatrix} 2 & 6 \\ -8 & -2 \end{pmatrix}_{(2,2)}.$$

The product of two matrices Reminder: Scalar product (or row-by-column product)

Example

We multiply a row vector (1,3) by a column vector (3,1) as follows:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}_{(1,3)} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{(3,1)} = x_1y_1 + x_2y_2 + x_3y$$

Matrix Product

To multiply two matrices A and B, the number of columns of the first matrix A must be equal to the number of rows of the second matrix B.

We must therefore have: $A_{(n,m)} \times B_{(m,p)}$. The result of this multiplication is a matrix of size $(M)_{(n,p)}$.

It is because of this condition that the matrix product is not commutative in general and that we have to pay attention to the "side" from where the multiplication occurs.

After checking the condition above, the multiplication is done as follows:

The element m_{ij} will be the result of the multiplication of the ith row of A with the Jth column of B. We write :

$$\forall i, j, (m)_{ij} = \sum_{k=1}^{m} a_{ik} .b_{kj}.$$

Example

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix}_{(2,2)} ; B = \begin{pmatrix} 1 & 2 & 3 \\ -5 & 0 & -1 \end{pmatrix}_{(2,3)}.$$

A is of size (2,2) and B is of size (2,3), so multiplication between them is possible and the result is a matrix M of size (2,3).

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}_{(2,3)}$$

Where:

$$\begin{cases} m_{11} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = 0.1 + 1.(-5) = -5 \\ m_{12} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 0.2 + 1.0 = 0 \\ m_{13} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0.3 + 1.(-1) = -1 \end{cases}$$

$$\begin{cases} m_{21} = \begin{pmatrix} -4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = -4.1 + (-1).(-5) = 1 \\ m_{22} = \begin{pmatrix} -4 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = -4.2 + (-1).0 = -8 \\ m_{23} = \begin{pmatrix} -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = -4.3 + (-1).(-1) = -11 \end{cases}$$

So,

$$M = \begin{pmatrix} -5 & 0 & -1 \\ 1 & -8 & -11 \end{pmatrix}_{(2,3)}.$$

Remarks

- We cannot multiply more than two matrices at a time.
- The matrix product is not commutative as the previous example prooves. We can easily check that $A.B = M$, but $B.A$ is not possible because the number of columns of B (which is 3), is not equal to the number of rows of A (which is 2).
- For any square matrix A, one has : $A.Id = Id.A = A$.
- $(A \times B)^t = (B)^t \times (A)^t$ (and never the contrary).
- $A \times B = 0$ does not necessarily mean that one of the two matrices is zero.

Example

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}_{(2,2)} \times \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}_{(2,2)} = \begin{pmatrix} 0.0+(-4).0 & 0.(-1)+(-4).0 \\ 0.0+0.0 & 0.(-1)+0.0 \end{pmatrix}_{(2,2)}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{(2,2)} = 0_{(2,2)}.$$

- By the same manner $A \times B = A \times C$ does not mean that $B = C$.

Raising a square matrix M to a power p

We define the nth power of a matrix M by :

$$M^p = M \times M \times ... \times M \qquad (p \text{ time}).$$

This multiplication is in the sense of the matrix product.

If $M_{(n,m)}$ is a matrix with $n \neq m$, we then cannot multiply $M_{(n,m)} \times M_{(n,m)}$, because the condition of the matrix product is not verified. It is therefore necessary that $n = m$. This is why, one can raise to a power only square matrices.

Example

Calculate A^2 where $A = \begin{pmatrix} 1 & 3 \\ -4 & -1 \end{pmatrix}_{(2,2)}$.

$$\begin{aligned} A^2 &= A \times A = \begin{pmatrix} 1 & 3 \\ -4 & -1 \end{pmatrix}_{(2,2)} \times \begin{pmatrix} 1 & 3 \\ -4 & -1 \end{pmatrix}_{(2,2)} \\ &= \begin{pmatrix} 1.1+3.(-4) & 1.3+3.(-1) \\ -4.1+(-1).(-4) & -4.3+(-1).(-1) \end{pmatrix}_{(2,2)} \\ &= \begin{pmatrix} -11 & 0 \\ 0 & -11 \end{pmatrix}_{(2,2)} \\ &= (-11)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(2,2)} \\ &= (-11)\, Id_{(2,2)}. \end{aligned}$$

Remark

$$(A+B)^2 = \left(A^2 + AB + BA + B^2\right)$$

So, in the general case, $(A+B)^2 \neq (A^2 + 2AB + B^2)$. We must first check that $AB = BA$ to have $(A+B)^2 = (A^2 + 2AB + B^2)$.

Particular case

When the matrix is diagonal square, we can calculate its nth power, by directly raising its diagonal elements to the nth power,

Example

$$N = \begin{pmatrix} \pi & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is diagonal, so} N^2 = \begin{pmatrix} \pi^2 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Matrix calculation

We now present some examples of matrix calculation. Let us first recall that:

$$\begin{aligned} AB &\neq BA \text{ (in general) and that} \\ AId &= IdA = A \text{ (for all } A) \end{aligned}$$

The division in matrices is not defined, consequently, it is better to write $\frac{1}{2}A$ than $\frac{A}{2}$!

Let A, B and C be three square matrices of the same order. We give the following calculation examples:

$$\begin{aligned} 1)\ A(B+C) &= AB + AC \text{ and,} \\ (B+C)A &= BA + CA \end{aligned}$$

$$\begin{aligned} 2)\ (2BA + AC - Id)A &= 2BA^2 + ACA - A \\ A(B - \frac{1}{2}Id)A &= AB - \frac{1}{2}A^2 \end{aligned}$$

$$\begin{aligned} 3) \quad A^2 + AB - A &= A(A + B - Id) \text{ and,} \\ A^3 - 2BA + A &= (A^2 - 2B + Id)A \end{aligned}$$

4) In $\left(BA^2 + AB\right)$ factorization is impossible, neither of A nor of B.

Invertible square matrices

Let A be a square matrix of order n.

We say that A is invertible if and only if we can find a matrix B of the same dimension as A, verifying:

$$AB = BA = Id.$$

We say that B is the inverse matrix of A, it is denoted A^{-1}.

Remarks

1) $\left(A^{-1}\right)^{-1} = A$.
2) $(AB)^{-1} = B^{-1}.A^{-1}$

Examples

Find each time the expression of A^{-1} in terms of A:

$$1) \ A^2 + 3A = Id \Rightarrow A(A + 3Id) = Id \Rightarrow A^{-1} = (A + 3Id).$$

$$\begin{aligned} 2) \quad 2A^3 - A &= 3Id \\ &\Rightarrow A(2A^2 - Id) = 3Id \\ &\Rightarrow \frac{1}{3}A(2A^2 - Id) = Id \\ &\Rightarrow A^{-1} = \frac{1}{3}(2A^2 - Id). \end{aligned}$$

$$\begin{aligned} 3) \quad A.B &= 2Id \\ &\Rightarrow A\left(\frac{1}{2}B\right) = Id \\ &\Rightarrow A^{-1} = \left(\frac{1}{2}B\right). \end{aligned}$$

1.1.3 Calculation of determinants

Definition

The determinant of a square matrix A, denoted $\det(A)$ (or sometimes Δ or even $|A|$), is a number that determines whether the matrix is invertible or not, thanks to the following rule:

$$\boxed{\det(A) \neq 0 \Leftrightarrow A \text{ is invertible.}}$$

1) Second order determinants

Let A be the following matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(2,2)}$, so

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

Example

$M = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}_{(2,2)}$, so

$$\begin{aligned} \det(M) &= \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} \\ &= 1.4 - (-1).0 \\ &= 4 \neq 0, \end{aligned}$$

$\det(M) \neq 0$ which means that M is invertible, or equivalently that M^{-1} exists.

2) **Co-factors et co-matrices**

Definition

Let :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}_{(n,n)}.$$

The cofactor of A associated with the ith line and the jth column, noted $cof(A)_{ij}$, is the multiplication of $(-1)^{i+j}$ by the determinant of order $(n-1)$, obtained by removing the ith line and the jth column of A.

$$cof(A)_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(i-1)} & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix}.$$

Example

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{so} \begin{cases} cof(M)_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} = 2 \\ cof(M)_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \end{cases} \quad ...etc.$$

Definition

The co-matrix of A, denoted coA, is the matrix formed by all the possible cofactors of A, that is:

$$coA = \begin{pmatrix} cof(A)_{11} & cof(A)_{12} & ... & cof(A)_{1n} \\ cof(A)_{21} & cof(A)_{22} & ... & cof(A)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ cof(A)_{n1} & cof(A)_{n2} & ... & cof(A)_{nn} \end{pmatrix}_{(n,n)}.$$

Remark

coA is of the same size as A.

Example

Here is a convenient method to calculate all possible co-factors of M, and coM by the same occasion:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{pmatrix}_{(3,3)},$$

$$
\begin{aligned}
coM &= \begin{pmatrix} +\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{pmatrix}
\end{aligned}
$$

3) Higher order determinants

Let be the matrix $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}_{(n,n)}$.

To calculate the determinant of A, we must first choose a line i, then apply the following formula:

$$\det(A) = \sum_{j=1}^{n} a_{ij} cof(A)_{ij}.$$

We say that we have developed the determinant with respect to the chosen line i.

Example

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{pmatrix}_{(3,3)}$$

*) Development relative to the first line:

$$
\begin{aligned}
\det(M) &= m_{11}coM_{11} + m_{12}coM_{12} + m_{13}coM_{13} \\
&= 1.2 + 0.0 + 1.0 \\
&= 2
\end{aligned}
$$

*) Development relative to the second line

$$\begin{aligned}\det(M) &= m_{21}coM_{21} + m_{22}coM_{22} + m_{23}coM_{23} \\ &= 0.1 + 2.0 + (-2).(-1) \\ &= 2\end{aligned}$$

Remarks

1) We can also expand the determinant with respect to the columns, by the following formula:

$$\det(A)=\sum_{i=1}^{n} a_{ij}cof(A)_{ij}.$$

For example, expanding with respect to the first column gives us:

$$\begin{aligned}\det(M) &= m_{11}coM_{11} + m_{21}coM_{21} + m_{31}coM_{31} \\ &= 1.2 + 0.1 + 0.(-2) \\ &= 2\end{aligned}$$

2) You should know that the value of the determinant does not depend on the row (or column) chosen.

3) By common sense, we expand the determinant with respect to the row or column that contains the most zeros.

4) If in a determinant, a row (or a column) is completely zero, then this determinant is zero.

5) If in a determinant two rows (or two columns) are identical or one is a multiple of the other, then this determinant is zero.

6)

$$\begin{aligned}\det(A) &= \det(A^t) \\ \det(A.B) &= \det(A).\det(B)\end{aligned}$$

4) **The Sarrus method (with respect to lines)**

It is an easy and fast method, unfortunately it only applies for the calculation of the determinants of order 3.

We copy the first and the second line then we multiply the elements of the descending diagonals and we add the result then we multiply the elements of the ascending diagonals and we subtract the result.

Example

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det M = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 1.2.0 + 0.1.1 + 0.0.(-2) - 0.0.0 - 1.1.(-2) - 0.2.1 \\ \det M &= 2. \end{aligned}$$

Remark

We can also use the Sarrus method with respect to columns, which consists in copying the first and the second column and writing them on the right of the matrix then restarting the calculation in the same way.

$$\det M = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1.2.0 + 0.(-2).0 + 1.0.1 - 0.2.1 - 1.(-2).1 - 0.0.0 \\ \det M &= 2. \end{aligned}$$

5) **Particular case**

To calculate the determinant of a diagonal square, upper triangular, or lower triangular matrix, simply multiply its diagonal elements.

Example:

$$N = \begin{pmatrix} \pi & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(3,3)}$$

Then, $\det(N) = \pi.\left(\frac{-1}{2}\right).1 = \left(\frac{-\pi}{2}\right) \neq 0$. We thus can conclude that N is invertible.

Question Give $\det Id$.

Answer Whatever the size of the Id matrix given, $\det Id = 1$.

1.1.4 The inverse matrix calculus

Let A be a square matrix of order n.

If A is invertible, (that is to say that $\det A \neq 0$), then its inverse matrix is given by the following (important) formula:

$$\boxed{A^{-1} = \frac{1}{\det A}(coA)^t}$$

Exercise

Calculate M^{-1}. How can we check the found result?

Correction

Since we have already calculated $\det M$ and coM, it remains to apply the formula:

$$\begin{aligned} M^{-1} &= \frac{1}{\det M}(coM)^t \\ &= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{pmatrix}^t \\ &= \frac{1}{2}\begin{pmatrix} 2 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix} \end{aligned}$$

To check the result, just multiply M by its inverse M^{-1}. We must then find the identity matrix.

Indeed, for our example:

$$\begin{aligned} MM^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= Id. \end{aligned}$$

important Remarks:

1) $Id^{-1} = Id$.

2) $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

3) To have the inverse of a matrix which is diagonal, it suffices to invert its diagonal elements.

For example

$$N = \begin{pmatrix} \pi & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(3,3)} \Rightarrow N^{-1} = \begin{pmatrix} \frac{1}{\pi} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(3,3)}$$

1.2 Lesson N°2 Vector spaces and linear maps

1.2.1 Internal laws and compositions

Definition : Internal composition law

A set E is endowed with an internal composition law if at certain evens (a, b) of elements of E, corresponds a well-determined element of E.

Examples

1) Addition is a law of composition internal to the set of real numbers.

2) Division is also a law of composition internal to the set of real numbers defined when the divisor is not zero.

Definition : External composition law

Consider two sets E and K. Suppose that at any element $a \in E$ and that to any element $\lambda \in K$ we associate an element $b \in E$.

We say that we have defined an external composition law on E.

Example

E is the set of vectors in three-dimensional space and $K = \mathbb{R}$.

The multiplication of a vector $\overrightarrow{u}$ by a real number λ is a vector $(\lambda \overrightarrow{u})$. This operation is an external composition law on the set E.

1.2.2 Vector spaces

In what follows the set K denotes $\mathbb{R}$ or $\mathbb{C}$.

Definition

We say that a set E is a vector space over the field K and we denote E a K-VS if we can define on the elements of E -which we now call vectors- two composition laws, one internal and one external satisfying the following axioms:

- **The intern operation:** noted $+$, it must be commutative, associative, having a neutral element 0_E and for which each vector u of E admits a reciprocal (or inverse) element u' that verifies: $u + u' = 0_E$.

  Our internal operation must therefore verify:

1. $\forall x, y \in E, (x + y) = (y + x)$
2. $\forall x, y, z \in E, (x + y) + z = x + (y + z)$
3. $\forall x \in E, (x + 0_E) = (0_E + x) = x$
4. $\forall x \in E, \exists x' \in E/(x + x') = (x' + x) = 0_E$.

- **The extern operation**: denoted $\cdot$, it is the multiplication by the (external) elements of K, which we will henceforth call scalars as opposed to the vectors of E.

  It must satisfy the following axioms:

  1. $\forall \alpha \in K, \forall x, y \in E, \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.(Distributivity over E).
  2. $\forall \alpha, \beta \in K, \forall x \in E, (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$. (Distributivity over K).
  3. $\forall \alpha, \beta \in K, \forall x \in E, (\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta x)$.
  4. $\forall x \in E, (1_K \cdot x) = x$. 1_K is the unity of K.

Operations 1. and 2. are called linear operations on the vectors of E.

Examples

The following sets are K-Vector Spaces:

- E the set of vectors of elementary geometry.
- P(n) the set of polynomials in one real variable of degree less than or equal to n.
- S the set of convergent numerical series.
- Ş the set of solutions of a linear differential equation without second member.

1.2.3 Immediate properties

1. $\forall x \in E, 0_K x = 0_E$.
2. $\forall \alpha \in K, \alpha.0_E = 0_E$.
3. $\forall \alpha \in K, \forall x \in E, (\alpha.x) = 0_E \Rightarrow \alpha = 0_K$ or $x = 0_E$.

1.2.4 Basic definitions and properties

Independent vectors, related families

By taking p elements $(u_1, u_2, ...u_p)$ from a K-VS, we constitute a family (or a system) of vectors.

Definiion

One says that the p vectors are independent or that the family of p vectors is free if and only if:

$$\forall \alpha_i \in K, \sum_{i=1}^{p} \alpha_i u_i = 0 \Rightarrow \alpha_i = 0, i = \overline{1,p}$$

Otherwise, that is, if there is p scalars α_i not all zero such as $\sum_{i=1}^{p} \alpha_i u_i = 0$ then the vectors $(u_1, u_2, ..., u_p)$ are dependent or form a related family (not free).

In this case, at least one of the α_i, say α_p is non-zero. We can then write:

$$\begin{aligned} \alpha_p u_p &= -\alpha_1 u_1 - \alpha_2 u_2 - ... - \alpha_{p-1} u_{p-1} \\ &\Longrightarrow \quad u_p = \sum_{i=1}^{p-1} \lambda_i u_i \text{ where } \lambda_i = -\frac{\alpha_i}{\alpha_p} \end{aligned}$$

We say that a vector of the family is a linear combination of the $(p-1)$ other vectors.

Vector subspaces

Let E be a K-VS let $F \subset E$ be non-empty.

Definition

For F to be a vector subspace of E, it is necessary and sufficient that any linear combination of vectors of F be a vector of F. We say that F is stable under linear combination:

$$\forall u, v \in F, \forall \alpha, \beta \in K, (\alpha u + \beta v) \in F$$

Examples

1) $\{0\}$ is a vector subspace of all vector spaces.

2) The space of vectors parallel to a given plane forms a vector subspace of free vectors of elementary geometry, it is called vector plane.

Generators, bases and coordinates

a) Consider a family of p vectors $(u_1, u_2, ..., u_p)$ of a K-VS E. The set of all possible linear combinations of these p vectors form a vector subspace F of E. We say that F is generated by the family $(u_1, u_2, ..., u_p)$ or that the u_i are generators of F.

b) If the family above is free, then we say that it is a basis of E. We also say generator free family of E.

Examples

1- Two independent, i.e. non-collinear, vectors of a plane constitute a basis of this plane.

2- Functions $y_1 = sinx$ and $y_2 = cosx$ form a basis of the VS of the solutions of the differential equation $y'' + y = 0$.

1.2.5 Theorem

For a family of n vectors ß=$(e_1, e_2, ..., e_n)$ of a K-VS E to be a basis of E, it is necessary and sufficient that any vector u of E admits a unique decomposition with respect to this family, of the form : $u = \sum_{i=1}^{n} x_i e_i$.

The scalars (x_i) are called the coordinates (or components) of u with respect to the basis ß= $(e_1, e_2, ..., e_n)$.

1.2.6 Incomplete basis theorem

Let ß= $(e_1, e_2, ..., e_n)$ be a basis of a K-VS E and let $(u_1, u_2, ..., u_p)$ be a family of independent vectors of E with $p < n$.

We can then complete the family $(u_1, u_2, ..., u_p)$ by $(n-p)$ suitably chosen vectors from ß to form a new basis of E.

Remarks
1/ Any basis of E has **exactly** n vectors.
2/ Any free family of E has **at most** n vectors.

1.2.7 Dimension

Definition

If there exists a natural number n such that the VS E has a basis made up of n vectors, then any other basis of E is also made up of n vectors and this number n is called dimension of E, denoted dim(E) .

$$\dim(E) = n$$

Remarks

1/ If n is finite, the vector space E is said to be finite-dimensional, like $\mathbb{R}^2, \mathbb{R}^3$.

2/ The space $\{0_E\}$ is of dimension 0 by convention.

Canonical basis
We often favour a basis that we call canonical and we represent it by the n-tuples : $\{(1,0,0,...,0), (0,1,0,...,0), (0,0,1,...,0), ..., (0,0,0,...,1)\}$.

1.2.8 Rank

Let be a system of p vectors $(u_1, u_2, ..., u_p)$ of a K-VS of $dim(E) = n$.

The maximum number of **free** vectors that can be extracted from the given system is called the rank r.

It is to highlight that :

$$r \leq n \quad \text{and} \quad r \leq p \qquad \text{i.e.} \qquad r \leq \inf(n, p).$$

1.2.9 Linear maps

Definitions

Linear maps

Let E and F two K-VS.
We call linear map of E into F the map $f : E \to F$ that verifies:

$$\forall \alpha, \beta \in K, \forall x, y \in E, f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Remark

It is easy to proove that, when f is a linear map, we then necessarily have : $f(0_E) = 0_F$.

Kernel (or core)

We call the kernel of f and we denote by $ker f$[1] the reciprocal image of the null vector of F, i.e.

$$\ker f = \{u \in E / f(u) = 0_F\}$$

Image

We call image of f and we denote by Imf, the vector set $f(u)$ such that u is in E, i.e:

$$\operatorname{Im} f = \{v \in F / \exists u \in E / f(u) = v\}$$

Or:

$$\operatorname{Im} f = \{f(u) \text{ with } u \in E\}$$

Theorem

If E is of finite dimension, and $f : E \to F$ is a linear map, then one has:

$$\dim(\ker f) + \dim(\operatorname{Im} f) = \dim E$$

[1] ker comes from the German word kern which means core.

Rank of a linear map

We call rank of a linear map $f : E \to F$ the dimension $dim(Imf)$ when it is finite.

Remember that when $dim(E)$ is finite, then $dim(Imf)$ is also finite.
One has :

$$\begin{aligned} rg(f) &\leq n \quad \text{and} \\ rg(f) &= n \Leftrightarrow \ker f = \{O_E\} \Leftrightarrow f \text{ is injective.} \end{aligned}$$

Reminder : Injection, surjection and bijection Let us recall the notions of injection (or one-to-one), surjection (onto) and bijection maps, since we are going to need them in the rest of this chapiter.

In mathematics, injections, surjections, and bijections are classes of functions distinguished by the manner in which arguments (input expressions from the domain) and images (output expressions from the codomain) are related or mapped to each other. A function maps elements from its domain to elements in its codomain. Given a function:

$$f : X \to Y$$

- The function is **injective**, or **one-to-one**, if each element of the codomain is mapped to by at most one element of the domain, or equivalently, if distinct elements of the domain map to distinct elements in the codomain. An injective function is also called an injection.

$$\forall x, x' \in X, f(x) = f(x') \Rightarrow x = x'$$

- The function is surjective, or onto, if each element of the codomain is mapped to by at least one element of the domain. That is, the image and the codomain of the function are equal. A surjective function is a surjection.

$$\forall y \in Y, \exists x \in X \quad / \quad f(x) = y$$

- The function is bijective (one-to-one and onto, one-to-one correspondence, or invertible) if each element of the codomain is mapped to by exactly one element of the domain. That is, the function is both injective and surjective. A bijective function is also called a bijection.

$$\forall y \in Y, \exists ! x \in X \quad / \quad f(x) = y$$

1.2.10 Matrix associated with a linear map

Let E be such that $dim(E) = n$ with basis $ß_E = (e_1, e_2, ..., e_n)$ and F be such that $dim(F) = p$ with basis $ß_F = \left(\varphi_1, \varphi_2, ..., \varphi_p\right)$.

Let $f : E \rightarrow F$ be a linear map.

f is completely defined by giving the image of the vectors of $ß_E$ into F : $(f(e_1), f(e_2), ..., f(e_n))$.

Since $f(e_j) = \alpha_{1j}\varphi_1 + \alpha_{2j}\varphi_2 + ... + \alpha_{pj}\varphi_p$ we take : a_{ij} the ith coordinate of $f(e_j)$ in the basis $\left(\varphi_1, \varphi_2, ..., \varphi_p\right)$.

Considering $u \in E/ \ u = \sum\limits_{i=1}^{n} x_i e_i$, one has:

$$f(u) = \sum_{i=1}^{n} x_i \sum_{j=1}^{p} a_{ij}\varphi_j$$

Put :

$$y = f(u)$$

This means that :

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \qquad \vdots \\ y_p = a_{p1}x_1 + a_{p2}x_2 + ... + a_{pn}x_n \end{cases}$$

f is therefore perfectly known if we know the following rectangular table which we call the matrix associated with the map f:

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & \cdots & \cdots & a_{pn} \end{pmatrix}$$

Remark

The number of columns of A_f is equal to $dim(E)$ i.e. to n and the number of lines of A_f is equal to $dim(F)$ i.e. to p.

Example

Let E be such that $dim(E) = 3$ with basis $ß_E = (e_1, e_2, e_3)$ and F be such that $dim(F) = 4$ with basis $ß_F = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$.

Let $f : E \to F$ be the following linear map:

$$f(x, y, z) = (x - 2y - 2z, -3x + y + z, -2x - 2y - 2z, -x + 3y + z)$$

Take: $(y_1, y_2, y_3, y_4) = f(x, y, z)$.

We get the following linear system:

$$\Leftrightarrow \begin{cases} y_1 = x - 2y - 2z \\ y_2 = -3x + y + z \\ y_3 = -2x - 2y - 2z \\ y_4 = -x + 3y + z \end{cases}$$

Which is written in matrix form as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ -3 & 1 & 1 \\ -2 & -2 & -2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So, the matrix associated with our linear map is:

$$A_f = \begin{pmatrix} 1 & -2 & -2 \\ -3 & 1 & 1 \\ -2 & -2 & -2 \\ -1 & 3 & 1 \end{pmatrix}_{(4\times 3)}$$

1.2.11 Linear map associated with a matrix

Let E be a K-VS of dimension n provided with the basis $B_E = \{e_1, e_2, ..., e_n\}$ and let F be a K-VS of dimension p provided with the basis $B_F = \{u_1, u_2, ..., u_p\}$.

Let f be a linear map from E into F of corresponding matrix given by :

$$A_f = \begin{pmatrix} a_{11} & ... & ... & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & ... & ... & a_{pn} \end{pmatrix}$$

Let's take : $(y_1, y_2, ..., y_p) = f(x_1, x_2, ..., x_n)$.

In matrix form, we can write:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & \cdots & \cdots & a_{pn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$$

So, in the form of a system, we will have:

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ y_p = a_{p1}x_1 + a_{p2}x_2 + ... + a_{pn}x_n \end{cases}$$

Which allows us to write:

$$\begin{aligned} f(x_1, x_2, ..., x_n) &= (y_1, y_2, ..., y_p) \\ &= \begin{matrix} (a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n, y_2\ , a_{21}x_1 + a_{22}x_2 \\ +... + a_{2n}x_n\ , ...\ ,\ a_{p1}x_1 + a_{p2}x_2 + ... + a_{pn}x_n) \end{matrix} \end{aligned}$$

Which is exactly the linear map associated with the matrix A_f.

Remarks

1. The matrix of a linear map depends on the choices of the bases of the starting and the arrival sets.

2. Expression $f(x_1, x_2, ..., x_n)$ also depends on the choices of the bases of the starting set and that of the finish.

3. Changing a basis changes the matrix and the expression $f(x_1, x_2, ..., x_n)$.

Example

Let E be a K-VS of dimension 3 provided with the basis $B_E = \{e_1, e_2, e_3\}$ et Let F be a K-VS of dimension 3 provided with the basis $B_F = \{u_1, u_2, u_3\}$.

Let f be a linear map from E towards F of a corresponding matrix, given by :

$$A_f = \begin{pmatrix} -1 & -2 & 1 \\ 2 & -2 & 1 \\ -2 & -2 & -1 \end{pmatrix}$$

Let us take: $(X, Y, Z) = f(x, y, z)$.

In matrix form, we can write:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 2 & -2 & 1 \\ -2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So, the linear map associated with the matrix A_f is given by :

$$f(x, y, z) = (-x - 2y + z, 2x - 2y + z, -2x - 2y - z)$$

1.3 Lesson N°3 Change of basis, transition matrix

It is necessary to keep in mind that the matrix of a linear map is a representation of it, which depends on the choice of the bases at the beginning and at the arrival set. It is useful to know how to move from one basis to another.

Definition

Let E be such that $dim(E) = n$ and consider two different basis of E, ß$= (e_1, e_2, ..., e_n)$ and ß$' = (\varphi_1, \varphi_2, ..., \varphi_n)$.

The vectors of ß$'$ can be expressed in the first basis ß as follows:

$$\begin{cases} \varphi_1 = a_{11}e_1 + a_{12}e_2 + ... + a_{1n}e_n \\ \varphi_2 = a_{21}e_1 + a_{22}e_2 + ... + a_{2n}e_n \\ \vdots \\ \varphi_n = a_{n1}e_1 + a_{n2}e_2 + ... + a_{nn}e_n \end{cases}$$

The square matrix P obtained by writing the above system of equations in matrix form is called the transition matrix from basis ß to basis ß'.

$$P = \begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & ... & a_{nn} \end{pmatrix}$$

Remark

1. P is necessarily invertible.

2. The matrix P^{-1} is the transition matrix from the ß$'$ basis to the ß basis.

Example

Let ß$(o, \overrightarrow{i}, \overrightarrow{j})$ and ß$'(o, \overrightarrow{i'}, \overrightarrow{j'})$ two orthonormal landmarks of $\mathbb{R}^2$. We consider the rotation on the origin o of angle θ.

So the transition matrix to pass from ß to ß$'$ is given by :

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

And its inverse is:

$$P^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Another way of basis change is the diagonalization of square matrices, which we will expose in the following.

1.3.1 Diagonalization of square matrices

Diagonalizing a square matrix M amounts to finding a passing matrix P which is invertible and a diagonal matrix D such that: $M = PDP^{-1}$.

Unfortunately, not all matrices are diagonalizable. To know if a given matrix is diagonalizable or not, we have to go through three steps:

1. Calculate its eigenvalues,
2. Calculate its eigenvectors,
3. Apply the Diagonalization Theorem.

In what follows, we will consider three examples on which we will apply the theory in order to assimilate the basic notions:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}_{(2,2)}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 6 & -3 \\ -1 & 4 & 0 \end{pmatrix}.$$

1.3.2 Eigenvalues

Let $\lambda \in \mathbb{R}$,

Characteristic polynomial

We call characteristic polynomial of a square matrix A and we denote it by $P(\lambda)$, the determinant of the matrix $(A - \lambda Id)$, in other words $P(\lambda) = \det(A - \lambda Id)$.

Eigenvalues

The eigenvalues of the square matrix A are the roots of its characteristic polynomial, i.e. the solutions of the equation

$$P(\lambda) = 0.$$

Example 1

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

Find the eigenvalues of A:
Calculate the characteristic polynomial, first :

$$\begin{aligned} \lambda Id &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \\ &\Rightarrow (A - \lambda Id) = \begin{pmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} P(\lambda) &= \det(A - \lambda Id) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda) - 12 \end{aligned}$$

$$P(\lambda) = \lambda^2 - 3\lambda - 10.$$

To find the roots of $P(\lambda)$, we must factor it, by calculating the discriminant:

$$
\begin{aligned}
\Delta &= (-3)^2 - 4(1)(-10) = 49, \\
\text{So } \lambda_1 &= -2 \text{ and } \lambda_2 = 5. \\
P(\lambda) &= (\lambda + 2)(\lambda - 5).
\end{aligned}
$$

$$P(\lambda) = 0 \Leftrightarrow \lambda_1 = -2 \text{ and } \lambda_2 = 5.$$

So, the set of **eigenvalues** of A , denoted $\sigma(A)$ is :

$$\sigma(A) = \{-2, 5\}$$

Example 2

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

Find the eigenvalues of B:
Calculate the characteristic polynomial:

$$(B - \lambda Id) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

$$
\begin{aligned}
P(\lambda) &= \det(B - \lambda Id) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 (2-\lambda).
\end{aligned}
$$

$$P(\lambda) = 0 \Leftrightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 2.$$

So, the set of **eigenvalues** of B , denoted $\sigma(B)$ is :

$$\sigma(B) = \{1, 2\}$$

Example 3

$$C = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 6 & -3 \\ -1 & 4 & 0 \end{pmatrix}$$

Find the eigenvalues of C:
Calculate the characteristic polynomial:

$$(C - \lambda Id) = \begin{pmatrix} 1-\lambda & 4 & -2 \\ 0 & 6-\lambda & -3 \\ -1 & 4 & -\lambda \end{pmatrix}.$$

$$\begin{aligned} P(\lambda) &= \det(C - \lambda Id) = \begin{vmatrix} 1-\lambda & 4 & -2 \\ 0 & 6-\lambda & -3 \\ -1 & 4 & -\lambda \end{vmatrix} \\ &= -\left(\lambda^3 - 7\lambda^2 + 16\lambda - 12\right). \end{aligned}$$

Note that $\lambda_1 = 2$ is an apparent root, we therefore make an Euclidean division by $(\lambda - \lambda_1)$:

$$\begin{array}{r|l} \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda - 2 \\ \hline -2\lambda^2 \quad\quad\quad\quad & \lambda^2 - 5\lambda + 6 \\ \hline -5\lambda^2 + 16\lambda - 12 & \\ 10\lambda \quad\quad\quad & \\ \hline 6\lambda - 12 & \\ 0 & \end{array}$$

We get :

$$P(\lambda) = -(\lambda - 2)\left(\lambda^2 - 5\lambda + 6\right).$$

It remains to factorize $(\lambda^2 - 5\lambda + 6)$ by calculating the discriminant Δ.

$$\Delta = (-5)^2 - 4(1)(6) = 1, \text{so}, \lambda_1 = 2 \ \text{ and } \ \lambda_2 = 3.$$

$$P(\lambda) = -(\lambda - 2)(\lambda - 2)(\lambda - 3) = -(\lambda - 2)^2(\lambda - 3)$$

$$P(\lambda) = 0 \Leftrightarrow \lambda_1 = 2 \ \text{ and } \ \lambda_2 = 3.$$

So, the set of **eigenvalues** of C , denoted $\sigma(C)$ is :

$$\sigma(C) = \{2, 3\}$$

Multiplicity of eigenvalues

The multiplicity of the eigenvalue λ is the number of times it is the root of the characteristic polynomial, that is, its power in expansion of the characteristic polynomial.

In example 1, the multiplicity of $\lambda_1 = -2$ and $\lambda_2 = 5$ is equal to 1, We say that they are **simple** eigenvalues.

In example 2, $\lambda_2 = 2$ is a **simple** eigenvalue, but the multiplicity of $\lambda_1 = 1$ is equal to 2, it is said to be a **double** eigenvalue.

In example 3, $\lambda_1 = 2$ is a **double** eigenvalue. $\lambda_2 = 3$ is a **simple** eigenvalue.

1.3.3 Eigenvectors

Definition the solution space and its dimension

Let (S) be a non-Cramerian system, where the solution is given by the parameter α as follows :

$$\begin{cases} x = a\alpha \\ y = b\alpha \\ z = c\alpha \end{cases}, \alpha \in \mathbb{R}.$$

The set $E = \{(x, y, z) \in \mathbb{R}^3 / x = a\alpha, y = b\alpha, z = c\alpha, \alpha \in \mathbb{R}\}$ is called the solution space of (S).

We say that the vector $V \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the vector that generates E the solution space of (S).

There are as many generating vectors as there are parameters in the expression of the solution.

We call dimension of the solution space and we denote $dimE$, the number of vectors generating the space E.

Definition eigenspace associated with an eigenvalue

Let λ_i be an eigenvalue of a matrix A,

The solution space of the non-Cramerian system $(A - \lambda_i Id)\, X = 0$ is called eigenspace associated with λ_i, it is noted E_{λ_i}.

The eigenvectors associated with the eigenvalue λ_i are the vectors which generate the eigenspace E_{λ_i}.

Calculation of the eigenvectors of A

*) For $\lambda_1 = -2$:

$$(A - \lambda_1 Id)\, X = 0 \Leftrightarrow (A - (-2)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{cases} 4x + 3y = 0 \\ 4x + 3y = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = \frac{-3}{4}y \\ y \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{-3}{4}\alpha \\ y = \alpha \end{cases}, \alpha \in \mathbb{R}.$$

The vector generating the eigenspace is $V_1 \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}$, so $dim E_{\lambda_1} = 1$.

*) For $\lambda_2 = 5$:

$$(A - \lambda_2 Id)\, X = 0 \Leftrightarrow (A - (5)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{cases} -3x + 3y = 0 \\ 4x - 4y = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = y \\ y \in \mathbb{R} \end{cases},$$

Or:

$$\begin{cases} x = \alpha \\ y = \alpha \end{cases}, \alpha \in \mathbb{R}.$$

The vector generating the eigenspace is $V_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $dim E_{\lambda_2} = 1$.

Calculation of the eigenvectors of B

*) For $\lambda_1 = 1$:

$$(B - \lambda_1 Id)\, X = 0 \Leftrightarrow (B - (1)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 0 = 0 \\ 0 = 0 \\ x - y + z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y - z \\ y \in \mathbb{R} \\ z \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \alpha - \beta \\ y = \alpha \\ z = \beta \end{cases} , \alpha \in \mathbb{R}, \beta \in \mathbb{R}.$$

The vectors that generate the eigenspace are $V_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,$V_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, so $dimE_{\lambda_1} = 2$.

*) For $\lambda_2 = 2$:

$$(B - \lambda_2 Id)\, X = 0 \Leftrightarrow (B - (2)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x = 0 \\ -y = 0 \\ x - y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \\ z \in \mathbb{R} \end{cases}$$

Or

$$\begin{cases} x = 0 \\ y = 0 \\ z = \alpha \end{cases}, \alpha \in \mathbb{R}.$$

The vector generating the eigenspace is $V_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so $dim E_{\lambda_2} = 1$.

Calculation of the eigenvectors of C

*) For $\lambda_1 = 3$:

$$(C - \lambda_1 Id)\, X = 0 \Leftrightarrow (C - (3)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} -2 & 4 & -2 \\ 0 & 3 & -3 \\ -1 & 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -2x + 4y - 2z = 0 & (1) \\ 3y - 3z = 0 & (2) \\ -x + 4y - 3z = 0 & (3) \end{cases}$$

Since $\Delta = \begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix} = -6 \neq 0$, we sacrifice the parameter z, and we keep equation (3) as verification equation: (2) $\Leftrightarrow y = z$.

Replacing in (1) $\Leftrightarrow -2x + 4\,(z) - 2z = 0 \Leftrightarrow x = z$.

(3) $\Leftrightarrow -(z) + 4(z) - 3z = 0$, is verified.

Finally

$$\begin{cases} x = z \\ y = z \\ z \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x = \alpha \\ y = \alpha \\ z = \alpha \end{cases}, \alpha \in \mathbb{R}.$$

The vector generating the eigenspace is $V_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $dim E_{\lambda_1} = 1$.

*) For $\lambda_2 = 2$:

$$(C - \lambda_2 Id)\, X = 0 \Leftrightarrow (C - (2)Id)\, X = 0$$

$$\Leftrightarrow \begin{pmatrix} -1 & 4 & -2 \\ 0 & 4 & -3 \\ -1 & 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x + 4y - 2z = 0 & (1) \\ 4y - 3z = 0 & (2) \\ -x + 4y - 2z = 0 & (3) \end{cases}$$

Since $\Delta = \begin{vmatrix} -1 & 4 \\ 0 & 4 \end{vmatrix} = -4 \neq 0$, we sacrifice the parameter z, and keep equation (3) as a verification equation: (2) $\Leftrightarrow y = \frac{3}{4}z$.

Replacing in (1) $\Leftrightarrow -x + 4\left(\frac{3}{4}z\right) - 2z = 0 \Leftrightarrow x = z$.

(3) $\Leftrightarrow -(z) + 4\left(\frac{3}{4}z\right) - 2z = 0$, verified.

Finally

$$\begin{cases} x = z \\ y = \frac{3}{4}z \\ z \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x = \alpha \\ y = \frac{3}{4}\alpha \\ z = \alpha \end{cases}, \alpha \in \mathbb{R}.$$

The vector generating the eigenspace is $V_2 \begin{pmatrix} 1 \\ \frac{3}{4} \\ 1 \end{pmatrix}$, so $dimE_{\lambda_2} = 1$.

1.3.4 Diagonalization theorems

Theorem 1

Let M be a square matrix $(n \times n)$.

If M has n **simple eigenvalues**, then M is diagonalizable.

Theorem 2

If each eigenvalue of M has an eigenspace of dimension equal to its multiplicity, then M is diagonalizable.

Example 1:

For matrix A, one has:

Eigenvalues	*multiplicity*	$\dim E_\lambda$
$\lambda_1 = -2$	$1(simple)$	1
$\lambda_2 = 5$	$1(simple)$	1

A is of order 2 and it has 2 simple eigenvalues, then A is diagonalizable, (by Theorem 1).

Example 2 :
For matrix B, one has:

Eigenvalues	*multiplicity*	$\dim E_\lambda$
$\lambda_1 = 1$	2(*double*)	2
$\lambda_2 = 2$	1(*simple*)	1

Which means that B is diagonalizable, (by Theorem 2).

Example 3 :
For matrix C, one has:

Eigenvalues	*multiplicity*	$\dim E_\lambda$
$\lambda_1 = 3$	2(*double*)	1
$\lambda_2 = 2$	1(*simple*)	1

so C is **not** diagonalizable.

Diagonalization

Theorem 3

if M is a diagonalizable square matrix, then $M = PDP^{-1}$, where D is the diagonal square matrix formed by all the eigenvalues of M and P is the square matrix formed by the associated eigenvectors, in an ordered way.

Example 1

$$A = PDP^{-1} \quad \text{where} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} -\frac{3}{4} & 1 \\ 1 & 1 \end{pmatrix}.$$

Example 2

$$B = PDP^{-1} \text{ where } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Usefulness of diagonalization

When we diagonalize M, we find P and D that verify $M = PDP^{-1}$, we can then calculate the nth powers of M easily, since:

$$\begin{aligned} M &= PDP^{-1} \\ M^2 &= M.M = PDP^{-1}PDP^{-1} = PD^2P^{-1}(\text{because } \ P^{-1}P = Id). \\ M^3 &= M^2.M = PD^2P^{-1}PDP^{-1} = PD^3P^{-1} \\ &\vdots \\ M^n &= PD^nP^{-1}. \end{aligned}$$

Knowing that D^n is easily obtained, one can caculate M^n.

Exercise

Calculate A^4.

Correction

We start by calculating P^{-1} :

$$P^{-1} = \frac{1}{\det P}\,(coP)^t = \frac{1}{7}\begin{pmatrix} -4 & 4 \\ 4 & 3 \end{pmatrix}.$$

Knowing that

$$\begin{aligned} P &= \begin{pmatrix} -\frac{3}{4} & 1 \\ 1 & 1 \end{pmatrix};\ \text{and} \\ D &= \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow D^4 = \begin{pmatrix} (-2)^4 & 0 \\ 0 & (5)^4 \end{pmatrix}\ \text{because } D \text{ is a diagonal matrix} \\ &\Rightarrow\ D^4 = \begin{pmatrix} 16 & 0 \\ 0 & 625 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{aligned} A^4 &= PD^4P^{-1} \\ &= \frac{1}{7}\begin{pmatrix} -\frac{3}{4} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 625 \end{pmatrix}\begin{pmatrix} -4 & 4 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 364 & 261 \\ 348 & 277 \end{pmatrix}. \end{aligned}$$

1.4 Series of exercises N°1

Exercise 1

I) Let A be the matrix : $A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$.

1/ Verify that : $A^2 - 3A + 2Id = 0$.

2/ Using the first question, calculate A^{-1}.

3/ Confirm the result found in question 2 of A^{-1} by another method.

II) Let be the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

Calculate $A^n, \forall n > 1$.

Exercise 2

1) Calculate the determinant of the following matrix:

$$K(x) = \begin{pmatrix} 2-x & 2 & 0 \\ 1 & 1-x & 0 \\ 0 & 0 & 4-x \end{pmatrix}$$

2) What are the values of x for which $K(x)$ is invertible?

Exercise 3

Let $\theta \in \mathbb{R}$. Let ß$(o, \overrightarrow{i}, \overrightarrow{j})$ and ß$'(o, \overrightarrow{i'}, \overrightarrow{j'})$ two orthonormal landmarks of $\mathbb{R}^2$. We consider the rotation on the origin o of angle θ.

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1) Calculate $A^t(\theta), A^2(\theta)$.

2) Show that $A(\theta)$ is invertible and calculate $A^{-1}(\theta)$.

Exercise 4

Check if the following spaces are vector subspaces of the K-VS $\mathbb{R}^3$:

1. $A = \{(x, y, z) \in \mathbb{R}^3 / x \geq 0\}$.
2. $B = \{(x, y, 1) \in \mathbb{R}^3 / x, y \in \mathbb{R}\}$.

3. $C = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 \leq 1\}$.

4. $D = \{(x, y, z) \in \mathbb{R}^3 / x = 2y \text{ and } z = 0\}$.

Exercise 5

1. For what value of k the following vector of $\mathbb{R}^3$, $u = (1, -2, k)$ is a linear combination of the two vectors $a = (1, 1, 1)$ and $b = (1, 2, 3)$?

2. Show that the vectors $a = (1, 1, 1), b = (1, 2, 1)$, and $c = (2, -1, 1)$ generate $\mathbb{R}^3$.

Exercise 6

1. Show that the functions f, g and h defined by:

$$f(x) = x, \qquad g(x) = \sin x, \qquad h(x) = \cos x$$

are linearly independent in the set of functions of $\mathbb{R}$ into $\mathbb{R}$, noted $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

2. Let the vectors $a = (0, 1, -1), b = (-1, 0, 1)$, and $c = (1, -1, 0)$ of $\mathbb{R}^3$.

a) Show that these vectors are independent in evens.
b) Is the family $\{a, b, c\}$ free?
c) What is the dimension of the vector space generated by this family?

Exercise 7

Let E be the vector subspace of $\mathbb{R}^4$ generated by the vectors :

$$a = (-1, 1, -3, 0), b = (1, 2, 0, 1), c = (2, 1, 3, 1)$$

Give a basis of E and determine $dim(E)$.

Exercise 8

Show that the vectors:

$$a = (1, 1, 2, 1), b = (1, -1, 0, 1), c = (0, 0, -1, 1), \text{ and } d = (1, 2, 2, 0)$$

form a basis of $\mathbb{R}^4$.

2/ Find the coordinates of the vector $u = (1, 1, 1, 1)$ in this basis.

Exercise 9

Say if the following maps are linear or not:

1. $f : \mathbb{R}^2 \to \mathbb{R}^3 / f(x, y) = (x + y, x - 2y, 0)$.
2. $g : \mathbb{R}^2 \to \mathbb{R}^3 / g(x, y) = (x + y, x - 2y, 1)$.
3. $h : \mathbb{R}^2 \to \mathbb{R} / h(x, y) = x^2 - y^2$.

Exercise 10

1/ Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined by

$$f(x, y) = (x + y, x - y, x + y).$$

Determine the kernel of f then find its image.

Can you tell if f is injective (one-to-one)? Surjective (onto)?

2/ Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ the linear map defined by :

$$f(x, y, z) = (-3x - y + z, \quad 8x + 3y - 2z, \quad -4x - y + 2z).$$

a) Determine a kernel basis of f along with its dimension.
b) Is f injective?
c) Find a basis of $Im(f)$.

Exercise 11

Let the two following matrices:

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & -1 \\ -2 & -2 & 1 \end{pmatrix}.$$

1/ Find the eigenvalues of A and M.
2/ Find the eigenvectors associated with the eigenvalues of the two matrices.
3/ Give the diagonal matrix D and the transition matrix P for both matrices.

1.5 Correction of the exercise series N°1

Exercise 1

I) Let A be the matrix : $A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$.

1/ Verify that : $A^2 - 3A + 2Id = 0$:

$$A^2 = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix}$$

$$-3A = -3 \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 3 \\ 9 & -12 & 9 \\ 3 & -3 & 0 \end{pmatrix}$$

$$2Id = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} A^2 - 3A + 2Id &= \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 3 \\ 9 & -12 & 9 \\ 3 & -3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

2/ Using the first question, calculate A^{-1} :

$$\begin{aligned} A^2 - 3A + 2Id &= 0 \\ &\Leftrightarrow A^2 - 3A = -2Id \\ &\Leftrightarrow A(A - 3Id) = -2Id \end{aligned}$$

so,

$$A\left[\frac{-1}{2}(A - 3Id)\right] = Id$$

Finally

$$\begin{aligned} A^{-1} &= \left[\frac{-1}{2}(A-3Id)\right] \\ &= \frac{-1}{2}\left[\begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\ &= \frac{-1}{2}\begin{pmatrix} -3 & 1 & -1 \\ -3 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix} \end{aligned}$$

3/ Confirm the result of A^{-1} by another method.

$$A^{-1} = \frac{1}{\det A}(coA)^t$$

$$\det A = \begin{vmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{vmatrix} = 2$$

$$\begin{aligned} coA &= \begin{pmatrix} +\begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -3 & -3 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & -1 \\ 4 & -3 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ -3 & -3 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -3 & 4 \end{vmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 1 \\ -1 & -1 & -1 \\ 1 & 3 & 3 \end{pmatrix} \end{aligned}$$

$$(coA)^t = \begin{pmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{pmatrix}.$$

so,

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} -3 & 1 & -1 \\ -3 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix}.$$

II) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

$$A^4 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\vdots$$

$$\forall n > 1, \qquad A^n = \begin{cases} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} & \text{if } n \text{ is odd} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n \text{ is even} \end{cases}.$$

Exercise 2

1) Calculate the determinant of the following matrix:

$$K(x) = \begin{pmatrix} 2-x & 2 & 0 \\ 1 & 1-x & 0 \\ 0 & 0 & 4-x \end{pmatrix}$$

$$det(K(x)) = (4-x) \begin{vmatrix} 2-x & 2 \\ 1 & 1-x \end{vmatrix} = (4-x)\left[(2-x)(1-x) - 2\right]$$

$$= (4-x)\left(x^2 - 3x\right) = x(4-x)(x-3)$$

2/ What are the values of x such that $K(x)$ is invertible?

$$det(K(x)) = 0 \Leftrightarrow x(4-x)(x-3) = 0 \Leftrightarrow x = 0, x = 4, \text{or } x = 3.$$

For $K(x)$ to be invertible, it is necessary that $x \in \mathbb{R}\setminus\{0, 3, 4\}$.

Exercise 3

Let $\theta \in \mathbb{R}$. Let ß$(o, \overrightarrow{i}, \overrightarrow{j})$ and ß$'(o, \overrightarrow{i'}, \overrightarrow{j'})$ two orthonormal landmarks of $\mathbb{R}^2$.

We consider the rotation on the origin o of angle θ.

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Knowing that:

$$\begin{aligned} \cos^2(\theta) - \sin^2(\theta) &= \cos(2\theta) \\ 2\cos\theta\sin\theta &= \sin(2\theta) \\ \cos^2(\theta) + \sin^2(\theta) &= 1 \end{aligned}$$

1)

$$\begin{aligned} A^t(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \\ A^2(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2(\theta) - \sin^2(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \end{aligned}$$

2) Demonstrate that $A(\theta)$ is invertible and calculate $A^{-1}(\theta)$.

$$\begin{aligned} \det A(\theta) &= \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2(\theta) + \sin^2(\theta) \\ &= 1 \neq 0. \end{aligned}$$

So $A(\theta)$ is invertible.

$$\begin{aligned} co\,(A(\theta)) &= \begin{pmatrix} +\cos\theta & -\sin\theta \\ -(-\sin\theta) & +\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ [co\,(A(\theta))]^t &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ A^{-1}(\theta) &= \frac{1}{\det A(\theta)}\,[co\,(A(\theta))]^t \\ &= \frac{1}{1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ A^{-1}(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

Exercise 4

Check if the following spaces are vector subspaces of the K-VS $\mathbb{R}^3$:

1. $A = \{(x, y, z) \in \mathbb{R}^3 / x \geq 0\}$.

   One has : $A \subset \mathbb{R}^3$ and A is non-empty since it contains the vector $(0, 0, 0)$ for example.

   Then just take $u = (4, 0, -1)$ and $\lambda = -1$ for example to have

   $$\lambda u = (-4, 0, 1) \notin A.$$

   So, no, A is not a vector subspace of $\mathbb{R}^3$.

2. $B = \{(x, y, 1) \in \mathbb{R}^3 / x, y \in \mathbb{R}\}$.

   B is not a vector subspace of $\mathbb{R}^3$ because $(0, 0, 0) \notin B$.

3. $C = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 \leq 1\}$.

   One has : $C \subset \mathbb{R}^3$ and C is non-empty since it contains the vector $u = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for example.

   Then just take $\lambda = 2$ for example to have $\lambda u = (1, 1, 1) \notin C$.

   so, C is not a vector subspace of $\mathbb{R}^3$.

4. $D = \{(x, y, z) \in \mathbb{R}^3 / x = 2y \text{ and } z = 0\}$. It is easy to see that $D \subset \mathbb{R}^3$ and that $D \neq \emptyset$.

   Let $\alpha, \beta \in K, u = (x, y, z), v = (x', y', z') \in D$. One has :

   $$\begin{aligned} \alpha x + \beta x' &= \alpha (2y) + \beta(2y') \\ &= 2(\alpha y + \beta y') \end{aligned}$$

And since $z = z' = 0$ then $\alpha z + \beta z' = 0$. So $(\alpha u + \beta v) \in D$. One may then conclud that D is a vector subspace of $\mathbb{R}^3$.

Exercise 5

1. For what value of k the following vector of $\mathbb{R}^3$, $u = (1, -2, k)$ is a linear combination of the two vectors $a = (1, 1, 1)$ and $b = (1, 2, 3)$?

The vector $u = (1, -2, k)$ is a linear combination of the two vectors a and b if we can find two scalars α and β that verify:

$$u = \alpha x + \beta y$$

We then have:

$$\begin{aligned}
(1,-2,k) &= \alpha(1,1,1)+\beta(1,2,3) \\
&= (\alpha+\beta,\quad \alpha+2\beta,\quad \alpha+3\beta) \\
&\Leftrightarrow \begin{cases} \alpha+\beta=1 \\ \alpha+2\beta=-2 \\ \alpha+3\beta=k \end{cases} \\
&\Leftrightarrow k=-5
\end{aligned}$$

2. Show that the vectors $a = (1,1,1), b = (1,2,1)$, and $c = (2,-1,1)$ generate $\mathbb{R}^3$.

We must show that any vector $u = (x,y,z)$ of $\mathbb{R}^3$ is written as a linear combination of a,b and c.

That is, there are scalars α, β, γ such that $u = \alpha a + \beta b + \gamma c$.

i.e:

$$\begin{aligned}
(x,y,z) &= (\alpha+\beta+2\gamma,\quad \alpha+2\beta-\gamma,\quad \alpha+3\beta+\gamma) \\
&\Leftrightarrow \begin{cases} \alpha+\beta+2\gamma=x \\ \alpha+2\beta-\gamma=y \\ \alpha+3\beta+\gamma=z \end{cases} \\
&\Leftrightarrow \begin{cases} \alpha+\beta+2\gamma=x \\ -\beta+3\gamma=x-y \\ -2\beta+\gamma=x-z \end{cases} \\
&\Leftrightarrow \begin{cases} \alpha+\beta+2\gamma=x \\ -\beta+3\gamma=x-y \\ -5\gamma=-x+2y-z \end{cases}
\end{aligned}$$

Which gives :

$$\begin{cases} \beta=\frac{1}{5}(-2x-y+3z) \\ \gamma=\frac{1}{5}(x-2y+z) \\ \alpha=x+y-z \end{cases}$$

Since the $\dim(\mathbb{R}^3)=3$ then the family $\{a,b,c\}$ form a basis of $\mathbb{R}^3$.

Exercise 6

1. Show that the functions f, g and h defined by :

$$f(x) = x, g(x) = \sin x, h(x) = \cos x$$

are linearly independent in the set of functions of $\mathbb{R}$ into $\mathbb{R}$, noted $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Let the scalars α, β, γ be such that $\alpha f + \beta g + \gamma h \equiv 0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$

let's remember that $0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$ is the zero function of $\mathbb{R}$ into $\mathbb{R}$. It is the neutral element of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

So, one has :

$$\forall x \in \mathbb{R}, \alpha f(x) + \beta g(x) + \gamma h(x) = 0$$

Especially for $x = 0$,π and $\frac{\pi}{2}$ one obtains :

$$\begin{cases} \alpha.0 + \beta.0 + \gamma.1 = 0 \\ \alpha\left(\frac{\pi}{2}\right) + \beta.1 + \gamma.0 = 0 \\ \alpha.\pi + \beta.0 + \gamma.(-1) = 0 \end{cases}$$

Which gives: $\alpha = \beta = \gamma = 0$.

Finally, we have the following equivalence:

$$\alpha f + \beta g + \gamma h \equiv 0_{\mathcal{F}(\mathbb{R},\mathbb{R})} \Leftrightarrow \alpha = \beta = \gamma = 0$$

So, the given functions are linearly independent.

2. Let be the vectors $a = (0, 1, -1), b = (-1, 0, 1)$, and $c = (1, -1, 0)$ of $\mathbb{R}^3$.

a) Show that these vectors are independent in evens.

Let's show that a and b are linearly independent:

Let the scalars α, β be such that $\alpha a + \beta b = 0$.

So,

$$\begin{aligned}
\alpha(0,1,-1)+\beta(-1,0,1) &= (0,0,0)\\
&\Leftrightarrow \begin{cases} -\beta = 0\\ \alpha = 0\\ -\alpha+\beta = 0\end{cases}\\
&\Leftrightarrow \alpha=\beta=0
\end{aligned}$$

This implies that a and b are linearly independent.

By the same way, we show that a and c are linearly independent and that b and c are linearly independent too.

b) Is the family $\{a,b,c\}$ free?

Suppose that $\alpha a+\beta b+\gamma c=0$ where α,β,γ are scalars. We will then have:

$$\begin{aligned}
\alpha(0,1,-1)+\beta(-1,0,1)+\gamma(1,-1,0) &= (0,0,0)\\
&\Leftrightarrow \begin{cases} -\beta+\gamma = 0\\ \alpha-\gamma = 0\\ -\alpha+\beta = 0\end{cases}\\
&\Leftrightarrow \alpha=\beta=\gamma
\end{aligned}$$

Therefore, we have an infinity of solutions, the scalars α,β,γ don't have to be all zero.

Family $\{a,b,c\}$ is consequently not free, it is dependent.

Remark

We could just notice from the beginning that : $\alpha=\beta=\gamma \Rightarrow a+b+c=0$ for example.

c) What is the dimension of the vector space generated by this family?
Let S be the set generated by the family $\{a,b,c\}$:

$$\begin{aligned}
S &= \{v\in\mathbb{R}^3/v=\alpha a+\beta b+\gamma c, \quad where \;\; \alpha,\beta,\gamma\in\mathbb{R}\}\\
&= \{v\in\mathbb{R}^3/v=\alpha(-b-c)+\beta b+\gamma c, \quad where \;\; \alpha,\beta,\gamma\in\mathbb{R}\}\\
&= \{v\in\mathbb{R}^3/v=(-\alpha+\beta)b+(-\alpha+\gamma)c, \quad where \;\; \alpha,\beta,\gamma\in\mathbb{R}\}
\end{aligned}$$

So b and c generate S and since they are two linearly independent vectors then they form a basis of S. Which means that $dim(S) = 2$.

Remark

Similarly, we can prove that $\{a, b\}$ and $\{a, c\}$ also form two different basis of S.

Exercise 7

Let E be the vector subspace of $\mathbb{R}^4$ generated by the vectors:

$$a = (-1, 1, -3, 0), b = (1, 2, 0, 1), c = (2, 1, 3, 1)$$

Give a basis of E and determine $dim(E)$.

One has :

$$E = \{v \in E^4 / v = \alpha a + \beta b + \gamma c, \quad where \quad \alpha, \beta, \gamma \in \mathbb{R}\}$$

If the vectors a, b, c are linearly independent and the family $\{a, b, c\}$ generates E then it forms a basis of E and therefore $dim(E) = 3$.

But, notice that $b = a + c$, which means that the vectors are not linearly independent, that's why $dim(E) < 3$.

So,

$$\begin{aligned} E &= \{v \in E^4 / v = \alpha a + \beta(a + c) + \gamma c, \quad where \quad \alpha, \beta, \gamma \in \mathbb{R}\} \\ &= \{v \in E^4 / v = (\alpha + \beta) a + (\beta + \gamma) c, \quad where \quad \alpha, \beta, \gamma \in \mathbb{R}\} \end{aligned}$$

This demonstrates that a and c generate E. It remains to show that a and c are linearly independent:
Let the scalars α, β such that $\alpha a + \beta c = 0$.
Yet:

$$\begin{aligned} \alpha(-1, 1, -3, 0) + \beta(2, 1, 3, 1) &= (0, 0, 0, 0) \\ &\Leftrightarrow \begin{cases} 2\alpha - \beta = 0 \\ \alpha + \beta = 0 \\ 3\alpha - 3\beta = 0 \\ \alpha = 0 \end{cases} \\ &\Leftrightarrow \alpha = \beta = 0 \end{aligned}$$

That way, a and c are linearly independent. They therefore form a basis for E. Therefre, it is concluded that $dim(E) = 2$.

Exercise 8

Demonstrate that the following vectors :

$$a = (1,1,2,1), b = (1,-1,0,1), c = (0,0,-1,1), d = (1,2,2,0)$$

form a basis of $\mathbb{R}^4$.

Since $dim(\mathbb{R}^4) = 4$ it is enough to show that the family$\{a, b, c, d\}$ is free or that it engenders $\mathbb{R}^4$.

We will show the second option, that is that it generates $\mathbb{R}^4$:

Let $X = (x, y, z, t) \in \mathbb{R}^4$, we need to find scalars $\alpha, \beta, \gamma, \delta$ such that

$$X = \alpha a + \beta b + \gamma c + \delta d.$$

i.e:

$$\begin{aligned} (x,y,z,t) &= \alpha(1,1,2,1) + \beta(1,-1,0,1) + \gamma(0,0,-1,1) + \delta(1,2,2,0) \\ &\Leftrightarrow \begin{cases} \alpha + \beta + \delta = x \\ \alpha - \beta + 2\delta = y \\ 2\alpha - \gamma + 2\delta = z \\ \alpha + \beta + \gamma = t \end{cases} \end{aligned}$$

After calculations, we find:

$$\begin{cases} \alpha = \frac{1}{4}(-4x - y + 3z + 3t) \\ \beta = \frac{1}{4}(4x - y - z - t) \\ \gamma = \frac{1}{2}(y - z + t) \\ \delta = \frac{1}{2}(2x + y - z - t) \end{cases}$$

2/ Find vector coordinates $u = (1, 1, 1, 1)$ in this basis.

Just replace (x, y, z, t) by $(1, 1, 1, 1)$, one finds:

$$\begin{cases} \alpha = \frac{1}{4} \\ \beta = \frac{1}{4} \\ \gamma = \frac{1}{2} \\ \delta = \frac{1}{2} \end{cases}$$

Which gives : $u = \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c + \frac{1}{2}d$.

Exercise 9

Say if the following maps are linear:

1. $f : \mathbb{R}^2 \to \mathbb{R}^3 / f(x, y) = (x + y, x - 2y, 0)$.

   Let $u = (x, y)$ and $v = (x', y')$ be in $\mathbb{R}^2$ and let λ be in $\mathbb{R}$:

$$\begin{aligned} f(u+v) &= ((x+x') + (y+y'), (x+x') - 2(y+y'), 0) \\ &= (x+y, x-2y, 0) + (x'+y', x'-2y', 0) \\ &= f(u) + f(v) \end{aligned}$$

   Likewise :

$$\begin{aligned} f(\lambda u) &= (\lambda x + \lambda y, \lambda x - \lambda(2y), 0) \\ &= \lambda(x+y, x-2y, 0) \\ &= \lambda f(u) \end{aligned}$$

   So, yes,, f is a linear map.

2. $g : \mathbb{R}^2 \to \mathbb{R}^3 / g(x, y) = (x + y, x - 2y, 1)$.

   Since $f(0, 0) \neq (0, 0, 0)$ then no, g is **not** a linear map.

3. $h : \mathbb{R}^2 \to \mathbb{R} / h(x, y) = x^2 - y^2$.

   One has :

$$\begin{aligned} f(1,0) &= 1, f(-1,0) = 1, f(0,0) = 0 \\ f(1,0) + f(-1,0) &= 2 \neq f(0,0) \end{aligned}$$

   Hence h is **not** a linear map.

Exercise 10

1/ Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined by:

$$f(x, y) = (x + y, x - y, x + y).$$

Find the kernel of f then its image. Is f injective? Surjective?

$$\ker f = \{u \in \mathbb{R}^2 / f(u) = 0\}$$

$$\begin{aligned} f(u) &= 0 \Leftrightarrow \begin{cases} x+y=0 \\ x-y=0 \\ x+y=0 \end{cases} \\ &\Leftrightarrow \begin{cases} x+y=0 \\ 2x=0 \end{cases} \\ &\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \end{aligned}$$

So: $\ker f = \{(0,0)\}$. Which means that f is injective.

$$\begin{aligned} \operatorname{Im} f &= \left\{v=(a,b,c) \in \mathbb{R}^3 / \exists u=(x,y) \in \mathbb{R}^2 / f(u)=v\right\} \\ f(u) &= v \Leftrightarrow \begin{cases} x+y=a \\ x-y=b \\ x+y=c \end{cases} \\ &\Leftrightarrow \begin{cases} x=\frac{a+b}{2} \\ y=\frac{a-b}{2} \\ c-a=0 \end{cases} \end{aligned}$$

So :

$$\operatorname{Im} f = \left\{v=(a,b,c) \in \mathbb{R}^3 / c-a=0\right\}$$

Just notice that $v=(1,1,0) \notin \operatorname{Im} f$ hence f is not surjective.

2/ Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by:

$$f(x,y,z) = (-3x-y+z, 8x+3y-2z, -4x-y+2z).$$

a) Determine a basis of the kernel of f and its dimension.

$$\ker f = \{u \in \mathbb{R}^3 / f(u) = 0\}$$

$$f(u)=0 \Leftrightarrow \begin{cases} -3x-y+z=0 \\ 8x+3y-2z=0 \\ -4x-y+2z=0 \end{cases}$$

After solving, we find:

$$\begin{aligned}(x,y,z) &= (x,-2x,x)\\ &= x(1,-2,1)\end{aligned}$$

Hence a kernel basis of f is the one-element family $\{(1,-2,1)\}$, which implies that $\dim(\ker f)=1$.

b) Is f injective?

As the kernel is not reduced to $0_{\mathbb{R}^3}$, f is not injective.

c) Determine a basis of $Im(f)$.

$$\operatorname{Im} f = \left\{v=(a,b,c)\in\mathbb{R}^3/\exists u=(x,y,z)\in\mathbb{R}^3/f(u)=v\right\}$$

$$f(u)=0 \Leftrightarrow \begin{cases} -3x-y+z=a\\ 8x+3y-2z=b\\ -4x-y+2z=c\end{cases}$$

After solving, we find:

$$\operatorname{Im} f = \{u=(a(-3,8,-4)+b(-1,3,-1)+c(1,-2,2)\}$$

Taking $u_1(-3,8,-4)$ and $u_2(-1,3,-1)$, it is easy to show that $\{u_1,u_2\}$ constitute a basis of $\operatorname{Im} f$.

Exercise 11

- Let be the matrix $A=\begin{pmatrix}2&2&0\\1&1&0\\0&0&4\end{pmatrix}$.

1/ Find eigenvalues of A.

$$\begin{aligned}\det(A-\lambda Id) &= \begin{vmatrix}2-\lambda&2&0\\1&1-\lambda&0\\0&0&4-\lambda\end{vmatrix}\\ &= (4-\lambda)\begin{vmatrix}2-\lambda&2\\1&1-\lambda\end{vmatrix}\\ &= (4-\lambda)\left[(2-\lambda)(1-\lambda)-2\right]\end{aligned}$$

$$\begin{aligned}\det(A-\lambda Id) &= (4-\lambda)\left[\lambda^2-3\lambda\right]\\ &= \lambda(4-\lambda)(\lambda-3).\end{aligned}$$

$\det(A-\lambda Id)=0 \Leftrightarrow \lambda_1=0$ or $\lambda_2=4$ or $\lambda_3=3$.
The eigenvalues of the matrix A are then:

$$\lambda_1=0, \lambda_2=4, \lambda_3=3$$

2/ Find the eigenvectors associated with the eigenvalues of A:
*/ For : $\lambda_1=0$:

$$(A-\lambda_1 Id)=\begin{pmatrix} 2-0 & 2 & 0\\ 1 & 1-0 & 0\\ 0 & 0 & 4-0\end{pmatrix}=\begin{pmatrix} 2 & 2 & 0\\ 1 & 1 & 0\\ 0 & 0 & 4\end{pmatrix}$$

$$(A-\lambda_1 Id)X=0 \Leftrightarrow \begin{pmatrix} 2 & 2 & 0\\ 1 & 1 & 0\\ 0 & 0 & 4\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}$$

$$\Leftrightarrow \left\{\begin{array}{r} 2x+2y=0\\ x+y=0\\ z=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=-y\\ z=0\end{array}\right.$$

$$\text{or } \left\{\begin{array}{l} x=-\alpha\\ y=\alpha\\ z=0\end{array}\right.$$

The first eigenvector is : $V_1=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$

*/ For : $\lambda_2=4$:

$$(A-\lambda_2 Id)=\begin{pmatrix} 2-4 & 2 & 0\\ 1 & 1-4 & 0\\ 0 & 0 & 4-4\end{pmatrix}=\begin{pmatrix} -2 & 2 & 0\\ 1 & -3 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{aligned} A-\lambda_2 Id)X &= 0 \Leftrightarrow \begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \left\{ \begin{array}{r} -2x+2y=0 \\ x-3y=0 \\ 0=0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x=0 \\ y=0 \\ z \end{array} \right. \\ &\text{or } \left\{ \begin{array}{l} x=0 \\ y=0 \\ z=\alpha \end{array} \right. \end{aligned}$$

The second eigenvector is : $V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

*/ For : $\lambda_3 = 3$:

$$(A-\lambda_3 Id) = \begin{pmatrix} 2-3 & 2 & 0 \\ 1 & 1-3 & 0 \\ 0 & 0 & 4-3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} (A-\lambda_2 Id)X &= 0 \Leftrightarrow \begin{pmatrix} -1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \left\{ \begin{array}{r} -x+2y=0 \\ x-2y=0 \\ z=0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x=2y \\ z=0 \end{array} \right. \\ &\text{or } \left\{ \begin{array}{l} x=2\alpha \\ y=\alpha \\ z=0 \end{array} \right. \end{aligned}$$

The third eigenvector is : $V_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

3/ Give the diagonal matrix D and the transition matrix P:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, P = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

• Consider the matrix :

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & -1 \\ -2 & -2 & 1 \end{pmatrix}$$

1/ Calculate the eigenvalues of M.

$$\det(M - \lambda Id) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -\lambda & -1 \\ -2 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-\lambda)(1-\lambda) - 2\right]$$

$$\det(M - \lambda Id) = (1-\lambda)\left[\lambda^2 - \lambda - 2\right] = (1-\lambda)(\lambda+1)(\lambda-2).$$

$$\det(M - \lambda Id) = 0 \Leftrightarrow \lambda_1 = 1 \text{ or } \lambda_2 = -1 \text{ or } \lambda_3 = 2.$$

The eigenvalues of the matrix M are then : $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$.

2/ Compute the eigenvectors associated with the eigenvalues of M.

*/ For : $\lambda_1 = 1$:

$$(M - \lambda_1 Id) = \begin{pmatrix} 1-\lambda_1 & 0 & 0 \\ -2 & -\lambda_1 & -1 \\ -2 & -2 & 1-\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -1 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

$$(M - \lambda_1 Id)X = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ -2 & -1 & -1 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 0 = 0 \\ -2x - y - z = 0 \\ -2x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y \\ y = z \end{cases}$$

Or :

$$\left\{\begin{array}{l} x=-\alpha \\ y=\alpha \\ z=\alpha \end{array}\right. .$$

The first eigenvector is : $V_1=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

*/ For : $\lambda_2=2$:

$$(M-\lambda_2 Id)=\begin{pmatrix} 1-\lambda_2 & 0 & 0 \\ -2 & -\lambda_2 & -1 \\ -2 & -2 & 1-\lambda_2 \end{pmatrix}=\begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix}$$

$$\begin{aligned} (M-\lambda_2 Id)X &= 0 \Leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \left\{\begin{array}{r} 2x=0 \\ -2x+y-z=0 \\ -2x-2y+2z=0 \end{array}\right. \Leftrightarrow \\ &\left\{\begin{array}{l} x=0 \\ y=z \\ y=z \end{array}\right. \end{aligned}$$

Or :

$$\left\{\begin{array}{l} x=0 \\ y=\alpha \\ z=\alpha \end{array}\right.$$

The second eigenvector is : $V_2=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

*/ For $\lambda_3=2$:

$$(A-\lambda_3 Id)=\begin{pmatrix} 1-\lambda_3 & 0 & 0 \\ -2 & -\lambda_3 & -1 \\ -2 & -2 & 1-\lambda_3 \end{pmatrix}=\begin{pmatrix} -1 & 0 & 0 \\ -2 & -2 & -1 \\ -2 & -2 & -1 \end{pmatrix}$$

$$(A-\lambda_2 Id)X=0 \Leftrightarrow \begin{pmatrix} -1 & 0 & 0 \\ -2 & -2 & -1 \\ -2 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x = 0 \\ -2x - 2y - z = 0 \\ -2x - 2y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ z = 2y \\ z = 2y \end{cases}$$

Or :

$$\begin{cases} x = 0 \\ y = \alpha \\ z = 2\alpha \end{cases}$$

The third eigenvector is : $V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

3/ Give the diagonal matrix D and the transition matrix P:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; P = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

2 Chapter 2 Systems of Linear Equations

2.1 Generalities

We call linear system of n equations with p unknown a system of the type:

$$(S):\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2p}x_p = b_2 \\ \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{np}x_p = b_n \end{cases}$$

Where the (a_{ij}) and the (b_{ij}) are real constants and the (x_{ij}) are the variables or unknowns of the system (S) for $1 \leq i \leq n$ and $1 \leq j \leq p$.

Solving the system (S) amounts to finding the unknowns (x_{ij}).

2.2 Study of the set of all solutions

The set of solutions of a given system is the set of all its possible solutions. It is generaly noted S. It does not change if the following elementary operations are performed on its equations:

*) Change the order of the equations.

*) Multiply an equation by a nonzero constant.

*) Add to an equation a multiple of another equation.

*) Change order of variables.

The matrix form

Any system of linear equations is written in the following matrix form:

$$(S) \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{np} \end{pmatrix}_{(n,p)} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}_{(p,1)} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{(n,1)}$$

Take:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix}_{(n,p)} = (a_{ij})_{(n,p)} \text{ the coefficient matrix.}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}_{(p,1)} = (x_{ij})_{(p,1)} \quad \text{the unknowns vector.}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{(n,1)} = (b_{ij})_{(n,1)} \text{ the vector of the second member.}$$

So, now one gets :

$$(S): \ AX = b.....(Eq1)$$

Matrix form Properties

- We can go from the matrix form to the "system" form by multiplication.
- In matrix form, the rows represent the equations and the columns represent the variables, for example, the first row corresponds to the first equation of the system, the second row to the second system equation...etc.
- Also, the column $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ is the column corresponding to variable x_1, the column $\begin{pmatrix} a_{12} \\ a_{22} \\ \\ a_{n2} \end{pmatrix}$ corresponding to variable x_2 ...etc.

Example

Consider the following system of linear equations, where the number of equations is equal to the number of unknowns.

We are going to use this example several times to present different methods in the subsequent paragraphs :

$$(S):\begin{cases} z+y+x-8=0 \\ x-3z+2y=5 \\ 3y-3x-z-2=0 \end{cases}$$

This given system is not ordered, we must first the ordinate before writing the matrix form, that is, we must write it correctly, as follows:

$$(S):\begin{cases} x+y-z=8 \\ x+2y-3z=5 \\ 3x-3y-z=2 \end{cases}.$$

To solve this system, we first turn to the matrix form...

We separate the coefficients of the variables in a matrix, the variables in a column vector and the second member in another column vector, (S) becomes:

$$(S):\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ 3 & -3 & -1 \end{pmatrix}_{(3,3)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{(3,1)} = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}_{(3,1)}.$$

Take:

$$A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ 3 & -3 & -1 \end{pmatrix}_{(3,3)}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{(3,1)}, b=\begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}_{(3,1)}.$$

So now, one gets :

$$(S):\ AX=b.....(Eq1)$$

Cramer's systems

Definition

A system of linear equations is said to be Cramer's if and only if:

1) We have a number of equations equal to the number of unknowns. That is, it is a square system.

2) The Matrix A is invertible (i.e. $\det A \neq 0$).

Theorem Every Cramerian system admits a unique solution.

2.3 Resolution of Cramerian Systems

2.3.1 Inverse matrix method

Given a Cramer system $AX = b$.

Since the matrix A is invertible (that is to say that A^{-1} exists), we can multiply the equation (Eq1) by A^{-1}, we find:

$$\begin{aligned}(Eq1) &\Leftrightarrow A^{-1}AX = A^{-1}b\\ &\Leftrightarrow IdX = A^{-1}b\\ &\Leftrightarrow \boxed{X = A^{-1}b}.\end{aligned}$$

To find the X, we have to calculate A^{-1} then multiply it by b.

Attention, we multiply by A^{-1} on the left because the product bA^{-1} is not possible (because of the dimensions).

$$\det A = \begin{vmatrix} 1 & 1 & -1\\ 1 & 2 & -3\\ 3 & -3 & -1\end{vmatrix} = -10, \qquad coA = \begin{pmatrix} -11 & -8 & -9\\ 4 & 2 & 6\\ -1 & 2 & 1\end{pmatrix}$$

So:

$$A^{-1} = \frac{1}{\det A}(coA)^t = \frac{-1}{10}\begin{pmatrix} -11 & 4 & -1\\ -8 & 2 & 2\\ -9 & 6 & 1\end{pmatrix}$$

Finally :

$$\begin{aligned}\boxed{X = A^{-1}b} &= \frac{-1}{10}\begin{pmatrix} -11 & 4 & -1\\ -8 & 2 & 2\\ -9 & 6 & 1\end{pmatrix}_{(3,3)} . \begin{pmatrix} 8\\ 5\\ 2\end{pmatrix}_{(3,1)}\\ &= \frac{-1}{10}\begin{pmatrix} (-11).8 + 4.5 + (-1).2\\ -(8).8 + 2.5 + 2.2\\ (-9).8 + 6.5 + 1.2\end{pmatrix}\\ &= \frac{-1}{10}\begin{pmatrix} -70\\ -50\\ -40\end{pmatrix}.\end{aligned}$$

$$\boxed{X = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}} \quad \text{or} \quad \boxed{\begin{cases} x = 7 \\ y = 5 \\ z = 4 \end{cases}}$$

So the solutions set of system (S) is given by :

$$S = \left\{ \begin{pmatrix} 7 & 5 & 4 \end{pmatrix} \right\}$$

2.3.2 Cramer's method (or determinants)

For this method, one takes :

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ 3 & -3 & -1 \end{vmatrix} = -10,$$

Called **main determinant**.

We call determinant associated with a given variable, the determinant found by replacing the column corresponding to this variable by the vector of the second member, so we have:

$$\Delta_x = \begin{vmatrix} \mathbf{8} & 1 & -1 \\ \mathbf{5} & 2 & -3 \\ \mathbf{2} & -3 & -1 \end{vmatrix} = -70, \ \Delta_y = \begin{vmatrix} 1 & \mathbf{8} & -1 \\ 1 & \mathbf{5} & -3 \\ 3 & \mathbf{2} & -1 \end{vmatrix} = -50, \ \Delta_z = \begin{vmatrix} 1 & 1 & \mathbf{8} \\ 1 & 2 & \mathbf{5} \\ 3 & -3 & \mathbf{2} \end{vmatrix} = -40$$

And we claim that :

$$\boxed{x = \tfrac{\Delta_x}{\Delta}, y = \tfrac{\Delta_y}{\Delta}, z = \tfrac{\Delta_z}{\Delta}.}$$

Which gives:

$$\boxed{x = \tfrac{-70}{-10} = 7, y = \tfrac{-50}{-10} = 5, z = \tfrac{-40}{-10} = 4.}$$

So we get the same solutions set of system (S) as by the previous method, which is given by :

$$S = \left\{ \begin{pmatrix} 7 & 5 & 4 \end{pmatrix} \right\}$$

2.3.3 Gauss' method

In this method, we want to simplify the system (S) by making the matrix A upper triangular. We are going to do linear operations on the matrix A to make 0 appear under its diagonal. Of course, these operations must also be done on the vector of the second member so as not to change the initial system.

To make this task easier for us, we will adopt a new writing of system (S):

Gauss' writing $[A|\,b]$

The matrix obtained by writing A then b next to it, on the right, is called the augmented matrix of the system (S). It simply allows us to do the Gaussian transformations on A and on b at the same time.

In our Example, the writing of Gauss gives the following augmented matrix:

$$[A|\,b] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 1 & 2 & -3 & 5 \\ 3 & -3 & -1 & 2 \end{array}\right]$$

The lines are numbered like this:

$$\begin{array}{c} l_1 \rightarrow \\ l_2 \rightarrow \\ l_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 1 & 2 & -3 & 5 \\ 3 & -3 & -1 & 2 \end{array}\right]$$

We need three zeros below the diagonal, so we'll do three steps, making one zero appear per step:

First step, to be done between l_1 **and** l_2 :

In our example, we calculate : $l_2' = (l_2 - l_1)$

$$\begin{array}{rcccc} l_2: & 1 & 2 & -3 & 5 \\ l_1: & 1 & 1 & -1 & 8 \\ l_2' = l_2 - l_1: & 0 & 1 & -2 & -3 \end{array} \Rightarrow \begin{array}{c} l_1 \rightarrow \\ l_2' \rightarrow \\ l_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 0 & 1 & -2 & -3 \\ 3 & -3 & -1 & 2 \end{array}\right]$$

Second step, to be done between l_1 **and** l_3 :

In our example, we calculate : $l'_3 = (l_3 - 3.l_1)$

$$\begin{array}{rcccc} l_3 : & 3 & -3 & -1 & 2 \\ 3.l_1 : & 3 & 3 & -3 & 24 \\ l'_3 = l_3 - 3.l_1 : & 0 & -6 & 2 & -22 \end{array} \Rightarrow \begin{array}{c} l_1 \rightarrow \\ l'_2 \rightarrow \\ l'_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 0 & 1 & -2 & -3 \\ 0 & -6 & 2 & -22 \end{array}\right]$$

Third step, to be done between l'_3 **and** l'_2**:**

In our example, we calculate : $l''_3 = (l'_3 + 6.l'_2)$

$$\begin{array}{rcccc} l'_3 : & 0 & -6 & 2 & -22 \\ 6.l'_2 : & 0 & 6 & -12 & -18 \\ l''_3 = l'_3 + 6.l'_2 : & 0 & 0 & -10 & -40 \end{array} \Rightarrow \begin{array}{c} l_1 \rightarrow \\ l'_2 \rightarrow \\ l''_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -40 \end{array}\right]$$

After these three steps, we return to the form of the system with the new matrix and the new second member:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -40 \end{pmatrix}$$

In system form, this gives : $\begin{cases} x + y - z = 8 & ...(1) \\ \quad y - 2z = -3 & ...(2) \\ \qquad - 10z = -40 & ...(3) \end{cases}$

And we solve going from the bottom to the top of variables:

$$\Leftrightarrow \begin{cases} (3) \Leftrightarrow z = 4 \\ (2) \Leftrightarrow y = -3 + 2(4) = 5 \\ (1) \Leftrightarrow x = 8 + (4) - (5) = 7 \end{cases}$$

Attention

The operations we perform must be linear, that is, we cannot multiply or divide two lines between them, nor multiply by 0.

Remarks

1. Cramer's systems have a unique solution, so we must find the same result (the same solution) regardless of the method used.

2. For homogeneous Cramer systems, i.e. when the second member is zero (b=0), then the only solution is the zero solution (also called trivial solution): $(x = 0, y = 0, z = 0)$.

3. If $a_{11} = 0$, we need to do a row swap at the first step between l_1 and l_2. Then we continue the procedure as before.

- **Example:**

$$\begin{array}{c} l_1 \rightarrow \\ l_2 \rightarrow \\ l_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 4 \\ 1 & 1 & -3 & 2 \\ 3 & -3 & -0 & 2 \end{array}\right] \Rightarrow \begin{array}{c} l_2 \rightarrow \\ l_1 \rightarrow \\ l_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 4 \\ 3 & -3 & -0 & 2 \end{array}\right]$$

2.4 Resolution of non-cramerian systems

The given linear system is said to be non-cramerian if we face one of the following two situations:

1) The number of equations is **different** from the number of unknowns.

Or

2) The system is indeed square but its matrix is **not invertible** (i.e. $detA = 0$).

For the resolution of this type of systems, we proceed as follows:

- **Step 1: framework**

From each non-Cramerian system with n equations and p unknowns $(n \neq p)$, we extract a square Cramerian subsystem of maximum order r. We note it (Sc).

The equations and the unknowns which form this subsystem are said to be main or primary.

Those outside the subsystem are said to be auxiliary or non-main.
The non-primary equations are the verification equations.

- **Step 2: Resolution**

We solve the system formed by the r main equations, where the non-main unknowns (if they exist) are considered as parameters and are transferred to the second member.

Recall that the subsystem (Sc) is Cramerian, so it can be solved either by the inverse matrix method, Cramer's method or by Gauss's method.

- **Step 3: System Compatibility Requirements**

We replace the found solutions of the subsystem (Sc) in the verification equations. There will be 2 possible cases:

Case 1: If the verification equations are respected, the main system has either a single solution or an infinity of solutions (this will be the case if the solutions are parameterized).

Case 2: If the verification equations present a contradiction, the system is impossible (it does not admit any solution), it is said to be incompatible.

2.4.1 Linear systems with number of equations strictly greater than the number of unknowns

In other words, the system verifies $(n > p)$, like this one :

$$(S1) : \begin{cases} x - 2y = 5 & (l_1) \\ 2x + 3y = 3 & (l_2) \\ 3x + 2y = 7 & (l_3) \end{cases} .$$

To solve $(S1)$, we must look for a minor of order 2 nonzero and leave the remaining equation outside this choice as a verification equation.

For example, we choose:

$$\Delta = \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = 7 \neq 0.$$

We then solve the reduced system

$$\begin{cases} x - 2y = 5 & (l_1) \\ 2x + 3y = 3 & (l_2) \end{cases} ,$$

And we keep (3) as verification equation.

By elimination of Gauss (for example, but we could have chosen any method appropriate to cramerian systems, since the minor is non-zero and therefore the associated system is cramerian), we have:

$$\begin{cases} 2 \times (l_1) \Rightarrow 2x - 4y = 10 \\ (l_2) \Rightarrow 2x + 3y = 3 \end{cases}$$

$$(l_2) - 2 \times (l_1) \Leftrightarrow -7y = 7 \Rightarrow y = -1.$$

Substituting in (1) (by Example, but could very well have replaced in (2)), we find:

$$(l_1) \Leftrightarrow x - 2(-1) = 5 \Rightarrow x = 3.$$

We found a candidate solution:

$$\begin{cases} x = 3 \\ y = -1 \end{cases}.$$

If this solution satisfies the equation (l_3) it is accepted, if not it is refused.

$$(l_3) \Leftrightarrow 3(3) + 2(-1) = 7.$$

Equation (l_3) holds, so the solution $\begin{cases} x = 3 \\ y = -1 \end{cases}$ is accepted.
The set of solutions of system $(S1)$ is given as follows :

$$S = \left\{ \begin{pmatrix} 3 & -1 \end{pmatrix} \right\}.$$

Remark

In the case of systems with number of equations strictly greater than the number of unknowns, one can have either a single solution (if the verification equation is satisfied) or no solution (if the verification equation is not satisfied).

2.4.2 Linear systems with number of equations strictly smaller than the number of unknowns

In other words, the system verifies $(n < p)$.

Example

$$(S2) : \begin{cases} x + 2y + 3z = 4 & (1) \\ -x + 2y - 7z = 4 & (2) \end{cases} .$$

In this case, it is impossible to find the 3 unknowns, some of them must be "sacrificed", they go from "unknown" status to "parameter" status, they pass with the constants in the second member .

The solution of the system will be according to those "parameters".

To solve (S2) we must look for a nonzero minor[2] of order r and leave the variables outside this choice as a parameter.

In our example, we choose:

$$\Delta = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4 \neq 0.$$

We then solve the reduced system:

$$(S2) \Leftrightarrow \begin{cases} x + 2y = 4 - 3z & (1) \\ -x + 2y = 4 + 7z & (2) \end{cases}$$

By the method of elimination of Gauss, we have:

$$(l_1) + (l_2) \Leftrightarrow 4y = 8 + 4z \Rightarrow y = 2 + z,$$

Then,

$$(l_1) \Leftrightarrow x = 4 - 3z - 2(2 + z) \Leftrightarrow x = -5z.$$

The solution is directly accepted:

[2] A minor is a determinant formed from the matrix of the given system but which is of order smaller than the order of the system.

$$\begin{cases} x = -5z \\ y = 2 + z \\ z \in \mathbb{R} \end{cases}.$$

Some authors replace z by a parameter $\alpha \in \mathbb{R}$ to accentuate the parameterization notion of the solution. This gives:

$$\begin{cases} x = -5\alpha \\ y = 2 + \alpha \\ z = \alpha \end{cases}, (\alpha \in \mathbb{R}).$$

Now, we set the solutions set of system $(S2)$ as follows :

$$S = \left\{ \begin{pmatrix} -5\alpha & 2+\alpha & \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

Remarks

1. When $n > p$, we always have an infinity of solutions, (because of the parameter). We say that we have a space of solutions rather than a set of solution.

2. In the previous example, since we have only one parameter, the space of solutions is of dimension 1.

3. We can have several parameters at the same time in the solution, for example, to solve the lying system formed by a single equation and the following three unknowns:

$$(Sys) : \{x - y + 2z = 1$$

We consider the coefficient of x which is here non-zero $(= 1)$ as the principal minor. x is now a main variable, but y and z are now parameters that must pass to the second member, so:

$$(Sys) : \begin{cases} x = 1 + y - 2z \\ y \in \mathbb{R} \\ z \in \mathbb{R} \end{cases} \quad \text{or} \quad \begin{cases} x = 1 + \alpha - 2\beta \\ \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{cases}$$

The space of solutions of system (Sys) is of dimension 2. It is given as follows :

$$S = \left\{ \begin{pmatrix} 1+\alpha-2\beta & \alpha & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}.$$

2.4.3 Augmented systems

These are square systems (i.e. the number of equations is equal to the number of unknowns, $(n = p)$), but where the main determinant is zero $(\det A = 0)$.

Example

$$(S3) : \begin{cases} x - 2y + 3z = 2 & (1) \\ 2x - 3y + 8z = 7 & (2) \\ 3x - 4y + 13z = 8 & (3) \end{cases}$$

In matrix form, we have:

$$\begin{array}{ccc} \begin{pmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ 3 & -4 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & = & \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix}, \\ AX & = & B \end{array}$$

But $\det A = 0$. This means that we might have at least a verification equation and a parameter.

To solve (S_3) we have to look for a nonzero minor of order 2, leave the remaining equation outside this choice as verification equation and the variable outside this choice as a parameter.

For example, we decide to choose the minor:

$$\Delta = \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = 1 \neq 0.$$

Equation (3) $3x - 4y + 13z = 8$ is left as the verification equation and the unknown z as the parameter.

Our system then becomes:

$$(S3) \Leftrightarrow \begin{cases} x - 2y = 2 - 3z & (1) \\ 2x - 3y = 7 - 8z & (2) \\ \boxed{3x - 4y + 13z = 8} & (3) \end{cases}$$

By Gauss elimination:

$$(2) - 2 \times (1) \Leftrightarrow y = 3 - 2z,$$

Then,

$$(1) \Leftrightarrow x = 2 - 3z + 2(3 - 2z) = 8 - 7z,$$

We have a candidate solution :$\begin{cases} x = 8 - 7z \\ y = 3 - 2z \end{cases}$.

We have to see if it verifies the equation(3):

$$(3) \Leftrightarrow 3(8 - 7z) - 4(3 - 2z) + 13z = 12 \neq 8.$$

Equation (3) is not respected, the candidate solution is then refused, so there is no solution.

We say that (S3) is an "impossible" or "incompatible" system. In this case :

$$S = \emptyset.$$

Remarks

1) In the case of augmented systems, we have either an infinity of solutions, or no solution.

2) Attention, if the second member is null in a non-cramerian system, the solution is not inevitably null.

Example

Solve the following non-cramerian system:

$$(S) \begin{cases} 2x + y = 0 \\ x + y + z = 0 \\ y + 2z = 0 \end{cases}$$

Correction

Note that the second member is all zero.

(S) in matrix form would be:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$AX = B$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \det A = 0$$

To solve (S) we have to look for a minor of order 2 nonzero, leave the remaining equation outside this choice as verification equation and the variable outside this choice as parameter.

For example, we decide to choose the minor:

$$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

Equation $y + 2z = 0$ is left as the verification equation and the unknown z as the parameter.

Our system becomes:

$$(S) \Leftrightarrow \begin{cases} 2x + y = 0 & (1) \\ x + y = -z & (2) \\ \boxed{y + 2z = 0} & (3) \end{cases}$$

By Gauss elimination:

$$(1) - (2) \Leftrightarrow x = z,$$

Thus,

$$(2) \Leftrightarrow y = -2z,$$

One obtains a candidate solution : $\begin{cases} x = z \\ y = -2z \end{cases}$.

Let's see if it verifies equation (3):

$$(3) \Leftrightarrow -2z + 2z = 0.$$

Clearly, equation (3) is respected, the candidate solution is then accepted.

$$\begin{cases} x = z \\ y = -2z \\ z \in \mathbb{R} \end{cases} .$$

Or :

$$\begin{cases} x = \alpha \\ y = -2\alpha \\ z = \alpha \end{cases} , (\alpha \in \mathbb{R}) .$$

The solutions set of system (S) is the following one dimensional space :

$$S = \left\{ \begin{pmatrix} \alpha & -2\alpha & \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\} .$$

2.5 Exercise series N°2

Exercise 1

1/ Let the matrices: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 1 & 2 \\ 3 & -3 & 0 \\ 6 & 2 & -2 \end{pmatrix}$.

Calculate AB then BA. Deduce accordingly A^{-1} the inverse matrix of A.

2/ Let f be the function defined on $\mathbb{R}$ by : $f(x) = x^3 + ax^2 + bx + c$.

Where $a, b, c \in \mathbb{R}$. Let the system (S): $\begin{cases} f(1) = 0 \\ f(-1) = 0 \\ f(2) = 10 \end{cases}$.

Solve the Cramerian system (S) by three different methods.

Exercise 2

Let $m \in \mathbb{R}$ and consider the following system:

$$(S_m) \begin{cases} mx + y + z = m - 2 \\ mx - y + z = m - 2 \\ (m-1)y + (m-1)z = m(m-1) \end{cases}$$

1) Write system (S_m) in matrix form $AX = B$.
2) For which value(s) of parameter m, the matrix A is invertible?
3) Solve system (S_2) by the method of your choice.

Exercise 3

Consider the following matrices :

$$M = \begin{pmatrix} x & -1 \\ 2 & 1 \end{pmatrix}, N = \begin{pmatrix} 4 & y \\ -1 & 2 \end{pmatrix}$$

1. We call Lie bracket between two matrices A and B the following operation :

$$[A, B] = AB - BA$$

   Calculate $[M, N]$.

2. We say that matrices A and B commute if and only if $AB = BA$.

   Find x and y so that M and N commute.

2.6 Correction of the exercise series N°2

Exercise 1

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 & 2 \\ 3 & -3 & 0 \\ 6 & 2 & -2 \end{pmatrix}.$$

Calculate AB then BA. Deduce accordingly A^{-1} the inverse matrix of A.

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix}_{(3,3)} . \begin{pmatrix} -3 & 1 & 2 \\ 3 & -3 & 0 \\ 6 & 2 & -2 \end{pmatrix}_{(3,3)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{(3,3)}.$$

$$a_{11} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} = 6, a_{12} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = 0...etc.$$

Finally : $AB = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}_{(3,3)} = 6Id_{(3,3)}$. By the same way, we obtain : $BA = 6Id_{(3,3)}$.

Hence, one has $AB = BA = 6Id_{(3,3)} \Leftrightarrow A\left[\frac{1}{6}B\right] = \left[\frac{1}{6}B\right]A = Id_{(3,3)}$, which gives $A^{-1} = \left[\frac{1}{6}B\right]$.

2/ Let f be the function defined on $\mathbb{R}$ par : $f(x) = x^3 + ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$. Let be the system (S):$\begin{cases} f(1) = 0 \\ f(-1) = 0 \\ f(2) = 10 \end{cases}$

Solve the Cramerian system (S) by three different methods.

We first have:

$$(S) \quad : \quad \begin{cases} f(1) = 0 \\ f(-1) = 0 \\ f(2) = 10 \end{cases} \Leftrightarrow \begin{cases} 1 + a + b + c = 0 \\ -1 + a - b + c = 0 \\ 8 + 4a + 2b + c = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b + c = -1 \\ a - b + c = 1 \\ 4a + 2b + c = 2 \end{cases}$$

In matrix form, this gives : $AX = B$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix},$$

1/ By the inverse matrix method:

Since we have already calculated the inverse of A in the first question, we directly apply : $X = A^{-1}B$.

That gives :

$$\begin{aligned} X &= \frac{1}{6}\begin{pmatrix} -3 & 1 & 2 \\ 3 & -3 & 0 \\ 6 & 2 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 8 \\ -6 \\ -8 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{3} \\ -1 \\ -\frac{4}{3} \end{pmatrix}. \end{aligned}$$

Or :

$$\begin{aligned} &\begin{cases} a = \frac{4}{3} \\ b = -1 \\ c = \frac{-4}{3} \end{cases} \\ S &= \left\{ \begin{pmatrix} \frac{4}{3} & -1 & \frac{-4}{3} \end{pmatrix} \right\} \end{aligned}$$

.

2/ By the Cramer method

One has :

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 6$$

$$\begin{aligned} \Delta_x &= \begin{vmatrix} \mathbf{-1} & 1 & 1 \\ \mathbf{1} & -1 & 1 \\ \mathbf{2} & 2 & 1 \end{vmatrix} = 8 \\ &\Rightarrow \quad x = \frac{\Delta_x}{\Delta} = \frac{8}{6} = \frac{4}{3}. \end{aligned}$$

$$\Delta_y = \begin{vmatrix} 1 & \mathbf{-1} & 1 \\ 1 & \mathbf{1} & 1 \\ 4 & \mathbf{2} & 1 \end{vmatrix} = -6$$
$$\Rightarrow \quad y = \frac{\Delta_y}{\Delta} = \frac{6}{-6} = -1.$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & \mathbf{-1} \\ 1 & -1 & \mathbf{1} \\ 4 & 2 & \mathbf{2} \end{vmatrix} = -8$$
$$\Rightarrow \quad z = \frac{\Delta z}{\Delta} = \frac{-8}{6} = \frac{-4}{3}.$$

Finally :

$$\begin{cases} a = \frac{4}{3} \\ b = -1 \\ c = \frac{-4}{3} \end{cases}$$
$$S = \{ \left(\tfrac{4}{3} \quad -1 \quad \tfrac{-4}{3} \right) \}$$

3/ By Gauss method:

Gauss' writing :

$$\begin{matrix} \to l_1 \\ \to l_2 \\ \to l_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 4 & 2 & 1 & 2 \end{array} \right].$$

First step: between l_1 and l_2 : $l_2 - l_1$

$$\begin{matrix} \to l_1 \\ \to l_2 \\ \to l_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 4 & 2 & 1 & 2 \end{array} \right].$$

Second step: between l_1 and l_3 : $l_3 - 4l_1$

$$\begin{matrix} \to l_1 \\ \to l_2 \\ \to l_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & 6 \end{array} \right].$$

Third step: between l_2 and l_3 : $l_3 - l_2$

$$\begin{array}{l} \to l_1 \\ \to l_2 \\ \to l_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & 4 \end{array}\right].$$

Our system is now written:

$$\left\{\begin{array}{r} a+b+c=-1 \\ -2b=2 \\ -3c=4 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a=\frac{4}{3} \\ b=-1 \\ c=\frac{-4}{3} \end{array}\right.$$

$$\Longrightarrow S=\left\{\left(\begin{array}{ccc} \frac{4}{3} & -1 & \frac{-4}{3} \end{array}\right)\right\}.$$

Exercise 2

Let $m \in \mathbb{R}$ and consider the following system :

$$(S_m)\left\{\begin{array}{r} mx+y+z=m-2 \\ mx-y+z=m-2 \\ (m-1)y+(m-1)z=m(m-1) \end{array}\right.$$

1) Write system (S_m) in matrix form $AX=B$.

$$\begin{pmatrix} m & 1 & 1 \\ 1 & -1 & 1 \\ 0 & m-1 & m-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} m-2 \\ m-2 \\ m(m-1) \end{pmatrix}$$

2) For which value(s) of parameter m, the matrix A is invertible?

$$A=\begin{pmatrix} m & 1 & 1 \\ 1 & -1 & 1 \\ 0 & m-1 & m-1 \end{pmatrix}$$

$$\begin{aligned} \det A &= \begin{vmatrix} m & 1 & 1 \\ 1 & -1 & 1 \\ 0 & m-1 & m-1 \end{vmatrix} \\ &= 2m-2m^2=2m(1-m). \end{aligned}$$

For A to be invertible, it is necessary and sufficient that $m \in \mathbb{R}-\{0,1\}$.
3) Solve system (S_2) by the method of your choice.

$$(S_2) \Leftrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Just apply what was learned before about the inverse matrix method, Cramer method or Gauss method to find the answer. The solution is given by :

$$X = \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}.$$

$$\Longrightarrow S = \left\{ \begin{pmatrix} -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \right\}.$$

Exercise 3

Consider the following matrices : $M = \begin{pmatrix} x & -1 \\ 2 & 1 \end{pmatrix}, N = \begin{pmatrix} 4 & y \\ -1 & 2 \end{pmatrix}.$

1. Calculate : $[M, N] = MN - NM$.

$$MN = \begin{pmatrix} x & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & y \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4x+1 & xy-2 \\ 7 & 2y+2 \end{pmatrix}.$$

$$NM = \begin{pmatrix} 4 & y \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4x+2y & y-4 \\ 4-x & 3 \end{pmatrix}.$$

Hence:

$$\begin{aligned} [M, N] &= \begin{pmatrix} 4x+1 & xy-2 \\ 7 & 2y+2 \end{pmatrix} - \begin{pmatrix} 4x+2y & y-4 \\ 4-x & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1-2y & xy-y+2 \\ x+3 & 2y-1 \end{pmatrix}. \end{aligned}$$

2. Now, find x and y such that M and N commute:

M and N commute means that $MN = NM$, which implies that $[M, N] = 0$.

$$[M, N] = 0 \Leftrightarrow \begin{pmatrix} 1-2y & xy-y+2 \\ x+3 & 2y-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 1-2y = 0 & (1) \\ xy - y + 2 = 0 & (2) \\ x + 3 = 0 & (3) \\ 2y - 1 = 0 & (4) \end{cases}.$$

Attention, it is not a linear system (because of the (xy) in the second equation)!

There is no unified method for solving a nonlinear system.

In this case, we can solve it intuitively: we first find x and y from the first and the third equation while keeping the second and the fourth equations as verification equations.

That gives : $\begin{cases} x = -3 \\ y = \frac{1}{2} \end{cases}$.

Then :

(2)$\Leftrightarrow xy - y + 2 = 0 \Leftrightarrow (-3)\frac{1}{2} - \frac{1}{2} + 2 = -\frac{4}{2} + 2 = 0.$

(4)$\Leftrightarrow 2y - 1 = 2(\frac{1}{2}) - 1 = 0.$

The two verification equations are respected. Therefore we can conclude that the solution $\begin{cases} x = -3 \\ y = \frac{1}{2} \end{cases}$ is accepted.

We may also point out that it is unique.

$$\Longrightarrow S = \left\{ \begin{pmatrix} -3 & \frac{1}{2} \end{pmatrix} \right\}.$$

3 Chapiter 3 Integrals

3.1 Lesson N°1 Indefined integrals, properties

Let f be a defined and continuous function on $[a, b]$. We say that the function F is a primitive of f on $[a, b]$ if, and only if for all $x \in [a, b]$, F is differentiable and :

$$F'(x) = f(x).$$

Example

The functions :

$$F_1(x) = x^2; F_2(x) = x^2 + 5 \text{ and } F_3(x) = x^2 - 4$$

are all primitives on $\mathbb{R}$ of the same function $f(x) = 2x$; because $\forall x \in \mathbb{R}, F_1, F_2$ and F_3 are differentiable and

$$\forall x \in \mathbb{R}, F_1'(x) = F_2'(x) = F_3'(x) = 2x,$$

So the primitive of a function is not unique.

Remark

a) Primitives of the same function differ by only one constant C, That is why, we write :

$$\int f(x)dx = F(x) + C, (C \in \mathbb{R}).$$

Where :

- $\int f(x)dx$ is called an indefinite integral (without bounds).
- dx is the differential that indicates that x is the variable of integration (and derivation too!).
- $C \in \mathbb{R}$ is the integration constant.

So :

$$I(x) = \int 2xdx = x^2 + C, \quad (C \in \mathbb{R}).$$

b) We may find the value of C when we know the value of the integral for a certain x. This specification is called the initial condition.

Example

Determine the antiderivative of $f(x) = 2x$ which is worth 3 for $x = -1$.

Since :

$$I(x) = \int 2xdx = x^2 + C,$$

then, the initial condition translates to:

$$I(-1) = 3 \Rightarrow (-1)^2 + C = 3 \Rightarrow C = 2.$$

Finally, the primitive that verifies the requested initial condition is:

$$I(x) = x^2 + 2.$$

Remark

The primitive that verifies the initial condition is unique.

3.1.1 Defined integration

Let be $a < b \in \mathbb{R}$ and let f be a defined and continuous function on $[a, b]$, and F a primitive of f.

So :

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

is called defined integral (i.e[3] with bounds) from a to b.

The result of a defined integration is therefore a constant number and not a function.

Example

$$\int_{-1}^{0} 2xdx = [x^2 + C]_{-1}^{0} = \left[0^2 + C\right] - \left[(-1)^2 + C\right] = -1.$$

As we have just noticed on this example, the constant C will always be eliminated at the time of the calculation of the bounded integrals, this is why we do not take it into account in this case. It is therefore more usual to write:

$$\int_{-1}^{0} 2xdx = [x^2]_{-1}^{0} = 0^2 - (-1)^2 = -1.$$

[3] i.e = "id is" expression widely used in mathematics and which means "that is to say".

3.1.2 Integration by the table of laws

Let's first learn how to handle integrals, we have the following calculation laws:

1) $\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$	2) $\int \lambda f(x)dx = \lambda \int f(x)dx$
3) $\int f(x).g(x)dx \neq \int f(x)dx. \int g(x)dx$	4) $\int \dfrac{f(x)}{g(x)}dx \neq \dfrac{\int f(x)dx}{\int g(x)dx}$

Some fundamental formulas of integration

Basic formula $(a, C \in \mathbb{R})$	**Generalization** $(\alpha \neq -1)$
1)$\int adx = ax + C$;	
2)$\int x^{\alpha}dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$ (when $\alpha \neq -1$)	$\int u'(x)\,[u(x)]^{\alpha}\, dx = \dfrac{[u(x)]^{\alpha+1}}{\alpha+1} + C$
3)$\int \dfrac{dx}{x} = \ln\lvert x\rvert + C \quad (x \neq 0)$	$\int \dfrac{u'(x)}{u(x)}dx = \ln\lvert u(x)\rvert + C \quad (u(x) \neq 0)$
4)$\int e^{x}dx = e^{x} + C$	$\int u'(x)e^{u(x)}dx = e^{u(x)} + C$
5)$\begin{cases} \int \cos x dx = \sin x + C \\ \int \sin x dx = -\cos x + C \end{cases}$	$\begin{cases} \int u'(x)\cos[u(x)]\,dx = \sin[u(x)] + C \\ \int u'(x)\sin[u(x)]\,dx = -\cos[u(x)] + C \end{cases}$
6)$\begin{cases} \int chxdx = shx + C \\ \int shxdx = chx + C \end{cases}$	$\begin{cases} \int u'(x)ch[u(x)]\,dx = sh[u(x)] + C \\ \int u'(x)sh[u(x)]\,dx = ch[u(x)] + C \end{cases}$
7)$\int \dfrac{1}{x^2+1}dx = \arctan(x) + C$	$\int \dfrac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\dfrac{x}{a}\right) + C$

Other useful laws :

$$1)\int \frac{1}{1-x^2}dx = \tfrac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C; \quad (C \in \mathbb{R}).$$

Generalization : $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C; \quad (a, C \in \mathbb{R})$

$$2)\int \frac{1}{\cos^2 x}dx = \int (1+\tan^2 x)dx = \tan x + C$$

$$3)\int \frac{1}{\sin^2 x}dx = -\frac{1}{\tan x} + C$$

$$4)\int \frac{1}{\cos x}dx = \begin{cases} \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C \\ \ln\left|\tan x + \frac{1}{\cos x}\right| + C \end{cases}$$

$$5)\int \frac{1}{\sin x}dx = \begin{cases} \ln\left|\tan\left(\frac{x}{2}\right)\right| + C \\ \ln\left|\frac{1}{\tan x} - \frac{1}{\sin x}\right| + C \end{cases}$$

$$6)\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$$

$$7)\int \ln x \; dx = x\ln x - x + C; (x > 0)$$

$$8)\int \tan x \; dx = -\ln|\cos x| + C$$

$$9)\int \frac{1}{x\sqrt{x^2-1}}dx = arcséc\,(x) + C$$

$$10)\int \frac{-1}{x\sqrt{x^2-1}}dx = \arccos éc\,(x) + C$$

$$11)\ \int \sec x dx = \ln|\tan x + \sec x| + C$$

$$12)\ \int \sec^3 x dx = \tfrac{1}{2}\left[\sec x \tan x + \ln|\tan x + \sec x|\right] + C$$

Examples

Calculate the following integrals:

- $I_1(x) = \int (2x+5)^{11}dx$

One takes $u(x) = 2x + 5$; so $u'(x) = 2$.

(Multiply the integral by 2 to show the derivative $u'(x) = 2$ and divide by 2 to recover the initial integral).

$$I_1(x) = \frac{1}{2}\int 2(2x+5)^{11}dx = \frac{1}{2}\int u'(x)(u(x))^{11}dx$$

By applying the generalization of the law 2) of the table, it comes:

$$\begin{aligned} I_1(x) &= \frac{1}{2}\frac{[u(x)]^{12}}{12} + C \\ &= \frac{(2x+5)^{12}}{24} + C, (C \in \mathbb{R}). \end{aligned}$$

- $I_2(x) = \int x^3\sqrt{x^4+3}dx = I_2(x) = \int x^3\left(x^4+3\right)^{\frac{1}{2}}dx$

  One takes $u(x) = (x^4+3)$; so $u'(x) = 4x^3$

$$I_2(x) = \frac{1}{4}\int 4x^3\left(x^4+3\right)^{\frac{1}{2}}dx = \frac{1}{4}\int u'(x)(u(x))^{\frac{1}{2}}dx$$

By applying the generalization of the law 2) of the table, it comes :

$$\begin{aligned} I_2(x) &= \frac{1}{4}\frac{[u(x)]^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= \frac{(x^4+3)^{\frac{3}{2}}}{6} + C, (C \in \mathbb{R}). \end{aligned}$$

- $I_3(x) = \int \frac{(\ln x)^{\,2}}{x}dx = \int \frac{1}{x}(\ln x)^{\,2}dx$

  One takes $u(x) = (\ln x)$; so $u'(x) = \frac{1}{x}$

$$I_3(x) = \int u'(x)\,(u(x))^{\,2}dx$$

By applying the generalization of the law 2) of the table, it comes :

$$\begin{aligned} I_3(x) &= \frac{[u(x)]^3}{3} + C \\ &= \frac{(\ln x)^3}{3} + C, (C \in \mathbb{R}). \end{aligned}$$

- $I_4(x) = \int \dfrac{6x-3}{x^2-x+3} dx$

  One takes $u(x) = x^2 - x + 3$, so $u'(x) = 2x - 1$, we can then write :

$$I_4(x) = \int \frac{3.u'(x)}{u(x)} dx$$

By applying the generalization of the law 3) of the table, it comes :

$$\begin{aligned} I_4(x) &= 3\ln|u(x)| + C \\ &= 3\ln\left|x^2 - x + 3\right| + C, (C \in \mathbb{R}). \end{aligned}$$

3.1.3 Integration by parts

When an expression to be integrated is formed from a product of two functions, one of which is easy to integrate, we can use integration by parts.

Let u and v be two differentiable functions. We derive the formula for integration by parts from the formula for derivation of a product of two functions as follows :

$$(u.v)' = u'.v + u.v'$$

Hence,

$$u'.v = (u.v)' - u.v'$$

Which gives :

$$\int u'.v = \int (u.v)' - \int u.v'$$

$$\int u'.v = [u.v] - \int u.v'$$

To apply this formula, we must separate under the sign $\int$ a part that will be noted u' and another part noted v, but beware :

1) The part chosen as u' must be easily integrated.

2)$\int u.v'$ must not be more difficult than the desired integral $\int u'v$.

Some tips for integration by parts

1) If we have under the sign $\int$ a product of :	Choice of u' and v
A polynomial $\times \begin{cases} \text{exponential function} \\ \text{or trigonometric function (sin, cos,...)} \end{cases}$	$\begin{cases} u'(x) \text{ The function} \\ v(x) \text{ A polynomial} \end{cases}$
A polynomial $\times$ logarithmic function.	$\begin{cases} u'(x) \text{ A polynomial} \\ v(x) \text{ The function} \end{cases}$

2) When we integrate by parts, we add the integration constant completely at the end of the calculations.

Examples

1/ Calculate $I = \int x \cos x dx$

One takes $\begin{cases} u'(x) = \cos x \\ v(x) = x \end{cases}$, to get $\begin{cases} u(x) = \int \cos x dx = \sin x \\ v'(x) = 1 \end{cases}$.

The integration by parts formula gives us :

$$\begin{aligned} I &= \int u'v = [uv] - \int v'u \\ &= x \sin x - \int \sin x dx \\ I &= x \sin x + \cos x + C, (C \in \mathbb{R}). \end{aligned}$$

2/ Calculate $I = \int xe^x dx$

One takes $\begin{cases} u'(x) = e^x \\ v(x) = x \end{cases}$, to get $\begin{cases} u(x) = \int e^x dx = e^x \\ v'(x) = 1 \end{cases}$.

The integration by parts formula gives us :

$$\begin{aligned} I &= \int u'v = [uv] - \int v'u \\ &= xe^x - \int e^x dx \\ I &= xe^x + e^x + C, (C \in \mathbb{R}). \end{aligned}$$

3/ Calculate $I = \int x \ln x dx$; (for $x > 0$).

One takes $\left\{ \begin{array}{l} u'(x) = x \\ v(x) = \ln x \end{array} \right.$, to get $\left\{ \begin{array}{l} u(x) = \int x dx = \frac{x^2}{2} \\ v'(x) = \frac{1}{x} \end{array} \right.$.

The integration by parts formula gives us :

$$\begin{aligned} I &= \int u'v = [uv] - \int v'u \\ &= \frac{x^2}{2} \ln x - \int \left(\frac{1}{x} \times \frac{x^2}{2} \right) dx \\ I &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx \\ I &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C, (C \in \mathbb{R}). \end{aligned}$$

4/ **Circular integrals :**

Calculate $I = \int \sin t.e^t dt$

One takes $\left\{ \begin{array}{l} u'(t) = e^t \\ v(t) = \sin t \end{array} \right.$, to get $\left\{ \begin{array}{l} u(t) = e^t \\ v'(t) = \cos t \end{array} \right.$.

The integration by parts formula gives us :

$$\begin{aligned} I &= \int u'v = [uv] - \int v'u \\ &= \left[\sin t.e^t\right] - \int \cos t.e^t dt \end{aligned}$$

One takes $J = \int \cos t.e^t dt$ and we have to proceed by parts again to calculate J:

One takes $\left\{ \begin{array}{l} u'(t) = e^t \\ v(t) = \cos t \end{array} \right.$, to get $\left\{ \begin{array}{l} u(t) = e^t \\ v'(t) = -\sin t \end{array} \right.$.

The integration by parts formula gives us :

$$\begin{aligned} J &= \int u'v = [uv] - \int v'u \\ &= \left[\cos t.e^t\right] + \int \sin t.e^t dt \end{aligned}$$

We therefore come to the initial integral $I = \int \sin t.e^t dt$.

This kind of integrals are called circular integrals (i.e. which return to the starting point), we have to cut the vicious circle as follows:

$$\begin{aligned} I &= \left[\sin t.e^t\right] - \left[\cos t.e^t + I\right] \\ &\Rightarrow\ 2I = \sin t.e^t - \cos t.e^t \\ &\Rightarrow\ I = \frac{1}{2}\left(\sin t - \cos t\right).e^t + C, (C \in \mathbb{R}). \end{aligned}$$

3.1.4 Integration by change of variables

There is a precise technique when doing an integration by change of variables that we must follow to the letter. We have to change the variable but also the dx according to the new variable. Do not forget to return to the primary variable at the end.

Examples

1) Calculate $I = \int \sin(\ln x)dx$.
One takes $t = \ln x$, so one gets $x = e^t$ and $dx = e^t dt$.

Hence :

$$I = \int \sin t.e^t dt$$

And we continue by parts, (see example 4 about **circular integrals** in the previous page, section "integration by parts").

2) Calculate $I = \int \frac{e^{2x}}{1+e^{4x}}dx$.

One takes $t = e^{2x}$, to get $x = \frac{1}{2}\ln t$ and $dx = \frac{1}{2}\frac{1}{t}dt$.

Hence,

$$\begin{aligned} I &= \int \frac{t}{1+t^2}.\frac{1}{2}.\frac{1}{t}dt \\ &= \frac{1}{2}\int \frac{1}{1+t^2}dt \\ &= \frac{1}{2}\arctan(t) + C, (C \in \mathbb{R}). \end{aligned}$$

3.1.5 Integration of the pattern "polynomial by exponential function"

Reminder

We call polynomial of degree n any function that can be written in the form :

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

Where $a_i, i = 0, ..., n$ are real constants, $a_n \neq 0$.

Example :

*/ $P(x) = 4x^5 + \frac{1}{2}x^4 - x^2$ is a polynomial of degree 5.
*/ The function $Q(x) = \sqrt{x^2 - 1}$ is not a polynomial !

Integration of the form $\int P(x)e^{\alpha x}dx$

Let be $P(x)$ a polynomial of degree n and $\alpha \in \mathbb{R}$.

We should notice that the integral $I(x) = \int P(x)e^{\alpha x}dx$ actually has the same form as the function to be integrated, i.e.:

$$\forall x \in \mathbb{R}, I(x) = \int P(x)e^{\alpha x}dx = Q(x)e^{\alpha x} + C, (C \in \mathbb{R}).$$

Where $Q(x)$ is a polynomial of degree n too.

Example

Calculate $I(x) = \int (x^3 - x^2 + x + 1)e^{-2x}dx$.
From the above, we have:

$$\forall x \in \mathbb{R}, I(x) = Q(x)e^{-2x} + C \quad, (C \in \mathbb{R}). \qquad (*)$$

Where $Q(x)$ is a polynomial of degree 3 too.

Let be $Q(x) = ax^3+bx^2+cx+d$, which implies that $Q^{'}(x) = 3ax^2+2bx+c$.

By differentiating (*), we find:

$$\begin{aligned} \forall x \in \mathbb{R}, I^{'}(x) &= Q^{'}(x)e^{-2x} - 2Q(x)e^{-2x} \\ &= \left[Q^{'}(x) - 2Q(x)\right] e^{-2x} \end{aligned}$$

So, on one hand, we've got :

$$\begin{aligned}\forall x &\in \mathbb{R}, \left[Q^{'}(x) - 2Q(x)\right] = 3ax^2 + 2bx + c - 2(ax^3 + bx^2 + cx + d) \\ &= -2ax^3 + (3a - 2b)x^2 + (2b - 2c)x + c - 2d\end{aligned}$$

And on the other hand, we've got :

$$\forall x \in \mathbb{R}, I^{'}(x) = (x^3 - x^2 + x + 1)e^{-2x}$$

Hence :

$$\forall x \in \mathbb{R}, (x^3 - x^2 + x + 1)e^{-2x} = \left[-2ax^3 + (3a - 2b)x^2 + (2b - 2c)x + c - 2d\right] e^{-2x}$$

By analogy, we find:

$$\begin{cases} -2a = 1 \\ 3a - 2b = -1 \\ 2b - 2c = 1 \\ c - 2d = 1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{-1}{2} \\ b = \frac{-1}{4} \\ c = \frac{-3}{4} \\ d = \frac{-7}{8} \end{cases}$$

Finally,

$$\forall x \in \mathbb{R}, I(x) = -\frac{1}{8}(4x^3 + 2x^2 + 6x + 7)e^{-2x} + C, (C \in \mathbb{R}).$$

3.1.6 Integration of trigonometric functions

1/ Patterns of the form : $\int \sin^n(x)\,dx$ **et** $\int \cos^n(x)\,dx, \quad (n \geq 2)$.
a) **If** n **is even** :

We use the following linearization formulas : $\forall x \in \mathbb{R}$,

$$\begin{aligned}\cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \\ \sin^2(x) &= \frac{1}{2}(1 - \cos(2x))\end{aligned}$$

Example : calculate $I = \int \cos^4 x dx$.
One has :

$$
\begin{aligned}
\cos^4 x &= \left[\cos^2(x)\right]^2 \\
&= \left[\frac{1}{2}(1+\cos(2x))\right]^2 \\
&= \frac{1}{4}\left[1+2\cos(2x)+\cos^2(2x)\right] \\
&= \frac{1}{4}\left[1+2\cos(2x)+\frac{1}{2}(1+\cos(4x))\right]
\end{aligned}
$$

Hence :

$$
\begin{aligned}
I &= \frac{1}{4}\left[\int 1dx+\int 2\cos(2x)dx+\frac{1}{2}\int 1dx+\frac{1}{2}\int \cos(4x)dx\right] \\
&= \frac{1}{4}x+\frac{1}{4}\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)+C \\
&= \frac{3}{4}x+\frac{1}{4}\sin(2x)+\frac{1}{8}\sin(4x)+C, (C\in\mathbb{R}).
\end{aligned}
$$

b) **If n is odd** :

We separate the power n in $n=(n-1)+1$. We then keep the power 1 which will play the role of the derivative, then we use the relation :

$$\forall x\in\mathbb{R}, \cos^2(x)+\sin^2(x)=1.$$

Example calculate $I=\int \sin^5(x)dx$.

One has :

$$
\begin{aligned}
\sin^5(x) &= \sin^4(x)\sin(x) \\
&= [1-\cos^2(x)]^2\sin(x) \\
&= \left[1-2\cos^2(x)+\cos^4(x)\right]\sin(x) \\
&= \sin(x)-2\cos^2(x)\sin(x)+\cos^4(x)\sin(x)
\end{aligned}
$$

Hence :

$$
\begin{aligned}
I &= \int \sin(x)dx-\int 2\cos^2(x)\sin(x)dx+\int \cos^4(x)\sin(x)dx \\
&= -\cos(x)+2\frac{\cos^3(x)}{3}-\frac{\cos^5(x)}{5}+C, (C\in\mathbb{R}).
\end{aligned}
$$

2/ Patterns of the form :$\int \cos^n(x)\sin^m(x)\,dx \quad (n,m \geq 1)$.

a) **If either** n **or** m **is odd** :

We proceed as before by decomposing the one that is odd then, we still use the relation :

$$\forall x \in \mathbb{R}, \cos^2(x) + \sin^2(x) = 1.$$

Example

*/ Calculate $I = \int \sin^3(x)\cos^4(x)\,dx$. One has:

$$\begin{aligned}
\sin^3(x)\cos^4(x) &= \sin^2(x)\cos^4(x)\sin x \\
&= (1-\cos^2 x)\cos^4(x)\sin x \\
&= \cos^4(x)\sin x - \cos^6(x)\sin x
\end{aligned}$$

Hence :

$$\begin{aligned}
I &= \int \cos^4(x)\sin x dx - \int \cos^6(x)\sin x dx \\
&= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C, (C \in \mathbb{R}).
\end{aligned}$$

*/ Calculate : $I = \int \sin^8(x)\cos^3 x dx$. One has:

$$\begin{aligned}
\forall x &\in \mathbb{R}, \quad \sin^8(x)\cos^3(x) = \sin^8(x)\cos^2(x)\cos(x) \\
&= \sin^8(x)\left[1-\sin^2(x)\right]\cos(x) \\
&= \sin^8(x)\cos x - \sin^{10}(x)\cos(x)
\end{aligned}$$

Hence :

$$\begin{aligned}
I &= \int \sin^8(x)\cos x\ dx - \int \sin^{10}(x)\cos(x)\ dx \\
&= \frac{\sin^9(x)}{9} - \frac{\sin^{11}(x)}{11} + C, (C \in \mathbb{R}).
\end{aligned}$$

b) **If** n **and** m **are both odd :**

We decompose the smallest between n and m, then we use the relation :

$$\forall x \in \mathbb{R}, \cos^2(x) + \sin^2(x) = 1.$$

Example calculate $I = \int \sin^5(x)\cos^7(x)\,dx$.
One has:

$$\begin{aligned}
\sin^5(x)\cos^7(x) &= \left[\sin^2(x)\right]^2 \cos^7(x)\sin(x) \\
&= \left[1-\cos^2(x)\right]^2 \cos^7(x)\sin(x) \\
&= \cos^7(x)\sin(x) - 2\cos^9(x)\sin(x) + \cos^{11}(x)\sin(x)
\end{aligned}$$

Hence :

$$\begin{aligned}
I &= \int \cos^7(x)\sin(x)\,dx - \int 2\cos^9(x)\sin(x)\,dx + \int \cos^{11}(x)\sin(x)\,dx \\
&= -\frac{\cos^8(x)}{8} + \frac{\cos^{10}(x)}{5} - \frac{\cos^{12}(x)}{12} + C, (C \in \mathbb{R}).
\end{aligned}$$

c) **If n and m are both even:**
We use the following relation $\sin(2x) = 2\sin x \cos x$.

Example calculate $I = \int \sin^4(x)\cos^2(x)\,dx$.
One has:

$$\begin{aligned}
\sin^4(x)\cos^2(x) &= [\sin(x)\cos(x)]^2 \sin^2(x) \\
&= \left[\frac{\sin(2x)}{2}\right]^2 \sin^2(x) \\
\text{Recall that} &: \forall x \in \mathbb{R}, \sin^2(x) = \frac{1}{2}(1-\cos(2x)), \text{ so} \\
\sin^4(x)\cos^2(x) &= \frac{1}{4}\sin^2(2x)\left[\frac{1}{2}(1-\cos(2x))\right] \\
&= \frac{1}{8}\sin^2(2x)\left[(1-\cos(2x))\right] \\
&= \frac{1}{8}\sin^2(2x) - \frac{1}{8}\sin^2(2x)\cos(2x) \\
&= \frac{1}{8}\left[\frac{1}{2}(1-\cos(4x))\right] - \frac{1}{8}\sin^2(2x)\cos(2x)
\end{aligned}$$

Hence :

$$\begin{aligned} I &= \frac{1}{16}\int(1-\cos(4x))dx - \frac{1}{8}\int \sin^{2}(2x)\cos(2x)dx \\ &= \frac{1}{16}\left[x-\frac{1}{4}\sin(4x)\right] - \frac{1}{16}\left[\frac{1}{2}\frac{\sin^3(2x)}{3}\right] + C \\ &= \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{96}\sin^3(2x) + C, (C \in \mathbb{R}). \end{aligned}$$

3/ Patterns of the form : $\int \sin(\alpha x)\cos(\beta x)\,dx$, $\int \sin(\alpha x)\sin(\beta x)\,dx$, $\int \cos(\alpha x)\cos(\beta x)\,dx$

For this type of primitives, we use the following trigonometric formulas :
$\forall \alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \cos(\alpha x)\cos(\beta x) &= \frac{1}{2}\left[\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right]. \\ \sin(\alpha x)\cos(\beta x) &= \frac{1}{2}\left[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x\right]. \\ \sin(\alpha x)\sin(\beta x) &= \frac{1}{2}\left[-\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right]. \end{aligned}$$

Example

Calculate : $\int \sin(5x)\sin(3x)\,dx$.

One has :

$$\begin{aligned} \sin(5x)\sin(3x) &= \frac{1}{2}\left[-\cos(5+3)x + \cos(5-3)x\right] \\ &= \frac{1}{2}\left[-\cos(8x) + \cos(2x)\right]. \end{aligned}$$

Hence :

$$\begin{aligned} \int \sin(5x)\sin(3x)\,dx &= \frac{1}{2}\int\left(-\cos(8x)+\cos(2x)\right)dx \\ &= \frac{1}{2}\left[\frac{-1}{8}\sin(8x) + \frac{1}{2}\sin(2x)\right] + C \\ &= \frac{-1}{16}\sin(8x) + \frac{1}{4}\sin(2x) + C, (C \in \mathbb{R}). \end{aligned}$$

4/ integration of fractions containing trigonometric functions

Let be the following integral $I = \int R(x)dx$ where $R(x)$ is a fraction that contains circular functions *sin*, *cos* and/or *tan*.

a) If $R(x) = \cos x R_1(\sin x)$

In this case, we put $t = \sin x$, and find $dt = \cos x dx$.

Example: $I = \int (1 + 3\sin x)^{\frac{1}{3}} \cos x dx$

Let's put $t = \sin x$, we then find $dt = \cos x dx$.

So :

$$\begin{aligned} I &= \int (1+3t)^{\frac{1}{3}} dt \\ &= \frac{1}{3}\left[\frac{(1+3t)^{\frac{4}{3}}}{\frac{4}{3}}\right] + C, (C \in \mathbb{R}) . \\ &= \frac{1}{4}\left[(1+3\sin x)^{\frac{4}{3}}\right] + C, (C \in \mathbb{R}) . \end{aligned}$$

b) If $R(x) = \sin x R_2(\cos x)$
In such cases, we put $t = \cos x$, and find $dt = -\sin x dx$.

Example: $I = \int \frac{\sin x}{6 - \sin^2 x} dx$

Since $\forall x \in \mathbb{R}, \cos^2(x) + \sin^2(x) = 1$ then :

$$\begin{aligned} I &= \int \frac{\sin x}{6 - (1 - \cos^2 x)} dx \\ &= \int \frac{\sin x}{5 + \cos^2 x} dx \end{aligned}$$

one sets $t = \cos x$, and find $dt = -\sin x dx$.

This gives :

$$
\begin{aligned}
I &= \int \frac{dt}{5+t^2} \\
&= \frac{1}{\sqrt{5}}\left[arctg\left(\frac{t}{\sqrt{5}}\right)\right] + C, (C \in \mathbb{R}). \\
&= \frac{\sqrt{5}}{5}\left[arctg\left(\frac{\sqrt{5}\cos x}{5}\right)\right] + C, (C \in \mathbb{R}).
\end{aligned}
$$

c) If $R(x) = R_3(\tan x)$

In similar cases, we put $t = \tan x$, and find :

$$dt = \left(1 + \tan^2 x\right) dx \Rightarrow dx = \frac{dt}{1+t^2}.$$

Example:

$$
\begin{aligned}
I &= \int \frac{\cos x + \sin x \tan x}{2\sin x + 3\cos x} dx \\
&= \int \frac{1+\tan x}{2\tan x + 3} dx
\end{aligned}
$$

Take $t = \tan x$, deduce that $dx = \dfrac{dt}{1+t^2}$.

$$
\begin{aligned}
I &= \int \frac{1+t^2}{2t+3}\frac{dt}{1+t^2} \\
&= \int \frac{dt}{2t+3} \\
&= \frac{1}{2}\left[\ln|2t+3|\right] + C, (C \in \mathbb{R}).
\end{aligned}
$$

d) If $R(x)$ is not written in any previous form :

In this case, there is a particular change of variable, very useful when integrating fractions in sin, cos, tan, etc...

We pose $y = \tan\left(\frac{x}{2}\right)$, $x \neq (2k+1)\frac{\pi}{2}, (k \in \mathbb{Z})$. Which gives :

$$x = 2\arctan(y) \quad \text{and} \quad dx = \frac{2}{1+y^2}dy.$$

And by the trigonometric formulas we can get $\sin x$, $\cos x$ and $\tan x$ depending on the new variable y.

Indeed, we have :

$$\begin{aligned}
\sin x &= \sin\left(2\left(\frac{x}{2}\right)\right) \\
\text{and since } \forall\alpha &\in \mathbb{R}, \sin(2\alpha) = 2\sin\alpha\cos\alpha \quad \text{we obtain :} \\
\sin x &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right). \\
\text{We multiply and divide by } \cos\left(\frac{x}{2}\right) \text{, so now} &: \\
\sin x &= 2\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\cos^2\left(\frac{x}{2}\right) \\
\text{and since } \cos^2\left(\frac{x}{2}\right) &= \frac{1}{1+\tan^2\left(\frac{x}{2}\right)} \text{ we get :} \\
\sin x &= 2\tan\left(\frac{x}{2}\right)\frac{1}{1+\tan^2\left(\frac{x}{2}\right)}. \\
\text{Hence} &: \\
\sin x &= \frac{2y}{1+y^2}.
\end{aligned}$$

Afterwards,

$$\begin{aligned}
\cos x &= \cos\left(2\left(\frac{x}{2}\right)\right) \\
\text{and since } \forall\alpha &\in \mathbb{R}, \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{we obtain} \\
\cos x &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right). \\
\text{Take out } \cos^2\left(\frac{x}{2}\right) \text{ as a common factor, we obtain} &: \\
\cos x &= \cos^2\left(\frac{x}{2}\right)\left(1 - \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}\right) \\
&= \cos^2\left(\frac{x}{2}\right)\left(1-\tan^2\left(\frac{x}{2}\right)\right) \\
&= \frac{1}{1+\tan^2\left(\frac{x}{2}\right)}\left(1-\tan^2\left(\frac{x}{2}\right)\right) \\
\cos x &= \frac{1-y^2}{1+y^2}
\end{aligned}$$

Finally:

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
&= \frac{\frac{2y}{1+y^2}}{\frac{1-y^2}{1+y^2}} \\
\tan x &= \frac{2y}{1-y^2}
\end{aligned}$$

Example

Calculate $I = \int \dfrac{dx}{\sin^3(x)}$.

Let's put $y = \tan\left(\frac{x}{2}\right)$, to get $x = 2\arctan(y)$ and $dx = \dfrac{2}{1+y^2}dy$.
Hence,

$$\begin{aligned}
I &= \int \frac{\frac{2}{1+y^2}}{\left(\frac{2y}{1+y^2}\right)^3} dy \\
&= \frac{1}{4}\int \frac{(1+y^2)^2}{y^3} dy \\
&= \frac{1}{4}\int \left(\frac{1}{y^3} + \frac{2}{y} + y^4\right) dy \\
&= \frac{1}{4}\left[-\frac{1}{2y^2} + 2\ln|y| + \frac{y^5}{5}\right] + C \\
&= -\frac{1}{8y^2} + \frac{1}{2}\ln|y| + \frac{1}{20}y^5 + C.
\end{aligned}$$

We now return to the initial variable x:

$$I = -\frac{1}{8\tan^2\left(\frac{x}{2}\right)} + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right| + \frac{1}{20}\tan^5\left(\frac{x}{2}\right) + C, (C \in \mathbb{R}).$$

Abelian integrals in what follows, we are going to learn how to integrate three patterns which we call Abelian integrals:

Form 1 : $I_1 = \int \sqrt{a^2 - x^2} dx$
Consider $f(x) = \sqrt{a^2 - x^2}$
$D_f = \{x \in \mathbb{R}/a^2 - x^2 \geq 0\} = [-a, a]$.
and take $x = a \sin y$, so $y = \arcsin\left(\frac{x}{a}\right)$ and $dx = a \cos y dy$.
Hence :

$$\begin{aligned} I_1 &= \int \sqrt{a^2 - a^2 \sin^2 y} a \cos y dy \\ &= a^2 \int \sqrt{1 - \sin^2 y} \cos y dy \\ &= a^2 \int \sqrt{\cos^2 y} \cos y dy \\ &= a^2 \int |\cos y| \cos y dy \end{aligned}$$

Note that when $x \in [-a, a]$, then $y \in [-\frac{\pi}{2}, -\frac{\pi}{2}]$, which implies that $cosy \geq 0$.

So,

$$\begin{aligned} I_1 &= a^2 \int \cos^2 y dy \\ &= a^2 \int \frac{\cos(2y) + 1}{2} dy \\ &= \frac{a^2}{2} \left[\frac{1}{2} \sin(2y) + y\right] + C, (C \in \mathbb{R}). \end{aligned}$$

and since $\forall \alpha \in \mathbb{R}, \sin(2\alpha) = 2 \sin \alpha \cos \alpha$ we obtain:

$$I_1 = \frac{a^2}{2} [\sin y \cos y + y] + C, (C \in \mathbb{R}).$$

Now just go back to the variable x by acknowledging that :

$$\begin{cases} \sin y = \left(\frac{x}{a}\right) \\ y = \arcsin\left(\frac{x}{a}\right) \\ \cos y = \sqrt{1 - \left(\frac{x}{a}\right)^2} = \frac{1}{a}\sqrt{a^2 - x^2} \end{cases}$$

So :

$$I_1 = \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{1}{2} x\sqrt{a^2 - x^2} + C, (C \in \mathbb{R}).$$

Example Calcuate $I = \int_0^1 \sqrt{1 - x^2} dx$

First method : Variable change

One poses $x = \sin y$, one finds $dx = \cos y.dy$
$x \in [0, 1]$ and $x = \sin y \Rightarrow y = \arcsin x$.

So,

$$\begin{cases} x_1 = 0 \Rightarrow y_1 = \arcsin 0 = 0 \\ x_2 = 1 \Rightarrow y_2 = \arcsin 1 = \frac{\pi}{2} \end{cases}.$$

Hence,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 y}\ \cos y\ dy \\ &= \int_0^{\frac{\pi}{2}} |\cos y| \cos y dy \end{aligned}$$

But since $y \in \left[0, \frac{\pi}{2}\right]$ then $\cos y \geq 0$. Which gives that :

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos^2 y\ dy \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\cos(2y) + 1)\ dy \\ &= \frac{1}{2} \left[\frac{1}{2} \sin(2y) + y\right]_0^{\frac{\pi}{2}} \\ I &= \frac{\pi}{4}. \end{aligned}$$

Second method: by parts

Take $\begin{cases} u'(x) = 1 \\ v(x) = \sqrt{1 - x^2} \end{cases}$, and get $\begin{cases} u(x) = x \\ v'(x) = \frac{1}{2} \dfrac{(-2x)}{\sqrt{1 - x^2}} = \dfrac{-x}{\sqrt{1 - x^2}} \end{cases}$.

Hence :

$$
\begin{aligned}
I &= \left[x\sqrt{1-x^2}\right]_0^1 - \int_0^1 \frac{-x^2}{\sqrt{1-x^2}}dx \\
&= -\int_0^1 \frac{1-x^2-1}{\sqrt{1-x^2}}dx \\
&= -\int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}dx \\
I &= -I + \int_0^1 \frac{1}{\sqrt{1-x^2}}dx \\
&\Rightarrow\ 2I = [\arcsin(x)]_0^1 \\
&\Rightarrow\ I = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}.
\end{aligned}
$$

Form 2 : $I_2 = \int \sqrt{a^2+x^2}dx$
Consider $f(x) = \sqrt{a^2+x^2}$
$D_f = \{x \in \mathbb{R}/a^2+x^2 \geq 0\} = \mathbb{R}$.
Take $x = ashy$, to get $y = \arg sh\left(\frac{x}{a}\right)$ and $dx = achydy$.

Hence :

$$
\begin{aligned}
I_2 &= \int \sqrt{a^2+a^2\ sh^2y}\ \ a\ chydy \\
&= a^2 \int |chy|\ \ chydy
\end{aligned}
$$

Recall that, we have $\forall y \in \mathbb{R}, chy \geq 0$. So:

$$
\begin{aligned}
I_2 &= a^2 \int ch^2ydy \\
&= a^2 \int \frac{ch(2y)+1}{2}dy \\
&= \frac{a^2}{2}\left[\frac{1}{2}sh(2y)+y\right] + C, (C \in \mathbb{R}).
\end{aligned}
$$

and since $\forall \alpha \in \mathbb{R}, sh(2\alpha) = 2\ sh\alpha\ ch\alpha$ we obtain :

$$
I_2 = \frac{a^2}{2}\left[shy\ chy + y\right] + C, (C \in \mathbb{R}).
$$

Now just go back to the variable x by acknowledging that :

$$\begin{cases} shy = \left(\frac{x}{a}\right) \\ y = \arg sh\left(\frac{x}{a}\right) \\ chy = \sqrt{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\sqrt{a^2+x^2} \end{cases}$$

Hence :

$$I_2 = \frac{a^2}{2}\arg sh\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2+x^2} + C, (C \in \mathbb{R}).$$

Example : Calculate the following integral : $I = \int \sqrt{9+x^2}dx$.

First method : change of variables

Put $x = 3shy$, get $dx = 3chy.dy$

Hence,

$$\begin{aligned} I &= \int \sqrt{9+(3shy)^2}\,.(3chy)\ dy \\ &= \int \sqrt{9\left(1+sh^2y\right)}\,.(3chy)\ dy \\ &= 9\int |chy|\ chydy\ . \end{aligned}$$

Since $\forall y \in \mathbb{R}$, $chy > 0$, then:

$$\begin{aligned} I &= 9\int ch^2y\ dy \\ &= \frac{9}{2}\int (ch(2y)+1)\ dy \\ &= \frac{9}{2}\left[\frac{1}{2}sh(2y)+y\right]+C \\ I &= \frac{9}{4}sh(2y)+\frac{9}{2}y+C, \end{aligned}$$

Then $sh(2y) = 2\ shy\ chy$ so $I = \frac{9}{2}sh(y)\ chy + \frac{9}{2}y + C$, and since $x = 3shy$ then:

$$\begin{cases} shy = \dfrac{x}{3} \\ y = \arg sh\left(\dfrac{x}{3}\right) \\ chy = \sqrt{1+sh^2y} = \sqrt{1+\left(\frac{x}{3}\right)^2} = \frac{1}{3}\sqrt{9+x^2} \end{cases}$$

Finalally :

$$I = \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\arg sh\left(\frac{x}{3}\right) + C, (C \in \mathbb{R}).$$

Note: we can use integration by part to solve this example, as follows :

Second method By parts :

One poses $\begin{cases} u'(x) = 1 \\ v(x) = \sqrt{9+x^2} \end{cases}$, to get $\begin{cases} u(x) = x \\ v'(x) = \frac{1}{2}\dfrac{(2x)}{\sqrt{9+x^2}} = \dfrac{x}{\sqrt{9+x^2}} \end{cases}$

So,

$$\begin{aligned} I &= \left[x\sqrt{9+x^2}\right] - \int \frac{x^2}{\sqrt{9+x^2}}dx \\ &= \left[x\sqrt{9+x^2}\right] - \int \frac{9+x^2-9}{\sqrt{9+x^2}}dx \\ &= \left[x\sqrt{9+x^2}\right] - \int \frac{9+x^2}{\sqrt{9+x^2}}dx + \int \frac{9}{\sqrt{9+x^2}}dx \\ I &= \left[x\sqrt{9+x^2}\right] - I + 9\int \frac{1}{\sqrt{9+x^2}}dx \\ &\Rightarrow\ 2I = \left[x\sqrt{9+x^2}\right] + 9\left[\arg sh\left(\frac{x}{3}\right)\right] + C \end{aligned}$$

$$\Rightarrow I = \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\arg sh\left(\frac{x}{3}\right) + C, (C \in \mathbb{R}).$$

Form 3 : $I_3 = \int \sqrt{x^2-a^2}dx$

Take $f(x) = \sqrt{x^2-a^2}$.

$D_f = \{x \in \mathbb{R}/x^2-a^2 \geq 0\} =]-\infty, -a] \cup [a, +\infty[$.

*/ On $[a, +\infty[$ one takes $x = achy$, to have $y = \arg ch\left(\frac{x}{a}\right)$ et $dx = ashydy$.

So,

$$\begin{aligned} I_3 &= \int \sqrt{a^2ch^2y - a^2}.ashydy \\ &= a^2\int |shy|\, shydy \end{aligned}$$

Note that $y = \arg ch\left(\frac{x}{a}\right)$, which implies that $y \geq 0$. So $shy \geq 0$,

$$\begin{aligned}
I_3 &= a^2 \int sh^2 y dy \\
&= a^2 \int \frac{ch(2y) - 1}{2} dy \\
&= \frac{a^2}{2}\left[\frac{1}{2} sh(2y) - y\right] + C, (C \in \mathbb{R}).
\end{aligned}$$

And since $\forall \alpha \in \mathbb{R}, sh(2\alpha) = 2sh\alpha ch\alpha$ one obtains :

$$I_3 = \frac{a^2}{2}\left[shychy + y\right] + C, (C \in \mathbb{R}).$$

Now just go back to the variable x by acknowledging that :

$$\begin{cases}
chy = \left(\frac{x}{a}\right) \\
y = \arg ch\left(\frac{x}{a}\right) \\
shy = \sqrt{ch^2\left(\frac{x}{a}\right) - 1} = \frac{1}{a}\sqrt{x^2 - a^2}
\end{cases}$$

Hence :

$$I_3 = \frac{a^2}{2} \arg ch\left(\frac{x}{a}\right) + \frac{1}{2} x\sqrt{x^2 - a^2} + C, (C \in \mathbb{R}).$$

*/ On $]-\infty, -a]$ either take $x = -achy$ and continue in the same manner, or conclude the result by symmetry since f is a even function.

Remark

The integrals of the inverse of the Abelian forms are very easy to calculate with the laws of the integration table.

Indeed

- **Form 1 :** $I_1^* = \int \frac{1}{\sqrt{a^2 - x^2}} dx$. It is very easy to calculate I_1^* as follows :

$$I_1^* = \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx, \text{ (recall that } a > 0\text{)}.$$

Just make the change of variables : $y = \frac{x}{a} \Rightarrow x = a.y$ and $dx = a.dy$ to get :

$$\begin{aligned} I_1^* &= \frac{1}{a}\int \frac{a}{\sqrt{1-y^2}}dy \\ &= \arcsin(y) + C \\ I_1^* &= \arcsin\left(\frac{x}{a}\right) + C, (C \in \mathbb{R}). \end{aligned}$$

- **Form 2 :** $I_2^* = \int \frac{1}{\sqrt{a^2+x^2}}dx$. In the same way, we calculate I_2^* :

$$I_2^* = \frac{1}{a}\int \frac{1}{\sqrt{1+\left(\frac{x}{a}\right)^2}}dx$$

  Just make the change of variables : $y = \frac{x}{a} \Rightarrow x = a.y$ and $dx = a.dy$ to get :

$$\begin{aligned} I_2^* &= \frac{1}{a}\int \frac{a}{\sqrt{1+y^2}}dy \\ &= \arg sh(y) + C \\ I_2^* &= \arg sh\left(\frac{x}{a}\right) + C, (C \in \mathbb{R}). \end{aligned}$$

- **Form 3 :** $I_3^* = \int \frac{1}{\sqrt{x^2-a^2}}dx$. Same variable change for I_3^*, one obtains :

$$I_3^* = \arg ch\left(\frac{x}{a}\right) + C, (C \in \mathbb{R}).$$

Other Abelian Integrals

For this part, we will very quickly introduce new trigonometric functions, which will be useful to us, particularly for calculating integrals of fractions with sines and cosines in the denominator.

1) **The secant and arcsecant functions**

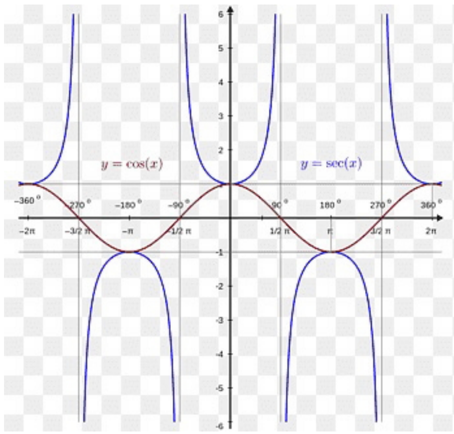
We define the secant function, noted sec as follows :

$$\begin{aligned} \sec : \ & \mathbb{R}\backslash\left\{\left(\tfrac{\pi}{2}+k\pi\right), k\in\mathbb{Z}\right\} \to \mathbb{R} \\ & x \mapsto \sec x = \frac{1}{\cos x} \end{aligned}$$

It is a $(2\pi)\,-$periodic function, even and continuous on its domain of definition. We can consider its restriction on the interval $[0, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi]$.

In addition, $\lim\limits_{\substack{x \to \frac{\pi}{2} \\ >}} \sec x = -\infty$ and $\lim\limits_{\substack{x \to \frac{\pi}{2} \\ <}} \sec x = +\infty$.

Finally, it is strictly increasing on $[0, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi]$ and its graph is symmetric with respect to the y-axis.

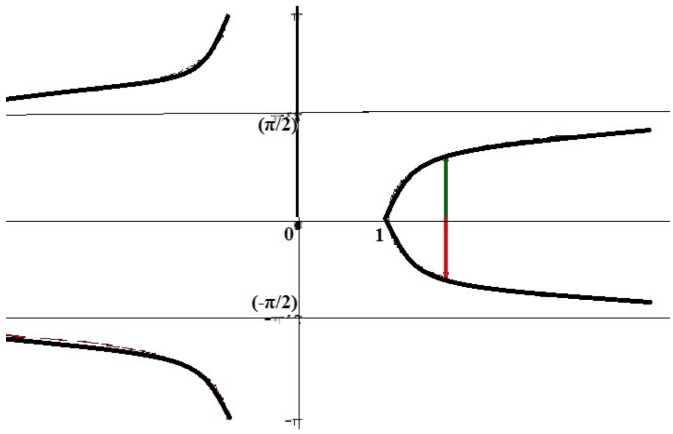


$Fig3.1$: Graph of cos and $secant$ functions.

- Since sec performs a bijection of $[0, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi]$ into $\mathbb{R}$, it admits an inverse function, called arcsec :

$$\begin{aligned} \text{arcsec}: \quad \mathbb{R} &\longrightarrow [0, \tfrac{\pi}{2}[\cup]\tfrac{\pi}{2}, \pi] \\ x &\longmapsto \text{arcsec}\, x \end{aligned}$$

Here is its graph:



$Fig3.2$: Graph of *arcsecant.*

One has :

$$(\sec x)' = \sec x \tan x \quad \text{and in general : } (\sec(u(x)))' = u'(x)\sec(u(x))\tan(u(x))$$

And

$$(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}. \quad \text{In general :} (\operatorname{arcsec}(u(x)))' = \frac{u'(x)}{(u(x))\sqrt{(u(x))^2-1}}$$

2) The cosecant and arccosecant functions

- We define the cosecant function, denoted *cosec* as follows :

$$\begin{aligned} cosec: \ & \mathbb{R}\backslash\{(k\pi), k\in\mathbb{Z}\} \to \mathbb{R} \\ & x \mapsto cosec(x) = \frac{1}{\sin x} \end{aligned}$$

It's a (2π) −periodic function, odd and continuous on its domain of definition. We can consider its restriction on the interval $]-\pi,\pi[$.

In addition $\lim\limits_{\substack{x\to 0\\ <}} cosec\ x = -\infty$ and $\lim\limits_{\substack{x\to 0\\ >}} cosec\ x = +\infty$.

Finally, it is strictly increasing on $]-\pi,\pi[$ and its graph is symmetric with respect to the origin.

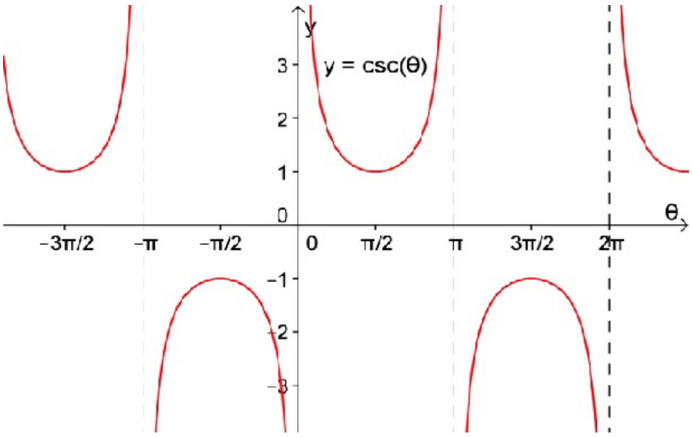


Fig 3.3 : Graph of *cosecant*.

- Since *cosec* performs a bijection $]-\pi, \pi[$ into $\mathbb{R}$, it admits an inverse function *arccosec*:

$$arccosec: \quad \begin{array}{l} \mathbb{R} \to]-\pi, \pi[\\ x \mapsto arccosec \end{array}$$

Here is its graph:

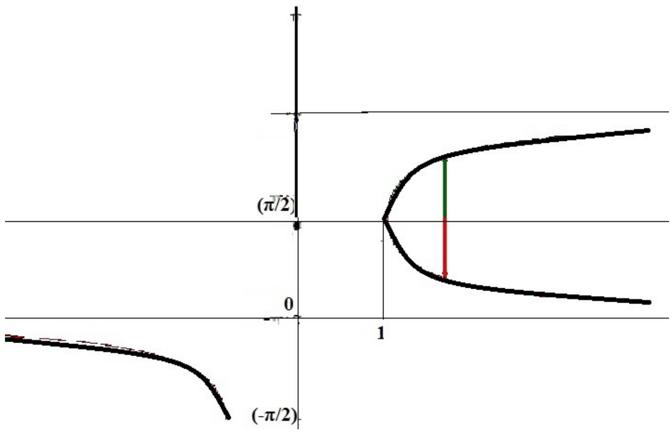


*Fig*3.4 : Graph of *arccosec*.

One has:

$$(cosecx)' = -cosec\,(x)\,cotan\,(x)$$

and in general $\quad (cosec\,(u(x)))' = -u'(x)cosec\,(u(x))\,cotan\,(u(x))$

And

$$(arccosec\,(x))' = \frac{-1}{x\sqrt{x^2-1}}. \quad \text{In general } (arccosec\,(u(x)))' = \frac{-u'(x)}{(u(x))\,\sqrt{(u(x))^{\,2}-1}}$$

The forms to be integrated that interest us here are the following :
Let be two real constants $a, b \neq 0$,
Form 1: $\quad a^2x^2 + b^2$
For the first form : $a^2x^2 + b^2$,
Put $x = \frac{b}{a}\tan y$ and find $dx = \frac{b}{a}\sec^2 y dy$.
Then we use the trigonometric formula: $\sec^2 y = 1 + \tan^2 y$.

Form 2: $\quad a^2x^2 - b^2$
For the second form : $a^2x^2 - b^2$,
Put $x = \frac{b}{a}\sec y$ and find $dx = \frac{b}{a}\sec y \tan y dy$.
Then we use the trigonometric formula : $\sec^2 y - 1 = \tan^2 y$.

Form 3: $\quad b^2 - a^2x^2$
For the third form : $b^2 - a^2x^2$,
Put $x = \frac{b}{a}\sin y$ and find $dx = \frac{b}{a}\cos y dy$.
Then we use the trigonometric formula : $1 - \sin^2 x = \cos^2 y$.

Examples

Integrate:

$$I_1 = \int \frac{dx}{x\sqrt{4x^2+9}}, \quad I_2 = \int \frac{dx}{\sqrt{9x^2-16}}, \quad I_3 = \int \frac{\sqrt{9-4x^2}}{x}dx, \quad I_4 = \int \frac{dx}{(9x^2-16)^{\frac{3}{2}}}.$$

Correction

•

$$I_1 = \int \frac{dx}{x\sqrt{4x^2+9}}$$

We are facing the first form : $a^2x^2 + b^2 = 4x^2 + 9$,
Put $x = \frac{3}{2}\tan y$, and find $dx = \frac{3}{2}\sec^2 y dy$.
Hence :

$$\begin{aligned} I_1 &= \int \frac{\frac{3}{2}\sec^2 y dy}{\frac{3}{2}\tan y\sqrt{4\left(\frac{3}{2}\tan y\right)^2 + 9}} \\ &= \int \frac{\sec^2 y dy}{3\tan y\sqrt{\tan^2 y + 1}} \end{aligned}$$

Then we use the trigonometric formula : $\sec^2 y = 1 + \tan^2 y$, we then obtains:

$$\begin{aligned} I_1 &= \frac{1}{3}\int \frac{\sec^2 y dy}{\tan y \sec y} \\ &= \frac{1}{3}\int \frac{\sec y}{\tan y} dy \\ &= \frac{1}{3}\int \frac{1}{\sin y} dy = \int cosec y dy \\ I_1 &= \frac{1}{3}\left[\ln\left|cotan\,(y) - cosec\,(y)\right|\right] + C. \text{ (See table of integration laws).} \end{aligned}$$

We get back to the initial variable:

$$I_1 = \frac{1}{3}\left[\ln\left|\frac{3}{2x} - cosec\left(\arctan(\frac{2}{3}x)\right)\right|\right] + C, (C \in \mathbb{R}).$$

- $$I_2 = \int \frac{dx}{\sqrt{9x^2 - 16}}$$

We are facing the second form : $a^2x^2 - b^2 = 9x^2 - 16$,
Taking $x = \frac{4}{3}\sec y$, to get $dx = \frac{4}{3}\sec y \tan y dy$.

$$
\begin{aligned}
I_2 &= \int \frac{dx}{\sqrt{9x^2-16}} \\
&= \int \frac{\frac{4}{3}\sec y \tan y dy}{\sqrt{9\left(\frac{4}{3}\sec y\right)^{2}-16}} \\
&= \frac{1}{3}\int \frac{\sec y \tan y dy}{\sqrt{\sec^{2} y-1}}
\end{aligned}
$$

Then we use the trigonometric formula : $\sec^{2} y - 1 = \tan^{2} y$. we get:

$$
\begin{aligned}
I_2 &= \frac{1}{3}\int \frac{\sec y \tan y dy}{\tan y} = \frac{1}{3}\int \sec y dy \\
&= \frac{1}{3}\left[\ln\left|\tan y + \sec y\right|\right] + C
\end{aligned}
$$

$$
I_2 = \frac{1}{3}\ln\left|\left(\frac{3x}{4}\right) + \sqrt{\left(\frac{3x}{4}\right)^{2} - 1}\right| + C, (C \in \mathbb{R}).
$$

$$
I_3 = \int \frac{\sqrt{9-4x^2}}{x} dx
$$

We are facing the third form : $b^2 - a^2x^2 = 9 - 4x^2$,
Taking $x = \frac{3}{2}\sin y$, to get $dx = \frac{3}{2}\cos y dy$.
One has:

$$
\begin{aligned}
I_3 &= \int \frac{\sqrt{9-4\left(\frac{3}{2}\sin y\right)^{2}}}{\frac{3}{2}\sin y}\left(\frac{3}{2}\cos y\right) dy. \\
&= 3\int \frac{\sqrt{1-\sin^{2} y}}{\sin y}\cos y dy
\end{aligned}
$$

Then we use the trigonometric formula : $1 - \sin^{2} x = \cos^{2} y$. we find:

$$
\begin{aligned}
I_3 &= 3\int \frac{\cos^2 y}{\sin y}dy \\
&= 3\int \frac{1-\sin^2 y}{\sin y}dy \\
&= 3\int \frac{1}{\sin y}dy - \int \sin y dy \\
&= 3\int cosec\,(y)\,dy + \cos y + C \\
&= 3\left[\ln\left|cotan\,(y) - cosec\,(y)\right| + \cos y\right] + C
\end{aligned}
$$

Getting back to the initial variable :

$$I_3 = 3\left[\ln\left|cotan\,(y) - cosec\,(y)\right| + \cos y\right] + C, (C \in \mathbb{R})\,.$$

- $$I_4 = \int \frac{dx}{(9x^2-16)^{\frac{3}{2}}}$$

We are facing the third form : $a^2x^2 - b^2 = 9x^2 - 16$,
So take $x = \frac{4}{3}\sec y$, and get $dx = \frac{4}{3}\sec y \tan y dy$.

$$
\begin{aligned}
I_4 &= \int \frac{dx}{(9x^2-16)^{\frac{3}{2}}} \\
&= \int \frac{\frac{4}{3}\sec y \tan y dy}{\left[9\left(\frac{4}{3}\sec y\right)^2 - 16\right]^{\frac{3}{2}}} \\
&= \frac{4}{3}\frac{1}{(16)^{\frac{3}{2}}}\int \frac{\sec y \tan y dy}{(\sec^2 y - 1)^{\frac{3}{2}}}
\end{aligned}
$$

Using the formula : $\sec^2 y - 1 = \tan^2 y$. One gets:

$$
\begin{aligned}
I_4 &= \frac{1}{48}\int \frac{\sec y \tan y dy}{[\tan^2 y]^{\frac{3}{2}}} \\
&= \frac{1}{48}\int \frac{\sec y}{\tan^2 y} dy \\
&= \frac{1}{48}\int \frac{\cos y}{\sin^2 y} dy \\
&= \frac{1}{48}\int \cos y . \sin^{-2} y dy \\
&= \frac{1}{48}\left[\frac{\sin^{-1} y}{(-1)}\right] \\
&= -\frac{1}{48}\frac{1}{\sin y} + C, (C \in \mathbb{R}) .
\end{aligned}
$$

We return to the initial variable:

$$
\frac{4}{3}\sec y = x \Rightarrow \cos y = \frac{4}{3x} \Rightarrow \sin y = \sqrt{1 - \left(\frac{4}{3x}\right)^2}
$$

$$
\begin{aligned}
I_4 &= -\frac{1}{48}\frac{1}{\sqrt{1 - \left(\frac{4}{3x}\right)^2}} + C \\
I_4 &= -\frac{1}{16}\frac{x}{\sqrt{9x^2 - 16}} + C, (C \in \mathbb{R}) .
\end{aligned}
$$

3.2 Lesson N°2 Integration of rational functions

3.2.1 Definition

We call rational function (or fraction) an expression of the type $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and x is such that : $Q(x) \neq 0$.

3.2.2 Integration of some basic forms of rational fractions

We must learn to integrate the following four basic forms of rational fractions:

$$\boxed{\text{Form (1)} : (a \neq 0), \quad \int \frac{1}{ax+b}dx = \frac{1}{a}\left[\ln|ax+b|\right] + C, (C \in \mathbb{R}).}$$

Examples

- $\int \dfrac{-1}{x+1}dx = -\ln|x+1| + C, (C \in \mathbb{R}).$
- $\int \dfrac{7}{3x+4}dx = \dfrac{7}{3}\ln|3x+4| + C, (C \in \mathbb{R}).$

$$\boxed{\text{Form (2)} : (a \neq 0), (n \neq 1), \quad \int \frac{1}{(ax+b)^n}dx = \tfrac{1}{a}\left[\tfrac{(ax+b)^{-n+1}}{-n+1}\right] + C, (C \in \mathbb{R}).}$$

Examples

- $\int \dfrac{-1}{(x+1)^3}dx = \dfrac{1}{(-2)(x+1)^2} + C, (C \in \mathbb{R}).$
- $\int \dfrac{7}{(3x+4)^2}dx = \dfrac{7}{(3)(-1)(3x+4)^1} + C = \dfrac{-7}{3(3x+4)} + C, (C \in \mathbb{R}).$

$$\boxed{\text{Form (3)} : (\alpha > 0), \quad \int \frac{1}{y^2+\alpha^2}dy = \frac{1}{\alpha}arctg\left(\frac{y}{\alpha}\right) + C, (C \in \mathbb{R}).}$$

Examples

- $\int \dfrac{1}{x^2+4}dx = \dfrac{1}{2}arctg\left(\dfrac{x}{2}\right) + C, (C \in \mathbb{R}).$

- $\int \frac{7}{3x^2+5}dx = \int \frac{7}{x^2+\left(\sqrt{\frac{5}{3}}\right)^2}dx = \frac{7}{\sqrt{\frac{5}{3}}} arctg\left(\frac{x}{\sqrt{\frac{5}{3}}}\right) + C = 7\sqrt{\frac{3}{5}} arctg\left(\sqrt{\frac{3}{5}}x\right) + C, (C \in \mathbb{R})$.

This third form that we have just learned is in fact used to integrate fractions where the denominator is a polynomial of the second order (i.e. $p(x) = ax^2 + bx + c$ with $a \neq 0$) that has a negative discriminant.
If the two previous examples are immediate, in the general case we must use the **canonical form**...

Canonical form: We can put any second order polynomial $p(x) = ax^2 + bx + c$ that has a negative discriminant ($\Delta < 0$) in a very precise form, which we call canonical form, as follows. One has :

$$\forall x \in \mathbb{R}, \quad ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{|\Delta|}{4a}. \text{ recall that } (a \neq 0).$$

Proof :

$$\begin{aligned}
\forall x \in \mathbb{R}, \quad ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{|\Delta|}{4a^2}\right], (a \neq 0).
\end{aligned}$$

- $\int \frac{3}{x^2+x+3}dx = ?$

One has $(x^2 + x + 3)$ is a second-order polynomial with $\Delta = -11 < 0$. So we can put it in canonical form as follows :

$$\left(x^2 + x + 3\right) = (x + \frac{1}{2})^2 + \frac{11}{4}.$$

So our integral becomes :

$$\begin{aligned}
\int \frac{3}{x^2+x+3}dx &= \int \frac{3}{(x+\frac{1}{2})^2+\frac{11}{4}}dx; \\
\text{One poses} &: \begin{cases} y = x+\frac{1}{2} \\ \alpha = \sqrt{\frac{11}{4}} \end{cases}, \text{ one gets : } dx = dy \text{ and :} \\
\int \frac{3}{y^2+\alpha^2}dy &= \frac{3}{\frac{\sqrt{11}}{2}} arctg\left(\frac{y}{\frac{\sqrt{11}}{2}}\right) + C \\
&= \frac{6}{\sqrt{11}} arctg\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C \\
&= \frac{6}{\sqrt{11}} arctg\left(\frac{2x+1}{\sqrt{11}}\right) + C, (C \in \mathbb{R}).
\end{aligned}$$

Form (4) : $\int \frac{Ax+B}{ax^2+bx+c}dx = ?$ (Where ax^2+bx+c is such that $\Delta < 0$).

To integrate form (4), we will partly use the form (3).

To integrate this form, we need to brighten up the derivative of the denominator in the numerator.

After this step, there will remain a simple integral of the form $\int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + C$, $(u(x) \neq 0)$ and another one of the type (3) to do.

Example

- $\int \frac{2x+4}{x^2+x+3}dx = \int \frac{2x+1+3}{x^2+x+3}dx = \int \frac{2x+1}{x^2+x+3}dx + \int \frac{3}{x^2+x+3}dx.$

Take $I = \int \frac{2x+1}{x^2+x+3}dx$ and $J = \int \frac{3}{x^2+x+3}dx.$

We integrate I by the well-known law of integration. (See table of integration).:

$$\int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + C, \ (u(x) \neq 0).$$

$$I = \ln\left|x^2+x+3\right| + C.$$

Then we integrate J by the canonical form of the form (3), (already done in previous page) :

$$J=\frac{6}{\sqrt{11}}arctg\left(\frac{2x+1}{\sqrt{11}}\right)+C.$$

Hence :

$$\int\frac{2x+4}{x^2+x+3}dx=\ln\left|x^2+x+3\right|+\frac{6}{\sqrt{11}}arctg\left(\frac{2x+1}{\sqrt{11}}\right)+C,(C\in\mathbb{R}).$$

Exercise

Calculate $I=\int\frac{3x+2}{2x^2+x+3}dx.$

One poses $Q(x)=2x^2+x+3,\quad \Delta=-23<0$ so $Q(x)=2\left[\left(x+\frac{1}{4}\right)^2+\frac{23}{16}\right]$
On the other hand : $Q^{'}(x)=4x+1.$
And :

$$\begin{aligned}3x+2&=3\left(x+\frac{2}{3}\right)\\&=\frac{3}{4}\left(4x+\frac{8}{3}\right)\\&=\frac{3}{4}\left(4x+1-1+\frac{8}{3}\right)\\&=\frac{3}{4}\left((4x+1)+\frac{5}{3}\right)\\&=\frac{3}{4}(4x+1)+\frac{3}{4}.\frac{5}{3}\\&=\frac{3}{4}(4x+1)+\frac{5}{4}\end{aligned}$$

Hence :

$$\begin{aligned}
I &= \int \frac{3x+2}{2x^2+x+3}dx \\
&= \int \frac{\frac{3}{4}(4x+1)+\frac{5}{4}}{2x^2+x+3}dx \\
&= \frac{3}{4}\int \frac{4x+1}{2x^2+x+3}dx + \frac{5}{4}\int \frac{1}{2x^2+x+3}dx \\
&= \frac{3}{4}\ln\left|2x^2+x+3\right| + \frac{5}{4}\int \frac{1}{2\left[\left(x+\frac{1}{4}\right)^2+\frac{23}{16}\right]}dx \\
&= \frac{3}{4}\ln\left|2x^2+x+3\right| + \frac{5}{4}.\frac{1}{2}\int \frac{1}{\left[\left(x+\frac{1}{4}\right)^2+\frac{23}{16}\right]}dx \\
&= \frac{3}{4}\ln\left|2x^2+x+3\right| + \frac{5}{8}.\frac{4}{\sqrt{23}}\arctan\left[\frac{\left(x+\frac{1}{4}\right)}{\frac{\sqrt{23}}{4}}\right] + C, (C\in\mathbb{R}). \\
I &= \frac{3}{4}\ln\left|2x^2+x+3\right| + \frac{5}{2\sqrt{23}}\arctan\left[\frac{(4x+1)}{\sqrt{23}}\right] + C, (C\in\mathbb{R}).
\end{aligned}$$

Form (5) : $\int \frac{Ax+B}{(ax^2+bx+c)^n}dx =?$
(where ax^2+bx+c is such that $\Delta<0$), $(A, a\neq 0)$, $n\geq 2$

To integrate form (5), we need to brighten up the derivative of the denominator in the numerator.

After this step, there will remain a simple integral of the form

$$\int u'(x)\left[u(x)\right]^n dx = \frac{\left[u(x)\right]^{n+1}}{n+1} + C,\ (n\neq -1)$$

and another integral as the following J, that we must learn how to integrate, since we are going to find it in the form (5) :

$$J = \int \frac{1}{(x^2+a^2)^n}dx,$$

A simple change of variable will give us the result of J:

$$J = \int \frac{1}{(x^2+a^2)^n}dx$$

Just take $x = a\tan y$, to find : $dx = a(1+\tan^2 y)dy$, that way :

$$\begin{aligned}
J &= \int \frac{1}{(x^2+a^2)^n} dx \\
&= \frac{1}{(a^2)^n} \int \frac{(1+\tan^2 y)}{(1+\tan^2 y)^n} a dy \\
&= \frac{1}{(a^2)^{n-1}} \int (1+\tan^2 y)^{1-n} dy \\
\text{But since } (1+\tan^2 y) &= \frac{1}{\cos^2 y}, \text{ we get :} \\
J &= \frac{1}{(a^2)^{n-1}} \int \cos^{n-1} y \; dy \; , \; (n \geq 1).
\end{aligned}$$

That we know how to integrate by trigonometric forms.

Example

- $\int \frac{2x+4}{(x^2+1)^2} dx = \int \frac{2x}{(x^2+1)^2} dx + \int \frac{4}{(x^2+1)^2} dx.$

Put $I = \int \frac{2x}{(x^2+1)^2} dx$ and $J = \int \frac{4}{(x^2+1)^2} dx.$

I is easily calculated by the laws of integration :

$$I = \left[\frac{(x^2+1)^{-1}}{-1} \right] + C.$$

J is calculated by the appropriate change of variables:
Take $x = \tan y$,to get : $dx = (1+\tan^2 y) dy$, so :

$$\begin{aligned}
J &= 4 \int \frac{(1+\tan^2 y)}{(1+\tan^2 y)^2} dy \\
&= 4 \int \frac{1}{(1+\tan^2 y)} dy \\
\text{But since } \cos^2 y &= \frac{1}{(1+\tan^2 y)}, \text{ one gets :} \\
J &= 4 \int \cos^2 y \; dy
\end{aligned}$$

We know how to calculate $\int \cos^2 y \; dy$ by linearization :

$$\int \cos^2 y \ dy = \frac{1}{2}\left[\sin(2y) + y\right] + C.$$

It remains for us to return to the first variable, noting that:

$$x = \tan y \Rightarrow y = \arctan x \text{ and } \sin(2y) = \sin(2\arctan x)$$

Finally :

$$\int \frac{2x+4}{(x^2+1)^{\,2}}dx = \frac{-1}{x^2+1} + 2\left[\sin(2\arctan x) + \arctan x\right] + C, (C \in \mathbb{R}).$$

3.2.3 Decomposition into simple elements

Let the rational function $\frac{P(x)}{Q(x)}$ where P and Q are polynomials satisfying $Q(x) \neq 0$ and $deg(P) < deg(Q)$.

Note We will discuss the case when $deg(P) \geq deg(Q)$ later on in this chapiter.

-) We call simple element of the first order of multiplicity α polynomials of the type $(ax+b)^{\alpha}.(a \neq 0)$.

-) We call simple element of the second order of multiplicity β polynomials of the type $(cx^2+dx+e)^{\beta}, (c \neq 0)$, that verify $\Delta = d^2 - 4ce < 0$. That is to say that it no longer admits any possible decomposition into simple elements of the first order.

To facilitate the integration of this rational function, we will write it as a sum of several rational fractions where the denominator is a simple element (of the first or of the second order). For this, we use the following methodology.

First step : Decompose the denominator

We will start by decomposing the denominator and having a product of simple elements of the first order of multiplicity α by simple elements of the second order of multiplicity β, as follows :

$$Q(x) = (ax+b)^{\alpha}(cx^2+dx+e)^{\beta}.$$

At the same time, we will establish D_f.

Second step : Preparing partial fractions

We prepare partial fractions by following the rule below:

A constant in the numerator for denominators of degree 1.

A polynomial of degree 1 for the denominators of degree 2.

Put a sum of partial fractions by increasing the multiplicity from the denominator until arriving at the multiplicity of simple elements when decomposing.

i.e:

$$\begin{aligned} \forall x \quad \in \quad & D_f, \frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^n(cx^2+dx+e)^m} \\ \Longrightarrow \quad & \forall x \in D_f, \frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{(ax+b)^i} + \sum_{i=1}^{m} \frac{(B_i x + C_i)}{(cx^2+dx+e)^i} \end{aligned}$$

Example 1

Suppose we have already decomposed the denominator $Q(x)$ in :

$$Q(x) = (x+2)^3(x^2+2x+2)^2.$$

The partial fractions are then written as follows :

$$\begin{aligned} \forall x \quad \in \quad & D_f, f(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(x+2)^3(x^2+2x+2)^2} \\ = \quad & \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{Dx+E}{(x^2+2x+2)} + \frac{Fx+G}{(x^2+2x+2)^2} \end{aligned}$$

We may highlight now that $D_f = \mathbb{R}\setminus\{-2\}$.
Third step : Finding constants A_i, B_i,C_i...

This is where the numerator $P(x)$ comes in.

To find the constants of rational fractions, several methods exist, the best-known being analogy. We will develop some of them with simple examples.

Example 1

We want to decompose the fraction $f(x) = \dfrac{1}{x^2+x}$ into a sum of partial fractions. We have :

$$Q(x) = x^2 + x = x(x+1).$$

We may highlight now that $D_f = \mathbb{R}\backslash\{-1, 0\}$.
So, we have two simple first-order elements, both of multiplicity 1.

This means that the partial rational fractions will be written:

$$\begin{aligned}\forall x \in D_f, \frac{1}{x^2+x} &= \frac{1}{x(x+1)} \\ \Longrightarrow \forall x \in D_f, \frac{1}{x^2+x} &= \frac{A}{x} + \frac{B}{(x+1)}.\end{aligned}$$

- Let us first use the analogy method, one has :

$$\begin{aligned}\forall x \in D_f, \frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{(x+1)} = \frac{A(x+1)+Bx}{x(x+1)} \\ \Longrightarrow \forall x \in D_f, \frac{1}{x(x+1)} &= \frac{(A+B)\,x + A}{x(x+1)}.\end{aligned}$$

By analogy between the two numerators, we find:

$$\begin{cases} A + B = 0 \\ A = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}.$$

This leads to :

$$\forall x \in D_f, \frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{(x+1)}.$$

- A second method consists in multiplying each time the following equality $(*)$ by a denominator, then give an appropriate value to x. One has :

$$\forall x \in D_f, \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{(x+1)} \ldots(*) \qquad (*)$$

Multiplying equation (*) by x, we get :

$$\forall x \in \mathbb{R}/\left\{-1\right\}, \frac{1}{(x+1)} = A + \frac{Bx}{(x+1)}.$$

As the above equality is true for all $x \in \mathbb{R}/\left\{-1\right\}$, we may take $x = 0$ to eliminate the term B, we hence get :

$$\begin{aligned} \frac{1}{(0+1)} &= A \\ &\Leftrightarrow A = 1. \end{aligned}$$

Similarly, we multiply the whole equation (*) by $(x+1)$, we find :

$$\forall x \in \mathbb{R}^*, \frac{1}{x} = \frac{A}{x}(x+1) + B.$$

We now choose to take $x = -1$ to eliminate the term A, we find directly:

$$\begin{aligned} \frac{1}{(-1)} &= B \\ i.e \;\; B &= -1. \end{aligned}$$

Which again gives that :

$$\forall x \in D_f, \frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{(x+1)}.$$

Fourth step : Integration

In our example, we will have :

$$\begin{aligned} \int \frac{1}{x(x+1)}dx &= \int \frac{1}{x}dx + \int \frac{-1}{(x+1)}dx \\ &= \ln|x| - \ln|x+1| + C, (C \in \mathbb{R}). \end{aligned}$$

Example 2
Calculate :

$$\begin{aligned} &\int \frac{4-x}{(x-1)(x-2)}dx \\ \Longrightarrow \;\; &D_f = \mathbb{R}\setminus\left\{1,2\right\}. \end{aligned}$$

One has :

$$\begin{aligned}\forall x \quad &\in \quad D_f, \frac{4-x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \\ &= \quad \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} \\ &\Longrightarrow \quad \forall x \in D_f, \frac{4-x}{(x-1)(x-2)} = \frac{(A+B)x-2A-B}{(x-1)(x-2)}\end{aligned}$$

By analogy, we get :

$$\begin{cases} (A+B) = -1 \\ -2A-B = 4 \end{cases}$$
$$\begin{cases} A = -3 \\ B = 2 \end{cases}$$

Hence,

$$\forall x \in D_f, \frac{4-x}{(x-1)(x-2)} = \frac{(-3)}{x-1} + \frac{2}{x-2}$$

So:

$$\begin{aligned}\int \frac{4-x}{(x-1)(x-2)}dx &= (-3)\int \frac{dx}{x-1} + 2\int \frac{dx}{x-2} \\ &= -3\ln|x-1| + 2\ln|x-2| + C, (C \in \mathbb{R}).\end{aligned}$$

Example 3 (harder) calulate the following integral

$$J = \int \frac{shx\ chx}{(sh^2x-1)(shx+sh^2x+1)}dx$$

One poses $y = shx$, to get $dy = chxdx$, so :

$$J = \int \frac{ydy}{(y^2-1)(y^2+y+1)} = \int f(y)dy$$

One has :

$$\begin{aligned}\frac{y}{(y^2-1)(y^2+y+1)} &= \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cy+D}{y^2+y+1} \quad \text{(Fraction)} \\ D_f &= \mathbb{R}/\{-1,1\}\end{aligned}$$

We multiply the whole equation above by $(y-1)$, we find :

$$\forall y \in \mathbb{R}/\{-1\}, \frac{y}{(y+1)(y^2+y+1)} = A + \frac{B(y-1)}{y+1} + \frac{(Cy+D)(y-1)}{y^2+y+1}.$$

Take $y = 1$ to get :

$$A = \frac{1}{6}$$

Similarly, we multiply the whole equation called $(Fraction)$ by $(y+1)$, we find :

$$\forall y \in \mathbb{R}/\{1\}, \frac{y}{(y-1)(y^2+y+1)} = \frac{A(y+1)}{y-1} + B + \frac{(Cy+D)(y+1)}{y^2+y+1}$$

We now choose to take $y = -1$, we find directly :

$$B = \frac{1}{2}$$

Now we come back to our equation given in $(Fraction)$:

$$\forall y \in D_f, \frac{y}{(y^2-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cy+D}{y^2+y+1}$$

Since we have already calculated A and B, we may take directly $y = 0$ to find the D, as follows :

$$\begin{aligned} 0 &= -A + B + D \\ &\Rightarrow D = -\frac{1}{3} \end{aligned}$$

And finally, we multiply the whole equation $(Fraction)$ by y and we pass to the limit when y tends to $+\infty$:

$$\begin{aligned} \forall y \in D_f, \quad (y.(Fraction)) &\Leftrightarrow \frac{y^2}{(y^2-1)(y^2+y+1)} = \frac{Ay}{y-1} + \frac{By}{y+1} + \frac{(Cy+D)y}{y^2+y+1} \\ &\Rightarrow \lim_{y\to+\infty} \frac{y^2}{(y^2-1)(y^2+y+1)} = \lim_{y\to+\infty} \left[\frac{Ay}{y-1} + \frac{By}{y+1} + \frac{(Cy+D)y}{y^2+y+1}\right] \\ &\Rightarrow 0 = A + B + C \\ &\Rightarrow C = -\frac{2}{3}. \end{aligned}$$

Hence :

$$\forall y \in D_f,\ \frac{y}{(y^2-1)(y^2+y+1)} = \frac{\frac{1}{6}}{y-1} + \frac{\frac{1}{2}}{y+1} + \frac{-\frac{2}{3}y - \frac{1}{3}}{y^2+y+1}$$

All that remains is to integrate:

$$I = \int \frac{ydy}{(y^2-1)(y^2+y+1)} = \frac{1}{6}\int \frac{dy}{y-1} + \frac{1}{2}\int \frac{dy}{y+1} - \frac{1}{3}\int \frac{2y+1}{y^2+y+1}dy$$

$$I = \frac{1}{6}\ln|y-1| + \frac{1}{2}\ln|y+1| - \frac{1}{3}\left[\ln\left|y^2+y+1\right|\right] + C, (C \in \mathbb{R}).$$

Finally,

$$I = \frac{1}{6}\ln|shx-1| + \frac{1}{2}\ln|shx+1| - \frac{1}{3}\ln\left(sh^2x + shx + 1\right) + C, (C \in \mathbb{R}).$$

Remark

When the rational function $x \to f(x) = \dfrac{P(x)}{Q(x)}$ is such that $Q(x) \neq 0$ and $deg(P) \geq deg(Q)$, we first start with an Euclidean division and then apply the method of rational fractions to the remainder of this division.

Example

Calculate $I = \int \dfrac{3x^5+2x^4+x^2+3x+2}{x^2+1}dx$

$$f(x) = \frac{3x^5+2x^4+x^2+3x+2}{x^2+1},\ D_f = \mathbb{R}.$$

After Euclidean division, we find

$$f(x) = 3x^3 + 2x^2 - 3x - 1 + 3\left(\frac{2x+1}{x^2+1}\right).$$

We then apply what we have seen on the fractions $\left(\dfrac{2x+1}{x^2+1}\right)$ because the degree of its numerator is strictly less than the degree of its denominator.

But in this particular example, it's very simple. Indeed :

$$
\begin{aligned}
I &= \int \left[3x^3 + 2x^2 - 3x - 1 + 3\left(\frac{2x+1}{x^2+1}\right)\right] dx \\
I &= \int \left[3x^3 + 2x^2 - 3x - 1 + 3\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)\right] dx \\
&= \frac{3}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 - x + 3\left[\ln\left(x^2+1\right) + \arctan x\right] + C, (C \in \mathbb{R}).
\end{aligned}
$$

3.3 Lesson N°3 Defined Integration

3.3.1 Surface calculation

Methodology The area is calculated after performing the following steps:

1/ Study of the given function (definition set, limits, sign of the derivative and variation table).

2/ Clearly specify the important points (critical points, points of intersection between the graphs given in the description as well as the points of intersection with the abscissa and ordinate axes if necessary).

3/ Draw the graph and delimit the area to be calculated.

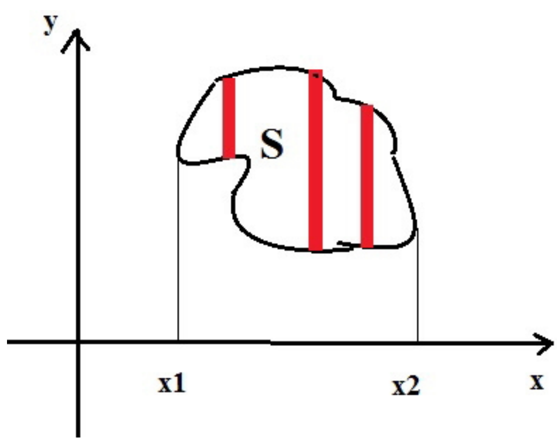
4/ Calculate The bounded integral which represents the required surface. We can calculate the area in two different ways, using the horizontal rectangle or using the vertical rectangle. We will see both methods on simple examples.

Remarks

*/ Area must be a positive quantity.

*/ Do not forget to follow the result by the mention SU which means **square units**.

Vertical rectangle method We use this method when y is given as a function of x. In this case, the bounds of the integral are given on the x-axis and the function to be integrated is given by the y's, as follows:



$Fig3.5$: Vertical rectangle for an area calculation.

Surface (S) is then given by the following formula :

$$S = \int_{x_1}^{x_2} (y_1(x) - y_2(x))dx$$

We obtain x_1 and x_2 thanks to the graph, provided that $x_1 < x_2$.

On the other hand, $y_1(x)$ and $y_2(x)$ are given in the problem statement, provided that

$$\forall x \in [x_1, x_2], y_1(x) \geq y_2(x)$$

That is to say that the graph of y_1 is above the graph of y_2.

Example 1

Calculate the area between the function given by $f(x) = 2 - x^2$ and the line $y = -x$.

1/ So we start by quickly studying the given function:

$$\begin{aligned} D_f &= \mathbb{R}, \lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = -\infty \\ f'(x) &= -2x. \end{aligned}$$

Hence :

x	$-\infty$		0		$+\infty$
$f\prime(x)$		$+$	0	$-$	
$f(x)$			2		
		$\nearrow$		$\searrow$	
	$-\infty$				$-\infty$

2/ The important points are:

*) The optimum $A(0, 2)$.

*) The intersection of the two graphs :

$$\begin{aligned} &\begin{cases} y = 2 - x^2 \\ y = -x \end{cases} \\ \Leftrightarrow\ & 2 - x^2 = -x \\ \Longleftrightarrow\ & x^2 - x - 2 = 0 \\ \Longleftrightarrow\ & x_1 = -1, x_2 = 2. \end{aligned}$$

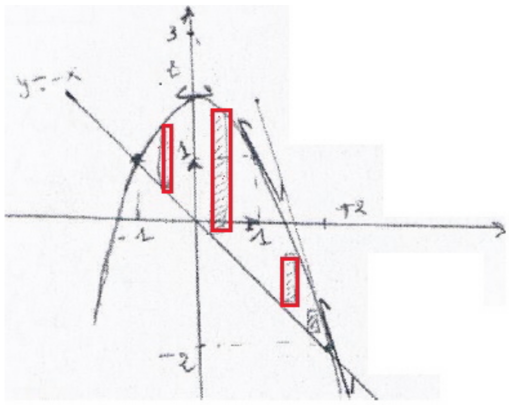
The points of intersection therefore are :

$$B(-1,1), C(2,-2).$$

*) Other points to draw the graph:
$D(1,1)$ and the equation of the tangent at this point : $y = -2x + 3$.
We can add the equation of the tangent at the point $C(2,-2)$,

$$y = -4x + 6.$$

3/ The graph with delimitation of the desired surface :

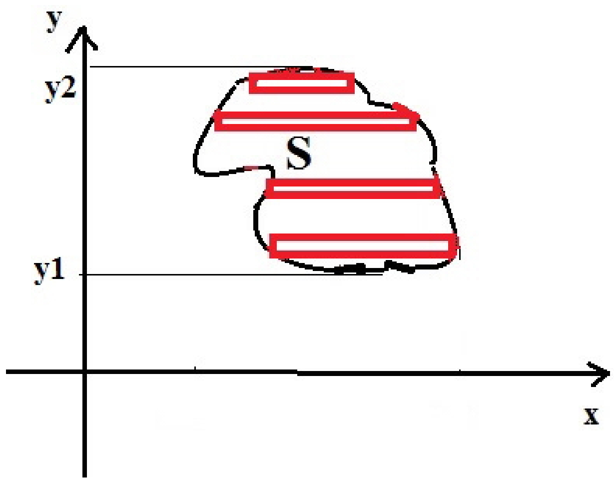


$Fig3.6$: Graph of the example 1.

4/ Calculation of the integral :

$$\begin{aligned} S &= \int_{x_1}^{x_2} (y_1 - y_2)dx \\ &= \int_{-1}^{2} (2 - x^2) - (-x)dx \\ &= \int_{-1}^{2} (2 + x - x^2)dx \\ &= 4.5 \quad SU. \end{aligned}$$

Horizontal rectangle method This method is most often used when x is given according to y.



$Fig3.7$: Horizontal rectangle for area calculation.

Here we have:

$$S = \int_{y_1}^{y_2} (x_1(y) - x_2(y))dy.$$

We obtain y_1 and y_2 thanks to the graph, provided that $y_1 < y_2$

The functions $x_1(y)$ and $x_2(y)$ are given in the problem statement, provided that you take :

$$\forall y \in [y_1, y_2], x_1(y) \geq x_2(y)$$

that is, the graph of $x_1(y)$ comes after that of $x_2(y)$ along the direction of the abscissa axis. We will develop all this on the following example.

Example 2

Calculate the area between the function of euation $y^2 = 6 - x$ and the line $y = -x$.

It is clear here that the vertical rectangle is more adequate. In this case, the bounds of the integral are given on the y-axis and the function to be integrated is given by the x's, as follows:

1/ We start by quickly studying the given function :

$$\begin{aligned} D_f &= \mathbb{R}, \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = -\infty \\ x'(y) &= -2y. \end{aligned}$$

Hence :

y	$-\infty$		0		$+\infty$
$x'(y)$		$+$	0	$-$	
$x(y)$	$-\infty$	$\nearrow$	6	$\searrow$	$-\infty$

2/ The important points are:
*) The optimum $A(0,6)$.
*) The intersection of the two graphs:

$$\begin{cases} x = 2 - y^2 \\ y = -x \end{cases}$$
$$\Leftrightarrow \quad 6 - y^2 = -y$$
$$\Longleftrightarrow \quad y^2 - y - 6 = 0$$
$$\Longleftrightarrow \quad y_1 = 3, y_2 = -2.$$

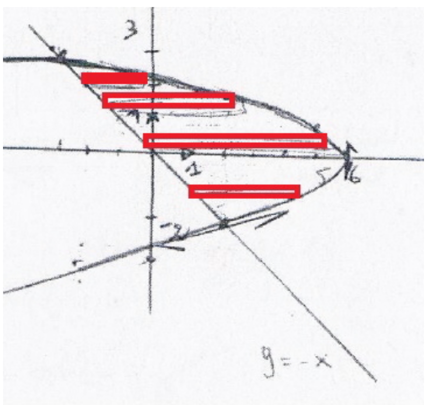
The points of intersection are therefore:

$$B(-3,3), C(2,-2).$$

*) Other points to draw the graph:
$D(5,1)$ and the equation of the tangent at this point : $x = -2y + 7$.
We can add the equation of the tangent at the point $C(2,-2)$,

$$x = 4y + 10.$$

3/ The graph with delimitation of the desired surface:



$Fig3.8$: Graph of example 2.

4/ Calculation of the integral :

$$\begin{aligned} S &= \int_{y_1}^{y_2} (x_1(y) - x_2(y))dy \\ &= \int_{-2}^{3} (6 - y^2) - (-y)dx \\ &= \int_{-2}^{3} (6 + y - y^2)dx \\ &= \frac{125}{6} \;\; SU. \end{aligned}$$

3.4 Exercise series N°3

Exercise 1

Let f be a function defined by

$$f(x) = e^{3x}\sin x.$$

1) Calculate $f'(x)$ and $f''(x)$.

2) Find two real numbers a, b such that :

$$f(x) = af'(x) + bf''(x).$$

3) Deduce an integral of the function f.

Exercise 2

1/ Using the integration table, calculate:

$$1)\int (2x^2 - 5x + 5)\,dx \quad ; \quad 2)\int \frac{x^3 - 5x^2 - 4}{x^2}dx.$$

$$3)\int (e^x - x^e)\,dx \quad ; \quad 4)\int (3x+4)^{15}\,dx.$$

$$5)\int (\tfrac{-7}{x} - \tfrac{1}{4}e^x + 3\sqrt{x} + 2\cos x)dx.$$

$$6)\int x^2\sqrt{x^3+2}dx \quad ; \quad 7)\int x\sqrt[3]{(1-x^2)}dx.$$

$$8)\int \tfrac{8x^2}{x^3+2}dx \quad ; \quad 9)\int \tfrac{x^2+2x}{(x+1)^2}dx.$$

$$10)\int e^x(e^x+1)^4dx \quad ; \quad 11)\int e^{(2x+1)}dx.$$

$$12)\int 5x\sin(x^2+1)dx \quad ; \quad 13)\int \frac{x^2}{\sqrt{x^3+1}}dx.$$

2/ By a change of variables, calculate:

$$I = \int \frac{x}{x^4+1}dx.$$

Exercise 3

Find the primitives of the following functions which vanish for $x = 1$.

$$\begin{aligned} f(x) &= x + \sqrt{x} + \frac{1}{x}. \\ g(x) &= \frac{x^2}{\sqrt{x^3+1}}. \\ h(x) &= \frac{2x+1}{(x^2+x-1)^3}. \end{aligned}$$

Exercise 4

Using the integration-by-part formula, calculate:

$$\begin{aligned} 1)I_1 &= \int x \sin x dx \quad ; \quad 2)I_2 = \int x^2 e^x dx. \\ 3)I_3 &= \int \ln x dx \quad ; \quad 4)I_4 = \int \left(xe^{3x}\right) dx. \end{aligned}$$

$$5)I_5 = \int \frac{t^2}{(t^2+1)^2} dt \quad ; \quad 6)I_6 = \int x^{(n+1)} \ln x dx.$$

Exercise 5

1. Get back to integrals I_2 and I_4 of exercise 4 and solve them by the method of integrating "polynomials by exponential functions".

2. By the latter method, calculate also :

$$I = \int (x^2 - x + 1)e^{-x} dx.$$

Exercise 6

1. Calculate $I_1 = \int_0^1 \frac{x}{x^2+1} dx$.
2. We Put $I_2 = \int_0^1 \frac{x^3}{x^2+1} dx$. Calculate $(I_1 + I_2)$ then deduce I_2.
3. Let be the two integrals :

$$I_1 = \int_0^{\pi} x^2 \cos^2(x)dx \quad \text{and} \quad I_2 = \int_0^{\pi} x^2 \sin^2(x)dx$$

a) Calculate $I_1 + I_2$.

b) Knowing that $\forall \alpha \in \mathbb{R}, \cos(2\alpha) = \dfrac{\cos^2\alpha - \sin^2\alpha}{2}$, calculate $I_1 - I_2$.

c) Deduce the value of I_1 and of I_2.

Exercise 7

1/ Find three reals a, b, c such that :

$$\frac{4x^2 - 5x + 1}{x + 3} = ax + b + \frac{c}{x + 3}.$$

2/ Calculate $\int \dfrac{4x^2 - 5x + 1}{x + 3}dx$, then deduce $\int_0^2 \dfrac{4x^2 - 5x + 1}{x + 3}dx$.

Exercise 8

Calculate the following trigonometric integrals :

$$1) \int \sin^2(2x)dx,$$

$$2) \int 3\cos^3(x)dx$$

$$3) \int \cos^2(x)\sin^2(x)\,dx,$$

$$4) \int \sin(2x)\cos(3x)dx.$$

Exercise 9

1. Point out the difference between the following integrals then calculate them :

$$1) \int \frac{1}{x^2 + 1}dx, \quad 2) \int \frac{x}{x^2 + 1}dx,$$

$$3) \int \frac{1}{x^2 - 1}dx, \quad 4) \int \frac{x}{x^2 - 1}dx.$$

2. Calculate the integrals of the following rational fractions:

$$1) \int \frac{A}{3x+1} dx, \quad 2) \int \frac{B}{(3x+1)^3} dx,$$
$$(A, B \neq 0).$$
$$3) \int \frac{x-2}{x^2+5x+7} dx.$$

3. Calculate the following integrals :

$$a/ \ \ I = \int \frac{dx}{x^2+10x+30}; \ \ b/ \ \ J = \int \sqrt{x^2-2x^4} dx$$

4. Calculate the following inegrals :

$$a) \ \ I = \int_0^1 \sqrt{1-x^2} dx \quad ; \quad b) \ \ J = \int \frac{x-1}{x^2+2x+3} dx$$

Exercise 10

Calculate the area between the following graphs :

$$\begin{cases} x^2+y^2=9 \\ x=y \\ x=-y \end{cases}.$$

Reminder

1/ The most useful integration formulas :
$(a, C \in \mathbb{R})$;$(\alpha \neq -1)$.

1)$\int a dx = ax + C$.

2)$\int u'(x) \left[u(x)\right]^{\alpha} dx = \frac{[u(x)]^{\alpha+1}}{\alpha+1} + C$.

Example :$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$.

3)$\int \frac{u'(x)}{u(x)} dx = \ln |u(x)| + C$,$(u(x) \neq 0)$.

Example :$\int \frac{dx}{x} = \ln |x| + C, (x \neq 0)$.

4)$\int u'(x) e^{u(x)} dx = e^{u(x)} + C$

Example : $\int e^x dx = e^x + C$.

5) $\int u'(x) \cos u(x) dx = \sin u(x) + C$.

Example : $\int \cos x dx = \sin x + C$.

6) $\int u'(x) \sin u(x) dx = -\cos u(x) + C$.

Example : $\int \sin x dx = -\cos x + C$.

7) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$.

Example : $\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$.

8) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \left(\frac{x}{a} \right) + C$

Example : $\int \frac{1}{x^2+1} dx = \arctan(x) + C$.

2/ Integration by parts formula :

$$\boxed{\int u'.v = [u.v] - \int u.v'}$$

3/ Canonicalal form :
If $ax^2 + bx + c$ with $\Delta < 0$, then:

$$\boxed{ax^2 + bx + c = a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{|\Delta|}{4a^2} \right].}$$

4/ **Defined integration :**

$$\boxed{\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a).}$$

3.5 Correction of the exercise series N°3

Exercise 1

Let f be the defined function $f(x) = e^{3x}\sin x$.

1) Calculate $f'(x)$ and $f''(x)$.

$$\begin{aligned} f'(x) &= 3e^{3x}\sin x + e^{3x}\cos x \\ &= e^{3x}\left[3\sin x + \cos x\right] \\ f''(x) &= 3e^{3x}\left[3\sin x + \cos x\right] + e^{3x}\left[3\cos x - \sin x\right] \\ &= 2e^{3x}\left[3\cos x + 4\sin x\right]. \end{aligned}$$

2) Find two reals a, b such that $f(x) = af'(x) + bf''(x)$.

$$\begin{aligned} f(x) &= af'(x) + bf''(x) \\ &= ae^{3x}\left[3\sin x + \cos x\right] + 2be^{3x}\left[3\cos x + 4\sin x\right] \\ &= e^{3x}(3a\sin x + a\cos x + 8b\sin x + 6b\cos x) \\ e^{3x}\sin x &= e^{3x}\left[(3a + 8b)\sin x + (a + 6b)\cos x\right]. \end{aligned}$$

After calculations and by identification, we find the system :

$$\begin{cases} a + 6b = 0 \\ 3a + 8b = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{3}{5} \\ b = -\frac{1}{10} \end{cases}.$$

Hence :

$$f(x) = \frac{3}{5}f'(x) - \frac{1}{10}f''(x).$$

3) Deduce an integral of the function f.

$$\begin{aligned} \int f(x)dx &= \int \left[\frac{3}{5}f'(x) - \frac{1}{10}f''(x)\right] dx \\ &= \frac{3}{5}f(x) - \frac{1}{10}f'(x) + C \\ &= e^{3x}\left[\frac{3}{5}\sin x - \frac{1}{10}(3\sin x + \cos x)\right] + C. \\ &= \frac{1}{10}e^{3x}(3\sin x - \cos x) + C, (C \in \mathbb{R}). \end{aligned}$$

Exercise 2

1/ Using the integration table, calculate :

$$1)\int \left(2x^2-5x+5\right)dx = 2\int x^2dx-5\int xdx+\int 5dx$$
$$= 2\left[\frac{x^3}{3}\right]-5\left[\frac{x^2}{2}\right]+5x+C,(C\in\mathbb{R}).$$

$$2)\int \frac{x^3-5x^2-4}{x^2}dx = \int\left(x-5-\frac{4}{x^2}\right)dx=\int xdx+\int -5dx+\int -\frac{4}{x^2}dx$$
$$= \frac{x^2}{2}-5x+\frac{4}{x}+C,(C\in\mathbb{R}).$$

$$3)\int \left(e^x-x^e\right)dx=e^x-\frac{x^{e+1}}{e+1}+C,(C\in\mathbb{R}).$$

$$4)\int (\frac{-7}{x}-\frac{1}{4}e^x+3\sqrt{x}+2\cos x)dx=-7\ln|x|-\frac{1}{4}e^x+3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]+2\sin x+C,(C\in\mathbb{R}).$$

$$5)\int (3x+4)^{15}dx = \frac{1}{3}\int 3(3x+4)^{15}dx$$
$$= \frac{1}{3}\left[\frac{(3x+4)^{16}}{16}\right]+C,(C\in\mathbb{R}).$$

$$6)\int x^2\sqrt{x^3+2}dx = \frac{1}{3}\int\left(3x^2\right)\left(x^3+2\right)^{\frac{1}{2}}dx$$
$$= \frac{1}{3}\left[\frac{(x^3+2)^{\frac{3}{2}}}{\frac{3}{2}}\right]+C,(C\in\mathbb{R}).$$

$$7)\int x\sqrt[3]{(1-x^2)}dx = \frac{-1}{2}\int(-2x)\left(1-x^2\right)^{\frac{1}{3}}dx$$
$$= \frac{-1}{2}\left[\frac{(1-x^2)^{\frac{4}{3}}}{\frac{4}{3}}\right]+C,(C\in\mathbb{R}).$$

$$8)\int \frac{8x^2}{x^3+2}dx = \frac{8}{3}\int \frac{3x^2}{x^3+2}dx = \frac{8}{3}\ln\left|x^3+2\right| + C, (C\in\mathbb{R}).$$

$$9)\int \frac{x^2+2x}{(x+1)^{\,2}}dx = \int \frac{x^2+2x+\mathbf{1}-\mathbf{1}}{(x+1)^{\,2}}dx = \int \frac{(x+1)^{\,2}-1}{(x+1)^{\,2}}dx = \int\left(1-\frac{1}{(x+1)^{\,2}}\right)dx = x+\frac{1}{(x+1)}+C, (C\in\mathbb{R}).$$

$$10)\int e^x(e^x+1)^4dx = \left[\frac{(e^x+1)^5}{5}\right]+C, (C\in\mathbb{R}).$$

$$11)\int e^{(2x+1)}dx = \frac{1}{2}\int 2e^{(2x+1)}dx = \frac{1}{2}\left[e^{(2x+1)}\right]+C, (C\in\mathbb{R}).$$

$$12)\int 5x\sin(x^2+1)dx = \frac{5}{2}\int (2x)\sin(x^2+1)dx = \frac{5}{2}\left[-\cos(x^2+1)\right]+C = -\frac{5}{2}\left[\cos(x^2+1)\right]+C, (C\in\mathbb{R}).$$

$$13)\int \frac{x^2}{\sqrt{x^3+1}}dx = \frac{1}{3}\int 3\frac{x^2}{\sqrt{x^3+1}}dx = \frac{2}{3}\sqrt{x^3+1}+C, (C\in\mathbb{R}).$$

2/

$$I=\int \frac{x}{x^4+1}dx.$$

One poses $y = x^2$, to get $dy = 2xdx$.

Hence,

$$\begin{aligned} I &= \int \frac{\frac{1}{2}dy}{y^2+1} \\ &= \frac{1}{2}arctg(y)+C \\ &= \frac{1}{2}arctg(x^2)+C, (C \in \mathbb{R}). \end{aligned}$$

Exercise 3

Find the primitive functions of the following functions which vanish for $x = 1$.

$$f(x) = x + \sqrt{x} + \frac{1}{x}, \quad g(x) = \frac{x^2}{\sqrt{x^3+1}}, \quad h(x) = \frac{2x+1}{(x^2+x-1)^3}.$$

*) $f(x) = x + \sqrt{x} + \frac{1}{x}$,

$$\begin{aligned} F(x) &= \int f(x)dx = \int \left(x + \sqrt{x} + \frac{1}{x} \right) dx \\ &= \frac{1}{2}x^2 + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \ln x + C. \end{aligned}$$

Hence :

$$F(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + \ln x + C.$$

If $F(1) = 0$ then $\frac{1}{2}(1)^2 + \frac{2}{3}(1)^{\frac{3}{2}} + \ln(1) + C = 0$ which gives : $C = -\frac{7}{6}$.

The primitive that verifies the requested initial condition is :

$$F(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + \ln x - \frac{7}{6}.$$

*) $g(x) = \dfrac{x^2}{\sqrt{x^3+1}}$,

$$\begin{aligned}
G(x) &= \int g(x)dx = \int \frac{x^2}{\sqrt{x^3+1}}dx \\
&= \int x^2\left(x^3+1\right)^{-\frac{1}{2}}dx. \\
&= \frac{1}{3}\left[\frac{(x^3+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]+C \\
&= \frac{2}{3}\sqrt{x^3+1}+C.
\end{aligned}$$

Hence :

$$G(x) = \frac{2}{3}\sqrt{x^3+1}+C.$$

If $G(1)=0$ then $\frac{2}{3}\sqrt{(1)^3+1}+C=0$ which gives : $C=-\frac{2}{3}\sqrt{2}$.

The primitive that verifies the requested initial condition is :

$$G(x) = \frac{2}{3}\sqrt{x^3+1}-\frac{2}{3}\sqrt{2}.$$

*) $h(x) = \dfrac{2x+1}{(x^2+x-1)^3}$,

$$\begin{aligned}
H(x) &= \int h(x)dx = \int \frac{2x+1}{(x^2+x-1)^3}dx \\
&= \int (2x+1)\,(x^2+x-1)^{-3}dx \\
&= \left[\frac{(x^2+x-1)^{-2}}{(-2)}\right]+C.
\end{aligned}$$

Hence :

$$H(x) = \frac{-1}{2(x^2+x-1)^2}+C.$$

If $H(1)=0$ then $\frac{-1}{2((1)^2+(1)-1)^2}+C=0$ which gives : $C=\frac{1}{2}$.

The primitive that verifies the requested initial condition is :

$$H(x) = \frac{-1}{2(x^2+x-1)^2}+\frac{1}{2}.$$

Exercise 4

1) One poses $u = x$ so $u' = 1$ and $v' = \sin x$, hence $v = \int \sin x dx = -\cos x$.

We apply the law of integration by parts : $\int uv' = uv - \int u'v$, we find :

$$\begin{aligned} I_1 &= x(-\cos x) - \int -\cos x dx \\ &= -x\cos x + \sin x + C, (C \in \mathbb{R}). \end{aligned}$$

2) One poses $u = x^2$ so $u' = 2x$ and $v' = e^x$, hence $v = e^x$.

We apply the law of integration by parts : $\int uv' = uv - \int u'v$, we find:

$$\begin{aligned} I_2 &= x^2(e^x) - \int 2xe^x dx \\ &= x^2e^x - 2\int xe^x dx. \end{aligned}$$

We need to apply integration by parts a second time on the new integral:

$$J = \int xe^x dx.$$

One poses $u = x$ so $u\prime = 1$ and $v\prime = e^x$, hence $v = e^x$.

We apply the law of integration by parts : $\int uv\prime = uv - \int u'v$, we find :

$$J = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x.$$

We get J to its place, we then get :

$$\begin{aligned} I_2 &= x^2e^x - 2(xe^x - e^x) + C \\ &= (x^2 - 2x + 2)e^x + C, (C \in \mathbb{R}). \end{aligned}$$

3) One poses $u = \ln x$ so $u' = \frac{1}{x}$ and $v' = 1$, hence $v = x$.

We apply the law of integration by parts : $\int uv' = uv - \int u'v$, we find :

$$\begin{aligned} I_3 &= x(\ln x) - \int 1dx \\ &= x\ln x - x + C, (C \in \mathbb{R}). \end{aligned}$$

4) One poses $u = x$ so $u' = 1$.
And $v' = e^{3x}$ so $v = \int e^{3x}dx = \frac{1}{3}\int 3e^{3x}dx = \frac{1}{3}e^{3x}$.

We apply the law of integration by parts : $\int uv\prime = uv - \int u'v$, we find:

$$\begin{aligned} I_3 &= x\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3}\left(\frac{1}{3}\int 3e^{3x}dx\right) \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C, (C \in \mathbb{R}). \end{aligned}$$

5)$I_5 = \int \frac{t^2}{(t^2+1)^2}dt$
One poses $u = t$ so $u' = 1$.
And $v' = \frac{t}{(t^2+1)^2}$ so $v = \int \frac{t}{(t^2+1)^2}dt = \frac{1}{2}\int \frac{2t}{(t^2+1)^2}dt = -\frac{1}{2}\frac{1}{(t^2+1)}$.

We apply the law of integration by parts : $\int uv\prime = uv - \int u'v$, we find :

$$\begin{aligned} I_5 &= -\frac{1}{2}\frac{t}{(t^2+1)} - \left(-\frac{1}{2}\right)\int \frac{dt}{(t^2+1)} \\ &= -\frac{1}{2}\frac{t}{(t^2+1)} + \frac{1}{2}arctg(t) + C, (C \in \mathbb{R}). \end{aligned}$$

6) $I_6 = \int x^{(n+1)} \ln x dx$,
One poses $u = \ln x$ so $u' = \dfrac{1}{x}$.
And $v' = x^{(n+1)}$ so $v = \int x^{(n+1)}dx = \dfrac{x^{(n+2)}}{n+2}$.

We apply the law of integration by parts : $\int uv\prime = uv - \int u'v$, we find :

$$\begin{aligned} I_6 &= \frac{x^{(n+2)}}{n+2}\ln x - \int \frac{x^{(n+2)}}{n+2}\frac{1}{x}dx \\ &= \frac{x^{(n+2)}}{n+2}\ln x - \frac{1}{n+2}\int x^{(n+1)}dx \\ &= \frac{x^{(n+2)}}{n+2}\ln x - \frac{1}{n+2}\left[\frac{x^{(n+2)}}{n+2}\right] + C, (C \in \mathbb{R}). \end{aligned}$$

Exercise 5
1.

- $I_2 = \int x^2 e^x dx = Q(x)e^x + C$ where $Q(x)$ is a polynomial of degree 2.

$$\begin{aligned} Q(x) &= ax^2 + bx + c \\ &\Rightarrow Q^{'}(x) = 2ax + b. \end{aligned}$$

On the other hand, since $I_2 = \int x^2 e^x dx = Q(x)e^x + C$.
By derivation we obtain :

$$\begin{aligned} I_2^{'} &= x^2 e^x = \left[Q^{'}(x) + Q(x)\right] e^x \\ &= \left[ax^2 + bx + c + 2ax + b\right] e^x \\ x^2 e^x &= \left[ax^2 + (b + 2a)x + c + b\right] e^x. \end{aligned}$$

And by identification, we get :

$$\begin{cases} a = 1 \\ b + 2a = 0 \\ c + b = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} a = 1 \\ b = -2 \\ c = 2 \end{cases}.$$

Which implies that : $Q(x) = x^2 - 2x + 2$. Hence,

$$I_2 = \left(x^2 - 2x + 2\right) e^x + C, (C \in \mathbb{R}).$$

- $I_4 = \int xe^{3x} dx = Q(x)e^{3x} + C$ where $Q(x)$ is a polynomial of degree 1.

$$\begin{aligned} Q(x) &= ax + b \\ &\Rightarrow Q^{'}(x) = a. \end{aligned}$$

On the other hand, since $I_4 = \int xe^{3x}dx = Q(x)e^{3x} + C$.
We derive this equality and we get :

$$\begin{aligned} I_4^{'} &= xe^{3x} = \left[Q^{'}(x) + 3Q(x)\right] e^{3x} \\ x^2e^x &= \left[(3a)\, x + (3b+a)\right] e^{3x}. \end{aligned}$$

By identification, one finds that :

$$\begin{cases} a = 1 \\ 3b + a = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} a = 1 \\ b = -\frac{1}{3} \end{cases}.$$

Which implies that : $Q(x) = x - \frac{1}{3}$. So,

$$I_4 = \left(x - \frac{1}{3}\right) e^{3x} + C, (C \in \mathbb{R}).$$

2. $I = \int (x^2 - x + 1)\, e^{-x}dx = Q(x)e^{-x} + C$ where $Q(x)$ is a polynomial of degree 2.

$$\begin{aligned} Q(x) &= ax^2 + bx + c \\ &\Rightarrow Q^{'}(x) = 2ax + b. \end{aligned}$$

Yet $I = \int (x^2 - x + 1)\, e^{-x}dx = Q(x)e^{-x} + C$,
Deriving gives :

$$\begin{aligned} I^{'} &= \left(x^2 - x + 1\right) e^{-x} = \left[Q^{'}(x) - Q(x)\right] e^{-x} \\ &= \left[ax^2 + bx + c - 2ax - b\right] e^{-x} \\ \left(x^2 - x + 1\right) e^{-x} &= \left[ax^2 + (b - 2a)x + c - b\right] e^{-x}. \end{aligned}$$

Again, by identification :

$$\begin{cases} a = 1 \\ b - 2a = -1 \\ c - b = 1 \end{cases}$$
$$\Leftrightarrow \begin{cases} a = 1 \\ b = 1 \\ c = 2 \end{cases}.$$

Thus : $Q(x) = x^2 + x + 2$. That way, we get :

$$I = \left(x^2 + x + 2\right) e^{-x} + C, (C \in \mathbb{R}).$$

Exercise 6

1. Calculate $I_1 = \int_0^1 \frac{x}{x^2+1} dx$.

   One has :

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 \frac{2x}{x^2+1} dx = \frac{1}{2} \left[\ln\left|x^2+1\right|\right]_0^1 \\ &= \frac{1}{2} [\ln 2 - \ln 1] \\ &= \frac{1}{2} \ln 2. \end{aligned}$$

2. Take $I_2 = \int_0^1 \frac{x^3}{x^2+1} dx$, Calculate $(I_1 + I_2)$ then deduce I_2.

$$\begin{aligned} (I_1 + I_2) &= \int_0^1 \frac{x}{x^2+1} dx + \int_0^1 \frac{x^3}{x^2+1} dx \\ &= \int_0^1 \left(\frac{x}{x^2+1} + \frac{x^3}{x^2+1} \right) dx \\ &= \int_0^1 \left(\frac{x + x^3}{x^2+1} \right) dx \\ &= \int_0^1 \frac{x(1+x^2)}{x^2+1} dx \\ &= \int_0^1 x dx \\ &= \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1}{2}. \end{aligned}$$

Since $(I_1 + I_2) = \dfrac{1}{2}$ then $I_2 = \dfrac{1}{2} - I_1 = \dfrac{1}{2} - \dfrac{1}{2} \ln 2$.

3. Let :

$$I_1 = \int_0^\pi x^2 \cos^2(x) dx \quad \text{and} \quad I_2 = \int_0^\pi x^2 \sin^2(x) dx$$

a) Calculate $I_1 + I_2$.

$$I_1 + I_2 = \int_0^\pi x^2 dx = \left[\frac{x^3}{3}\right]_0^\pi = \frac{\pi^3}{3}.$$

b) Knowing that $\forall \alpha \in \mathbb{R}, \cos(2\alpha) = \dfrac{\cos^2\alpha - \sin^2\alpha}{2}$, calculate $I_1 - I_2$.

$$I_1 - I_2 = \int_0^\pi 2x^2 \cos(2x) dx$$

We first calculate $\int 2x^2 \cos(2x)dx$ without bounds, by parts:

$$\text{One poses} \begin{cases} u'(x) = 2\cos(2x) \\ v(x) = x^2 \end{cases}, \text{ one finds} \begin{cases} u(x) = \sin(2x) \\ v'(x) = 2x \end{cases}.$$

The integration by parts formula $\int u'v = [uv] - \int v'u$, gives us:

$$\int 2x^2 \cos(2x)dx = x^2 \sin(2x) - \int (2x)\sin(2x)dx.$$

One calculates $\int (2x)\sin(2x)dx$. a second time by parts, which leads to:

$$\text{Take} \begin{cases} u'(x) = 2\sin(2x) \\ v(x) = x \end{cases}, \text{ get} \begin{cases} u(x) = -\cos(2x) \\ v'(x) = 1 \end{cases}.$$

Applying the integration by parts formula again gives us:

$$\begin{aligned} \int (2x)\sin(2x)dx &= \int u'v = [uv] - \int v'u \\ &= -x\cos(2x) - \int -\cos(2x)dx. \\ \int (2x)\sin(2x)dx &= -x\cos(2x) + \frac{1}{2}\sin(2x) + C. \end{aligned}$$

Finally,

$$\begin{aligned} \int 2x^2 \cos(2x)dx &= x^2 \sin(2x) - \left[-x\cos(2x) + \frac{1}{2}\sin(2x) + C\right] \\ \int 2x^2 \cos(2x)dx &= \left(x^2 - \frac{1}{2}\right)\sin(2x) + x\cos(2x) + C. \end{aligned}$$

Yields :

$$\begin{aligned} I_1 - I_2 &= \left[\left(x^2 - \frac{1}{2}\right)\sin(2x) + x\cos(2x)\right]_0^{\pi} \\ &= \pi. \end{aligned}$$

c) Deduce the value of I_1 and of I_2.

One has :

$$\begin{cases} I_1 + I_2 = \frac{\pi^3}{3} \\ I_1 - I_2 = \pi \end{cases}.$$

$$\boxed{\Leftrightarrow \begin{cases} I_1 = \frac{1}{2}\left(\frac{\pi^3}{3} + \pi\right) \\ I_2 = \frac{1}{2}\left(\frac{\pi^3}{3} - \pi\right) \end{cases}}$$

Exercise 7

1/ Find the three reals a, b, c such that $\dfrac{4x^2 - 5x + 1}{x + 3} = ax + b + \dfrac{c}{x + 3}$.

We can proceed either by analogy or by Euclidean division, both methods are easy, we find:

$$a = 4, b = -17, c = 52.$$

2/ Calculate $\int \dfrac{4x^2 - 5x + 1}{x + 3}dx$, then deduce $\int_0^2 \dfrac{4x^2 - 5x + 1}{x + 3}dx$.

$$\begin{aligned} \int \frac{4x^2 - 5x + 1}{x + 3}dx &= \int \left(4x - 17 + \frac{52}{x + 3}\right) dx \\ &= 4\left(\frac{x^2}{2}\right) - 17x + 52\ln|x + 3| + C, (C \in \mathbb{R}). \end{aligned}$$

Hence:

$$\begin{aligned} \int_0^2 \frac{4x^2 - 5x + 1}{x + 3}dx &= \left[2x^2 - 17x + 52\ln|x + 3|\right]_0^2 \\ &= -24 + 52\ln 5 - 52\ln 3. \end{aligned}$$

Exercise 8

1) $\int \sin^2(2x)\, dx$.

We are facing the form $\int \sin^n(x)\, dx$ with n even. We use the following linearization formula:

$$\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha)).$$

Hence:

$$\sin^2(2x) = \frac{1}{2}(1 - \cos(4x)).$$

$$\begin{aligned} I &= \int \sin^2(2x)\, dx \\ &= \frac{1}{2}\int (1 - \cos(4x)) \\ &= \frac{1}{2}\left[x + \frac{1}{4}\sin(4x)\right] + C \\ &= \frac{1}{2}x + \frac{1}{8}\sin(4x) + C, (C \in \mathbb{R}). \end{aligned}$$

2) $I = \int 3\cos^3(x)dx$

We have the same pattern here, but n is odd. We separate the power 3 to $3 = 2+1$. We then keep the power 1 which will play the role of derivative, then we use the relation $\cos^2(x) + \sin^2(x) = 1$.

Hence :

$$\begin{aligned} \cos^3(x) &= \cos^2(x)\cos(x) \\ &= [1 - \sin^2(x)]\cos(x). \end{aligned}$$

And so :

$$\begin{aligned} I &= \int 3\cos^3(x)dx \\ I &= 3\left[\int \cos(x)dx - \int \sin^2(x)\cos(x)dx\right] \\ &= 3\sin(x) - \sin^3(x) + C, (C \in \mathbb{R}). \end{aligned}$$

3) $I=\int \sin^2(x)\cos^2(x)\,dx$.
Here, the two powers are even. We use the relation

$$\sin(2x)=2\sin x\cos x.$$

One has:

$$\begin{aligned}\sin^2(x)\cos^2(x) &= [\sin(x)\cos(x)]^2\\ &= \left[\frac{\sin(2x)}{2}\right]^2\\ &= \frac{1}{4}\sin^2(2x)\\ &= \frac{1}{4}\left[\frac{1}{2}(1-\cos(4x))\right].\end{aligned}$$

So :

$$\begin{aligned}I &= \frac{1}{8}\int(1-\cos(4x))dx\\ &= \frac{1}{8}\left[x-\frac{1}{4}\sin(4x)\right]+C,(C\in\mathbb{R}).\end{aligned}$$

4) $\int \sin(2x)\cos(3x)\,dx$

We have the form **:**$\int \sin(\alpha x)\cos(\beta x)\,dx$

We use the following trigonometric formula:

$$\sin(\alpha x)\cos(\beta x)=\frac{1}{2}\left[\sin(\alpha+\beta)x+\sin(\alpha-\beta)x\right].$$

Gives :

$$\begin{aligned}\sin(2x)\cos(3x) &= \frac{1}{2}\left[\sin(5x)+\sin(-x)\right]\\ &= \frac{1}{2}\left[\sin(5x)-\sin(x)\right].\end{aligned}$$

So:

$$\begin{aligned}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{2}\int(\sin(5x)-\sin(x))\,dx\\ &= \frac{1}{2}\left[\frac{-1}{5}\cos(5x)+\cos(x)\right]+C,(C\in\mathbb{R}).\end{aligned}$$

Exercise 9

1. We notice that, for the second and the fourth integral, we can show the derivative of the denominator in the numerator. On the other hand, we cannot do this for the first and the third integral. We therefore use the table of laws to integrate 2) and 4), as follows:

$$\begin{aligned}2)\int\frac{x}{x^2+1}dx &= \frac{1}{2}\int\frac{2x}{x^2+1}dx\\ &= \frac{1}{2}\ln(x^2+1)+C,(C\in\mathbb{R}).\end{aligned}$$

$$\begin{aligned}4)\int\frac{x}{x^2-1}dx &= \frac{1}{2}\int\frac{2x}{x^2-1}dx\\ &= \frac{1}{2}\ln\left|x^2-1\right|+C,(C\in\mathbb{R}).\end{aligned}$$

Then, we can see that the denominator of the first integral verifies $\Delta<0$, so it is the third form of the basic forms of rational fractions, while that of the third integral verifies $\Delta>0$. This is why we do not treat the two forms in the same way.

The result of these two integrals can be found in the table of laws:

$$1)\int\frac{1}{x^2+1}dx = arctg(x)+C,(C\in\mathbb{R}).$$

$$3)\int\frac{1}{x^2-1}dx = ??$$

One has :

$$
\begin{aligned}
\frac{1}{x^2-1} &= \frac{1}{(x+1)(x-1)} \\
&= \frac{A}{x+1} + \frac{B}{x-1} \\
&= \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} \\
\frac{1}{x^2-1} &= \frac{(A+B)x-A+B}{x^2-1}
\end{aligned}
$$

By analogy, one gets :

$$
\begin{cases} (A+B)=0 \\ -A+B=1 \end{cases}
$$

$$
\begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \end{cases}
$$

Hence,

$$
\frac{1}{x^2-1} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}
$$

So :

$$
\begin{aligned}
\int \frac{1}{x^2-1}dx &= \int \frac{-\frac{1}{2}}{x+1}dx + \int \frac{\frac{1}{2}}{x-1}dx \\
&= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\
&= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.
\end{aligned}
$$

2. Calculate the integrals of the following rational fractions:

$$
\begin{aligned}
1) \int \frac{A}{3x+1}dx &= \frac{A}{3}\ln|3x+1| + C, (C \in \mathbb{R}). \\
2) \int \frac{B}{(3x+1)^3}dx &= \frac{B}{3}\frac{1}{(-2)\,(3x+1)^{\,2}} + C \\
&= \frac{B}{(-6)\,(3x+1)^{\,2}} + C, (C \in \mathbb{R}).
\end{aligned}
$$

$$3)I = \int \frac{x-2}{x^2+5x+7}dx = \int \frac{\frac{1}{2}(2x+5)-\frac{5}{2}-2}{x^2+5x+7}dx$$

$$\begin{aligned} &= \frac{1}{2}\int \frac{2x+5}{x^2+5x+7}dx + \int \frac{\frac{-9}{2}}{x^2+5x+7}dx \\ &= \frac{1}{2}\int \frac{2x+5}{x^2+5x+7}dx - \frac{9}{2}\int \frac{1}{x^2+5x+7}dx \\ I &= \ln\left|x^2+5x+7\right| - \frac{9}{2}J. \end{aligned}$$

With $J = \int \dfrac{1}{x^2+5x+7}dx$, this is form 3 of rational fractions.
Canonical form : $x^2+5x+7 = (x+\frac{5}{2})^2 + \frac{3}{4}$, $(\triangle = -3)$. So,

$$\begin{aligned} J &= \int \frac{1}{(x+\frac{5}{2})^2+\frac{3}{4}}dx \\ &= \frac{2}{\sqrt{3}}arctg\frac{2(x+\frac{5}{2})}{\sqrt{3}} + C. \end{aligned}$$

Finally :

$$I = \ln\left|x^2+5x+7\right| - \frac{9}{\sqrt{3}}arctg\left(\frac{2x+5}{\sqrt{3}}\right) + C, (C \in \mathbb{R}).$$

3) Calculate the following integrals :

$$a/\ \ I = \int \frac{dx}{x^2+10x+30};\quad b/\ \ J = \int \sqrt{x^2-2x^4}dx$$

a/ $I = \int \dfrac{dx}{x^2+10x+30}$

$$\Delta = -20 < 0.$$

Canonical form :

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \frac{|\Delta|}{4a}.$$

Hence :

$$x^2 + 10x + 30 = (x+5)^2 + 5.$$

Yields to :

$$I = \int \frac{dx}{x^2 + 10x + 30} = \int \frac{dx}{(x+5)^2 + 5} = \frac{1}{\sqrt{5}} arctg \left(\frac{x+5}{\sqrt{5}} \right) + C.$$

b/ $J = \int \sqrt{x^2 - 2x^4} dx$.

$$\begin{aligned} J &= \int \sqrt{x^2 - 2x^4} dx = \int \sqrt{x^2(1 - 2x^2)} dx \\ &= \int |x| \sqrt{(1 - 2x^2)} dx \\ &= \pm \int x \sqrt{(1 - 2x^2)} dx \\ &= \pm \frac{1}{(-4)} \left[\frac{(1 - 2x^2)^{\frac{3}{2}}}{\frac{3}{2}} \right] + C. \end{aligned}$$

$$J = \pm \frac{1}{6} (1 - 2x^2)^{\frac{3}{2}} + C.$$

4/

a) $I = \int_0^1 \sqrt{1 - x^2} dx$
One poses $x = \sin y$, to get $dx = \cos y . dy$
$x \in [0, 1]$ and $x = \sin y \Rightarrow y = \arcsin x$. Yields $\begin{array}{l} x_1 = 0 \Rightarrow y_1 = \arcsin 0 = 0 \\ x_2 = 1 \Rightarrow y_2 = \arcsin 1 = \frac{\pi}{2} \end{array}$

.

Hence,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 y} \ \cos y \ dy \\ &= \int_0^{\frac{\pi}{2}} |\cos y| \cos y dy \end{aligned}$$

But since $y \in \left[0, \frac{\pi}{2}\right]$ then $\cos y \geq 0$. Which gives :

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \cos^2 y \; dy \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos(2y)+1) \; dy \\
&= \frac{1}{2}\left[\frac{1}{2}\sin(2y)+y\right]_0^{\frac{\pi}{2}} \\
I &= \frac{\pi}{4}.
\end{aligned}
$$

b)

$$J = \int \frac{x-1}{x^2+2x+3}dx$$

One poses $P(x) = x^2+2x+3$. Note that : $\Delta = -8 < 0$.
By the canonicl form, one rewrite P as : $\forall x \in \mathbb{R},\;\; P(x) = (x+1)^2 + 2$.
And since $P^{'}(x) = 2x+2$, one gets : $\forall x \in \mathbb{R}$,

$$
\begin{aligned}
(x-1) &= \frac{1}{2}(2x-2) \\
&= \frac{1}{2}(2x+2-2-2) \\
&= \frac{1}{2}[(2x+2)-4] \\
(x-1) &= \frac{1}{2}(2x+2)-2.
\end{aligned}
$$

So :

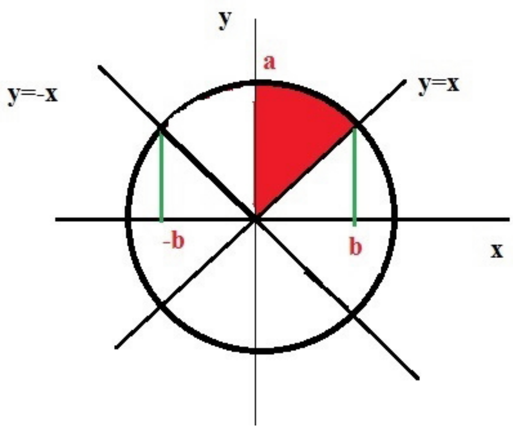
$$
\begin{aligned}
J &= \int \frac{x-1}{x^2+2x+3}dx \\
&= \int \frac{\frac{1}{2}(2x+2)-2}{x^2+2x+3}dx \\
&= \frac{1}{2}\int \frac{(2x+2)}{x^2+2x+3}dx - 2\int \frac{dx}{(x+1)^2+2} \\
J &= \frac{1}{2}\ln\left|x^2+2x+3\right| - 2\left(\frac{1}{\sqrt{2}}\arctan\frac{(x+1)}{\sqrt{2}}\right) + C, \quad (C \in \mathbb{R}).
\end{aligned}
$$

Exercise 10

Calculate the upper area between the following graphs:

$$\begin{cases} x^2 + y^2 = 9 \\ x = y \\ x = -y \end{cases}.$$

The first equation is the equation of a circle that is centered at the origin and whose radius is $R = 3$.



$Fig3.9$: Requested area in exercise 10.

We can know the result in advance since the area of the entire circle is given by :

$$\begin{aligned} S' &= \pi R^2 \\ &= 9\pi \quad SU \end{aligned}$$

This means that the requested area is equal to $S = \frac{9}{4}\pi \quad SU$.

On the other hand, as there is symmetry, we can calculate the surface coloured in red S_1 then multiply by 2.

By the vertical rectangle, we find:

$$x^2 + y^2 = 9 \Rightarrow y = \pm\sqrt{9 - x^2}.$$

But since we are working on the upper quarter of the circle, the y's there are positive, so we take $y = \sqrt{9 - x^2}$.

The point $a(0, y_a)$ verifies : $0^2 + {y_a}^2 = 9 \Rightarrow y = 3$, so a is the point $(0, 3)$.

Afterwards, b is on the circle but also on the line $y = x$, so b is a coordinate point of the form (x_b, x_b).

Hence : ${x^2}_b + {x^2}_b = 9 \Rightarrow x_b = \frac{3}{\sqrt{2}}$.

$$S_1 = \int_0^{\frac{3}{\sqrt{2}}} \left(\sqrt{9 - x^2} - x\right) dx.$$

Yet :

$$I = \int \left(\sqrt{9 - x^2}\right) dx.$$

For the third form : $b^2 - a^2x^2 = 9 - x^2$, one poses $3\sin y = x$, to get $dx = 3\cos y\ dy$.

Which implies that :

$$\begin{aligned} I &= \int \sqrt{9 - (3\sin y)^{\,2}}\ \ (3\cos y)\, dy. \\ &= 9\int \sqrt{1 - \sin^{\,2} y}\ \ \cos y\ dy. \end{aligned}$$

Then we use the trigonometric formula : $1 - \sin^{\,2} x = \cos^{\,2} y$. We then have :

$$\begin{aligned} I &= 9\int \cos^{\,2} y dy \\ &= 9\int \frac{1 + \cos(2y)}{2} dy \\ &= \frac{9}{2}\left[y + \frac{1}{2}\sin(2y)\right]. \end{aligned}$$

Now we go back to the first variable:

One has : $x = 3\sin y \Rightarrow y = \arcsin\left(\frac{x}{3}\right)$. But for $\sin(2y)$ we are going to use the following formula :

$$\begin{aligned} \sin(2y) &= 2\sin y \cos y \\ &= 2\sin y\sqrt{1 - \sin^{\,2} y} \end{aligned}$$

This gives : $\sin(2y) = 2\left(\frac{x}{3}\right)\sqrt{1-\left(\frac{x}{3}\right)^2} = 2\left(\frac{x}{9}\right)\sqrt{9-x^2}$.

$$I = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}.2\left(\frac{x}{9}\right)\sqrt{9-x^2}\right].$$

Finally :

$$\begin{aligned} S_1 &= \left[\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + x\sqrt{9-x^2}\right) - \frac{x^2}{2}\right]_0^{\frac{3}{\sqrt{2}}} \\ &= \frac{9}{2}\left[\arcsin\left(\frac{\sqrt{2}}{2}\right) - \frac{3\sqrt{2}}{2}\sqrt{9-\left(\frac{3\sqrt{2}}{2}\right)^2}\right] - \frac{9}{4} \\ &= \frac{9}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] - \frac{9}{4} = \frac{9}{2}\left(\frac{\pi}{4}\right) = \frac{9\pi}{8}. \end{aligned}$$

And so, as expected, we find :

$$\begin{aligned} S &= 2S_1 \\ &= \frac{9\pi}{4}SU. \end{aligned}$$

4 Chapiter 4 Ordinary Differential Equations

4.1 Introduction

In mathematics, an ordinary differential equation is an equation whose unknown(s) are functions; it is presented in the form of a relation between these unknown functions and their successive derivatives. It is a special case of functional equations. General equations involve dependent and Independent variables, but those equation which involves variables as well as derivative of dependent variable (y) with respect to independent variable (x) are known as Differential Equation.

The solution of a differential equation is a function, that represents a relationship between the variables, independent of derivatives. The solution of a differential equation is also known as its primitive.

There are generally two types of differential equations:

1) Ordinary Differential Equations (ODE), where the unknown function(s) depend on only one variable;

2) Then the partial differential equations (PDEs), where the unknown functions can depend on several independent variables.

Without further precision, the term differential equation most often refers to ordinary differential equations.

We will learn how to solve ODEs through practical examples, without going into theory. The solution methods are actually recipes that we have to learn in order to solve the requested ODEs. Each type of EDO has its own solution recipe!

4.2 First order differential equations

Let y be a differentiable function of the variable x.

We call first-order ordinary differential equation (ODE) any relation between the variable x, the function $y(x)$ and its first derivative with respect to x, $y\prime(x)$. We note the ODE therefore the relation $f(x, y, y\prime(x)) = 0$.

Examples

$$1) x(lnx)y\prime = (3lnx + 1)y$$

$$2) x^2 y\prime + xy = y^2 + x^2, (x \neq 0)$$

$$3) (1 + x^2)y' + xy = \sqrt{1 + x^2}$$

We can distinguish several types of first-order ODE depending on the form of the function f.

The resolution method is appropriate for each type differently.
In this course, we will learn to solve three forms of first-order ODEs, which are: separable, homogeneous and linear.

4.2.1 Separable ODEs

Separable ODEs are expressed in terms of (x, y) such that, the x-terms and y-terms can be separated to different sides of the equation. Thus each variable separated can be integrated easily to form the solution of differential equation.
The equations can be written as :

$$f(x)dx = g(y)dy$$

To solve this type of ODE, we consider x and y as independent variables. Then simply separate the two variables and then integrate, as follows:

Example 1

$$\begin{aligned} 2) x^2\ y + \sqrt{1 + x^3}\ y' &= 0. \\ &\Rightarrow \sqrt{1 + x^3}\ \frac{dy}{dx} = -x^2\ y \\ &\Rightarrow \frac{-x^2 dx}{\sqrt{1 + x^3}} = \frac{dy}{y} \\ &\Rightarrow \int \frac{-x^2 dx}{\sqrt{1 + x^3}} = \int \frac{dy}{y} \\ &\Rightarrow -\frac{1}{3}\left[\frac{\sqrt{1 + x^3}}{\frac{1}{2}}\right] + C = \ln|y|. \quad (C \in \mathbb{R}). \end{aligned}$$

$$y(x) = K.e^{-\frac{2}{3}\sqrt{1+x^3}}.\,(K \in \mathbb{R})\,.$$

Example 2

$$\begin{aligned} 2)\ x(\ln x)y\prime &= (3\ln x + 1)y \\ &\Rightarrow x(\ln x)dy = (3\ln x + 1)ydx \\ &\Rightarrow \frac{dy}{y} = \frac{(3\ln x + 1)dx}{x \ln x} \\ &\Rightarrow \int \frac{dy}{y} = \int \left(\frac{3}{x} + \frac{\frac{1}{x}}{\ln x}\right) dx \\ &\Rightarrow \ln y = 3\ln x + \ln|\ln x| + C, (C \in \mathbb{R})\,. \\ &\Rightarrow y = Kx^3 \ln x., (K \in \mathbb{R})\,. \end{aligned}$$

4.2.2 Homogeneous ODEs

Homogeneous ODEs can be summerized in the following form :

$$y' = f\left(\frac{y}{x}\right)$$

They require that $(x \neq 0)$. We may discuss the case when $(x = 0)$ separately.

To solve them, we perform a variable change as follows:

Take $u(x) = \dfrac{y(x)}{x}$, to get $y(x) = u(x)x$ and $y'(x) = u'(x)x + u(x)$. Note that u is, as y, a function of the variable x.

When replacing by the new variable in the ODE, we should obtain a seperable ODE to be soled as usuel.

Example

Solve the following first order homogeneous ODE :

$$3)\ x^2y' + xy = y^2 + x^2, (x \neq 0)$$

Dividing equation (3) by x^2, (remember that $x \neq 0$), one finds :

$$
\begin{aligned}
(3) \quad \Leftrightarrow \quad & y^{'} + \frac{y}{x} = \frac{y^2 + x^2}{x^2}, (x \neq 0) \, . \\
\Leftrightarrow \quad & y^{'} + \frac{y}{x} = \frac{y^2}{x^2} + 1 \ldots . (*)
\end{aligned}
$$

Take $u = \dfrac{y}{x}$, to get $y = ux$ and $y' = u'x + u$.
Replacing in $(*)$, one obtains:

$$
\begin{aligned}
u^{'}x + u + u &= u^2 + 1 \\
u^{'}x &= u^2 - 2u + 1 \\
&= (u-1)^2
\end{aligned}
$$

Which is an ODE with separable variables with respect to u since :

$$\frac{dx}{x} = \frac{du}{(u-1)^2}$$

We will solve it easily by :

$$\ln|x| + C = -\frac{1}{u-1}, (C \in \mathbb{R}) \, .$$

First we will find u then return to the variable y, this way :

$$
\begin{aligned}
u \quad = \quad & -\frac{1}{\ln|x| + C} + 1 \\
\Leftrightarrow \quad & \frac{y}{x} = -\frac{1}{\ln|x| + C} + 1 \\
\Leftrightarrow \quad & y = -\frac{x}{\ln|x| + C} + x, (C \in \mathbb{R}) \, .
\end{aligned}
$$

4.2.3 Linear ODE

These are ODEs that we can write in the form:

$$y' + a(x)y = b(x)$$

We can then solve them by two methods, which we will explain on a concrete example.

Example

$$4)\ xy' + 2y = x^2 + 1. \quad (x > 0).$$

$$\Longrightarrow y' + \frac{2}{x}y = \frac{x^2+1}{x}; (x > 0).$$

First method : variation of the constant :

We first solve the so-called equation without second member, that is that we consider $b(x) = 0$. We then obtain an ODE with separable variables, as follows:

Equation without second member :

$$\begin{aligned} y' + \frac{2}{x}y &= 0 \\ &\Leftrightarrow \frac{dy}{dx} = -\frac{2}{x}y \\ &\Leftrightarrow \frac{dy}{y} = -\frac{2}{x}dx \\ &\Leftrightarrow \int \frac{dy}{y} = \int -\frac{2}{x}dx \\ &\Leftrightarrow \ln y = \ln(x)^{-2} + C. \end{aligned}$$

$$\Leftrightarrow y = \frac{K}{x^2}.$$

Equation with second member :

Now, we are going to vary the constant! That is, the constant found when solving the first step is now considered a function of x. Which gives the following derivation:

$$\begin{aligned} y &= \frac{K(x)}{x^2} \\ &\Leftrightarrow y' = \frac{K'(x)x^2 - K(x)(2x)}{x^4} \\ &= \frac{K'(x)}{x^2} - \frac{2K(x)}{x^3}. \end{aligned}$$

Replacing y and y' in the initial equation, we find:

$$\begin{aligned}
y' + \frac{2}{x}y &= \frac{x^2+1}{x} \\
&\Leftrightarrow \frac{K'(x)}{x^2} - \frac{2K(x)}{x^3} + \frac{2}{x}\left[\frac{K(x)}{x^2}\right] \\
&= \frac{x^2+1}{x} \\
&\Leftrightarrow \frac{K'(x)}{x^2} = \frac{x^2+1}{x} \\
&\Leftrightarrow K'(x) = x(x^2+1) \\
&\Leftrightarrow K(x) = \int \left(x^3 + x\right) dx \\
&\Leftrightarrow K(x) = \frac{x^4}{4} + \frac{x^2}{2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
y(x) &= \frac{K}{x^2} + \frac{\frac{x^4}{4} + \frac{x^2}{2}}{x^2} \\
y(x) &= \frac{K}{x^2} + \frac{x^2}{4} + \frac{1}{2}
\end{aligned}$$

Second method : by the integrating factor:

$$y' + \frac{2}{x}y = \frac{x^2+1}{x}; (x > 0).$$

The equation is written in the form

$$y' + a(x)y = b(x); (x > 0).$$

With $a(x) = \dfrac{2}{x}$ et $b(x) = \dfrac{x^2+1}{x}$.

Step 1 : One calculates

$$\begin{aligned}
\int a(x)dx &= \int \frac{2}{x}dx \\
&= 2\ln x \\
&= \ln x^2.
\end{aligned}$$

Then find $R(x)$ the integrating factor :

$$\begin{aligned} R(x) &= e^{\int a(x)dx} \\ &= e^{\ln x^2} \\ R(x) &= x^2. \end{aligned}$$

Step 2 : One calculates :

$$\begin{aligned} I(x) &= \int R(x).b(x)dx \\ &= \int \left(x^2\right).\left(\frac{x^2+1}{x}\right)dx \\ &= \int \left(x^3+x\right)dx \\ &= \frac{x^4}{4}+\frac{x^2}{2}. \end{aligned}$$

Then find the expression of the solution as follows :

$$R(x)y(x) = I(x) + C.$$

Hence :

$$x^2y(x) = \frac{x^4}{4} + \frac{x^2}{2} + C.$$

Which gives :

$$y(x) = \frac{x^2}{4} + \frac{1}{2} + \frac{C}{x^2}.$$

4.3 Second-order differential equations

Definition

We call ordinary differential equation (ODE) of the second order any relation between the variable x, the function $y(x)$, its first derivative with respect to x, $y\prime(x)$ and its second derivative with respect to x, $y\prime\prime(x)$.

A second-order ODE given in its explicit form is therefore the relation $f(x, y, y\prime(x), y\prime\prime(x)) = 0$.

Examples :

$$\begin{aligned} 1)y'' &= x \\ 2)y'' + y' &= 2x^2 \end{aligned}$$

$$\begin{aligned} 3)y'' - 2y' + y &= \cos x. \\ 4)y'' - 4y' + 3y &= e^x. \\ 5)y'' + 3y' + 2y &= x^2 + \sin x + e^{-x}. \end{aligned}$$

4.3.1 Pattern $y^{''} = f(x)$

To solve this kind of second order ODE, It is sufficient to integrate two times.

$$\begin{aligned} y^{''} &= f(x) \\ &\Rightarrow \; y^{'} = \int f(x)dx + C_1 \\ \text{Then, } \; y &= \int \left(\int f(x)dx + C_1 \right) dx \\ &\Rightarrow \; y(x) = \int \left(\int f(x)dx \right) dx + C_1 x + C_2; \quad (C_1, C_2 \in \mathbb{R}). \end{aligned}$$

Example
Solve the following second order differential equation $y^{''} = 3x + 1$.

It is sufficient to integrate two times, as follows :

$$\begin{aligned} y^{''} &= 3x + 1 \\ &\Rightarrow \; y^{'} = \int (3x + 1)\, dx \\ &\Rightarrow \; y^{'} = \frac{3}{2}x^2 + x + C_1 \\ \text{Then, } \; y &= \int \left(\frac{3}{2}x^2 + x + C_1 \right) dx \\ &\Rightarrow \; y(x) = \frac{1}{2}x^3 + \frac{1}{2}x^2 + C_1 x + C_2; \quad (C_1, C_2 \in \mathbb{R}). \end{aligned}$$

4.3.2 Pattern $f(x, y', y'') = 0$

To solve this kind of second order ODE, we need to do a varible change by taking $z = y^{'}$, and getting $z^{'} = y^{''}$. Note that z is, as y, a function of the variable x.

So that the equation under investigation becomes a linear first order ODE of the variable z, that we have already learned how to solve in previous sections.

Example

Solve the following second order ODE:

$$xy^{''} + y^{'} + x = 0; (x > 0)...(1)$$

One poses $z = y^{'}$, one finds $z^{'} = y^{''}$. So that equation (1) becomes:

$$(1) \Leftrightarrow xz^{'} + z + x = 0...(2)$$

Which is a linear first order ODE:

$$(2) \Leftrightarrow z^{'} + \frac{1}{x}z = -1$$

With $a(x) = \dfrac{1}{x}$ and $b(x) = -1$.

Let's solve it by the integrating factor method :
$\int a(x)dx = \ln x$ so the integrating factor is $R(x) = e^{-\int a(x)dx} = e^{-\ln x} = \dfrac{1}{x}$.

Furthermore, $K(x) = \int \frac{b(x)}{R(x)}dx = \int -xdx = \dfrac{-x^2}{2}$.

The solution is then given by :

$$\begin{aligned} z(x) &= R(x)\left(K(x) + C_1\right). \\ &= \frac{1}{x}\left(\frac{-x^2}{2} + C_1\right) \\ &= \frac{-x}{2} + \frac{C_1}{x} \end{aligned}$$

And since $y(x) = \int z(x)dx$ then $y(x) = \int \left(\frac{-x}{2} + \frac{C_1}{x}\right) dx$

$$\Rightarrow y(x) = \frac{-x^2}{4} + C_1 \ln x + C_2. \quad \left(C_1, C_2 \in \mathbb{R}\right).$$

4.3.3 Second-order differential equations with constant coefficients

Definition

We call second order ordinary differential equation (ODE) with constant coefficients any relation between the variable x, the function $y(x)$, its first derivative with respect to x, $y\prime(x)$ and its second derivative with respect to x, $y\prime\prime(x)$ that takes the subsequent pattern :

$$\begin{aligned} ay'' + by' + cy &= f(x) \\ a, b, c &\in \mathbb{R}. \end{aligned}$$

To solve this kind of ODEs, we proceed in two steps, we first solve the equation without second member and note its solution by y_h , then we look for a particular solution that verifies the hole equation with its second member, and note it by y_p. The final solution of the overall euation is the summ of its homogenous one and particular one, that is to say that :

$$y(x) = y_h(x) + y_p(x)$$

Step 1 : We first solve the equation without second member also called the homogenous equation, where we consider the second membre as zero.

$$\begin{aligned} ay'' + by' + cy &= 0 \\ a, b, c &\in \mathbb{R}. \end{aligned}$$

Set the caracteristic equation (CE):

$$\begin{aligned} ar^2 + br + c &= 0 \ldots (CE) \\ \Delta &= b^2 - 4ac \end{aligned}$$

Solutions of this quadratic equation lead to solutions of the differential equation as follows :

Case 1 If $\Delta > 0$ there are two different real solutions of the algebric equation (CE):

$$r_1 = \frac{-b - \sqrt{\Delta}}{2a}, r_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

In this case, the homogenous differential equation solution is given by :

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \qquad C_1, C_2 \in \mathbb{R}.$$

Case 2 If $\Delta = 0$ there is only one solution of the algebric equation (CE), that is a double one :

$$r = \frac{-b}{2a},$$

In this case, the homogenous differential equation solution is given by :

$$y_h(x) = (C_1 + C_2 x)\, e^{rx}, \qquad C_1, C_2 \in \mathbb{R}.$$

Case 3 If $\Delta < 0$ there are two different complex solutions of the algebric equation (CE):

$$r_1 = \frac{-b}{2a} - i\frac{\sqrt{|\Delta|}}{2a}, r_2 = \frac{-b}{2a} + i\frac{\sqrt{|\Delta|}}{2a}$$

Set :

$$\alpha = \frac{-b}{2a}, \beta = \frac{\sqrt{|\Delta|}}{2a}$$

In this case, the homogenous differential equation solution is given by :

$$y_h(x) = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right), \qquad C_1, C_2 \in \mathbb{R}.$$

Step 2 : We look for a particular solution y_p that verifies the hole equation (with its second member).

Note that there are no added constants in the particular solution and that it depends on the form of the function $f(x)$.

So, if $f(x) = e^{\lambda x}[P_1(x)\cos(\theta x) + P_2(x)\sin(\theta x)], \lambda, \theta \in \mathbb{R}$, then

$$y_p(x) = e^{\lambda x} x^m \left[Q_1(x)\cos(\theta x) + Q_2(x)\sin(\theta x)\right].$$

*) If $(\lambda + i\theta)$ is a solution of the (CE), then $m = 1$, otherwise $m = 0$.

*) Polynomials Q_1, Q_2 are to be determined knowing that :$\deg(Q_1(x)) = \deg(Q_2(x)) = \max\left(\deg(P_1(x)), \deg(P_2(x)\right).$

Examples

1/ Solve $y'' + y = \cos x ...(1)$.

Equation without second member (homogenous equation) : $y'' + y = 0$

$$(CE) \quad : \quad r^2 + 1 = 0$$
$$\Rightarrow \quad r = \pm i = \alpha \pm i\beta \quad \text{with} \quad \begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}.$$

Let us note y_h the solution of the homogeneous equation associated with (1).

Then :

$$y_h(x) = C_1 \cos x + C_2 \sin x; (C_1, C_2 \in \mathbb{R}).$$

Equation with second member : One poses

$$\begin{aligned} f(x) &= \cos x \\ &= e^{\lambda x}[P_1(x) \cos(\theta x) + P_2(x) \sin(\theta x)] \\ &\quad \text{With} \begin{cases} \lambda = 0 \\ \theta = 1 \\ P_1(x) = 1 \Rightarrow \deg(P_1(x)) = 0 \\ P_2(x) = 0 \Rightarrow \deg(P_2(x)) = 0 \end{cases} \end{aligned}$$

Let us note y_p the solution of the homogeneous equation associated with (1).

Then y_p follows the subsequent pattern :

$$y_p(x) = e^{\lambda x} x^m [Q_1(x) \cos(\theta x) + Q_2(x) \sin(\theta x)].$$

*) Since $\lambda + i\theta = i$ is a solution of the (CE), then $m = 1$.

*) Since $\max(\deg(P_1(x)), \deg(P_2(x))) = 0$ then $\deg(Q_1(x)) = \deg(Q_2(x)) = 0$.

That is to say that $y_p(x) = x[a \cos x + b \sin x]$ where a and b are constants to be determined.

One has :

$$y_p'(x) = (a + bx)\cos x + (b - ax)\sin x.$$

Then :

$$y_p''(x) = (2b - ax)\cos x - (2a + bx)\sin x.$$

Substituting into equation (2.), we find :

$$(2b - ax)\cos x - (2a + bx)\sin x + x\left[a\cos x + b\sin x\right] = \cos x$$

$$\Rightarrow 2b\cos x - 2a\sin x = \cos x$$

By identification, we get: $\begin{cases} a = 0 \\ b = \frac{1}{2} \end{cases}$.

Hence :

$$y_p(x) = \frac{1}{2}x\sin x.$$

Finally :

$$y(x) = y_h(x) + y_p(x) = C_1\cos x + \left(C_2 + \frac{1}{2}x\right)\sin x;\ \ (C_1, C_2 \in \mathbb{R}).$$

4.4 Exercise Series N°4

Exercise 1

Solve the following first-order ODEs :

1. **With separable variables :**

$$1)\ y\prime \tan x = y \ln y.$$

$$2)\ x^2(1+y^2) + y\sqrt{1+x^3}y' = 0.$$

2. **Homogeneous:**

$$3)\ xy' = y + x\cos^2\left(\frac{y}{x}\right).\ (x \neq 0)$$

3. **Linear :**

$$\begin{aligned} 4)\ y' + \frac{1}{x}y &= \sin x; (x>0)\,. \\ 5)\ y' + \frac{1}{x\ln x}y &= \frac{1}{\ln x}, (x>1). \end{aligned}$$

Exercise 2

Solve the following second-order ODE :

$$y'' + 2y' + 2y = e^{-x}\sin x ...(1).$$

4.5 Correction of the exercise Series N°4

Exercise 1

1. **Separable ODEs**

$$1) y\prime \tan x = ylny$$

$$\begin{aligned}
&\Rightarrow \quad \frac{dy}{y\ln y} = \frac{dx}{\tan x} \\
&\Rightarrow \quad \int \frac{dy}{y\ln y} = \int \frac{\cos x}{\sin x} dx \\
&\Rightarrow \quad \int \frac{\frac{dy}{y}}{\ln y} = \ln|\sin x| + C, (C \in \mathbb{R}). \\
&\Rightarrow \quad \ln(\ln y) = \ln|\sin x| + C \\
&\Rightarrow \quad \ln y = K \sin x, (K \in \mathbb{R}). \\
&\Rightarrow \quad y = e^{K \sin x}, (K \in \mathbb{R}).
\end{aligned}$$

$$\begin{aligned}
2) x^2(1+y^2) + y\sqrt{1+x^3}y' &= 0 \\
x^2(1+y^2) &= -y\sqrt{1+x^3}\frac{dy}{dx} \\
\frac{-x^2dx}{\sqrt{1+x^3}} &= \frac{ydy}{(1+y^2)} \\
\int \frac{-x^2dx}{\sqrt{1+x^3}} &= \int \frac{ydy}{(1+y^2)} \\
-\frac{1}{3}\left[\frac{\sqrt{1+x^3}}{\frac{1}{2}}\right] + C &= \frac{1}{2}\ln(1+y^2).
\end{aligned}$$

2. **Homogenous:**

$$3) xy' = y + x\cos^2\left(\frac{y}{x}\right). (x \neq 0).$$

Dividing equation (3) by x, (remember that $x \neq 0$), one finds :

$$(3) \Leftrightarrow y' = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right), ...(*).$$

Take $u=\dfrac{y}{x}$, to get $y=ux$ and $y'=u'x+u$.

Replacing in $(*)$, one obtains:

$$\begin{aligned} u^{'}x+u &= u+\cos^2 u \\ u^{'}x &= \cos^2 u \end{aligned}$$

Which is an ODE with separable variables with respect to u since :

$$\frac{dx}{x}=\frac{du}{\cos^2 u}$$

We will solve it easily by :

$$\ln|x|+C=\tan u, (C\in\mathbb{R}).$$

First we will find u then return to the variable y, this way :

$$\begin{aligned} u &= \arctan(\ln|x|+C) \\ &\Leftrightarrow \frac{y}{x}=\arctan(\ln|x|+C) \\ &\Leftrightarrow y=x\arctan(\ln|x|+C), (C\in\mathbb{R}). \end{aligned}$$

3. **Linear** :

$$4) y'+\frac{1}{x\ln x}y=\frac{1}{\ln x}, (x>1).$$

First method : **constant variation** :

Equation without second member (Homogenous equation):

$$\begin{aligned} 4) &\Leftrightarrow y'+\frac{1}{x\ln x}y=0...(*) \\ &\Leftrightarrow \frac{dy}{dx}=-\frac{1}{x\ln x}y \\ &\Leftrightarrow \frac{dy}{y}=-\frac{1}{x\ln x}dx \\ &\Leftrightarrow \int\frac{dy}{y}=\int-\frac{\frac{1}{x}}{\ln x}dx \\ &\Leftrightarrow \ln y=-\ln|\ln(x)|+C, (C\in\mathbb{R}). \end{aligned}$$

$$\Leftrightarrow y = \frac{K}{\ln(x)}, (K \in \mathbb{R}), (x > 1).$$

Equation with second member :

One has :

$$\begin{aligned} y &= \frac{K(x)}{\ln(x)} \\ &\Leftrightarrow \ y' = \frac{K'(x)\ln(x) - K(x)(\frac{1}{x})}{\ln^2(x)} \end{aligned}$$

Replacing y and y' in equation (*) one gets :

$$\begin{aligned} y' + \frac{1}{x\ln x}y &= \frac{1}{\ln x} \\ &\Leftrightarrow \ \frac{K'(x)\ln(x) - K(x)(\frac{1}{x})}{\ln^2(x)} + \frac{1}{x\ln x}\left[\frac{K(x)}{\ln(x)}\right] = \frac{1}{\ln x} \\ &\Leftrightarrow \ \frac{K'(x)}{\ln(x)} = \frac{1}{\ln x} \\ &\Leftrightarrow \ K'(x) = 1 \\ &\Leftrightarrow \ K(x) = \int dx \\ &\Leftrightarrow \ K(x) = x \end{aligned}$$

Finally,

$$\begin{aligned} y(x) &= \frac{K}{\ln(x)} + \frac{x}{\ln(x)} \\ y(x) &= \frac{K + x}{\ln(x)}, (K \in \mathbb{R}), (x > 1). \end{aligned}$$

$$5) y' + \frac{1}{x}y = \sin x; (x > 0).$$

Second method : The integrating factor :

$$y' + \frac{1}{x}y = \sin x; (x > 0)\,.$$

The equation is written in the form:

$$y' + a(x)y = b(x).$$

With $a(x) = \dfrac{1}{x}$ and $b(x) = \dfrac{\sin x}{x}$.

Step 1 calculate :

$$\begin{aligned} \int a(x)dx &= \int \frac{1}{x}dx \\ &= \ln x; (x > 0) \end{aligned}$$

Then we set the integrating factor:

$$\begin{aligned} R(x) &= e^{\int a(x)dx} \\ &= e^{\ln|x|} \\ R(x) &= x. \end{aligned}$$

Step 2 : Calculate :

$$\begin{aligned} I(x) &= \int R(x).b(x)dx \\ &= \int x.\left(\sin x\right)dx \end{aligned}$$

By parts, one gets :

$$I(x) = -x\cos x + \sin x$$

And we write the solution as:

$$R(x).y(x) = I(x) + C.$$

Hence,

$$x.y(x) = -x\cos x + \sin x + C.$$

Which gives :

$$y(x) = \frac{-x\cos x + \sin x + C}{x}.$$

Exercise 2

$$y^{''} + 2y^{'} + 2y = e^{-x}\sin x...(1).$$

Equation without second member : $y^{''} + 2y^{'} + 2y = 0$

$$\begin{array}{lcl} (EC) & : & r^2 + 2r + 2 = 0 \\ & \Rightarrow & r = -1 \pm i = \alpha \pm i\beta \quad \text{with} \quad \begin{cases} \alpha = -1 \\ \beta = 1 \end{cases}. \end{array}$$

Let us note y_h the solution of the homogeneous equation associated with (1).

Then :

$$y_h(x) = e^{-x}\left[C_1\cos x + C_2\sin x\right]; (C_1, C_2 \in \mathbb{R}).$$

Equation with second member One poses

$$\begin{aligned} f(x) &= e^{-x}\sin x \\ &= e^{\lambda x}[P_1(x)\cos(\theta x) + P_2(x)\sin(\theta x)] \\ &\quad \text{With} \begin{cases} \lambda = -1 \\ \theta = 1 \\ P_1(x) = 0 \Rightarrow \deg(P_1(x)) = 0 \\ P_2(x) = 1 \Rightarrow \deg(P_2(x)) = 0 \end{cases} \end{aligned}$$

Let us note y_p the solution of the homogeneous equation associated with (1).

Hence y_p is written as :

$$y_p(x) = e^{\lambda x}x^m\left[Q_1(x)\cos(\theta x) + Q_2(x)\sin(\theta x)\right].$$

*) Since $\lambda + i\theta = -1 + i$ is a solution of the (CE), then $m = 1$.

*) Since $\max\left(\deg(P_1(x)), \deg(P_2(x))\right) = 0$ then $\deg(Q_1(x)) = \deg(Q_2(x)) = 0$.

Which means that $y_p(x) = e^{-x}x\left[a\cos x + b\sin x\right]$ where a and b are constants to be determined.

One has :

$$y_p^{'}(x) = e^{-x}\left[(a + (b-a)\,x)\cos x + (b - (a+b)\,x)\sin x\right].$$

And :

$$y_p^{''}(x) = e^{-x}\left[(2b - 2a - 2bx)\cos x - (2a + 2b - 2ax)\sin x\right].$$

Substituting into equation (2.), we find:

$$2b\cos x - 2a\sin x = \sin x$$

By identification, we get: $\begin{cases} a = -\frac{1}{2} \\ b = 0 \end{cases}$.

Which means that :

$$y_p(x) = -\frac{1}{2}e^{-x}x\cos x.$$

Finally,

$$y(x) = y_h(x) + y_p(x) = e^{-x}\left[\left(C_1 - \frac{1}{2}x\right)\cos x + C_2\sin x\right];\ (C_1, C_2 \in \mathbb{R}).$$

List of Figures