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Analysis of some nonlinear elliptic and parabolic problems involving fractional operator

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This is dedicated to whom ALLAH said: " *My Lord have mercy on them as they brought me up when i was young*",

to my beloved mother and to my father, may ALLAH have mercy on him.

They were my support and strength throughout every stage of my life.

To my sister Narimane, my brother Amine and my prince Rayane.

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Introduction

The logistic equation introduced by Pierre-François Verhulst in 1838, models population growth with carrying capacity (see [18, 58, 106]). Building on Thomas Robert Malthus's qualitative insights in [76] (1798) and Adolphe Quetelet's statistical contributions in [92](1835). Verhulst formalized this idea mathematically as

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{1}$$

where $P(t)$ represents the population at time t, r is the intrinsic growth rate and K is the carrying capacity. (for the reader's convenience see [106]).

Many researchers have extended Verhulst's work by including spatial effects through reaction-diffusion equations of the form

$$\frac{dP}{dt} = \Delta P + rP(1 - \frac{P}{K}) \tag{2}$$

where $P(x,t)$ represents the population density at location x and time t, and the Laplacian is introduced to model spatial distribution of population with local diffusion effects (see [108]). This type is known as "The classical reaction-diffusion equation with logistic growth", and is widely studied in mathematical biology because of its importance in modelling biological phenomena. For instance, Gurtin et al. [58] explored Verhulst's applications and described the evolution and the spatial diffusion of biological populations over time. Also, in [7] Afrouzi et al. studied the birth rate and diffusion extent interact to determine the existence or nonexistence of nontrivial steady-state distributions of population in a reaction-diffusion equation with logistic source. In [39], Cantrell et al. used this equation to model the dynamics of a population in a heterogeneous environment. Also, logistic evolutionary systems model the biological phenomenon of chemotaxis [104]. Also in ecology [80], physics [81] and other fields to study dynamical systems, where both growth and spatial spread occur.

The extension of classical logistic equation to the nonlocal setting, by involving the fractional Laplacian which corresponds to processes like Lévy flights [15], allows the model to account for long-range interaction and anomalous diffusion.

The equation under consideration is

$$\frac{dP}{dt} = (-\Delta)^s P + rP(1 - \frac{P}{K}) \tag{3}$$

where $(-\Delta)^s$ represents the fractional Laplacian, defined for any $u \in \mathcal{S}(\mathbb{R}^N)$

$$(-\Delta)^s u(x) := c_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,\, s \in (0,1) \tag{4}$$

with

$$c_{N,s} = \frac{s4^s\Gamma(s+\frac{N}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{5}$$

being a normalizing constant and $P.V.$ is the principal value of Cauchy. This operator generalizes the classical one to the fractional order $0 < s < 1$, and it is of significant interest and has a close connection to mathematical biology. An introduction to this operator and more details can be found in [44, 82].

To motivate the mathematical, biological and physical aspects, we introduce fractional operators into a logistic problems to lead to a nonlocal scalar field problem of the prototype form

$$\begin{cases} u_t + \mathcal{L}_s u &= f_\lambda(x,u) & \text{in} \quad \Omega \times (0,T), \\ u(x,0) &= u_0(x) > 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases} \tag{6}$$

where $\mathcal{L}_s$ represents a nonlocal operator and $f_\lambda(u) = \lambda u^{p-1} - u^{m-1}$ is of a logistic type.

Such a class of problems has been addressed by several authors, we mention in the case of fractional laplacian [56], and for fractional $p-$Laplacian see [57, 85]. Regarding its steady-state version, that was described by the following elliptic problem

$$\begin{cases} \mathcal{L}_s u &= f_\lambda(x,u) & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{7}$$

It has been the focus of numerous authors, among these works, we highlight in the case of fractional Laplacian [4, 26, 38], as well the problem involving p-Laplacian was addressed in [61, 64, 67].

This thesis is devoted to the study of a class of nonlocal scalar field problems of logistic type in the form of (6) and (7), involving a linear or nonlinear nonlocal operators.

The logistic nonlinearity in the second term exhibits an increasing power. In particular, this thesis focus on the asymptotic behavior of solutions as the exponent m diverges. The behavior of the solutions will be thoroughly investigated and characterized through a class of free boundary problems.

Structure of the Thesis

This thesis is organized as follows:

We begin by presenting some definitions and techniques used throughout this work, which is covered in the first chapter.

Chapter 2: Notations and Preliminaries

The objective is to recall the fundamental tools and theoretical framework necessary for the study of the three fractional problems.

The chapter is organized as follows:

- We begin with the definition of functional spaces including: Lebesgue spaces, fractional Sobolev spaces , Hölderian spaces and the fractional parabolic spaces.
- We introduce the key results to demonstrate existence such that: minimization method, Mountain Pass Theorem and Perron's method, presenting the essential background about the eigenvalue problems.
- We recall the fundamental concepts and some integration theory tools.

The thesis is then divided into two main parts:

The first part: Elliptic Problems

The first part is dedicated to the study of two types of nonlocal elliptic problems in bounded domains. Specifically, we are interested in the following elliptic problems involving a nonlocal operators and a logistic type nonlinearity:

- **First Problem**: the first problem is governed by the linear operator, the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, where the logistic reaction term is given by $f_\lambda(x,u) = \lambda u^{p-1} - u^{m-1}$ where that $2 < p < m$. In this context, we investigate the existence, nonexistence and multiplicity of solutions according the value of λ, p and m, emphasizing how the parameter p influence the study. Specifically, we categorize the case taken by the exponent p into subcritical, critical and supercritical.
- **Second Problem**: it is a fractional elliptic problem involving the nonlinear operator, the fractional p-Laplacian $(-\Delta)^s_p$ where $s \in (0,1)$ and $p > 1$. We focus on a logistic type nonlinearity

$f_\lambda(x,u)=\lambda u^{q-1}-u^{m-1}$ where $\lambda>0$, examining the key types of logistic problem, through the lens of diffusion regime: super-diffusive, equidiffusive and sub-diffusive, captured by: convex, linear and concave cases respectively, treating each case separately.

In both problems, we perform an asymptotic analysis to the solutions obtained in each case, as the exponent m diverges, determining their limit profiles as a solution of a free boundary problems will be specified later.

This part is organized into two chapters:

Chapter 3: A nonlocal scalar field problem: Existence, multiplicity and asymptotic behavior [22]

The main goal is to study the nonlocal scalar field problem

$$\begin{cases} (-\Delta)^s u &= \lambda u^{p-1}-u^{m-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{in} \quad \mathbb{R}^N\backslash\Omega, \end{cases} \tag{8}$$

under the condition $2<p<m$ and λ is a positive parameter. We address the questions of existence, non-existence, and multiplicity of solutions, for the largest possible range of the parameters λ, p, m.

Our approach is variational based on critical points theory. We define the energy functional

$$\mathcal{J}_{\lambda,m}(u)=\frac{1}{2}\iint_{D_\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy+\frac{1}{m}\int_\Omega |u|^m\,dx-\frac{\lambda}{p}\int_\Omega |u|^p\,dx.$$

We begin our study by proving a crucial regularity result, we provide that every solution of Problem (8) is uniformaly bounded. This result based on Stampacchia method.

Before stating the main results, we examine the geometry of the associated energy functional, noting that its behavior varies according to the increase of λ.

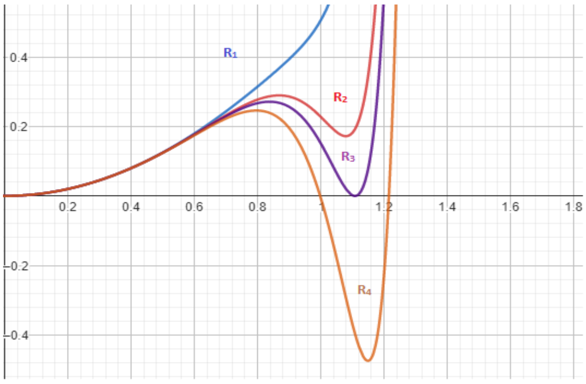

Figure 1: Existence and multiplicity of solutions according to the growth in value of λ

This analysis allows us to identify a critical thresholds for the parameter λ, which influence the existence, nonexistence and the energy behaviors. Namely, we define the threshold of existence

$$\Lambda(m) := \inf \Big\{ \lambda \in \mathbb{R}_+ : \text{ such that Problem (8) has a nontrivial solution} \Big\}.$$

such that

① If $0 < \lambda < \Lambda(m)$, Problem (8) does not admit a nontrivial solution.(see Region R_1, Figure (1))

② If $\lambda > \Lambda(m)$, Problem (8) has a different structure of solution set. Namely, (see Regions R_2, R_3, R_4, Figure (1)).

- ▶ We establish the existence of maximal solution denoted $w_{\lambda,m}$ by applying sub-supersolution method. This solution is characterized by the fact that it dominates all other solutions of Problem (8). Moreover, it is increasing to the respect to λ.
- ▶ We observe that, for sufficiently large values of λ, there exists a global minimum since the energy is negative. However, when λ close to $\Lambda(m)$, the energy level can be positive and the global minimum of $\mathcal{J}_{\lambda,m}$ becomes the trivial solution. To further analyse this, we determine a second threshold, exceeding $\Lambda(m)$, which plays a crucial role in investigating the different behaviors of $\mathcal{J}_{\lambda,m}$

$$\underline{\lambda}(m) = \inf\{\lambda : \mathcal{J}_{\lambda,m} \text{ has a local minimum } u_{\lambda,m} \text{ such that } \mathcal{J}_{\lambda,m}(u_{\lambda,m}) < 0\}.$$

In particular, from Figure (1) we observe that the energy exhibits a different levels according the increase of λ, it can be positive, zero or negative. Precisely,

- ✔ If $\Lambda(m) < \lambda < \underline{\lambda}(m)$, then $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) > 0$.(see Region R_2).
- ✔ If $\lambda = \underline{\lambda}(m)$, then $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) = 0$.(see Region R_3).
- ✔ If $\lambda > \underline{\lambda}(m)$, then $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) < 0$.(see Region R_4).

So it is not obvious that a critical point that is a local minimum. In order to overcome this difficulty, we use Perron's method to show that $\mathcal{J}_{\lambda,m}$ admits a minimizer in a closed convex set. Inspired by a variant of Alama method [4], we establish that this solution remains a local minimizer in $H_0^s(\Omega)$.

Now, Thanks to the L^∞ estimate, we are ready to study the asymptotic behavior of the sequence solutions as m tends to ∞. We begin by determining the limit of $\underline{\lambda}(m)$. We show that

$$\lim_{m\to\infty} \underline{\lambda}(m) = \lambda_*. \tag{9}$$

Then, we establish the existence of a second solution denoted $v_{\lambda,m}$, when the energy functional has the geometry of Mountain Pass Theorem for

$$2 < p \leqslant 2_s^\star < m.$$

Notice that for $\lambda > \lambda_* \geqslant \underline{\lambda}(m)$, Problem (8) admits a global minimum and a mountain pass solution, preciseley, we are in the fourth rigion R_4, see Figure (1). Hence, we study the asymptotic behavior of the minimum sequence of solutions and the sequence of mountain pass solutions, determining a limiting profile as a solution of free boundary problem of the form

$$\begin{cases} (-\Delta)^s w + \mathbf{F}_w(x)\chi_{\{w=1\}} & = & \lambda w^{p-1} & \text{in} & \Omega, \\ w & \geqslant & 0 & \text{in} & \Omega, \\ w & = & 0 & \text{in} & \mathbb{R}^N\setminus\Omega, \end{cases} \tag{10}$$

where $0 < \mathbf{F}_w(x) \leqslant \lambda$, $\mathbf{F}_w(1-w) = 0$ a.e. Ω. $\mathbf{F}_w(x)$ determines the asymptotic behavior of the nonlinear terms as $m \to \infty$ for both solutions $u_{\lambda,m}$ and $v_{\lambda,m}$. We reserve a more detailed discussion about this function for later chapters.

Chapter 4: A nonlocal p-Logistic problem: Existence, multiplicity and asymptotic behavior [16]

This chapter extends and improves upon the previous one [Chapter 2, Theorem 2.4.1] to a nonlinear framework, addressing the questions existence, nonexistence and multiplicity of positive solutions of the following problem

$$\begin{cases} (-\Delta)_p^s u &= \lambda u^{q-1} - u^{m-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{11}$$

In the context of nonlocal p-logistic problem, we classify the problem into three cases based on the value of q with respect to $p > 1$, where we treat separately:

① The superdiffusive case: $1 < p < q < m$.

② The equidiffusive case: $p = q < m$.

③ The subdiffusive case: $q < p < m$.

In our analysis we use variational and sub supersolution methods combined with suitable truncation techniques. Let us define the associated energy functional to Problem (11)

$$\mathcal{I}_{\lambda,m}(u) = \frac{1}{p} \iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy + \frac{1}{m}\int_\Omega |u|^m\, dx - \frac{\lambda}{q}\int_\Omega |u|^q\, dx.$$

➥ The superdiffusive case: $1 < p < q < m$

In the first part of this chapter, we address the superdiffussive case and show that Problem (11). Precisely, we aim to generalize the results obtained in Chapter 2 to the fractional p-Laplacian framework.

Firstly, we identify a critical threshold $\Lambda(m)$, beyond which we will discuss the existence results.

➢ **Existence**

The main existence result is summarized as follow:

① We prove that if $0 < \lambda < \Lambda(m)$, Problem (11) has only the trivial solution.

② To assure the existence of nontrivial solutions, we take $\lambda > \Lambda(m)$ such that

- ► Problem (11) has a maximal solution denoted $z_{\lambda,m} \in X^s$, using sub-supersolution method.

- ▶ we prove the existence of $\bar{\lambda}(m) > 0$ a new threshold such that $\lambda > \bar{\lambda}(m) > \Lambda(m)$, which will play a crucial role to investigate the existence of at least two positive solutions in three different scenarios:
  - ➙ a local minimum $u_{\lambda,m}$ in X^s such that
    - ✔ If $\Lambda(m) < \lambda < \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) > 0$.
    - ✔ If $\lambda = \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) = 0$.
    - ✔ If $\lambda > \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) < 0$.
  - ➙ a mountain pass solution $v_{\lambda,m}$ in X^s in every minimax level.

  Inspired by Iannizotto et. al [61, 66], we first prove that Problem (11) has a local minimizer in $C_0^s(\Omega)$ and then demonstrating that every local minimizer in $C_0^s(\Omega)$ is also a local minimizer in the X^s-topology too. In addition, from this local minimizer we obtain the multiplicity result by using Mountain Pass Theorem.

➢ **Asymptotic analysis**

Now, we perform an asymptotic analysis to the sequences of solutions obtained as m tends to ∞, proving the existence of $\lambda^{\star\star} > 0$ which determin the asymptotic behavior of $\bar{\lambda}(m)$ such that for $\lambda > \lambda^{\star\star} > \Lambda(m)$, the sequence $(u_{\lambda,m})_m$ of minimum solutions and the sequence of mountain pass solution $(v_{\lambda,m})_m$ converges strongly in $X^s(\Omega)$ to u and v, the solutions of particular free boundary problems will be precise later.

➥ The equidiffusive case: $p = q < m$

In this special case, we show that the existence threshold is the first eigenvalue associated to the fractional Dirichlet p-Laplacian problem denoted $\lambda_{s,p}$.

➢ **Existence and uniqueness**

Existence of the solution $w_{\lambda,m}$ will be assured when $\lambda > \lambda_{s,p}$, using a direct minimization method, and in this case we can achieve the uniqueness applying comparison principle and Picone's inequality.

➢ **Asymptotic analysis**

We stydy the asymptotic behavior of the solution $w_{\lambda,m}$, determining its behavior as a solution of free boundary problem.

➥ The subdiffusive case: $q < p < m$

This case coressponds to subdiffussive case, it is similar to the equidiffussive case.

- **Existence and uniqueness**

  We address the same question as the linear case, and we prove existence a unique solution $w_{\lambda,m}$ for all $\lambda > 0$.

- **Asymptotic analysis**

  The asymptotic behavior is also determined by a profile limite that satisfies a free boundary problem.

The second part: Parabolic Problems

In this part, we explore the evolutionary version of the elliptic problem addressed in the first chapter of Part 1.

Chapter 4: A class of fractional parabolic logistic problem[17]

The final chapter is dedicated to the study of the parabolic version of (8), answering the general questions about the global existence, uniqueness of weak positive solution and asymptotic behavior.

- **Existence and uniqueness**

  We prove the existence of global weak solution u to the evolutionary version of Problem (8) with $u_0 \in L^1(\Omega)$, using approximation method and comparison principle for uniqueness.

- **Asymptotic analysis**

  We investigate the asymptotic behavior of the solution

  - When $m \to \infty$, we obtain the limiting problem, which determines the asymptotic behavior of the solution.
  - When $t \to \infty$, we analyze the large time behavior of the solution to the parabolic free boundary problem obtained as $m \to \infty$.

1

Notations and Preliminaries

This preliminary chapter aims to provide the necessary background for the rest of the thesis. We start by defining the functional framework, introducing Sobolev spaces which play a crucial role in the analysis of nonlocal problems, as well the Hölder spaces which are necessary to obtain the fine regularity. Then, we focus on the class of nonlinear nonlocal elliptic problems, presenting some basic methods for proving the existence of minimizers and critical points. After that, we recall some necessary tools.

1.1 Introduction

Nonlocal elliptic problems have widely interest in pure and applied mathematics, due to the large extent of their applications to model complex phenomena in various fields such as physics, geometry, and finance. This class of elliptic problem can involve the linear operator the fractional Laplacian $(-\Delta)^s$ or its nonlinear generalization the fractional p-Laplacian $(-\Delta)_p^s$. Their presence introduces a significant theoretical and practical challenges, for instance, the nonlocal nature of those operators makes the standard techniques no directly applicable and force us to seek generalisation in a non-local context. Similarly, the fractional p-Laplacian, combining non-linearity with non-locality, which enhanced the difficulties.

In the study of such problems, it is essential to work in fractional Sobolev spaces to define the necessary functional framework which extend the classical Sobolev spaces that allow the inclusion of fractional derivatives.

The variational structure of nonlocal elliptic problem plays a crucial role to analyse the principal questions of existence and multiplicity of positive solutions, by using minimization method and mountain pass theorem, suitably adapted to the nonlocal framework.

The lack of appropriate tools compared to the local case necessitates the use of some tools from fractional analysis in both linear and nonlinear scenarios.

We will survey the functional framework from fractional spaces to the standard lemmas used in analysis. In addition, we explore some key definitions, emphasizing nonlocal operators and variational methods, and provide the tools needed in the nonlocal case to address any difficulties.

1.2 Functional spaces

Let Ω be an open and bounded domain of $\mathbb{R}^N$.

1.2.1 Classical Lebesgue space

Definition 1.2.1. *Let $1 \leqslant p < \infty$, we define the Lebesgue space*

$$L^p(\Omega) \overset{def}{=} \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } : |u|_p := \Big(\int\limits_{\Omega} |u|^p dx \Big)^{\frac{1}{p}} < +\infty \Big\},$$

and

$$L^\infty(\Omega) \overset{def}{=} \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } : |u|_\infty := ess \sup_{x \in \Omega} |u(x)| < +\infty \Big\},$$

where

$$ess \sup_{x \in \Omega} |u(x)| \overset{def}{=} \inf\{M > 0 : |u(x)| \leqslant M \text{ a.e. in } \Omega\}.$$

The classical Lebesgue space is a Banach space for $1 \leqslant p \leqslant \infty$ and separable for $1 \leqslant p < \infty$. Moreover, $L^p(\Omega)$ is also reflexive for $1 < p < \infty$.

We begin with the folloing useful inequality.

Theorem 1.2.1 (Hölder's inequality). *Let $1 \leqslant p \leqslant \infty$ and $p^{'} \geqslant 1$ be the conjugate exponent of p (i.e. $\frac{1}{p} + \frac{1}{p^{'}} = 1$). If $u \in L^p(\Omega)$ and $v \in L^{p^{'}}(\Omega)$, then $u.v \in L^1(\Omega)$ and*

$$|uv|_1 \leqslant |u|_p |v|_{p'}$$

In the case of bounded domain, the following proposition follows from Hölder's inequality.

Proposition 1.2.1. *If Ω is a bounded domain and $1 \leqslant p_1 \leqslant p_2 \leqslant +\infty$, then*

$$L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega) \text{ (continous embedding)}.$$

In particular

$$L^\infty(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow ... \hookrightarrow L^1(\Omega), \text{ for all } 1 \leqslant p \leqslant \infty.$$

Theorem 1.2.2 (Interpolation inequality). *Let $1 \leqslant p \leqslant q \leqslant +\infty$. If $u \in L^p(\Omega) \cap L^q(\Omega)$, then $u \in L^r(\Omega)$, for $r \in [p, q]$. Moreover*

$$|u|_r \leqslant |u|_p^\theta |v|_q^{1-\theta}$$

where $\frac{\theta}{p} + \frac{1-\theta}{q} = \frac{1}{r}$, $\theta \in (0, 1)$.

1.2.2 Hölderian spaces

Let Ω be an open, bounded domain of $\mathbb{R}^N$, $\alpha \in (0, 1)$.

- $\mathcal{C}^0(\bar{\Omega})$ **:** the space of uniformly continuous function $u : \bar{\Omega} \to \mathbb{R}$, endowed by the norm

$$|u|_{\mathcal{C}^0(\bar{\Omega})} = \sup_{x \in \bar{\Omega}} |u(x)|.$$

- $\mathcal{C}^{0,\alpha}(\bar{\Omega})$ **:** the space of fucntions u such that

  1. u in $L^\infty(\bar{\Omega})$.
  2. There exists $C_\alpha > 0$ such that for all $x, y \in \bar{\Omega}$;

$$|u(x) - u(y)| \leqslant \mathcal{C}_\alpha |x - y|^\alpha.$$

Endowed by the following norm

$$[u]_{0,\alpha} = \|u\|_{\mathcal{C}^0(\bar{\Omega})} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

$\mathcal{C}^{0,\alpha}(\bar{\Omega})$ is Banach space and if $\alpha \leqslant \alpha'$, then the embedding $\mathcal{C}^{0,\alpha'}(\bar{\Omega}) \subset \mathcal{C}^{0,\alpha}(\bar{\Omega})$ is continuous. Moreover, if $\alpha < \alpha'$, the embedding $\mathcal{C}^{0,\alpha'}(\bar{\Omega}) \subset \mathcal{C}^{0,\alpha}(\bar{\Omega})$ is compact.

We are now in position to define the weighted Hölder space we will work with in this thesis.

For all $x \in \bar{\Omega}$, we set $\delta^s(x) := \text{dist}(x, \partial\Omega)$, and we define

$$\mathcal{C}_0^s(\bar{\Omega}) \overset{\text{def}}{=} \left\{ u \in \mathcal{C}^0(\bar{\Omega}) : \frac{u}{\delta^s} \text{ admits a continuous extension to } \bar{\Omega} \right\}$$

is a Banach space endowed with the norm $\|u\|_{0,s} = \sup_{x \in \Omega} \left| \dfrac{u}{\delta^s} \right|$.

And for all $\alpha \in (0, 1)$,

$$\mathcal{C}_\delta^{0,\alpha}(\bar{\Omega}) \overset{\text{def}}{=} \left\{ u \in \mathcal{C}^0(\bar{\Omega}) : \frac{u}{\delta^s} \text{ has an } \alpha\text{-Hölder continuous extension to } \Omega \right\},$$

equipped by the norm

$$\|u\|_{0,\alpha} = \|u\|_{\mathcal{C}^0(\Omega)} + \sup_{x \neq y} \frac{\left| \frac{u(x)}{\delta^s(x)} - \frac{u(y)}{\delta^s(y)} \right|}{|x - y|^\alpha}.$$

The embedding $\mathcal{C}_\delta^{0,\alpha}(\bar{\Omega}) \hookrightarrow \mathcal{C}_0^s(\bar{\Omega})$ is compact for all $\alpha \in (0, 1)$. Moreover, the positive cone $\mathcal{C}_0^s(\bar{\Omega})^+$ of $\mathcal{C}_s^0(\bar{\Omega})$ has a nonempty interior given by

$$\mathrm{int}(\mathcal{C}_0^s(\bar{\Omega})^+) \overset{\text{def}}{=} \left\{ u \in \mathcal{C}_\delta^{0,\alpha}(\bar{\Omega}) : \inf_{\bar{\Omega}} \frac{u}{\delta^s} > 0 \right\}.$$

For more details see [66, 64].

1.2.3 Fractional Sobolev space

Let $s \in (0,1)$ and $1 \leqslant p \leqslant \infty$.

We define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) \overset{\text{def}}{=} \left\{ u \in L^p(\mathbb{R}^N) \;\; with \;\; \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\} \tag{1.1}$$

endowed with the norm

$$||u||_{W^{s,p}(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} |u|^p dx + \int\int_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx\, dy \Big)^{\frac{1}{p}} \tag{1.2}$$

For more details we refer the convenient reader to Hitchiker's guide [45, 82]. We will work in $W_0^{s,p}(\Omega)$, which is defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{s,p}(\mathbb{R}^N)}$.

$$W_0^{s,p}(\Omega) \overset{\text{def}}{=} \left\{ u \in W^{s,p}(\mathbb{R}^N) \text{ with } u = 0 \; a.e. \; in \; \mathbb{R}^N \setminus \Omega \right\}$$

endowed with the Gagliardo semi-norm

$$\|u\|_{s,p} = \Big(\int\int_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy \Big)^{\frac{1}{p}}.$$

where $D_\Omega = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. From now on, we will only denote $d\nu = \dfrac{dxdy}{|x-y|^{N+ps}}$.

For $p=2$, we denote $W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ the classical fractional Sobolev space,

$$H^s(\mathbb{R}^N) \overset{\text{def}}{=} \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}, \tag{1.3}$$

endowed with the norm

$$\|u\|_s^2 := \int_{\mathbb{R}^N} |u|^2\, dx + \frac{c_{N,s}}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^2 d\mu.$$

where $d\mu = \dfrac{dxdy}{|x-y|^{N+2s}}$.

For a given bounded domain $\Omega \subset \mathbb{R}^N$, let

$$\mathcal{H}_0^s(\Omega) \overset{\text{def}}{=} \left\{ u \in H^s(\mathbb{R}^N) : \; u = 0 \text{ in } \mathbb{R}^N \backslash \Omega \right\}.$$

Notice that $\mathcal{H}_0^s(\Omega)$ is a Hilbert space with the norm $\| \cdot \|_s$. Moreover, for all $u,\, v \in \mathcal{H}_0^s(\Omega)$,

$$\int_\Omega (-\Delta)^s uv \, dx = \frac{c_{N,s}}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(v(x) - v(y)) \, d\mu. \tag{1.4}$$

See for instance [45], [35] for more properties of fractional Sobolev spaces.

Now, we recall the embedding of the fractional Sobolev spaces into Lebesgue space $L^{q_s^*}$. .

Theorem 1.2.3 (Theorem 6.5, [45])**.** *Let* $s \in (0,1)$, $p \in [2, +\infty[$ *be such that* $N > ps$. *Then, there exists* $S = S(N,p,s) > 0$ *such that*

$$\|u\|^p_{L^{p_s^*}(\Omega)} \leqslant S \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx \, dy. \tag{1.5}$$

where $p_s^* = \dfrac{Np}{N - ps}$. *Moreover, we have*

$$W^{s,p}(\Omega) \hookrightarrow L^r(\Omega) \; continuously \; for \; r \in [p, p_s^*].$$

and

$$W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \; compactly \; for \; r \in [1, p_s^*[.$$

1.2.4 Parabolic Lebesgue space

Let $T \geqslant 0$ and $1 \leqslant p, q < \infty$. The parabolic Lebesgue space $L^p(0,T; L^q(\Omega))$ can be defined as

$$L^p(0,T;L^q(\Omega)) \overset{\text{def}}{=} \left\{ u : (0,T) \times \Omega \to \mathbb{R} \; : \; \|u\|_{L^p(0,T;L^q(\Omega))} := \left(\int_0^T \left(\int_\Omega |u(t,x)|^q \, dx \right)^{p/q} dt \right)^{1/p} < +\infty \right\}.$$

In the particular case $q = p$, this definition reduces to $L^p(\Omega_T)$. Moreover, if $p = q = \infty$, $L^\infty(\Omega_T)$ space is defined by

$$L^\infty(\Omega_T) \overset{\text{def}}{=} \left\{ u : (0,T) \times \Omega \to \mathbb{R} \; : |u|_{\infty,\Omega_T} := ess \sup_{t \in (0,T)} \; ess \sup_{x \in \Omega} |u(t,x)| < \infty \right\}$$

1.2.5 Fractional parabolic spaces

We now introduce the fractional parabolic framework, which corresponds to the parabolic problems we consider in this paper.

- The fractional parabolic Sobolev space $L^p(0,T;W_0^{s,p}(\Omega))$

$$L^p(0,T;W_0^{s,p}(\Omega)) \overset{\text{def}}{=} \Big\{\phi \in L^p(\Omega_T) : \|\phi\|_{L^p(0,T;W_0^{s,p}(\Omega))} := \Big(\int_0^T \|\phi(.,t)\|_{s,p}^p dt\Big)^{\frac{1}{p}} < +\infty\Big\},$$

It is clear that $L^p(0,T;W_0^{s,p}(\Omega))$ is a Banach space for $p \geqslant 1$, and since $W_0^{s,p}(\Omega)$ is separable for $1 \leqslant p < \infty$ and reflexive for $1 < p < \infty$, then $L^p(0,T;W_0^{s,p}(\Omega))$ is separable and reflexive under the same conditions on p. Moreover, its dual space is identified by $L^{p'}(0,T;W_0^{-s,p'}(\Omega))$.

For the particular case $p = 2$, this definition reduces precisely to $L^2(0,T;\mathcal{H}_0^s(\Omega))$ such that

$$L^2(0,T;H_0^s(\Omega)) \overset{\text{def}}{=} \Big\{\phi \in L^2(\Omega_T) : \|\phi\|_{L^2(0,T;H_0^s(\Omega))} := \Big(\int_0^T \|\phi(.,t)\|_s^2 dt\Big)^{\frac{1}{2}} < +\infty\Big\},$$

Its dual space is $L^2(0,T;H^{-s}(\Omega))$.

1.3 Non-linear fractional elliptic problem

1.3.1 Non-local operators

Recent studies have increasly focused on nonlocal operators, highlighting their crucial role in various fields. Especially, in both pure mathematics and practical applications. These operators provide a powerful framework for analyzing systems with long-range interactions, modeling phenomena where local interactions are insufficient. But what is a nonlocal operator?

Definition 1.3.1 (Non-local operator). *A non-local operator is an operator where the value at a point depends not only on the behavior of the function on local neighborhood, but also on its values over the whole* $\mathbb{R}^N$*. For instance, there is two classes of nonlocal operators:*

- ***Integrodifferential operators:*** *which involves integrating function (Kernel function) over a domain.*
- ***Fractional differential operators:*** *which generalize classical derivatives to fractional orders.*

In this thesis, we are primarily concerned with two types of fractional differential operators: fractional Laplacian and fractional p-Laplacian.

Fractional p-Laplacian

Let $s \in (0,1)$ and $1 < p < \infty$, then we define the fractional p-Laplacian, for all $u : \mathbb{R}^N \to I\!R$ smooth enough and for all $u \in \mathcal{C}_0^\infty(\Omega)$

$$\begin{aligned}(-\Delta)_p^s u(x) &:= 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy \\ &= 2\, P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \text{ for all } x \in \mathbb{R}^N.\end{aligned}$$

Where

- $B_\varepsilon(x)$ is the ball centred at $x \in \mathbb{R}^N$ and of radius ε.
- $P.V.$ denotes the Cauchy principal value.

Properties of the fractional p-Laplacian

- **Nonlocality:** The operator $(-\Delta)_p^s$ is non-local, in the sens that, the value of $(-\Delta)_p^s u(x)$ for all $x \in \Omega$ depends not only on the value of x on the set Ω but in fact on the whole $\mathbb{R}^N$.
- **Nonlinearity:** The operator $(-\Delta)_p^s$ is non-linear, for $p \neq 2$.
- **Extension:**
  - For $p = 2$, the operator $(-\Delta)_p^s$ generalizes the fractional Laplacian by introducing a nonlinearity controlled by the exponent p.
  - In the local case $s = 1$, the operator $(-\Delta)_p^s$ generalizes the classical p-Laplacian

    $$\lim_{s\to 1}(-\Delta)_p^s = (-\Delta)_p,$$

    where

    $$(-\Delta)_p u := div(|\nabla u|^{p-2}\nabla u).$$

    into the fractional framework.
- **Continuity:** The operator

  $$(-\Delta)_p^s : W_0^{s,p}(\Omega) \to (W_0^{s,p}(\Omega))'$$

is continuous. Moreover, for all $u, \varphi \in W_0^{s,p}(\Omega)$, we have

$$\langle (-\Delta)_p^s u, \varphi \rangle = \iint\limits_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\, dx\, dy,$$

satisfies

$$\langle (-\Delta)_p^s u, \varphi \rangle \leqslant \|u\|_{s,p}^{p-1} \|\varphi\|_{s,p}, \tag{1.6}$$

in particular, if $\varphi = u$ we get

$$\langle (-\Delta)_p^s u, u \rangle = \|u\|_{s,p}^{p}. \tag{1.7}$$

- **Compactness:** Since $W_0^{s,p}(\Omega)$ is uniformly convex (see [52],[53],[87]), then $(-\Delta)_p^s$ satisfies the following compactness property.

  Property 1. *If $(u_n)_n$ is a sequence in $W_0^{s,p}(\Omega)$, such that*

  - $u_n \rightharpoonup u$ *weakly in* $W_0^{s,p}(\Omega)$.
  - $\langle (-\Delta)_p^s u_n, u_n - u \rangle \to 0$.

  Then,

  $$u_n \to u \text{ strongly in } W_0^{s,p}(\Omega).$$

Fractional Laplacian

Let $s \in (0,1)$ and $u \in \mathcal{S}(\mathbb{R}^N)$. Then, the fractional Laplacian is defined as

$$(-\Delta)^s u(x) := c_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \tag{1.8}$$

with the normalization constant given as

$$c_{N,s} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|} \tag{1.9}$$

with $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}\, \mathrm{e}^{-t} dt$. For the reader's convenience, we refer to [33, 44] for a detailed explanation of this normalization constant. Here, $P.V.$ is interpreted as in the definition of the fractional p-Laplacian.

1.3.2 An eigenvalue problems

In this subsection, we investigate eigenvalue problems associated with the fractional p-Laplacian and the fractional Laplacian. Our focus is on the spectral properties of these operators, particularly the characterization of their first eigenvalue and corresponding eigenfunctions.

Definition 1.3.2. *Let $s \in (0,1)$, $\lambda > 0$ and $p > 1$. If $\varphi \in W_0^{s,p}(\Omega)$ is a nontrivial weak solution of the nonlinear problem*

$$\begin{cases} (-\Delta)_p^s \varphi & = & \lambda|\varphi|^{p-2}\varphi & \text{in} & \Omega, \\ \varphi & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.10}$$

Then,

- *λ is an eigenvalue of $(-\Delta)_p^s$.*
- *φ is a λ-eigenfunction.*
- *$\sigma(s,p)$ is the spectrum of $(-\Delta)_p^s$.*

It is well known that the complete structure of the spectrum of $(-\Delta)_p^s$ is less explicit to be described. However, severeral proprties have been established, see for instance [51, 68, 75]. In the following proposition, we recall the relevant results needed in this study about the first eigenvalue of the fractional p-Laplacian given by

$$\lambda_{s,p} = \inf_{\varphi_{s,p} \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^N \times \mathbb{R}^N} |\varphi_{s,p}(x) - \varphi_{s,p}(y)|^p d\nu}{\displaystyle\int_{\mathbb{R}^N} |\phi_{s,p}|^p dx} \tag{1.11}$$

and its associated eigenfunction $\varphi_{s,p}$

Proposition 1.3.1. *The eigenvalue and the eigenfunction of $(-\Delta)_p^s$ satisfies the following proprties:*

① *$\lambda_{s,p} > 0$ and it is an isolated and simple.*

② *$\phi_{s,p}$ is a unique positive eigenfunction.*

③ *$\phi_{s,p}$ belongs to $L^\infty(\Omega)$.*

To ensure clarity of notations, we dedicate separately the special linear case as $p = 2$ corresponding to the fractional Laplacian, since it will be used in the forthcoming chapters.

The linear fractional eigenvalue problem with Dirichlet conditions

$$\begin{cases} (-\Delta)^s\varphi = \lambda\varphi & \text{in} \quad \Omega, \\ \varphi = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.12}$$

Recall that if Problem (1.12) admits a nontrivial weak solution $\varphi \in H_0^s(\Omega)$, then λ is an eigenvalue and φ is λ-eigenfunction. In the following proposition, we integrate some basic properties of the eigenvalue and the eigenfunction of the fractional Laplacian, established in [Proposition 9, Appendix A, [98]] with a regularity result of the eigenfunction proved in [Proposition 4, [97]].

Proposition 1.3.2. *Let $s \in (0,1)$, and Ω be a bounded domain in $\mathbb{R}^N$, with $N > 2s$. Then*

① *Problem (1.12) admits an eigenvalue $\lambda_{1,s}$ which is positive and characterized by*

$$\lambda_{1,s} = \inf_{\varphi_{1,s} \in H_0^s(\Omega)\setminus\{0\}} \frac{\displaystyle\iint_{D_\Omega} |\varphi_{1,s}(x) - \varphi_{1,s}(y)|^2 d\mu}{\displaystyle\int_{\mathbb{R}^N} \varphi_{1,s}^2(x)dx}. \tag{1.13}$$

② *There exists $\varphi_{1,s}$ is the associated eigenfunction such that*

$$\varphi_{1,s} > 0 \text{ and } \varphi_{1,s} \in H_0^s(\Omega).$$

③ *The eigenfunction $\varphi_{1,s}$ is in $L^\infty(\Omega)$.*

1.3.3 Resolution methods for elliptic problems

In this subsection, we present various variational methods for solving fractional elliptic problems.

Minimization method

We begin by introducing some concepts used in this thesis.

Definition 1.3.3 (Global minimum). *Let $\mathcal{J}$ be a functional defined on a Banach space $\mathbb{X}$. We say that $u^* \in \mathbb{X}$ is a global minimum of $\mathcal{J}(u)$, if*

$$\mathcal{J}(u^*) \leqslant \mathcal{J}(u), \quad \forall u \in \mathbb{X}.$$

Definition 1.3.4 (local minimum). *Let $\mathcal{J}$ be a functional defined on a Banach space $\mathbb{X}$. We say that*

$\tilde{u} \in \mathbb{X}$ is a local minimum of $\mathcal{J}(u)$, if there exists a neighborhood $\mathcal{V} \subset \mathbb{X}$ around $\tilde{u}$, such that

$$\mathcal{J}(\tilde{u}) \leqslant \mathcal{J}(u), \quad \forall u \in \mathcal{V}.$$

Definition 1.3.5 (Coercivity). *A functional $\mathcal{J}$ defined on Banach space $\mathbb{X}$ is said to be coercive, if there exist two constants $\alpha > 0$ and $\beta \in \mathbb{R}$, suc that*

$$\mathcal{J}(x) \geqslant \alpha \|x\|_{\mathbb{X}} + \beta.$$

In addition, we have

$$\lim_{\|x\|_{\mathbb{X}} \to \infty} \mathcal{J}(x) = +\infty.$$

Remark 1.3.1. *It is obvious to see that if $\mathcal{J}$ is coercive, it is bounded below and every minimising sequence is bounded.*

Definition 1.3.6 (Weak lower semi-continuity). *Let $\mathcal{J}$ be a functional defined on a Banach space $\mathbb{X}$ is said to be weakly lower semi-continuous on x, if for every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{X}$ converging weakly to $x \in \mathbb{X}$, we have*

$$\mathcal{J}(x) \leqslant \lim_{k\to\infty} \inf \mathcal{J}(x_k).$$

We are now in a position to state the main result.

Theorem 1.3.1 (Minimization method [70]). *Let $\mathbb{X}$ be a reflexive Banach space. If $\mathcal{J} : \mathbb{X} \to I\!R$ is weakly lower semi-continuous and coercive on $\mathbb{X}$. Then, there exists $u^\star \in \mathbb{X}$ such that*

$$\mathcal{J}(u^\star) = \inf_{v\in\mathbb{X}} \mathcal{J}(v).$$

Mountain Pass Theorem versions

The Mountain Pass Theorem is a powerful tool in variational methods, used to find the critical points of energy functional which is not bounded inferiorly and, indeed, to prove the existence of solutions to nonlinear elliptic problems.

Before presenting the main result of this part, it is necessary to introduce a few definitions.

Palais Smale condition

In 1970 Richard S. Palais and Stephen Smale in [91], introduced the fundamental tool on variational calculus, a compactness condition that assure the minimizing sequences of the energy functional are convergent under certain compactness conditions.

Definition 1.3.7. *Let $\mathbb{X}$ be a Banach space and $\mathcal{J}$ Gateaux differentiable functional. We say that $\mathcal{J}$ satisfies the Palais Smale condition, if for every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{X}$ such that*

$$\mathcal{J}(x_k) \text{ is bounded} \quad \text{ and } \quad \mathcal{J}^{'}(x_k) \to 0 \text{ in } \mathbb{X}^{'}$$

has a subsequence $(x_{k_\ell})_{\ell\in\mathbb{N}}$ convergent.

In 1980, a variant of Palais Smale condition at specific level was introduced by Brezis, Coron, Nirenberg (see [37]).

Definition 1.3.8 (Palais Smale condition). *Let $\mathbb{X}$ be a Banach space and $\mathcal{J}$ Gateaux differentiable functional. If $c \in I\!R$, we say that $\mathcal{J}$ satisfies the Palais Smale condition (at the level $c \in \mathbb{R}$), if and only if, for every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{X}$ that satisfies*

$$\mathcal{J}(x_k) \to c \quad \text{ and } \quad \mathcal{J}^{'}(x_k) \to 0 \text{ in } \mathbb{X}^{'}$$

has a subsequence $(x_{k_\ell})_{\ell\in\mathbb{N}}$ convergent.

In 1989, in order to obtain information about the location of the critical points, Ghoussoub-preiss (see [55]) introduce a variant of Palais Smale condition given as follow

Definition 1.3.9 (Ghoussoub-Preiss Palais Smale condition). *Let $\mathbb{X}$ be a Banach space and $\mathcal{J}$ Gateaux differentiable functional. if $c \in I\!R$, we say that $\mathcal{J}$ satisfies the Palais Smale condition around the closed subset $\mathcal{F}$ of $\mathbb{X}$ (at the level c), if every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{X}$ satisfies*

① $\lim\limits_{k\to\infty} dist(x_k, \mathcal{F}) = 0,$

② $\lim\limits_{k\to\infty} \mathcal{J}(x_k) = c,$

③ $\lim\limits_{k\to\infty} \|\mathcal{J}'(x_k)\|_{\mathbb{X}'} = 0.$

has a convergent subsequence.

Now, we can formulate the Mountain Pass theorem by the sequence defined in definition 1.3.7 or 1.3.8 since they are equivalent. We begin by the original version of Ambrosetti and Rabinowitz [13], applies to the case where the mountain pass level is positive.

Theorem 1.3.2 (Ambrosetti- Rabinowitz Theorem). *Let $\mathbb{X}$ be a Banach space, $\mathcal{J} \in C^1$ on $\mathbb{X}$ satisfying the Palais Smale condition. Suppose that*

① $\mathcal{J}(0) = 0$.

② *There exists* $\rho > 0$ *and* $\alpha > 0$ *such that* $\|u\|_{\mathbb{X}} = \rho$, *then* $\mathcal{J} > \alpha$.

③ *There exists* $u_1 \in \mathbb{X}$ *such that* $\|u_1\| > \rho$ *and* $\mathcal{J} < \alpha$.

Then, $\mathcal{J}$ *has a critical value* $c \geqslant \alpha$. *More precisely, if we define*

$$\Gamma \overset{def}{=} \{\gamma : [0,1] \to \mathbb{X}, \gamma \textit{ is continuous and } \gamma(0) = 0, \gamma(1) = u_1\}, \tag{1.14}$$

then

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)). \tag{1.15}$$

In contrast, Pucci and Serrin extended this theorem to the case of mountains of *zero altitude* (i.e. critical value is zero), as detailed in [90, 89]. Instead of presenting their main result, we highlight the key corollary that provides a direct application.

Corollary 1.3.1 ([90])**.** *If Palais Smale condition holds and* $\mathcal{J}$ *has two different local minimum points, then* $\mathcal{J}$ *possesses a third critical point.*

Subsequently, Ghoussoub and Preiss provide a more generalized framework for understanding the structure of critical sets and provide more informations about the location of critical points. In this context, we now present their refinement of the mountain pass theorem, as established in [Theorem 1, [55]], using the concept of Palais smale sequence introduced in Definition 1.3.9.

Theorem 1.3.3 (Mountain Pass principle of Ghoussoub-Preiss). *Let* $\mathbb{X}$ *be a Banach space,* $\mathcal{J}$ *be a* C^1 *functional on* $\mathbb{X}$. *We define*

$$c := \inf_{\gamma\in\Gamma_{u_1}^{u_2}} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)), \tag{1.16}$$

where

$$\Gamma_{u_1}^{u_2} \overset{def}{=} \{\gamma : [0,1] \to \mathbb{X}(\Omega), \gamma \textit{ is continuous and } \gamma(0) = u_1, \gamma(1) = u_2\}. \tag{1.17}$$

Assume that $\mathcal{F}$ *is a closed subset of* $\mathbb{X}$ *such that for each* $\gamma \in \Gamma_{u_1}^{u_2}$, *one has* $\mathcal{F} \cap \{x \in \mathbb{X} : \mathcal{J}(x) \geqslant c\} \neq \emptyset$. *Suppose that* $\mathcal{J}$ *satisfies the Ghoussoub-Preiss Palais Smale condition, then there exists a critical point* $u_0 \in \mathcal{F}$ *such that* $\mathcal{J}(u_0) = c$ *and* $\mathcal{J}'(u_0) = 0$.

Perron's method

Perron's method can be seen as the variatonal version of sub supersolution method. From Struwe's book [[103], Chapter 1, Theorem 2.4], we state the following result which consists to moving from constrained minimization to a minimization extended over the entire space.

Theorem 1.3.4. *Suppose that* $\underline{u}$ *and* $\overline{u}$ *are a subsolution (supersolution respectively) in* $\mathbb{X}$ *to the problem*

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \textit{in} \quad \Omega, \\ u &= 0 \qquad\quad \textit{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.18}$$

and there exists $\alpha, \beta \in I\!R$ *such that*

$$-\infty < \alpha \leqslant \underline{u} \leqslant \bar{u} \leqslant \beta < \infty \qquad \textit{a.e. } \Omega.$$

Consider the bounded convex set

$$\mathcal{M} := \Big\{ u \in \mathbb{X} : \quad \underline{u} \leqslant u \leqslant \bar{u} \Big\}.$$

Assume that, the energy functional associated to (1.18)*,* $\mathcal{J} : \mathcal{M} \to \mathbb{R}_+^*$ *satisfies*

① $\mathcal{J}$ *is coercive on* $\mathcal{M}$*,*

② $\mathcal{J}$ *is weakly lower semi-continuous on* $\mathcal{M}$*.*

Then, $\mathcal{J}$ *is bounded from below on* $\mathcal{M}$*, and there exists* $u \in \mathcal{M}$ *such that*

$$\mathcal{J}(u) = \inf_{v \in \mathcal{M}} \mathcal{J}(v).$$

Moreover, u *is a solution of Problem* (1.18) *in* $\mathbb{X}$*.*

PROOF. The proof of this theorem will be given in the context of our study in Proposition 2.4.3. For more details see [8, 60, 83, 103]. ■

1.4 Some basic tools

1.4.1 Integration theory

Theorem 1.4.1. *Let* $1 \leqslant p < \infty$ *and*

$$f_n \rightharpoonup f \quad \text{weakly in} \quad L^p(\Omega) \quad (\overset{\star}{\rightharpoonup} \quad \text{in } L^\infty(\Omega) \quad \text{if } p = \infty).$$

Then, f_n *is bounded in* $L^p(\Omega)$ *and*

$$|f|_p \leqslant \liminf_{n\to\infty} |f_n|_p.$$

Theorem 1.4.2. *Let* $1 < p < \infty$ *and the sequence* $(f_n)_n$ *is bounded in* $L^p(\Omega)$. *Then there is a subsequence, still denoted by* $(f_n)_n$, *and a function* $f \in L^p(\Omega)$ *such that*

$$f_n \rightharpoonup f \quad \text{in} \quad L^p(\Omega).$$

If $p = \infty$,

$$f_n \overset{\star}{\rightharpoonup} f \quad \text{in} \quad L^\infty(\Omega).$$

Lemma 1.4.1 (Fatou's lemma [109]). *Let* Ω *be an open set of* $\mathbb{R}^N$. *If* $(f_n)_n$ *is a sequence of nonnegative measurable functions. Then*

$$\int_\Omega \lim_{n\to\infty} \inf f_n dx \leqslant \lim_{n\to\infty} \inf \int_\Omega f_n dx.$$

Theorem 1.4.3 (Vitali's theorem[107]). *Let* $1 \leqslant p < \infty$, Ω *be a set with finite measure for the Lebesgue measure on* $\mathbb{R}^N$. *Let* $(f_n)_n$ *be a sequence of functions in* $L^p(\Omega)$, *such that it satisfies the following conditions:*

① $f_n \to f$ *a.e. in* Ω,

② $(f_n)_n$ *is equi-integrable; For every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any measurable set* $E \subset \Omega$ *with* $|E| < \delta$,

$$\int_E |f_n|^p \, dx < \epsilon \quad \text{for all } n.$$

Then, $f \in L^p(\Omega)$ *and* $f_n \to f$ *strongly in* $L^p(\Omega)$.

Theorem 1.4.4. *Let* $(f_n)_n \subset L^1(\Omega)$ *and* $(g_n)_n \subset L^\infty(\Omega)$ *be two sequences such that:*

① $f_n \to f$ *weakly in* $L^1(\Omega)$, ② *$g_n \to g$ weakly-$\star$ in $L^\infty(\Omega)$ and a.e. in Ω.*

Then

$$\lim_{n\to\infty} \int_\Omega f_n g_n \, dx = \int_\Omega f g \, dx.$$

Theorem 1.4.5 (Dominated convergence theorem[70]). *Let $(f_n)_n \subset X$ be a sequence of measurable functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. Suppose that, there exists $g \in L^1(\Omega)$ such that for all $n \geqslant 1$, we have $|f_n| \leqslant g$ a.e. Ω. Then, $f \in L^1(\Omega)$ and f_n converges strongly to f in $L^1(\Omega)$.*

Theorem 1.4.6 (Ascoli Arzelà's Theorem [86]). *Let (K, d) be a compact metric space. Let us consider $\mathcal{A}$ be a subset of $\mathcal{C}(K)$, which satisfies the following hypothesis:*

① *Let the set $\mathcal{A}$ be bounded in $\mathcal{C}(K)$, i.e.*

$$(\exists M > 0)(\forall f \in \mathcal{A}), |f|_\infty \leqslant M < \infty.$$

② *$\mathcal{A}$ is uniformly equicontinuous, i.e.*

$$(\forall \varepsilon > 0)(\exists \eta > 0)(\forall x, y \in K), d(x, y) < \eta \Rightarrow (\forall f \in \mathcal{A})|f(x) - f(y)| < \varepsilon.$$

Then, $\mathcal{A}$ is relatively compact.

1.4.2 Elementary inequalities

In this study, we will based on certain algebraic inequalities.

Lemma 1.4.2. *[21] For all $\eta, \xi \in \mathbb{R}$ there exists $C_1, C_2 > 0$ such that*

- $p \in (1, 2]$

$$\Big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \Big| \leqslant C_2 |\xi - \eta|^{p-1}. \tag{1.19}$$

- $p > 2$

$$2^{p-2} \Big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \Big| (\xi - \eta) \geqslant |\xi - \eta|^p \tag{1.20}$$

Lemma 1.4.3 (Minkowski's inequality [63]). *Let $1 < p < +\infty$ and $a, b \geqslant 0$,*

$$(a + b)^p \leqslant 2^{p-1}(a^p + b^p) \tag{1.21}$$

Lemma 1.4.4 (Young's inequality[63]). *Let* $a, b > 0$ *and* $1 < p, p^{'} < \infty$ *such that* $\frac{1}{p} + \frac{1}{p^{'}} = 1$. *Then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p^{'}}}{p^{'}}.$$

Now, we present several algebraic inequalities that will be used in our study. Before stating these inequalities, we need to define the truncated function T_k given for all $k > 0$ by

$$T_k(r) = \begin{cases} r & \text{if} \quad |r| \leqslant k, \\ k & \text{if} \quad |r| > k. \end{cases}$$

and its primitive function $\Theta_k(r) = \int_0^r T_k(\tau) d\tau$.

Lemma 1.4.5. *Assume that* $a, b \in \mathbb{R}$ *, then for all* $k > 0$

$$(a - b)(T_k(a) - T_k(b)) \geqslant |T_k(a) - T_k(b)|^2 \tag{1.22}$$

Lemma 1.4.6.

① *Assume* $p \geqslant 1$ *and* $a, b \geqslant 0$

$$(a - b)(a^p - b^p) \leqslant p \max\{a^{p-1}, b^{p-1}\}(a - b)^2.[30] \tag{1.23}$$

② *Assume that* $p < m$ *and* $a, b \geqslant 0$. *Then, for any* $\varepsilon > 0$ *there exists* $c_\varepsilon > 0$, *such that*

$$a^{p-1} \leqslant a^{m-1} + c_\varepsilon, \tag{1.24}$$

$$a^{p-1} \leqslant c_1 a^{m-1} + c_2 a, \tag{1.25}$$

$$|a^p - b^p|(a - b) \leqslant \epsilon |a^m - b^m|(a - b) + c_\epsilon (a - b)^2. \tag{1.26}$$

1.4.3 Picone's inequality for p-Laplacian and application

In this subsection, We present the general version of Picone's inequality and an extension of the classical Brezis-Kamin principle to the fractional p-Laplacian setting, as adapted in [3]

Lemma 1.4.7. *(**Picone inequality**[3]) Let* $w \in W_0^{s,p}(\Omega)$, *such that* $w > 0$ *in* Ω. *For all* $u \in \mathcal{C}_0^\infty(\Omega)$, *we have*

$$\frac{1}{2}\int\int_{D_\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \geqslant \langle(-\Delta)^s_p w, \frac{|u|^p}{w^{p-1}}\rangle.$$

Lemma 1.4.8 (Comparison principle). *Let $p \geqslant 2$, Ω a bounded domain and f nonnegative continuous function for all $t > 0$. Assume that*

$$t \longmapsto \frac{f(x,t)}{t^{p-1}} \text{ is decreasing, for all } t > 0.$$

Let $u, v \in W^{s,p}_0(\Omega)$ such that $u, v > 0$ in Ω. If $v \leqslant u$ in $\mathbb{R}^N \setminus \Omega$ and

$$\begin{cases} (-\Delta)^s_p u & \geqslant & f(x,u) & \text{in} & \Omega, \\ (-\Delta)^s_p v & \leqslant & f(x,v) & \text{in} & \Omega. \end{cases}$$

Then, $u \geqslant v$ in Ω.

Part I

Nonlocal Elliptic Problems

2

A Nonlocal scalar field problem: Existence, multiplicity and asymptotic behavior

The aim of this chapter is to study a class of scalar field problems involving the fractional Laplacian, and a large growth term. Specifically, we investigate the existence, non-existence and multiplicity of positive solutions to the following elliptic problem involving fractional Laplacian with an homogeneous Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} & \text{in} & \Omega, \\ u &> 0 & \text{in} & \Omega, \\ u &= 0 & \text{in} & \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.1}$$

for suitable ranges of λ, p and m, with $0 < s < 1$, $2 < p < m$, $\lambda > 0$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$. Our particular interest is to understand how the behavior of the problem changes based on the value of λ. Furthermore, we focus on studying the asymptotic behavior of positive solutions for m sufficiently large, and describe this behavior through a free boundary problem.

This chapter is an extended version of the work published [22].

2.1 Introduction

The aim of this chapter is to shed light on the main questions of existence, non-existence and multiplicity of positive solutions to the following problem with an homogeneous Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.2}$$

for suitable ranges of λ, p and m. In addition, we investigate the asymptotic behavior of the sequence of solutions as m tends to ∞.

To our knowledge, such kind of problem, has not been extensively addressed in the last few years. Let us dive into the available literature and see how our results contribute to the existing knowledge.

In the local case:

Let us start with a brief survey of the literature concerning existence of positive solutions of Problem (2.2)

- In 1977, Strauss studied for $\Omega = \mathbb{R}^N$ the elliptic equation

  $$-\Delta u = F(u)$$

  has non-trivial solutions that decay to 0 at infinity, and are known as solitary waves (see [102]). Then, in 1983, Berestycki and Lions have proved under general conditions on F, the existence of a ground state solution (see [24]).

- The case of bounded domains was considered by Merle and Peletier in [79, 78], where they studied the following problem

  $$\begin{cases} -\Delta u &= \lambda u^p - \varepsilon u^m \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

  with $m > p \geqslant 2^\star - 1$ and $\varepsilon > 0$, considering a different domains. In the first work [79], they studied the problem in unit ball and proved the existence of two solutions. In the case of star-shaped domain [78], they established the existence of variational solution. In both framework, an asymptotic analysis was performed as $\varepsilon \to 0$. This work aligns with our study with a change $\lambda = \dfrac{1}{\varepsilon}$, which allows us to better understand the behavior of the solutions of Problem (2.2) when λ sufficiently large.

- Recently, Boccardo et al.[27], have studied the following problem with divergence operator

$$\begin{cases} -div(M(x)\nabla u) + u^{m-1} & = & \lambda u^{p-1} & \text{in} & \Omega, \\ u & > & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2.4}$$

where Ω is bounded domain, $2 < p \leqslant 2^\star < m$, $\lambda > 0$ and the matrix $M(x) = (m_{ij}(x))$ is symmetric, bounded, and positive definite, i.e.,

$$\text{there exist positive constants } 0 < \beta < \gamma \text{ such that } \beta|\xi|^2 \leqslant M(x)\xi \cdot \xi \leqslant \gamma|\xi|^2.$$

They proved the existence of at least two positive sequences of solutions for λ sufficiently large. The first one is a global minimum with negative energy and the second is a solution obtained by Mountain Pass Theorem. Then, they studied their asymptotic behavior when m tends to ∞, determining their behavior by a limiting problem. This phenomenon was studied for the first time by the first author and Murat (see [28]), in the case of equations with Leray-Lions type operator and with $\lambda = 0$. They proved that the solution of the nonlinear elliptic problem converges to the solution of the free boundary problem when the power tends to infinity.

In the nonlocal case:

We explored the key works that provided valuable insights in the nonlocal framework, to start our study.

- Bhakta et.al.[25, 26] have extended the results obtained by Merle and Peletier in [79, 78], to the fractional setting. They established the existence of a variational solution, adapting the arguments of [78]. Moreover, they characterize the asymptotic behavior of the solution as $\varepsilon \to 0$, and extended the asymptotic analysis in the critical case.

- In [94] Xavier Ros-Oton has extensively explored the Dirichlet problem for nonlocal operator with critical and discontinuous nonlinearity, proving existence and maximum principle, including regularity properties of solutions.

- Many authors have studied nonlocal semilinear elliptic problem for a different kind of nonlinearities. See, for instance,
    - [4, 99] for nonlinearities with subcritical growth.
    - [19, 20] for the critical case for the fractional Laplacian.
    - [74, 79, 100] for problems involving more general nonlocal operator .

Motivated by the above works, the main innovation point of this work is the effect of the increasing power term on the existence and multiplicity of positive solutions to Problem (2.2) even if $p \geqslant 2_s^\star$. The influence was studied in X.Ros-Oton's paper, when they proved that the problem ($\mathcal{P}$) without the nonlinear absorbing term has only the trivial solution whenever Ω is star shaped domain, due to the lack of compactness by applying fractional Pohozaev identity given in [96]. This fact motivates the perturbation term u^{m-1} since $m > 2_s^\star$.

Our work extends the results obtained in [27] to the fractional setting. In some way, we enhance the results obtained in the local framework and filling the gap in, by proving the nonexistence results for λ very small. We have been inspired by the ideas developed in [27] and provide a finer asymptotic analysis and a more detailed study of the solution set satisfying the non-existence, existence, and multiplicity of positive solutions of the Problem (2.2), for the largest possible ranges of the parameters λ, m, p. Then, we studied the asymptotic behavior of positive solutions obtained in the main existence results, determining the behavior as a solutions of a free boundary problems.

Plan of the chapter:

- ✓ We begin our study by section 2.3, where we prove an L^∞ regularity estimate, which will play a crucial role in the existence result and in the asymptotic analysis.
- ✓ In Section 2.4, we prove the existence and non-existence results, we show the existence of a non-negative solution in suitable Sobolev spaces according to the values of λ, more precisely, we demonstrate the existence of minimum solution with negative energy, zero energy, and positive energy.
- ✓ Section 2.5 presents the asymptotic analysis, when $m \to \infty$, of a sequence of minimal solutions and also gives the behavior of $\underline{\lambda}(m)$. This will lead to the proof of Theorem 2.5.2.
- ✓ Section 2.6 is devoted to the proof of the multiplicity result Theorem 2.6.2 and to understand the behavior of the Mountain pass solutions with respect to m, this yields to Theorem 2.6.3.

✓ Finally, in Section 2.7 we give some remarks and examples answering some important questions concerning the limiting problems (2.63) and (2.78).

2.2 Functional Framework and Variational Formulation

We introduce, in what follows, some notations and preliminary results needed throughout this chapter.

In order to obtain the existence of a weak solution of Problem (2.2), let us present some results on the functional framework

$$\mathcal{H}_0^s(\Omega)\cap L^m(\Omega) \overset{\text{def}}{=} \{u\in\mathcal{H}_0^s(\Omega) : \int\limits_{\Omega}|u|^m<+\infty\}$$

where $m>2$, endowed with the norm

$$\|\cdot\| = \|\cdot\|_s + |\cdot|_m.$$

For the simplicity of notation, in the whole chapter we will set

$$d\mu = \frac{dx\,dy}{|x-y|^{N+2s}} \text{ and } D_\Omega = \mathbb{R}^N\times\mathbb{R}^N\backslash\mathcal{C}\Omega\times\mathcal{C}\Omega.$$

Now, let make precise the notion of silution to Problem (2.2).

Definition 2.2.1. *We say that $u\in\mathcal{H}_0^s(\Omega)\cap L^m(\Omega)$ is a weak solution to Problem* (2.2) *if*

$$\iint\limits_{D_\Omega}(u(x)-u(y))(\varphi(x)-\varphi(y))\,d\mu + \int\limits_{\Omega}|u|^{m-2}u\varphi\,dx = \lambda\int\limits_{\Omega}|u|^{p-2}u\varphi\,dx, \tag{2.5}$$

for all $\varphi\in\mathcal{H}_0^s(\Omega)\cap L^m(\Omega)$.

Our approach is variational, notice that u is a weak solution to Problem (2.2), is equivalent to being a critical point of the energy functional

$$\mathcal{J}_{\lambda,m}(u) = \frac{1}{2}\iint_{D_\Omega}|u(x)-u(y)|^2\,d\mu + \frac{1}{m}\int_\Omega|u|^m\,dx - \frac{\lambda}{p}\int_\Omega|u|^p\,dx. \tag{2.6}$$

$\mathcal{J}_{\lambda,m}$ is well defined and of class $\mathcal{C}^1$ on $\mathcal{H}_0^s(\Omega)\cap L^m(\Omega)$. Furthermore, it's derivative is given by,

$$\left\langle\mathcal{J}'_{\lambda,m}(u),v\right\rangle = \iint\limits_{D_\Omega}(u(x)-u(y))(v(x)-v(y))\,d\mu + \int\limits_{\Omega}|u|^{m-2}uv\,dx - \lambda\int\limits_{\Omega}|u|^{q-2}uv\,dx. \tag{2.7}$$

We aim to establish the existence of non-trivial critical points of the functional $\mathcal{J}_{\lambda,m}$.

2.3 A priori estimate

The purpose of this part is to prove the main ingrediant of our study, the uniform a priori estimate for the weak positive solutions of Problem (2.2).

Theorem 2.3.1. *Let $\lambda > 0$ and $2 < p < m$, . If u is a solution to (2.2), then u is bounded and we have*

$$|u|_\infty \leqslant \lambda^{\frac{1}{m-p}}. \tag{2.8}$$

Proof. We first prove that any solution of Problem (2.2) is bounded. Indeed, let u be a solution of Problem (2.2) and let $h(t) = \lambda t_+^{p-1} - t_+^{m-1}$. Then, for any $\varepsilon < 0$, there exists $C_\varepsilon > 0$

$$h(t) \leqslant \varepsilon t + C_\varepsilon. \tag{2.9}$$

Hence, u is a sub-solution to the following problem

$$\begin{cases} (-\Delta)^s w - \varepsilon w & = & C_\varepsilon & \text{in} & \Omega, \\ w & \geqslant & 0 & \text{in} & \Omega, \\ w & = & 0 & \text{in} & \mathbb{R}^N \backslash \Omega. \end{cases}$$

Choosing $\varepsilon < -\bar{\varepsilon} < 0$, then we use the maximum principle to deduce the boundedness of $|u|_\infty$.

Let us define Ψ_ε, for $k, \varepsilon > 0$

$$\Psi_\varepsilon(s) := \begin{cases} 0 & \text{if} & s < k, \\ \dfrac{s-k}{\varepsilon} & \text{if} & k < s < k+\varepsilon, \\ 1 & \text{if} & s > k+\varepsilon. \end{cases}$$

We also introduce the set

$$E_k = E_k^1 \cup E_k^2 = \{x \in \Omega : \ w(x) > k\},$$

where

$$E_k^1 = \{x \in \Omega : \ k < w(x) < k+\varepsilon\}, \ E_k^2 = \{x \in \Omega : \ w(x) > k+\varepsilon\},$$

using $\Psi_\varepsilon(u)$ as a test function in (2.2), we obtain

$$\iint\limits_{D_\Omega} (w(x) - w(y))(\Psi_\varepsilon(w(x)) - \Psi_\varepsilon(w(y)))\, d\mu + \int\limits_\Omega w^{m-1}\Psi_\varepsilon(w)\, dx = \lambda \int\limits_\Omega w^{p-1}\Psi_\varepsilon(w)\, dx.$$

Notice that $D_\Omega = E_k \cup E_k^c$ is the union of E_k and its complement E_k^c . Subsequently, we split the set into four subsets denoted as D_i, $1 \leqslant i \leqslant 4$

$$\begin{aligned} D_1 &= \{(x,\, y) \in D_\Omega :\, x,\, y \in E_k\}, \qquad D_2 = \{(x,\, y) \in D_\Omega :\, x,\, y \in E_k^c\}, \\ D_3 &= \{(x,\, y) \in D_\Omega : x \in E_k \text{ and } y \in E_k^c\}, \quad D_4 = \{(x,\, y) \in D_\Omega : x \in E_k^c \text{ and } y \in E_k\}. \end{aligned}$$

Then

$$\begin{aligned} \iint\limits_{D_\Omega} (w(x) - w(y))(\Psi_\varepsilon(x) - \Psi_\varepsilon(y))\, d\mu &= \iint\limits_{D_1} (w(x) - w(y))(\Psi_\varepsilon(x) - \Psi_\varepsilon(y))\, d\mu \\ &+ \iint\limits_{D_2} (w(x) - w(y))(\Psi_\varepsilon(x) - \Psi_\varepsilon(y))\, d\mu \\ &+ \iint\limits_{D_3} (w(x) - w(y))(\Psi_\varepsilon(x) - \Psi_\varepsilon(y))\, d\mu \\ &+ \iint\limits_{D_4} (w(x) - w(y))(\Psi_\varepsilon(x) - \Psi_\varepsilon(y))\, d\mu \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We analyze each term separately, and we prove the posivity of each one.

In D_1, we have:

$$\begin{aligned} I_1 &= \iint\limits_{E_k^1 \times E_k^1} (w(x) - w(y))(\Psi_\varepsilon w(x) - \Psi_\varepsilon w(y))\, d\mu + \iint\limits_{E_k^2 \times E_k^1} (w(x) - w(y))(\Psi_\varepsilon w(x) - \Psi_\varepsilon w(y)) d\mu \\ &+ \iint\limits_{E_k^1 \times E_k^2} (w(x) - w(y))(\Psi_\varepsilon w(x) - \Psi_\varepsilon w(y)) d\mu \\ &= \frac{1}{\varepsilon} \iint\limits_{E_k^1 \times E_k^1} |w(x) - w(y)|^2\, d\mu + \frac{1}{\varepsilon} \iint\limits_{E_k^2 \times E_k^1} (w(x) - w(y))(\varepsilon - w(y) + k) d\mu \\ &+ \frac{1}{\varepsilon} \iint\limits_{E_k^1 \times E_k^2} (w(x) - w(y))(w(x) - k - \varepsilon) d\mu \geqslant 0. \end{aligned}$$

When $(x, y) \in D_2$, we observe that $\Psi_\varepsilon(w)(x) = \Psi_\varepsilon(w)(y) = 0$, then $I_2 = 0$.

On the other hand, since $w(x) \geqslant k$ in E_k^1, it follows that

$$\begin{aligned} I_3 :&= \frac{1}{\varepsilon} \iint\limits_{E_k^1 \times E_k^c} (w(x) - w(y))(w(x) - k)\, d\mu + \iint\limits_{E_k^2 \times E_k^c} (w(x) - w(y))\, d\mu \\ &\geqslant \frac{1}{\varepsilon} \iint\limits_{E_k^1 \times E_k^c} |w(x) - k|^2\, d\mu + \iint\limits_{E_k^2 \times E_k^c} (w(x) - k)\, d\mu \geqslant 0. \end{aligned}$$

Likewise, we deduce that

$$\begin{aligned} I_4 : &= \frac{1}{\varepsilon}\iint\limits_{E_k^c\times E_k^1} (w(x)-w(y))(k-w(y))\,d\mu - \iint\limits_{E_k^c\times E_k^2} (w(x)-w(y))\,d\mu \\ &\geqslant \frac{1}{\varepsilon}\iint\limits_{E_k^c\times E_k^1} |w(x)-w(y)|^2\,d\mu + \iint\limits_{E_k^c\times E_k^2} (w(y)-k)\,d\mu > 0. \end{aligned}$$

Since $w(y)-k>0$ in E_k^2, and $w(x)<k$ in E_k^c. Consequently

$$\begin{aligned} \int\limits_{E_k} w^{m-1}\Psi_\varepsilon(w)\,dx &\leqslant \iint\limits_{D_\Omega}(w(x)-w(y))(\Psi_\varepsilon(x)-\Psi_\varepsilon(y))\,d\mu + \int\limits_{\Omega} w^{m-1}\Psi_\varepsilon(w)\,dx \\ &= \lambda\int\limits_{\Omega} w^{p-1}\Psi_\varepsilon(w)\,dx \\ &\leqslant \lambda\int\limits_{\Omega} w^{p-1}\,dx. \end{aligned}$$

Moreover, applying Fatou's Lemma, we obtain

$$\int_{E_k} \liminf_{\varepsilon\to 0} w^{m-1}\Psi_\varepsilon(w)\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{E_k} w^{m-1}\Psi_\varepsilon(w) \leqslant \lambda\int\limits_{E_k} w^{p-1}\,dx,$$

and then, by Hölder's inequality

$$\int\limits_{E_k} w^{m-1}\,dx \leqslant \lambda\int\limits_{E_k} w^{p-1}\,dx \leqslant \lambda\Big[\int\limits_{E_k} w^{m-1}\,dx\Big]^{\frac{p-1}{m-1}}|E_k|^{1-\frac{p-1}{m-1}}.$$

Using the definition of the set E_k

$$k^{m-1}|E_k| \leqslant \int\limits_{E_k} w^{m-1}\,dx \leqslant \lambda^{\frac{m-1}{m-p}}|E_k|.$$

Hence, considering that $|E_k|\neq 0$, yields that

$$k\leqslant \lambda^{\frac{1}{m-p}},\ \forall k\in(0,|w|_\infty).$$

Particularly, if we take $k=|w|_\infty$, then

$$|w|_\infty \leqslant \lambda^{\frac{1}{m-p}}.$$

Thus (2.8) is established. □

Remark 2.3.1. *The same argument allows to obtain* $L^\infty(\Omega)$ *estimate for the solution of the problem*

$$\begin{cases} (-\Delta)^s w + w^{m-1} &= f \quad \text{in} \quad \Omega, \\ w &\geqslant 0 \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $m > 1$ *and* $f \in L^\infty(\Omega)$*. More precisely, we have*

$$|w|_\infty \leqslant |f|_\infty^{\frac{1}{m-1}}.$$

2.4 Existence and Non-existence results

The main existence result is outlined in the following theorem

Theorem 2.4.1. *Let* $2 < p < m$*. Then, there exist positive constants* $\Lambda(m)$, $\underline{\lambda}(m)$ *with* $0 < \Lambda(m) \leqslant \underline{\lambda}(m) < \infty$ *such that:*

① *If* $0 < \lambda < \Lambda(m)$*, then Problem* (2.2) *does not have a positive weak solution in* $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$*.*

② *If* $\lambda > \Lambda(m)$*, then Problem* (2.2) *has a maximal solution* $w_{\lambda,m} \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$*, in the sense that* $w_{\lambda,m} \geqslant u$ *for any solution* u *to problem* (2.2)*. Moreover, these solutions are ordered, namely, if* $\lambda_1 < \lambda_2$ *then* $w_{\lambda_1,m} < w_{\lambda_2,m}$*.*

③ *If* $\lambda > \Lambda(m)$*, then Problem* (2.2) *has a non trivial solution* $u_{\lambda,m} \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ *which is a local minimum of* $\mathcal{J}_{\lambda,m}$*. Furthermore,*

- *If* $\Lambda(m) < \lambda < \underline{\lambda}(m)$*, then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) > 0$*.*
- *If* $\lambda = \underline{\lambda}(m)$*, then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) = 0$*.*
- *If* $\lambda > \underline{\lambda}(m)$*, then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) < 0$*.*

④ $u_{\lambda,m}$ *is increasing in* λ*. That is, if* $\lambda_1 < \lambda_2$ *then* $u_{\lambda_1,m} < u_{\lambda_2,m}$*.*

Outline of proof:

We briefly outline the main steps of the proof:

① We determine $\Lambda(m)$ the critical threshold for the existence of solutions to Problem (2.2), and prove the following:

(a) For $0 < \lambda < \Lambda(m)$, the problem does not admits any solution in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$.

(b) For $\lambda \geqslant \Lambda(m)$, the existence of solutions is assured. Specifically:

- We demonstrate the existence of maximal solution, which dominates the other solutions, using sub-supersolution method.
- We prove the existence of a local minimizer using variational formulation of Perron's method and a variant of Alama's method.
    - We analyze the energy of the local minimum and study its behavior as function of the growth of λ.

2.4.1 Non-existence result

The objective of this subsection is to present the non-existence result for λ small enough.

Proposition 2.4.1. *If $0 < \lambda < \Lambda(m)$, then Problem (2.2) has only the trivial solution in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$.*

Proof. Assume that λ very small. Assume that Problem (2.2) admits a positive solution $u_\lambda \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$. Taking u_λ as test function, by the L^∞-estimate (2.8) and the definition of $\lambda_{1,s}$, we get

$$\|u_\lambda\|_s^2 \leqslant \|u_\lambda\|_s^2 + |u_\lambda|_m^m = \lambda |u_\lambda|_p^p \quad \leqslant \quad \lambda |u_\lambda|_\infty^{p-2} |u_\lambda|_2^2 \leqslant \frac{\lambda}{\lambda_{1,s}} \lambda^{\frac{p-2}{m-p}} \|u_\lambda\|_s^2,$$

which implies that

$$\|u_\lambda\|_s^2 \leqslant \frac{\lambda^{\frac{m-2}{m-p}}}{\lambda_{1,s}} \|u_\lambda\|_s^2.$$

Then, we find that $\lambda \geqslant [\lambda_{1,s}]^{\frac{m-p}{m-2}}$, which is a contradiction. □

2.4.2 Existence results

In contrast to the previous subsection, we aim to prove the principal existence results.

Let us define the threshold of existence.

$$\Lambda(m) := \inf \Big\{ \lambda \in \mathbb{R}_+ : \text{ such that Problem (2.2) has a nontrivial solution in } \mathcal{H}_0^s(\Omega) \cap L^m(\Omega) \Big\}. \tag{2.10}$$

We show the following crucial result regarding $\Lambda(m)$.

Lemma 2.4.1. *For $2 < p < m$, we have $0 < \Lambda(m) < \infty$.*

Proof. **<u>Part 01:</u>** We first check that $\Lambda(m)$ is finite.

Let us consider the following intermediary problem

$$\begin{cases} (-\Delta)^s\xi + \xi^{m-1} & = & g & \text{in} & \Omega, \\ \xi & > & 0 & \text{in} & \Omega, \\ \xi & = & 0 & \text{in} & \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.11}$$

with $g = 1$ the existence of $\xi \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$ is assured by variational method. Taking ξ as test function in (2.11), we obtain that

$$\|\xi\|_s^2 + |\xi|_m^m = |\xi|_1$$

Let us evaluate the energy functional associated to (2.2)

$$\begin{aligned} \mathcal{J}_{\lambda,m}(\xi) &= \frac{1}{2}\|\xi\|_s^2 + \frac{1}{m}|\xi|_m^m - \frac{\lambda}{p}|\xi|_p^p \\ &\leqslant max\{\frac{1}{2}, \frac{1}{m}\}(\|\xi\|_s^2 + |\xi|_m^m) - \frac{\lambda}{p}|\xi|_p^p \\ &\leqslant \int_\Omega \xi \, dx - \frac{\lambda}{p}|\xi|_p^p \leqslant |\xi|_p(1 - \frac{\lambda}{p}|\xi|_p^{p-1}). \end{aligned}$$

Hence, for λ large enough, the negative term dominates, ensuring that $\mathcal{J}_{\lambda,m}(\xi) < 0$.
Notice that, for λ very large and for such p and m such that $m > p$, Problem (2.2) admits a global minimum solution. In fact, the energy functional $\mathcal{J}_{\lambda,m}$ is coercive, indeed we apply the Hölder inequality and the L^∞-estimate (2.8) to obtain

$$|u|_p^p \leqslant C(\Omega)|u|_\infty^p \leqslant C(\Omega)\lambda^{\frac{p}{m-p}}$$

Therefore, as $\|u\| \to \infty$

$$\mathcal{J}_{\lambda,m}(u) \geqslant \alpha\|u\|^2 - C_1 \to \infty.$$

Clearly, $\mathcal{J}_{\lambda,m}$ is bounded. Next, we prove the weak lower semi-continuity of $\mathcal{J}_{\lambda,m}$ on $\mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$; Let $(v_k)_k \subset \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$ be a minimizing sequence for $\mathcal{J}_{\lambda,m}$, due to the coercivity of $\mathcal{J}_{\lambda,m}$, we immediately see that $(v_k)_k$ is bounded in $\mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$, therefore there exists $v \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$ such that, up to a subsequence still denoted $(v_k)_k$

$$\begin{aligned} v_k &\rightharpoonup v \quad \text{weakly in} \quad \mathcal{H}^s_0(\Omega), \\ v_k &\to v \quad \text{strongly in} \quad L^q(\Omega) \text{ for all } q \in (1, +\infty) \text{ as } k \to +\infty. \cdot \\ v_k &\to v \quad \text{a.e.} \quad \Omega, \end{aligned}$$

Besides, by the weak lower semi-continuity of the norm $\|.\|$, we obtain

$$\begin{aligned}
\mathcal{J}_{\lambda,m}(v) &= \frac{1}{2}\|v\|_s^2 + \frac{1}{m}|v|_m^m - \frac{\lambda}{p}|v|_p^p \\
&\leqslant \frac{1}{2}\lim_{k\to+\infty}\inf \|v_k\|_s^2 + \frac{1}{m}\lim_{k\to+\infty}\inf |v_k|_m^m - \frac{\lambda}{p}\lim_{k\to+\infty}|v_k|_p^p \\
&\leqslant \lim_{k\to+\infty}\inf(\frac{1}{2}\|v_k\|_s^2 + \frac{1}{m}|v_k|_m^m - \frac{\lambda}{p}|v_k|_p^p) \\
&= \lim_{k\to+\infty}\inf \mathcal{J}_{\lambda,m}(v_k)
\end{aligned}$$

So, there exists $u_\lambda \in \mathcal{H}_0^s(\Omega)\cap L^m(\Omega)$ such that

$$\mathcal{J}_{\lambda,m}(u_\lambda) = \inf_{\xi\in\mathcal{H}_0^s(\Omega)\cap L^m(\Omega)} \mathcal{J}_{\lambda,m}(\xi).$$

Therefore, it attains its global minimum u_λ .

It remains to us to prove that $u_\lambda \neq 0$. Indeed, we observe that for λ large enough,

$$\mathcal{J}_{\lambda,m}(u_\lambda) \leqslant \mathcal{J}_{\lambda,m}(\xi) < 0.$$

Therefore $u_\lambda \neq 0$, proving that for λ very large the Problem (2.2) admits a nontrivial minimizer. To prove that u_λ is positive we consider the truncated energy functional $\mathcal{I}_{\lambda,m}(u)$

$$\mathcal{I}_{\lambda,m}(u) = \frac{1}{2}\iint_{D_\Omega} |u(x)-u(y)|^2\,d\mu - \int_\Omega F_\lambda(u)\,dx. \tag{2.12}$$

where $F_\lambda(t) = \int_0^t f_\lambda(\sigma)\,d\sigma$ and

$$f_\lambda(t) = \begin{cases} \lambda t^{p-1} - t^{m-1} & \text{if} \quad 0\leqslant t\leqslant \lambda^{\frac{1}{m-p}}, \\ 0 & \text{otherwise .} \end{cases} \tag{2.13}$$

Arguing as above, we can show, by using ξ the solution to problem (2.11) with $g=\lambda$ (large enough), as test function, that there exists $u_\lambda \in \mathcal{H}_0^s(\Omega)\cap L^m(\Omega)$ such that

$$\mathcal{I}_{\lambda,m}(u_\lambda) = \inf_{\xi\in\mathcal{H}_0^s(\Omega)\cap L^m(\Omega)} \mathcal{I}_{\lambda,m}(\xi) < 0.$$

Hence, $\mathcal{I}_{\lambda,m}$ has a nontrivial global minimizer u_λ. Thanks to the L^∞-estimate we have that $\mathcal{J}_{\lambda,m}(u_\lambda) =$

$\mathcal{I}_{\lambda,m}(u_\lambda)$ and u_λ is a solutions to the problem

$$(-\Delta)^s u = f_\lambda(u) \text{ in } \Omega, \quad \text{and} \quad u = 0 \text{ in } \mathbb{R}^N \backslash \Omega. \tag{2.14}$$

Furthermore, since $f_\lambda(u_\lambda) \geqslant 0$, by the maximum principle we conclude that $u_\lambda > 0$.

Part 02: We prove that $\Lambda(m)$ is nonzero. As a direct consequence of the nonexistence result, it follows that $\Lambda(m) \geqslant [\lambda_{1,s}]^{\frac{m-p}{m-2}}$, which demonstrates that $\Lambda(m) > 0$. □

Existence of maximal solution

The purpose of this subsection is to prove the existence of maximal positive solution to Problem (2.2). For convenience, we will clarify the following concept.

Definition 2.4.1. *If $w_{\lambda,m}$ is a maximal solution of Problem (2.2), then for any other solution u to Problem (2.2), we have*

$$u \leqslant w_{\lambda,m}, \text{ a.e. in } \Omega.$$

Proposition 2.4.2. *For every $\lambda > \Lambda(m)$, Problem (2.2) has a maximal solution denoted $w_{\lambda,m} \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$. Moreover, these solutions are increasing to the respect to λ. Namely, if $\lambda_1 < \lambda_2$, then $w_{\lambda_1,m} < w_{\lambda_2,m}$.*

Proof. The proof is based on an iterative scheme.

Let us define

$$f(t) = \begin{cases} \lambda t^{p-1} - t^{m-1} & \text{if} \quad 0 \leqslant t \leqslant \bar{z}, \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

such that $\bar{z} := \lambda^{\frac{1}{m-p}}$. Moreover, it is possible to construct a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ to the iterated problems

$$\begin{cases} (-\Delta)^s u_{n+1} + M u_{n+1} = M u_n + f(u_n) \text{ in } \Omega \\ u_{n+1} = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.16}$$

where M is chosen such that, $t \mapsto Mt + f(t)$ is increasing for all $t \geqslant 0$ and $u_0 = \bar{z}$. Furthermore, let $\underline{u}$ be a solution of Problem (2.2), such that by the L^∞-estimate (2.8) we have that $\underline{u} < \bar{z}$, the idea now is to prove that

$$u_0 \geqslant u_1 \geqslant \ldots \geqslant \underline{u}$$

so, we have

$$(-\Delta)^s(u_0-u_1)+M(u_0-u_1) \;=\; (-\Delta)^s u_0+Mu_0 \geqslant 0.$$

By maximum principle $u_0 \geqslant u_1$. On the other hand, since $\underline{u} \leqslant u_0$, and $Mt+f(t)$ is increasing, we obtain

$$(-\Delta)^s(u_1-\underline{u})+M(u_1-\underline{u}) \;=\; f(u_0)+Mu_0-(f(\underline{u})+M\underline{u}) \geqslant 0$$

In the sequel, by the maximum principle, we obtain that $(u_n)_{n\in\mathbb{N}}$ is deacreasing and $\underline{u} \leqslant u_n \leqslant \bar{z}$. Then, by standard argument, since u_n is bounded there exists w_λ such that $u_n \to w_\lambda$ in $\mathcal{H}^s_0(\Omega)\cap L^\infty(\Omega)$ and $\underline{u} \leqslant w_\lambda \leqslant \bar{z}$. As consequence, w_λ is a maximal solution of Problem (2.2).

It remains for us to prove that the family of maximal solutions increasing with respect to λ. Let $\Lambda < \lambda_1 < \lambda_2$, we observe that w_{λ_1} is a subsolution to the followig problem

$$\begin{cases} (-\Delta)^s w_{\lambda_2} = \lambda_2 w_{\lambda_2}^{p-1} - w_{\lambda_2}^{m-1} & \text{in} \quad \Omega, \\ w_{\lambda_2} > 0 & \text{in} \quad \Omega, \\ w_{\lambda_2} = 0 & \text{in} \quad \mathbb{R}^N\backslash\Omega, \end{cases} \tag{2.17}$$

then, using the strong comparison principle, we deduce that $w_{\lambda_1} \leqslant w_{\lambda_2}$. □

Existence of local minimizer

Let us define,

$$\underline{\lambda}(m) = \inf\{\lambda : \mathcal{J}_{\lambda,m} \text{ has a local minimum } u_\lambda \in \mathcal{H}^s_0(\Omega)\cap L^m(\Omega) \text{ such that } \mathcal{J}_{\lambda,m}(u_\lambda) < 0\}. \tag{2.18}$$

The main goal in this section is to prove the existence of local minimizer. Indeed, we establish the existence of positive solution via Perron's method introduced by Struwe [103], and a variant of Alama method [8] adapted on the nonlocal framework by Abdellaoui et. all [4]. Subsequently, we prove that this solution, in fact, defines a local minimizer.

Firstly, we use Theorem 1.3.4 to assure the existence of positive solution to Problem (2.2) for all $\lambda > \Lambda(m)$.

Proposition 2.4.3. *Let $2 < p < m$. If $\lambda > \Lambda(m)$, Problem (2.2) has a non-trivial solution $\vartheta \in \mathcal{H}^s_0(\Omega)\cap L^m(\Omega)$.*

Proof. Let $\lambda > \lambda_0(m) > \Lambda(m)$, we take $\bar{u} = w_\lambda$ the maximal solution of Problem (2.2) and $\underline{u}$ the maximal solution to Problem (2.2) associated to λ_0, such that $\underline{u} < \bar{u}$.

We define

$$\mathcal{M} = \{\phi \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega) : \underline{u} \leqslant \phi \leqslant \bar{u}\}. \tag{2.19}$$

By the L^∞-estimate (2.8) we have $\underline{u}$, $\bar{u} \in L^\infty(\Omega)$, then $\mathcal{M} \in L^\infty(\Omega)$. Moreover, $\mathcal{M}$ is a closed convex set of $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$. Since $\mathcal{M}$ is bounded, then $\mathcal{J}_{\lambda,m}$ satisfies

$$\mathcal{J}_{\lambda,m} \geqslant \alpha \|u\|^2 - \lambda^{\frac{p}{m-p}} |\Omega|$$

which implies that $\mathcal{J}_{\lambda,m}$ is coercive and essentially bounded on $\mathcal{M}$. Next, we can simply show that $\mathcal{J}_{\lambda,m}$ is weakly lower semi-continuous over $\mathcal{M}$. Indeed, let $(v_k)_k$ a sequence in $\mathcal{M}$, by the coercivity of the functional over $\mathcal{M}$, we deduce that $(v_k)_k$ is bounded in $\mathcal{M}$, which implies, up to a subsequence, that

$$v_k \rightharpoonup v \text{ weakly in } \mathcal{H}_0^s(\Omega) \cap L^m(\Omega),$$

and strongly in every Lebesgue space, $v_k \to v, \ a.e \ in \ \Omega$. Then, it follows from the weak lower semicontinuity of the norm in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$, and the Lebesgue's dominated convergence theorem that

$$\mathcal{J}_{\lambda,m}(v) \leqslant \lim_{k \to +\infty} \inf \mathcal{J}_{\lambda,m}(v_k).$$

So by Theorem (1.3.4), there exists a minimizer ϑ of $\mathcal{J}_{\lambda,m}$ on $\mathcal{M}$.

Claim: ϑ is a weak solution of Problem (2.2).

Let $\varepsilon > 0$ and $\varphi \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$, then we define

$$v_\varepsilon = \min\{\bar{u}, \max\{\underline{u}, \vartheta + \varepsilon\varphi\}\} = \vartheta + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in \mathcal{M}$$

where

$$\varphi^\varepsilon = \max\{0, \vartheta + \varepsilon\varphi - \bar{u}\}, \varphi_\varepsilon = \max\{0, \underline{u} - \vartheta - \varepsilon\varphi\} \ in \ \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega) \tag{2.20}$$

Notice that $\mathcal{J}_{\lambda,m}$ is Gateâu differentiable at ϑ in the direction $(v_\varepsilon - \vartheta)$, then for all $t \in (0,1)$, $\vartheta +$

$t(v_\varepsilon - \vartheta) \in \mathcal{M}$, we get

$$\begin{aligned}\lim_{t\to 0}\frac{\mathcal{J}_{\lambda,m}(\vartheta + t(v_\varepsilon - \vartheta)) - \mathcal{J}_{\lambda,m}(\vartheta)}{t} &= \int\limits_\Omega (-\Delta)^s\vartheta(v_\varepsilon - \vartheta)dx + \int\limits_\Omega |\vartheta|^{m-1}(v_\varepsilon - \vartheta)\,dx \\ &\quad -\lambda\int\limits_\Omega |\vartheta|^{p-1}(v_\varepsilon - \vartheta)\,dx \geqslant 0\end{aligned}$$

which implies that

$$\int\limits_\Omega (-\Delta)^s\vartheta(\varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon)dx + \int\limits_\Omega |\vartheta|^{m-1}(\varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon)\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}(\varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon)\,dx \geqslant 0.$$

Hence

$$\int\limits_\Omega (-\Delta)^s\vartheta\varphi dx + \int\limits_\Omega |\vartheta|^{m-1}\varphi\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}\varphi\,dx \geqslant \frac{1}{\varepsilon}(\mathcal{E}^\varepsilon + \mathcal{E}_\varepsilon), \tag{2.21}$$

such that

$$\begin{aligned}\mathcal{E}^\varepsilon(\vartheta) &= \int\limits_\Omega (-\Delta)^s\vartheta\varphi^\varepsilon dx + \int\limits_\Omega |\vartheta|^{m-1}\varphi^\varepsilon\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}\varphi^\varepsilon\,dx \\ &= \int\limits_\Omega (-\Delta)^s(\vartheta - \bar{u})\varphi^\varepsilon dx + \int\limits_\Omega \varphi^\varepsilon(-\Delta)^s\bar{u}dx \\ &\quad + \int\limits_\Omega |\vartheta|^{m-1}\varphi^\varepsilon\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}\varphi^\varepsilon\,dx\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\mathcal{E}_\varepsilon(\vartheta) &= \int\limits_\Omega (-\Delta)^s\vartheta\varphi_\varepsilon dx + \int\limits_\Omega |\vartheta|^{m-1}\varphi_\varepsilon\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}\varphi_\varepsilon\,dx \\ &= \int\limits_\Omega (-\Delta)^s(\vartheta - \underline{u})\varphi_\varepsilon dx + \int\limits_\Omega (-\Delta)^s\underline{u}\varphi_\varepsilon dx + \int\limits_\Omega |\vartheta|^{m-1}\varphi_\varepsilon\,dx - \lambda\int\limits_\Omega |\vartheta|^{p-1}\varphi_\varepsilon\,dx\end{aligned}$$

We define

$$\Omega^\varepsilon = \{x \in \Omega : (\vartheta + \varepsilon\varphi)(x) \geqslant \bar{u} > \vartheta(x)\},$$

so that $|\Omega^\varepsilon| \to 0$ as $\varepsilon \to 0^+$ and also

$$\Omega^{\varepsilon c} = C\Omega^\varepsilon := \Omega \setminus \Omega^\varepsilon \subset \{x \in \Omega : (\vartheta + \varepsilon\varphi)(x) < \bar{u}\}$$

which implies that $|C\Omega^\varepsilon| \to 0$ as $\varepsilon \to 0^+$.

Now, we consider the term

$$\begin{aligned}\int_\Omega \varphi^{\epsilon}(-\Delta)^s(\vartheta-\bar{u})\,dx \;&= \iint_{D_\Omega} |(\vartheta-\bar{u})(x)-(\vartheta-\bar{u})(y)|^2(\varphi^{\varepsilon}(x)-\varphi^{\varepsilon}(y))d\mu\\
&= \iint_{\Omega^\varepsilon\times\Omega^\varepsilon} |(\vartheta-\bar{u})(x)-(\vartheta-\bar{u})(y)|^2 d\mu\\
&+\varepsilon\iint_{\Omega^\varepsilon\times\Omega^\varepsilon}((\vartheta-\bar{u})(x)-(\vartheta-\bar{u})(y))(\varphi(x)-\varphi(y))d\mu\\
&+\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})^2(x)d\mu+\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(x)d\mu\\
&-\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(x)(\vartheta-\bar{u})(y)d\mu-\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(x)d\mu\\
&+\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})^2(y)d\mu+\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(y)d\mu\\
&-\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(y)(\vartheta-\bar{u})(x)d\mu-\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(y)d\mu.\end{aligned}\tag{2.23}$$

Using the positivity of integrals, to obtain

$$\begin{aligned}\int_\Omega \varphi^{\epsilon}(-\Delta)^s(\vartheta-\bar{u})\,dx \;&\geqslant \varepsilon\iint_{\Omega^\varepsilon\times\Omega^\varepsilon}((\vartheta-\bar{u})(x)-(\vartheta-\bar{u})(y))(\varphi(x)-\varphi(y))d\mu\\
&+\varepsilon\iint_{\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(x)\phi(x)d\mu-\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(x)(\vartheta-\bar{u})(y)d\mu\\
&-\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(x)d\mu+\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(x)d\mu\\
&-\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(y)(\vartheta-\bar{u})(x)d\mu\\
&+\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(y)-\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(y)d\mu.\end{aligned}\tag{2.24}$$

We simplify the calculus

$$\begin{aligned}\int_\Omega \varphi^{\epsilon}(-\Delta)^s(\vartheta-\bar{u})\,dx \;&\geqslant \varepsilon\iint_{\Omega^\varepsilon\times\Omega^\varepsilon}((\vartheta-\bar{u})(x)-(\vartheta-\bar{u})(y))(\varphi(x)-\varphi(y))d\mu\\
&+\varepsilon\iint_{\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(x)\phi(x)d\mu-\varepsilon^2\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}\varphi(x)\varphi(y)d\mu\\
&-\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(x)d\mu+\varepsilon\iint_{\Omega^\varepsilon\times C\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(x)d\mu\\
&-\varepsilon^2\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}\varphi(y)\varphi(x)d\mu\\
&+\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(y)\varphi(y)-\varepsilon\iint_{C\Omega^\varepsilon\times\Omega^\varepsilon}(\vartheta-\bar{u})(x)\varphi(y)d\mu.\end{aligned}\tag{2.25}$$

We passe to the limit $\varepsilon\to 0$, we get

$$\frac{1}{\varepsilon}\int_\Omega \varphi^{\varepsilon}(-\Delta)^s(\vartheta-\bar{u})\,dx\geqslant o(1).$$

By the definition of the maximal solution $\bar{u}$, we have

$$\begin{aligned}\int\limits_{\Omega^\varepsilon} \varphi^\varepsilon(-\Delta)^s \bar{u} dx + \int\limits_{\Omega^\varepsilon} |\vartheta|^{m-1}\varphi^\varepsilon \, dx - \lambda \int\limits_{\Omega^\varepsilon} |\vartheta|^{p-1}\varphi^\varepsilon \, dx \quad &= \int\limits_{\Omega^\varepsilon} (-|\bar{u}|^{m-1} + |\vartheta|^{m-1})\varphi^\varepsilon dx \\ &\quad \lambda \int\limits_{\Omega^\varepsilon} (|\bar{u}|^{p-1} - |\vartheta|^{p-1})\varphi^\varepsilon dx \\ &\geqslant \int\limits_{\Omega^\varepsilon} (-|\bar{u}|^{m-1} - |\vartheta|^{m-1})\varphi^\varepsilon \, dx.\end{aligned} \tag{2.26}$$

Recalling $\varphi^\varepsilon = \max\{0, \vartheta + \varepsilon\varphi - \bar{u}\}$, and using inequality (1.20)

$$\begin{aligned}\int\limits_{\Omega^\varepsilon} (-|\bar{u}|^{m-1} + |\vartheta|^{m-1})\varphi^\varepsilon \, dx \quad &= \int\limits_{\Omega^\varepsilon} (-|\bar{u}|^{m-1} + |\vartheta|^{m-1})(\vartheta + \varepsilon\varphi - \bar{u}) dx \\ &\geqslant \varepsilon \int\limits_{\Omega^\varepsilon} \varphi(-|\bar{u}|^{m-1} + |\vartheta|^{m-1}) dx = o(\varepsilon).\end{aligned}$$

Then, we can conclude that

$$\mathcal{E}^\varepsilon(\vartheta) \geqslant o(\varepsilon)$$

and similarly

$$\mathcal{E}_\varepsilon(\vartheta) \leqslant o(\varepsilon)$$

hence

$$\frac{1}{\varepsilon}(\mathcal{E}^\varepsilon(\vartheta) + \mathcal{E}_\varepsilon(\vartheta)) = \frac{o(\varepsilon)}{\varepsilon} \to 0 \; as \; \varepsilon \to 0.$$

Therefore, by (2.21) for all $\varphi \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$

$$\int\limits_{\Omega} (-\Delta)^s \vartheta\varphi dx + \int\limits_{\Omega} |\vartheta|^{m-1}\varphi \, dx - \lambda \int\limits_{\Omega} |\vartheta|^{p-1}\varphi \, dx \geqslant o(1) \; as \; \varepsilon \to 0. \tag{2.27}$$

To conclude, by the same approach, we take $-\varphi$ instead of φ and letting $\varepsilon \to 0$, we get that ϑ is a weak solution of (2.15).

□

In the spirit of Alama's method, we prove that the weak solution ϑ obtained in Proposition 2.4.3 forms a local minimizer.

Proposition 2.4.4. *Let ϑ be such that*

$$\mathcal{J}_{\lambda,m}(\vartheta) = \inf\{\mathcal{J}_{\lambda,m}(u) : u \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega), \underline{u} \leqslant u \leqslant \bar{u} \;\; a.e. \; \Omega\}. \tag{2.28}$$

Then, ϑ is a local minimum of $\mathcal{J}_{\lambda,m}$ in $\mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$.

For our purpose, it is necessary to include the following lemma into our calculus.

Lemma 2.4.2. *Let* $h(t) := \lambda t^{p-1} - t^{m-1}$. *Then*

① $\langle \eta_n,\, \xi_n\rangle_s \leqslant 0$

② $\langle u_n - \bar{u},\, \eta_n\rangle_s \geqslant 0$

③ $\langle u_n - \underline{u},\, \xi_n\rangle_s \leqslant 0.$

④ $\langle u_n,\, \eta_n\rangle_s \geqslant \langle \bar{u},\, \eta_n\rangle_s = \int_\Omega h(\bar{u})\eta_n\, dx$

⑤ $\langle u_n,\, \xi_n\rangle_s \leqslant \langle \underline{u},\, \xi_n\rangle_s = \int_\Omega h(\underline{u})\xi_n\, dx$

Proof. From the definitions of the support of the functions ξ_n and η_n, and by a simple calculation, we get

①

$$\langle \eta_n,\, \xi_n\rangle_s = \iint_{\mathcal{D}_\Omega} (\eta_n(x)-\eta_n(y))(\xi_n(x)-\xi_n(y))d\mu = -\iint_{\mathcal{D}_\Omega} (\eta_n(x)\xi_n(y)+\eta_n(y)\xi_n(x))\, d\mu \leqslant 0$$

②

$$\langle u_n - \bar{u},\, \eta_n\rangle_s = -\int_{S_n^c}\int_{S_n} (u_n - \bar{u})(x)\eta_n(y)\, d\mu - \int_{S_n}\int_{S_n^c} (u_n - \bar{u})(y)\eta_n(x)\, d\mu \geqslant 0.$$

③

$$\langle u_n - \underline{u},\, \xi_n\rangle_s = -\int_{T_n^c}\int_{T_n} (u_n - \underline{u})(x)\xi_n(y)\, d\mu - \int_{T_n}\int_{T_n^c} (u_n - \underline{u})(y)\xi_n(x)\, d\mu \leqslant 0.$$

④

$$\langle u_n,\, \eta_n\rangle_s = \langle u_n - \bar{u},\, \eta_n\rangle_s + \langle \bar{u},\, \eta_n\rangle_s.$$

Since $\langle u_n - \bar{u},\, \eta_n\rangle_s \geqslant 0$, we get

$$\langle u_n,\, \eta_n\rangle_s \geqslant \langle \bar{u},\, \eta_n\rangle_s = \int_\Omega h(\bar{u})\eta_n.$$

⑤ $\langle u_n,\, \xi_n\rangle_s = \langle u_n - \underline{u},\, \xi_n\rangle_s + \langle \underline{u},\, \xi_n\rangle_s$, since $\langle u_n - \underline{u},\, \xi_n\rangle_s \leqslant 0$, we conclude

$$\langle u_n,\, \xi_n\rangle_s \leqslant \langle \underline{u},\, \xi_n\rangle_s = \int_\Omega h(\underline{u})\xi_n$$

□

We are now ready to demonstrate Proposition 2.4.4.

Proof. [**Proposition 2.4.4**] We argue by contradiction. Suppose that ϑ is not a local minimum of $\mathcal{J}_{\lambda,m}$. Then there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ such that

$$\|v_n - \vartheta\| \to 0 \; as \; n \to \infty \; and \; \mathcal{J}_{\lambda,m}(v_n) < \mathcal{J}_{\lambda,m}(\vartheta). \tag{2.29}$$

Let $\underline{u}$ and $\bar{u}$ are defined as in the proof of Proposition 2.4.3. Additionally, we define

$$u_n(x) = \max\{\underline{u}, \min\{v_n, \bar{u}\}\} = \begin{cases} \underline{u} & \text{if} \quad v_n(x) \leqslant \underline{u}, \\ v_n(x) & \text{if} \quad \underline{u} \leqslant v_n(x) \leqslant \bar{u}(x), \\ \bar{u}(x) & \text{if} \quad \bar{u}(x) \leqslant v_n(x). \end{cases}$$

$\eta_n = (v_n - \bar{u})_+ \geqslant 0$ and $\xi_n = (v_n - \underline{u})^- \geqslant 0$. So that $v_n = u_n + \eta_n - \xi_n$ and $u_n \in \mathcal{M}$. Correspondingly, we define the measurable sets

$$\begin{aligned} R_n &= \{x \in \Omega : \; u_n(x) = v_n(x)\} \\ T_n &= \{x \in \Omega : \; v_n(x) \leqslant \underline{u}\} := supp\, \xi_n \\ S_n &= \{x \in \Omega : \; v_n(x) \geqslant \bar{u}\} := supp\, \eta_n. \end{aligned}$$

Note that $\Omega = R_n \cup T_n \cup S_n$. We claim that

$$|S_n| \to 0 \text{ as } n \to \infty \text{ and } |T_n| \to 0 \text{ as } n \to \infty. \tag{2.30}$$

Indeed, let $\epsilon > 0$. For $\delta > 0$ to be suitably chosen, we set

$$\begin{aligned} E_n &= \{x \in \Omega : \; v_n(x) \geqslant \bar{u}(x) > \vartheta(x) + \delta\} \\ F_n &= \{x \in \Omega : \; v_n(x) \geqslant \bar{u}(x) \text{ and } \bar{u}(x) \leqslant \vartheta(x) + \delta\}. \end{aligned}$$

Since

$$\begin{aligned} 0 =& |\{x \in \Omega : \; \bar{u}(x) < \vartheta(x)\}| = \left| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega : \; \bar{u}(x) \leqslant \vartheta(x) + \frac{1}{j} \right\} \right| \\ &= \lim_{j\to\infty} \left| \left\{ x \in \Omega : \; \bar{u}(x) \leqslant \vartheta(x) + \frac{1}{j} \right\} \right|, \end{aligned}$$

then there exists $\delta_0 = \frac{1}{j_0}$ such that if $\delta < \delta_0$, then

$$|\{x \in \Omega : \; \bar{u}(x) \leqslant \vartheta(x) + \delta_0\}| \leqslant \frac{\epsilon}{2}.$$

Thus $|F_n| \leqslant \frac{\epsilon}{2}$. Alternatively, since $|v_n - \vartheta|_2 \to 0$ as $n \to \infty$, there exists $\eta = \frac{\delta^2\epsilon}{2}$ such that for all $n \geqslant n_0$,

$$\frac{\delta^2 \epsilon}{2} \geqslant \int_\Omega |v_n - \vartheta|^2 dx \geqslant \int_{E_n} |v_n - \vartheta|^2 dx \geqslant \delta^2 |E_n|.$$

Hence, $|E_n| \leqslant \frac{\epsilon}{2}$. Since $S_n \subset F_n \cup E_n$, we have

$$|S_n| \leqslant |E_n| + |F_n| \leqslant \epsilon$$

we conclude that $|S_n| \leqslant \epsilon$ for $n \geqslant n_0$ that is $|S_n| \to 0$. Using the same arguments we also deduce that $|T_n| \to 0$. Consequently, the assertion can be derived.

Now, we define

$$H(u) = \frac{\lambda}{p} u_+^p - \frac{1}{m} u_+^m := \int_0^u h(s)\, ds.$$

Let us recall the intermediary problem

$$\begin{cases} (-\Delta)^s w &= \varepsilon w + C_\varepsilon & \text{in} & \Omega, \\ w &\geqslant 0 & \text{in} & \Omega, \\ w &= 0 & \text{in } \mathbb{R}^N \setminus & \Omega, \end{cases}$$

notice that for any $\varepsilon < \lambda_{1,s}$ there exists $C_\varepsilon > 0$ such that

$$h(t) \leqslant \varepsilon t + C_\varepsilon \tag{2.31}$$

then by the definition of the function $H(t)$ we get

$$H(t) \leqslant \frac{\varepsilon}{2} t^2 + C_\varepsilon t.$$

Which implies that

$$H(\eta_n + \bar{u}) \leqslant \frac{\varepsilon}{2}(\eta_n + \bar{u})^2 + C_\varepsilon(\eta_n + \bar{u}). \tag{2.32}$$

$$H(\underline{u} - \xi_n) \leqslant \frac{\varepsilon}{2}(\underline{u} - \xi_n)^2 + C_\varepsilon(\underline{u} - \xi_n). \tag{2.33}$$

As $(\eta_n + \bar{u})^2 = \eta^2 + \bar{u}^2 + 2\eta\bar{u}$, we get

$$\int_{S_n} H(\eta_n + \bar{u})\, dx \leqslant \frac{\varepsilon}{2}\int_{S_n} \eta^2 dx + \int_{S_n} \bar{u}^2 dx + 2\int_{S_n} \eta\bar{u} dx + C_\varepsilon \int_{S_n} (\eta_n + \bar{u}) dx. \tag{2.34}$$

Using Hölder's and Poincaré's inequalities with the fact that $\bar{u}$ is uniformly bounded, and $|S_n| \to 0$ as $n \to \infty$, yields that

$$\begin{aligned}
\int_{S_n} H(\eta_n + \bar{u})\, dx &\leqslant \frac{\varepsilon}{2}\int_{S_n} |\eta_n|^2\, dx + \int_{S_n} |\eta_n|\, dx + o(1)\\
&\leqslant \frac{\varepsilon}{2}\int_{S_n} |\eta_n|^2\, dx + o(1)\|\eta_n\|_s^2 + o(1)\\
&\leqslant \frac{\varepsilon}{2\lambda_{1,s}}\|\eta_n\|_s^2 + o(1)\|\eta_n\|_s^2 + o(1).
\end{aligned} \tag{2.35}$$

Similarly, since $|T_n| \to 0$ we also get

$$\begin{aligned}
\int_{T_n} H(\underline{u} - \xi_n)\, dx &\leqslant \frac{\varepsilon}{2}\int_{T_n} |\xi_n|^2\, dx + \frac{\varepsilon}{2}\int_{T_n} \underline{u}^2 dx - \varepsilon\int_{T_n} \underline{u}\xi_n + \int_{T_n} \underline{u} - C_\varepsilon \int_{T_n} |\xi_n|\, dx.\\
&\leqslant \frac{\varepsilon}{2}\int_{T_n} |\xi_n|^2\, dx + o(1)\\
&\leqslant \frac{\varepsilon}{2\lambda_{1,s}}\|\xi_n\|_s^2 + o(1).
\end{aligned} \tag{2.36}$$

Where $\lambda_{1,s}$ is defined in (1.13).

Now, we can move on to the proof

$$\begin{aligned}
\mathcal{J}_{\lambda,m}(v_n) =& \frac{1}{2}\|v_n\|_s^2 - \int_\Omega H(v_n) dx \geqslant \frac{1}{2}\|u_n + \eta_n - \xi_n\|_s^2 - \int_\Omega H(v_n) dx\\
\geqslant& \frac{1}{2}\|u_n + \eta_n\|_s^2 + \frac{1}{2}\|\xi_n\|_s^2 - \langle u_n + \eta_n,\, \xi_n\rangle_s - \int_\Omega H(v_n) dx\\
\geqslant& \mathcal{J}_{\lambda,m}(u_n) + \frac{1}{2}\|u_n + \eta_n\|_s^2 - \frac{1}{2}\|u_n\|_s^2 + \frac{1}{2}\|\xi_n\|_s^2 - \langle u_n + \eta_n, \xi_n\rangle_s \quad - \int_\Omega H(v_n) dx + \int_\Omega H(u_n) dx\\
\geqslant& \mathcal{J}_{\lambda,m}(u_n) + \frac{1}{2}\|\eta_n\|_s^2 + \langle u_n,\, \eta_n\rangle_s - \langle u_n + \eta_n,\, \xi_n\rangle_s + \frac{1}{2}\|\xi_n\|_s^2 - \int_\Omega H(v_n) dx + \int_\Omega H(u_n) dx\\
\geqslant& \mathcal{J}_{\lambda,m}(u_n) + \frac{1}{2}\|\eta_n\|_s^2 + \langle u_n,\, \eta_n\rangle_s - \langle u_n + \eta_n,\, \xi_n\rangle_s + \frac{1}{2}\|\xi_n\|_s^2 - \int_{S_n} H(\eta_n + \bar{u}) dx\\
&- \int_{T_n} H(\underline{u} - \xi_n) dx + \int_{S_n} H(\bar{u}) dx + \int_{T_n} H(\underline{u}) dx\\
\geqslant& \mathcal{J}_{\lambda,m}(u_n) + \frac{1}{2}\|\eta_n\|_s^2 + \langle u_n,\, \eta_n\rangle_s - \langle u_n + \eta_n,\, \xi_n\rangle_s + \frac{1}{2}\|\xi_n\|_s^2\\
&- \int_{S_n} H(\eta_n + \bar{u}) dx - \int_{T_n} H(\underline{u} - \xi_n) dx.
\end{aligned}$$

By using Lemma 2.4.2 and 2.36 in the last inequality we obtain

$$\mathcal{J}_{\lambda,m}(v_n) \geqslant \mathcal{J}_{\lambda,m}(u_n) + (\frac{1}{2} - \frac{\varepsilon}{2\lambda_{1,s}} + o(1))\|\eta_n\|_s^2 + (\frac{1}{2} - \frac{\varepsilon}{2\lambda_{1,s}})\|\xi_n\|_s^2 + o(1).$$

Hence, recallig that $\mathcal{J}_{\lambda,m}(\vartheta) = \inf\limits_{u_n \in \mathcal{M}} \mathcal{J}_{\lambda,m}(u_n)$ we get

$$0 > \mathcal{J}_{\lambda,m}(v_n) - \mathcal{J}_{\lambda,m}(\vartheta) \geqslant (\frac{1}{2} - \frac{\varepsilon}{2\lambda_{1,s}} + o(1))\|\eta_n\|_s^2 + (\frac{1}{2} - \frac{\varepsilon}{2\lambda_{1,s}})\|\xi_n\|_s^2 + o(1).$$

Since $\varepsilon < \lambda_{1,s}$, we conclude that $\eta_n = \xi_n = 0$ for n large, so $v_n \in \mathcal{M}$ and then $\mathcal{J}_{\lambda,m}(v_n) \geqslant \mathcal{J}_{\lambda,m}(\vartheta)$, which is an contradiction with (2.29). This concludes point (3) of Theorem 2.4.1. $\square$

Now, we focus on an interesting question which naturally comes out from looking at the geometry of the functional $\mathcal{J}_{\lambda,m}$, where we observe that the energy of the local minimizer takes on different behaviors for a different ranges of λ compared to the threshold of negative energy level $\underline{\lambda}(m)$.

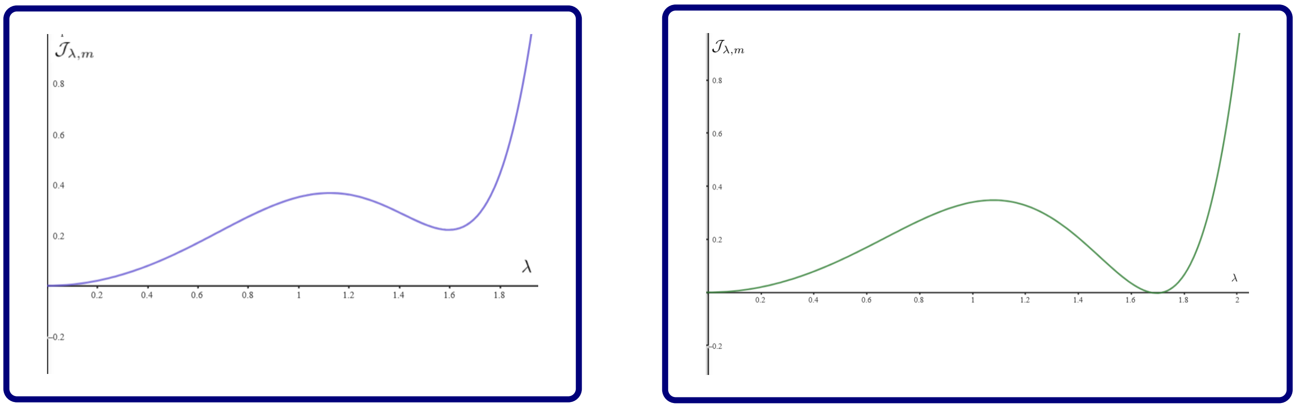

Figure 2.1: Case $\Lambda(m) < \lambda < \underline{\lambda}(m)$.

Figure 2.2: Case $\lambda = \underline{\lambda}(m)$.

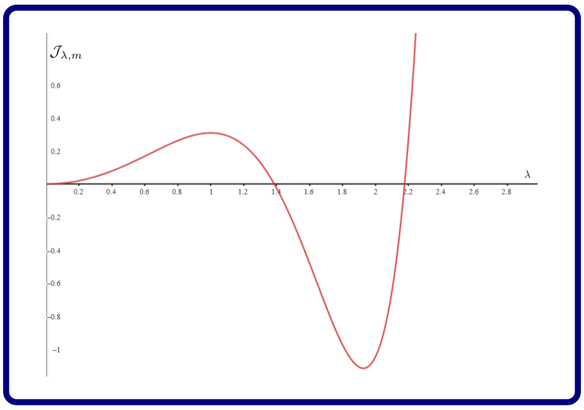

Figure 2.3: Case $\lambda > \underline{\lambda}(m)$.

Figure 2.4: Different energy behaviors according the value of λ

In this direction, the principal results is presented herewith

Proposition 2.4.5. *Assume* $0 < \Lambda(m) \leqslant \underline{\lambda}(m) < \infty$. *Then, Problem* (2.2) *has a local minimizer* $u_\lambda \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ *satisfying the following properties:*

(i) If $\Lambda(m) < \lambda < \underline{\lambda}(m)$, *then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) > 0$.

(ii) If $\lambda = \underline{\lambda}(m)$*, then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) = 0$.

(iii) If $\lambda > \underline{\lambda}(m)$*, then* $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) < 0$.

Proof. For the proof of property (i), by contradiction, assume that

$$\mathcal{J}_{\underline{\lambda},m}(\underline{u}) \leqslant 0. \tag{2.37}$$

Let us define $\lambda_\delta = \underline{\lambda}(m) - \delta$, therefore $\Lambda(m) < \lambda_\delta < \underline{\lambda}(m)$ and we have

$$\begin{aligned} 0 \geqslant \mathcal{J}_{\lambda_\delta,m}(\underline{u}) &= \frac{1}{2}\|\underline{u}\|_s^2 + \frac{1}{m}|\underline{u}|_m^m - \frac{\lambda_\delta}{p}|\underline{u}|_p^p \\ &= \mathcal{J}_{\underline{\lambda},m}(\underline{u}) + \frac{\delta}{p}|\underline{u}|_p^p \geqslant 0. \end{aligned}$$

Which is a contradiction. This proves that $\mathcal{J}_{\underline{\lambda},m}(\underline{u}) > 0$.

Next, in the borderline case $\lambda = \underline{\lambda}(m)$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n > \underline{\lambda}(m)$ and $\lambda_n \to \underline{\lambda}(m)$, as $n \to \infty$, and u_{λ_n} is the minimal solution of the problem $((2.15))_{\lambda=\lambda_n}$. Then,

$$\mathcal{J}_{\lambda_n,m}(u_{\lambda_n}) = \frac{1}{2}\|u_{\lambda_n}\|_s^2 + \frac{1}{m}|u_{\lambda_n}|_m^m - \frac{\lambda_n}{p}|u_{\lambda_n}|_p^p < 0. \tag{2.38}$$

It follows from Theorem 2.3.1 that there exists $C = C(\underline{\lambda}, m, p)$ such that

$$\|u_{\lambda_n}\|_s \leqslant C. \tag{2.39}$$

Hence, $(u_{\lambda_n})_n$ is bounded in $\mathcal{H}_0^s(\Omega)$. Thus, there exists $\underline{u} \in \mathcal{H}_0^s(\Omega)$ such that $u_{\lambda_n} \rightharpoonup \underline{u}$ weakly in $\mathcal{H}_0^s(\Omega)$, up to a subsequence $u_{\lambda_n} \to \underline{u}$ in $L^2(\Omega)$ and $u_{\lambda_n} \to \underline{u}$ a.e in Ω.

Now, we prove that $\underline{u} \neq 0$. We suppose by contradiction that $\underline{u} = 0$. Given that $|u_{\lambda_n}|_\infty \leqslant C$, and using the regularity results findings in [95], we deduce

$$u_{\lambda_n} \to 0 \text{ in } \mathcal{C}^{0,\beta}(\Omega), \text{ for some } \beta \in (0,1), \text{ and then } |u_{\lambda_n}|_\infty \to 0. \tag{2.40}$$

Recalling that $\mathcal{J}_{\lambda_n,m}(u_{\lambda_n}) < 0$, then

$$\frac{1}{2}\|u_{\lambda_n}\|_s^2 + \frac{1}{m}|u_{\lambda_n}|_m^m < \frac{\lambda_n}{p}|u_{\lambda_n}|_p^p. \tag{2.41}$$

By using Theorem 2.3.1 and Poincaré's inequality we get

$$\begin{aligned}\frac{1}{2}\|u_{\lambda_n}\|_s^2 <& \frac{\lambda_n}{p}|u_{\lambda_n}|_p^p < \frac{\lambda_n}{p}|u_{\lambda_n}|_\infty^{p-2}|u_{\lambda_n}|_2^2\\ <& \frac{\lambda_n}{p\lambda_{1,s}}|u_{\lambda_n}|_\infty^{p-2}\|u_{\lambda_n}\|_s^2.\end{aligned}$$

From this we obtain

$$\Big(\frac{1}{2}-\frac{\lambda_n}{p\lambda_{1,s}}|u_{\lambda_n}|_\infty^{p-2}\Big)\|u_{\lambda_n}\|_s^2 < 0. \tag{2.42}$$

Since $|u_{\lambda_n}|_\infty \to 0$, by (2.40), there exists $n_0 > 0$ such that

$$(\frac{1}{2}-\frac{\lambda_n}{p\lambda_{1,s}}|u_{\lambda_n}|_\infty^{p-2}) > 0 \text{ for all } n \geqslant n_0. \tag{2.43}$$

Thus, we conclude that $\|u_{\lambda_n}\|_s^2 = 0$ for all $n \geqslant n_0$. This is a contradiction with $\mathcal{J}_{\lambda_n,m}(u_{\lambda_n}) < 0$. At this point, let us prove that $\mathcal{J}_{\underline{\lambda},m}(\underline{u}) = 0$. By contradiction, suppose that

$$\mathcal{J}_{\underline{\lambda},m}(\underline{u}) < 0. \tag{2.44}$$

And we define $\lambda_\varepsilon := \underline{\lambda}(m) - \varepsilon$ then by the definition of $\underline{\lambda}(m)$ we have $\mathcal{J}_{\lambda_\varepsilon,m}(\underline{u}) \geqslant 0$. Then

$$\begin{aligned}\mathcal{J}_{\lambda_\varepsilon,m}(\underline{u}) &= \frac{1}{2}\|\underline{u}\|_s^2 + \frac{1}{m}|\underline{u}|_m^m - \frac{\lambda_\varepsilon}{p}|\underline{u}|_p^p\\ &= \mathcal{J}_{\underline{\lambda},m}(\underline{u}) + \frac{\varepsilon}{p}|\underline{u}|_p^p \geqslant 0.\end{aligned}$$

By choosing ε small enough, and since we have $\mathcal{J}_{\underline{\lambda},m}(\underline{u}) < 0$, we get that $\mathcal{J}_{\underline{\lambda},m}(\underline{u}) + \frac{\varepsilon}{p}|\underline{u}|_p^p < 0$. A contradiction is obtained. So $\mathcal{J}_{\underline{\lambda},m}(\underline{u}) = 0$.

Finally, we prove the last point of Proposition 2.4.5. Let $\lambda_\eta > \underline{\lambda}(m)$ where $\lambda_\eta = \underline{\lambda}(m) + \eta$ then by the definition of $\underline{\lambda}(m)$ we have $\mathcal{J}_{\lambda_\eta,m}(\underline{u}) < 0$.

□

2.5 Asymptotic Analysis

In this section, we are going to perform the asymptotic analysis of the sequence of minimum (resp. maximal) solutions $(u_{\lambda,m})_m$(resp. $(w_{\lambda,m})_m$) introduced in Proposition 2.4.2. In each case, we obtain a free boundary problem which asymptotically determines the behavior of solutions when $m \to \infty$.

First, we determine the threshold λ_* for the existence of solutions to (2.2). Exactly, at a certain point where we can give a uniform estimate of the energy level, under which the minimum solutions have a uniform upper bound for m sufficiently large.

Recalling that

$$\underline{\lambda}(m) = \inf\{\lambda : \mathcal{J}_{\lambda,m} \text{ has a local minimum } u_\lambda \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega) \text{ such that } \mathcal{J}_{\lambda,m}(u_\lambda) < 0\}. \quad (2.45)$$

Let

$$\sigma(\phi) := \frac{p}{2}\frac{\|\phi\|_s^2 |\phi|_\infty^{p-2}}{|\phi|_p^p},$$

and define

$$\lambda_* = \inf_{\phi \in \mathcal{H}_0^s(\Omega)\cap L^\infty(\Omega)} \sigma(\phi). \quad (2.46)$$

We will see that λ_* plays a crucial role in describing the asymptotic behavior of $\underline{\lambda}(m)$, and by using the results in Theorem 2.5.1 we will show that the infimum in (2.46) is attained.

Our first main result in this section is as follows.

Theorem 2.5.1. *We have*

$$\underline{\lambda}(m) = \lambda_* + o(1) \text{ as } m \to \infty. \quad (2.47)$$

Where

$$\lambda_* = \inf_{\xi \in \mathcal{H}_0^s(\Omega)\cap L^\infty(\Omega)} \frac{p}{2}\frac{\|\xi\|_s^2 |\xi|_\infty^{p-2}}{|\xi|_p^p}, \quad (2.48)$$

Proof. We will proceed in several steps.

Step 01: λ_* is well-defined.

We have $\sigma(\phi_1) < +\infty$, where ϕ_1 is the eigenfunction associated to the first eigenvalue of the fractional Laplacian with Dirichlet exterior condition denoted $\lambda_{1,s}$. In addition, for all $\phi \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ by using the L^∞-estimate (2.8), we obtain

$$\int_\Omega |\phi|^p\,dx = \int_\Omega |\phi|^{p-2}|\phi|^2\,dx \leqslant |\phi|_\infty^{p-2}\int_\Omega |\phi|^2\,dx \leqslant \frac{1}{\lambda_{1,s}}|\phi|_\infty^{p-2}\|\phi\|_s^2.$$

Thus, $\lambda_* \geqslant \frac{p}{2}\lambda_{1,s} > 0$ and then λ_* is positive.

We denote also $\mathcal{I}_\lambda$ the limit functional on $\mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ defined by

$$\mathcal{I}_\lambda(\xi) = \frac{1}{2}\|\xi\|_s^2 - \frac{\lambda}{p}|\xi|_p^p. \quad (2.49)$$

Step 02: We prove that

$$\underline{\lambda}(m) \leqslant \lambda_* + o(1), \tag{2.50}$$

where, $o(1) \to 0$ as $m \to \infty$.

Let $\xi \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ and $\kappa > 0$. Then

$$\mathcal{J}_{\lambda,m}(\kappa\xi) \leqslant g_m(\kappa),$$

where

$$g_m(\kappa) = \frac{\alpha}{\kappa^{p-2}} + \beta_m \kappa^{m-p}$$

and

$$\alpha = \frac{1}{2}\frac{\|\xi\|_s^2}{|\xi|_p^p}, \quad \beta_m = \frac{1}{m}\frac{|\xi|_m^m}{|\xi|_p^p}.$$

Since $m > p > 2$, we have $\lim\limits_{\kappa\to+\infty} g_m(\kappa) = \lim\limits_{\kappa\to 0} g_m(\kappa) = +\infty$, so that g_m reaches its global minimum at κ_m given by

$$\kappa_m = \left[\frac{p-2}{2(m-p)}\frac{\alpha}{\beta_m}\right]^{\frac{1}{m-2}} = \left[\frac{m(p-2)}{m-p}\|\xi\|_s^2\right]^{\frac{1}{m-2}} |\xi|_m^{\frac{-m}{m-2}},$$

we have

$$\mathcal{J}_{\lambda,m}(\kappa_m\xi) < 0 \Leftrightarrow \lambda > \sigma_m(\xi) := p \min g_m(\kappa) = p g_m(\kappa_m), \tag{2.51}$$

and

$$g_m(\kappa_m) = \left[\frac{1}{2}\right]^{\frac{m-p}{m-2}} \left[\frac{p-2}{m-p}\right]^{\frac{2-p}{m-2}} m^{\frac{2-p}{m-2}} \left[\frac{m-2}{m-p}\right] |\xi|_m^{\frac{-m(2-p)}{m-2}} |\xi|_p^{-p} \|\xi\|_s^{\frac{2(m-p)}{m-2}}.$$

Furthermore, we have

$$\lim_{m\to\infty} \kappa_m = \frac{1}{|\xi|_\infty}, \tag{2.52}$$

and

$$\lim_{m\to\infty} \sigma_m(\xi) = \frac{p}{2}\frac{\|\xi\|_s^2 |\xi|_\infty^{p-2}}{|\xi|_p^p} = \sigma(\xi). \tag{2.53}$$

Hence,

$$\begin{aligned}\lim_{m\to+\infty} \mathcal{J}_{\lambda,m}(\kappa_m\xi) &= \frac{1}{2}|\xi|_\infty^{-2}\|\xi\|_s^2 - \frac{\lambda}{p}|\xi|_p^p|\xi|_\infty^{-p} \\ &= \mathcal{I}_\lambda\Big(\frac{\xi}{|\xi|_\infty}\Big),\end{aligned}$$

where $\mathcal{I}_\lambda$ is defined in (2.49). Moreover, using (2.51) and (2.53), we obtain

$$\mathcal{I}_\lambda\Big(\frac{\xi}{|\xi|_\infty}\Big) < 0 \Leftrightarrow \lambda > \sigma(\xi) \geqslant \lambda_*. \tag{2.54}$$

Thus, for every $\lambda > \lambda_*$, we can fix $\xi \in \mathcal{H}_0^s(\Omega)\cap L^\infty(\Omega)$ such that (2.54) is satisfied. As a consequence,

$$\text{For all } \varepsilon > 0, \text{ there exists } m_0 \text{ such that: } \mathcal{J}_{\lambda,m}(\kappa_m\xi) < \mathcal{I}_\lambda\Big(\frac{\xi}{|\xi|_\infty}\Big) + \varepsilon.$$

Choosing ε small enough, one has

$$\mathcal{J}_{\lambda,m}(\kappa_m\xi) < 0.$$

This yields that $\lambda_* \geqslant \underline{\lambda}(m)$.

Step 03: For $\lambda = \underline{\lambda}(m)$, we have

$$\mathcal{J}_{\lambda,m}(u_m) = \frac{1}{2}\|u_m\|_s^2 + \frac{1}{m}|u_m|_m^m - \frac{\underline{\lambda}(m)}{p}|u_m|_p^p = 0, \tag{2.55}$$

then

$$\frac{p}{2}\frac{\|u_m\|_s^2}{|u_m|_p^p} \leqslant \underline{\lambda}(m). \tag{2.56}$$

Multiplying both sides of the inequality by $|u_m|_\infty^{p-2}$ then

$$\lambda_* \leqslant \frac{p}{2}\frac{\|u_m\|_s^2|u_m|_\infty^{p-2}}{|u_m|_p^p} \leqslant \underline{\lambda}(m)|u_m|_\infty^{p-2}, \tag{2.57}$$

using (2.50), we get

$$\lambda_* \leqslant \underline{\lambda}(m)|u_m|_\infty^{p-2} \leqslant \lambda_*|u_m|_\infty^{p-2}, \tag{2.58}$$

passing to the limit as $m \to \infty$ and using the fact that $|u|_\infty = 1$, we obtain

$$\underline{\lambda}(m) = \lambda_* + o(1). \tag{2.59}$$

This completes the proof of Theorem 2.5.1. □

Remark 2.5.1. *As an immediate consequence of the proof of Theorem 2.5.1, we have*

$$\lambda_* \leqslant \frac{p}{2}\frac{\|u_m\|_s^2|u_m|_\infty^{p-2}}{|u_m|_p^p} \leqslant \lambda_*|u_m|_\infty^{p-2}, \tag{2.60}$$

passing to the limit as $m \to \infty$ and using that $|u|_\infty \leqslant 1$, we end up with

$$\lambda_* \leqslant \frac{p}{2}\frac{\|u\|_s^2 |u|_\infty^{p-2}}{|u|_p^p} \leqslant \lambda_*. \tag{2.61}$$

Therefore, $(u_m)_m$ is a minimizing sequence of σ and u realize the minimum, that is $\lambda_* = \sigma(u)$.

Now, we provide an a priori upper bound at the level of the global minimum on the $\mathcal{H}_0^s$-norm.

Lemma 2.5.1. *Let $\lambda \geqslant \underline{\lambda}(m)$ and $(u_m)_m$ be a sequence of minimum solutions to Problem (2.15). Then, there exists a constant $C > 0$, such that*

$$\|u_m\|_s \geqslant C, \text{ for all } m > p. \tag{2.62}$$

Proof. Since $\lambda \geqslant \underline{\lambda}(m)$, then $\mathcal{J}_{\lambda,m}(u_m) \leqslant 0$. Hence,

$$\frac{1}{2}\|u_m\|_s^2 \leqslant \frac{1}{2}\|u_m\|_s^2 + \frac{\lambda}{m}|u_m|_m^m \leqslant \frac{\lambda}{p}|u_m|_p^p.$$

- **In the subcritical case:** if $p \leqslant \frac{2N}{N-2s}$, by Hölder inequality

  $$\frac{1}{2}\|u_m\|_s^2 \leqslant \frac{\lambda}{p}|u_m|_p^p \leqslant |u_m|_{2_s^\star}^p |\Omega|^{1-\frac{p}{2_s^\star}}.$$

  Sobolev inequality implies

  $$\frac{p}{2S\lambda|\Omega|^{1-\frac{2_s^\star}{p}}} \leqslant \|u_m\|_s^{p-2}.$$

- **In the supercritical case:** if $p > \frac{2N}{N-2s}$ then, by Theorem 2.3.1 we have

  $$\frac{1}{2}\|u_m\|_s^2 \leqslant \frac{\lambda}{p}|u_m|_p^{p-2_s^\star}|u_m|_p^{2_s^\star} \leqslant \frac{\lambda}{p}|u_m|_\infty^{p-2_s^\star}|u_m|_p^{2_s^\star} \leqslant \frac{\lambda}{p}|u_m|_p^p \leqslant \lambda^{\frac{p-2_s^\star}{m-p}+1}|u_m|_{2_s^\star}^{2_s^\star}.$$

  Using Sobolev inequality, we obtain

  $$\frac{pS}{2}\lambda^{-\frac{m-2_s^\star}{m-p}} \leqslant \|u_m\|_s^{2_s^\star-2}.$$

  Where S represents the Sobolev constant. This concludes the Lemma.

□

2.5.1 Limit of $u_{\lambda,m}$ as $m \to \infty$

Now, we state a very interesting result in our study that: as m tends to ∞, the limit function that determines the asymptotic behavior is in fact a solution of a free boundary problem with a non-linear right-hand side. More precisely, the main result is stated in the following theorem.

Theorem 2.5.2. *Assume that $2 < p < m$. Then, there exists $\lambda_* \geqslant \underline{\lambda}(m)$ such that for $\lambda > \lambda_*$, there exists $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ and*

$$u \in \mathcal{K} := \{u \in \mathcal{H}_0^s(\Omega) : 0 \leqslant u \leqslant 1\}, u \neq 0$$

such that the sequence of minimum solutions $(u_{\lambda,m})_m$ converges strongly to u in $\mathcal{H}_0^s(\Omega)$ and in every Lebesgue space. Moreover, u is a solution of the following limit problem

$$\begin{cases} (-\Delta)^s u + \mathbf{c}_u(x)\chi_{\{u=1\}} & = & \lambda u^{p-1} & \text{in} & \Omega, \\ u & \geqslant & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{in} & \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.63}$$

where $0 < \mathbf{c}_u(x) \leqslant \lambda$, $\mathbf{c}_u(1-u) = 0$ and $\Big|\{u=1\}\Big| > 0$.

To prove this Theorem 2.5.2, we will devide the proof in propositions. To this end, we start by the core of this Asymptotic analysis, the following Proposition which gives some properties of any sequence of solutions of Problem (2.15) when m tends to ∞ and shows that this problem converges to a bilateral problem posed on a convex set to be determined later. To be more precise, the sequence of solutions u_m belong to the following set

$$\mathcal{K}_m := \{u_m \in \mathcal{H}_0^s(\Omega) : 0 \leqslant u_m \leqslant \lambda^{\frac{1}{m-p}}\}$$

We first prove some a priori estimates for m fixed.

Proposition 2.5.1. *Let $u_m \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ be a weak solution of problem (2.2) then*

$$|u_m|_m^m \leqslant \lambda^{\frac{m}{m-p}}|\Omega| \tag{2.64}$$

$$\|u_m\|_s^2 + |u_m|_m^m \leqslant \lambda^{\frac{m}{m-p}}|\Omega| \tag{2.65}$$

Proof. By taking u_m as test function in (2.2), we have

$$\int_\Omega |u_m|^m\,dx \leqslant \iint_{D_\Omega} |u_m(x)-u_m(y)|^2\,d\mu + \int_\Omega u_m^m\,dx = \lambda \int_\Omega u_m^p\,dx.$$

Using Hölder's inequality

$$\int\limits_\Omega |u_m|^m\,dx \leqslant \lambda \int\limits_\Omega |u_m|^p\,dx \leqslant \lambda\Big[\int\limits_\Omega u_m^m\,dx\Big]^{\frac{p}{m}} |\Omega|^{1-\frac{p}{m}},$$

hence (2.64) is proved.

In the same way, we choose u_m as a test function in (2.2), and by Hölder's inequality and estimate (2.64), we get

$$\begin{aligned}\iint\limits_{D_\Omega} |u_m(x)-u_m(y)|^2\,d\mu + \int\limits_\Omega u_m^m\,dx &= \lambda \int\limits_\Omega u_m^p\,dx \leqslant \lambda[\lambda^{\frac{m}{m-p}}]^{\frac{p}{m}} |\Omega|^{\frac{p}{m}} |\Omega|^{1-\frac{p}{m}} \\ &= \lambda^{\frac{m}{m-p}} |\Omega|.\end{aligned}$$

☐

As a result of the a priori estimates, we can conclude the following convergence results as m tends to ∞.

Proposition 2.5.2. *Assume $\lambda \geqslant \lambda^*$, then there exists $u \in \mathcal{K}$ such that the sequence of solutions $(u_m)_m$ to Problem (2.2) satisfies the following convergence properties:*

(i) $u_m \to u$ strongly in $\mathcal{H}_0^s(\Omega)$.

(ii) $u_m \to u$ strongly in $L^q(\Omega)$, for all $q \geqslant 1$.

Where

$$u \in \mathcal{K} := \{u \in \mathcal{H}_0^s(\Omega) : 0 \leqslant u \leqslant 1\}, u \neq 0.$$

Then, u satisfies the following limit problem

$$\begin{cases} (-\Delta)^s u + \mathbf{c}_u(x)\chi_{\{u=1\}} &= \lambda u^{p-1} \quad \textit{in} \quad \Omega, \\ u &\geqslant 0 \qquad\;\; \textit{in} \quad \Omega, \\ u &= 0 \qquad\;\; \textit{in} \quad \mathbb{R}^N\backslash\Omega, \end{cases}$$

where

$$\mathbf{c}_u \in L^\infty(\Omega), 0 < \mathbf{c}_u \leqslant \lambda, \quad \mathbf{c}_u \neq 0, \quad \mathbf{c}_u(x)(1-u(x)) = 0 \text{ a.e. in } \Omega \tag{2.66}$$

Proof. As a consequence of $\mathcal{H}_0^s$-estimate in Proposition 2.5.1 and the L^∞-estimate given in Theorem (2.3.1), we deduce that u_m is bounded in $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$, then up to a subsequence, there exists $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ such that

- $u_m \rightharpoonup u$ weakly in $\mathcal{H}_0^s(\Omega)$.
- $u_m \to u$ strongly in $L^q(\Omega)$ for all $q \geqslant 1$

To prove the strong convergence of u_m, taking $(u_m - u)$ as test function in (2.15) we get

$$\langle u_m,\, u_m - u\rangle_s = \lambda \int_\Omega u_m^{p-1}(u_m - u)\, dx - \int_\Omega u_m^{m-1}(u_m - u)\, dx.$$

Using Theorem 2.3.1 once more, for any $r \geqslant 1$ we have

$$\left| \int_\Omega |u_m|^r (u_m - u)\, dx \right| \leqslant \lambda^{\frac{r}{m-p}} \int_\Omega |u_m - u|\, dx \to 0, \text{ as } m \text{ goes to } \infty.$$

Hence,

$$\langle u_m,\, u_m - u\rangle_s = o(1).$$

Therefore,

$$\begin{aligned} \|u_m - u\|_s &= \langle u_m,\, u_m - u\rangle_s - \langle u,\, u_m - u\rangle_s \\ &= o(1). \end{aligned}$$

Thus it follows that $u_m \to u$ strongly in $\mathcal{H}_0^s(\Omega)$. Let us prove that $u \neq 0$. To do this, we use (2.62), so

$$\|u_m\|_s \geqslant C > 0.$$

Letting $m \to \infty$, we obtain that $\|u\|_s \geqslant C$ and therefore $u \neq 0$.

From Theorem 2.3.1 we obtain

$$u \in \mathcal{K} := \left\{ v \in \mathcal{H}_0^s(\Omega) : 0 < v(x) \leqslant 1 \right\}.$$

Next, we show that the limit function u is a solution of the following bilateral obstacle problem posed on the convex set denoted $\mathcal{K}$

$$\langle u,\, (v-u)\rangle_s \geqslant \lambda \int_\Omega u^{p-1}(v-u)\, dx, \quad \forall v \in \mathcal{K}. \tag{2.67}$$

Let $v \in \mathcal{K}$, and θ be any real number such that $0 < \theta < 1$. Using $\theta v - u_m$ as test function in (2.15)

$$\langle u_m,\, \theta v - u_m\rangle_s + \int_\Omega u_m^{m-1} \left(\theta v - u_m\right)\, dx = \lambda \int_\Omega u_m^{p-1} \left(\theta v - u_m\right)\, dx. \tag{2.68}$$

We write the second term as

$$\begin{aligned}\int_\Omega u_m^{m-1} \left(\theta v - u_m\right)\, dx = &\int_{\{0\leqslant u_m < \theta v\}} u_m^{m-1} \left(\theta v - u_m\right)\, dx \\ &+ \int_{\{u_m \geqslant \theta v\}} u_m^{m-1} \left(\theta v - u_m\right)\, dx.\end{aligned}$$

We estimate each terms separately, and we get

$$u_m^{m-1} \left(\theta v - u_m\right) \leqslant [\theta v]^{m-1}(\theta v + \theta v) \leqslant 2\theta^m = o(1) \text{ as } m \to \infty, \tag{2.69}$$

for the second term, we have

$$u_m^{m-1} \left(\theta v - u_m\right) \leqslant 0 \text{ on } \{x : u_m(x) \geqslant \theta v\}$$

and this implies that

$$\limsup_{m\to\infty} \int_\Omega u_m^{m-1} \left(\theta v - u_m\right)\, dx \leqslant 0. \tag{2.70}$$

Returning to (2.68), we obtain

$$\begin{aligned}\theta\langle u_m,\, v\rangle_s + &\int_{\{x:0\leqslant u_m(x)<\theta v\}} u_m^{m-1} \left(\theta v - u_m\right)\, dx \\ &\geqslant \lambda \int_\Omega u_m^{p-1} \left(\theta v - u_m\right)\, dx + \|u_m\|_s^2.\end{aligned}$$

Passing to the limit as $m \to \infty$, and recalling that u_m strongly converges to u in $\mathcal{H}_0^s(\Omega)$. After using (2.69), and exploiting the semi-continuity of $\mathcal{H}_0^s-$norm, we derive

$$\theta\langle u,\, v\rangle_s \geqslant \lambda \int_\Omega u^{p-1} \left(\theta v - u\right)\, dx + \|u\|_s^2,$$

for any v in $\mathcal{K}$ and any $\theta \in (0, 1)$. To conclude, letting $\theta \to 1$, and (2.67) follows.

The question that arises: What kind of equation solves the limit function u?

To answer this question, we pass to the limit in the second term u_m^{m-1} and we determine the behavior as a solution of a free boundary problem.

From Theorem 2.3.1 shows that

$$|u_m|_\infty^{m-1} \leqslant \lambda^{\frac{m-1}{m-p}},$$

so, there exists $\mathbf{c}_u \in L^\infty(\Omega)$ (we note $\mathbf{c}_u$ because it depends on u) such that , up to a subsequence, $\left\{(u_m)^{m-1}\right\} \overset{*}{\rightarrow} \mathbf{c}_u$ weakly-$\star$ in $L^\infty(\Omega)$. As a consequence, we obtain that $\mathbf{c}_u \geqslant 0$. Furthermore, let us consider χ_E the characteristic function of the set

$$E := \{x \in \Omega : \mathbf{c}_u > \lambda\}$$

and using again the L^∞-estimate (2.8), we get

$$\int_\Omega u_m^{m-1}\chi_E \, dx \leqslant \lambda^{\frac{m-1}{m-p}}|E|,$$

which converge to

$$\lambda|E| < \int_E \mathbf{c}_u \, dx \leqslant \lambda|E|.$$

Hence, we conclude that $\mathbf{c}_u \leqslant \lambda$. Taking $\varphi \in \mathcal{H}_0^s(\Omega)$ as a test function in (2.15) and passing to the limit as $m \to \infty$, it follows that u satisfies

$$\langle u,\, \varphi\rangle_s + \int_\Omega \mathbf{c}_u\varphi \, dx = \lambda \int_\Omega u^{p-1}\varphi \, dx, \qquad \forall \varphi \in \mathcal{H}_0^s(\Omega). \tag{2.71}$$

Then, u is a solution of Problem (2.63). In the end, we just need to prove that $\mathbf{c}_u(x)\Big(1 - u(x)\Big) = 0$ a.e. in Ω. and $c_u \neq 0$. Let us take $\varphi = (v-u)$ as test function in (2.71), with $v \in \mathcal{K}$

$$\langle u,\, (v-u)\rangle_s + \int_\Omega \mathbf{c}_u(v-u)\, dx - \lambda\int_\Omega u^{p-1}(v-u)\, dx = 0, \quad \text{for all } v \in \mathcal{K}.$$

Using (2.67) we get

$$\int_\Omega \mathbf{c}_u(u-v)\, dx \geqslant 0, \quad \forall v \in \mathcal{K}.$$

Then, for any sequence $(v_j)_j \in \mathcal{K}$ such that $v_j \to 1$ in $L^1(\Omega)$, we reach that

$$\int_\Omega \mathbf{c}_u(u-1)\,dx \geqslant 0.$$

Hence, $\mathbf{c}_u \geqslant 0$ and $u \leqslant 1$, then $\mathbf{c}_u(x)\Big(1-u(x)\Big) = 0$ a.e. in Ω.
In order to conclude, it is only left to show that $\mathbf{c}_u \neq 0$. Thanks to the negativity of $\mathcal{J}_{\lambda,m}(u)$, which allows us to get

$$\lambda \int_\Omega u^p\,dx > \frac{p}{2}\|u\|_s^2.$$

On the other hand, by choosing $\varphi = u$ in (2.71), we obtain

$$\lambda \int_\Omega u^p\,dx = \int_\Omega \mathbf{c}_u u\,dx + \|u\|_s^2,$$

so that

$$\int_\Omega \mathbf{c}_u u\,dx > \left(\frac{p}{2}-1\right)\|u\|_s^2 > 0,$$

and since $\mathbf{c}_u \geqslant 0$, implies that $\mathbf{c}_u \neq 0$. □

Proof of Theorem 2.5.2 completed.

As a direct result of the previous Propositions 2.5.1 and 2.5.2, we arrive at the proof.

As an immediate consequence of the previous analysis, we conclude the following result concerning the asymptotic behavior of the sequence of the maximal solutions.

Corollary 2.5.1. *Let $(w_{\lambda,m})_m$ be the sequence of maximal solution of Problem (2.2). Then there exists $w \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$, such that $w_{\lambda,m}$ converges to w strongly in $\mathcal{H}_0^s(\Omega)$ and in every Lebesgue space. Furthermore, the same results as in Theorem 2.5.2 hold for w.*

2.6 Multiplicity & Asymptotic results

This section is dedicated to study the existence of a second solution, then we investigate the asymptotic behavior of this sequence of solutions as $m \to \infty$.

In order to obtain this multiplicity result, we first prove that for λ large enough, we can uniformly upper bound the energy level of $u_{\lambda,m}$ away from 0. More precisely we have

Theorem 2.6.1. *Let $p > 2$, then for all $\lambda > \lambda_*$ there exists $\bar{m} > p$ and $\rho > 0$, such that*

$$\mathcal{J}_{\lambda,m}(u_\lambda) < -\rho < 0, \quad \forall m \geqslant \bar{m}, \tag{2.72}$$

where u_λ *is a minimum solution to* (2.15).

Proof. We argue as in the proof of Lemma 2.5.1. By choosing $\xi = u$ in (2.54) we obtain

$$\mathcal{I}_\lambda\Big(\frac{u}{|u|_\infty}\Big) < 0 \Leftrightarrow \lambda > \sigma(u) \geqslant \lambda_*,$$

and, then

$$\mathcal{J}_{\lambda,m}(\kappa_m u) = \mathcal{I}_\lambda\Big(\frac{u}{|u|_\infty}\Big) + o(1).$$

So that, for ε small enough, we have

$$\mathcal{J}_{\lambda,m}(\kappa_m u) = \mathcal{I}_\lambda\Big(\frac{u}{|u|_\infty}\Big) + \varepsilon < 0.$$

Choosing ε small enough and letting

$$-\rho = \mathcal{I}_\lambda\Big(\frac{u}{|u|_\infty}\Big) + \varepsilon,$$

one has $\mathcal{J}_{\lambda,m}(\kappa_m \xi) < -\rho < 0$. ☐

2.6.1 Existence of the second solution

In this subsection, we will state the main multiplicity result of Problem (2.2).

Definition 2.6.1. *The critical points found at Mountain pass levels are called mountain pass solutions.*

Theorem 2.6.2. *Let* $0 < s < 1$ *and* $2 < p \leqslant 2^\star_s$. *Then, if* $\lambda > \lambda_*$ *there exists* $m_0 > 2^\star_s$ *for every* $m \geqslant m_0$, *the Problem* (2.2) *has a Mountain pass solution* $v_{\lambda,m} \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$ *such that* $v_{\lambda,m} \neq u_{\lambda,m}$.

The proof of this theorem is divided into several lemmas. First, we will show that the functional $\mathcal{J}_{\lambda,m}$ has the Mountain Pass Geometry (see Ambrostti Rabinowitz Theorem 1.3.2).

Lemma 2.6.1. *There exists* r_λ, $\rho_{m,\lambda} > 0$ *such that* $\mathcal{J}_{\lambda,m}(u) \geqslant \rho_{m,\lambda}$ *for* $u \in \mathcal{H}^s_0(\Omega) \cap L^m(\Omega)$, *if* $\|u\| = r_\lambda$.

Proof. By applying Sobolev and Hölder inequalities, we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,m}(u) &= \frac{1}{2}\|u\|_s^2 + \frac{1}{m}|u|_m^m - \frac{\lambda}{p}|u|_p^p \\ &\geqslant \frac{1}{2}\|u\|_s^2 + \frac{1}{m}|u|_m^m - \frac{\lambda S}{p}|\Omega|^{1-\frac{p}{2_s^\star}}\|u\|_s^p \\ &\geqslant \frac{1}{m}|u|_m^m + \|u\|_s^2\Big(\frac{1}{2} - \frac{\lambda S}{p}|\Omega|^{1-\frac{p}{2_s^\star}}\|u\|_s^{p-2}\Big).\end{aligned}$$

Let $R_0 = \|u\|_s$, and there exists p_0 such that $m \geqslant p_0 > 2_s^\star$ then to have

$$\frac{1}{2} - \frac{\lambda S}{p}|\Omega|^{1-\frac{p}{2_s^\star}}R_0^{p-2} > \frac{1}{p_0} > \frac{1}{m}.$$

We need

$$R_0 \leqslant \left[\frac{p}{C_0\lambda S|\Omega|^{1-\frac{p}{2_s^\star}}}\right]^{\frac{1}{p-2}}.$$

Therefore, by taking

$$r_\lambda = \min\left\{1, \left[\frac{p}{C_0\lambda S|\Omega|^{1-\frac{p}{2_s^\star}}}\right]^{\frac{1}{p-2}}\right\},$$

with $C_0(p_0) = \dfrac{2p_0}{p_0-2}$, we obtain that

$$\begin{aligned}\mathcal{J}_{\lambda,m}(u) &\geqslant \frac{1}{m}|u|_m^m + \frac{1}{p_0}\|u\|_s^2 \geqslant \frac{1}{m}\Big(|u|_m^m + \|u\|_s^m\Big) \\ &\geqslant \frac{1}{m2^{m-1}}\Big(|u|_m + \|u\|_s\Big)^m \\ &\geqslant \frac{1}{m2^{m-1}}r_\lambda^m =: \rho_{m,\lambda},\end{aligned}$$

for $r_\lambda \leqslant 1$, we get $\rho_{m,\lambda} > 0$. We conclude that there exists $r_\lambda > 0$ such that

$$\mathcal{J}_{\lambda,m}(u) \geqslant \rho_{m,\lambda}, \forall u \in \mathcal{H}_0^s(\Omega) \text{ with } \|u\| = r_\lambda.$$

□

Lemma 2.6.2. *There exists $m_2 \geqslant p$ such that $\|u\| > r_\lambda$ and $\mathcal{J}_{\lambda,m}(u) < 0$ for all $m \geqslant m_0$. Here u is a minimizer for λ_*.*

Proof. Let $(u_m)_m$ be a sequence of minimal solutions of (2.15). In the case $r_\lambda = \left(\frac{p}{C_0\lambda S|\Omega|^{1-\frac{p}{2_s^\star}}}\right)^{\frac{1}{p-2}}$, we have that $\kappa_m\|u_m\| > r_\lambda$ if

$$\lambda > \tau \left[\frac{m(p-2)}{2(m-p)}\right]^{\frac{p-2}{m-2}} \left[\frac{|u_m|_m}{\|u_m\|_s}\right]^{\frac{m(p-2)}{m-2}}, \tag{2.73}$$

where $\tau = \frac{p}{C_0 S|\Omega|^{1-\frac{p}{2_s^\star}}}$.

Observe that the right-hand side of (2.73) goes to $M := \tau \left[\frac{|u|_\infty}{\|u\|_s}\right]^{p-2}$, as $m \to +\infty$. Now, since $\lambda_* \geqslant M$, it follows that there exists $m_2 > p$ such that

$$\|u\| > r_\lambda \text{ and } \mathcal{J}_{\lambda,m}(u) < 0, \text{ for all } m > m_2.$$

This ends the proof of the Lemma. $\square$

Now, we define

$$c_m = \inf_{\gamma\in\Gamma_m} \max_{t\in[0,1]} \mathcal{J}_{\lambda,m}(\gamma(t)), \tag{2.74}$$

where

$$\Gamma_m = \{\gamma : [0,1] \to \mathcal{H}_0^s(\Omega) \cap L^m(\Omega), \gamma \text{ is continuous and } \gamma(0) = 0, \gamma(1) = \kappa_m u\}. \tag{2.75}$$

Then we have that $c_m \geqslant \rho_{m,\lambda}$. Therefore, the geometric hypotheses of the Mountain Pass lemma are fulfilled provided that

$$m \geqslant m^* := \max\{m_1(\lambda), m_2(\lambda)\} \text{ and } \lambda > \lambda_*. \tag{2.76}$$

Finally, we need to check that $\mathcal{J}_{\lambda,m}$ satisfies the Palais-Smale condition.

Lemma 2.6.3. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ such that*

$$\mathcal{J}_{\lambda,m}(v_n) \to c_m, \quad \mathcal{J}'_{\lambda,m}(v_n) \to 0 \textit{ in } (\mathcal{H}_0^s(\Omega) \cap L^m(\Omega))'.$$

Then $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$.

Proof. Assume that $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ is a Palais-Smale sequence of $\mathcal{J}_{\lambda,m}$ i.e.

$$\mathcal{J}_{\lambda,m}(v_n) \to c_m, \qquad \mathcal{J}'_{\lambda,m}(v_n) \to 0 \text{ in } (\mathcal{H}_0^s(\Omega) \cap L^m(\Omega))'.$$

This implies that there exists a constant $C > 0$ such that

$$|\mathcal{J}_{\lambda,m}(v_n)| \leqslant C,$$

since $\mathcal{J}_{\lambda,m}$ is coercive in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$ it implies that v_n is bounded in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$. Thus, $(v_n)_n$ weakly converges to some $v \in \mathcal{H}_0^s(\Omega)$ and strongly to v in $L^r(\Omega)$ for all $r \in [2, m)$. Hence, we have, as $n \to \infty$

$$\mathcal{J}_{\lambda,m}^{'}(v_n)(v_n - v) \to 0. \tag{2.77}$$

As a matter of fact, we have then

$$\begin{aligned} o(1) &= \langle \mathcal{J}_{\lambda,m}^{'}(v_n) - \mathcal{J}_{\lambda,m}^{'}(v), (v_n - v) \rangle \\ &= \|v_n - v\|_s^2 + \int_\Omega (|v_n|^{m-2}v_n - |v|^{m-2}v)(v_n - v)\, dx \\ &- \lambda \int_\Omega (|v_n|^{p-2}v_n - |v|^{p-2}v)(v_n - v)\, dx. \end{aligned}$$

Using the strong convergence of $(v_n)_{n\in\mathbb{N}}$ in $L^r(\Omega)$ and Hölder inequality we get

$$\begin{aligned} \left| \int_\Omega (|v_n|^{p-2}v_n - |v|^{p-2}v)(v_n - v)\, dx \right| &\leqslant \int_\Omega |v_n|^{p-1}|v_n - v|\, dx + \int_\Omega |v|^{p-1}|v_n - v|\, dx \\ &\leqslant |v_n|_p^{p-1}|v_n - v|_p + |v|_p^{p-1}|v_n - v|_p \to 0, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |v_n - v|^m\, dx &\leqslant | \int_\Omega (|v_n|^{m-2}v_n - |v|^{m-2}v)(v_n - v)|\, dx \\ &\leqslant |v_n|_m^{m-1}|v_n - v|_m + |v|_m^{m-1}|v_n - v|_m \to 0. \end{aligned}$$

As a consequence, we obtain

$$\|v_n - v\|_s^2 + |v_n - v|_m^m \to 0 \Rightarrow \|v_n - v\| \to 0, \text{ as } n \to +\infty.$$

This shows that $v_n \to v$ strongly in $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$, and we conclude that $\mathcal{J}_{\lambda,m}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$.

□

Proof of Theorem 2.6.2 completed. The Ambrosetti-Rabinowitz Theorem 1.3.2 implies the existence of a mountain pass critical point $v_{\lambda,m}$ with critical level $\mathcal{J}_{\lambda,m}(v_{\lambda,m}) = c_m \geqslant 0$. Since $\mathcal{J}_{\lambda,m}(v_{\lambda,m}) = c_m \geqslant 0 > \mathcal{J}_{\lambda,m}(u_{\lambda,m})$ we conclude that $v_{\lambda,m} \neq u_{\lambda,m}$. Finally, arguing in a way similar to that in the proof of Lemma 2.4.1, by considering the truncated energy $\mathcal{I}_{\lambda,m}$, we conclude that problem (2.2) has a second positive solution $v_{\lambda,m}$.

2.6.2 Asymptotic behavior of the mountain pass solution

In this subsection, we are interested by the asymptotic analysis for the sequence of mountain pass solutions constructed in Theorem 2.6.2 when $m \to \infty$, determining the limit profile as a solution of problem of free boundary nature introducing in the following theorem.

Theorem 2.6.3. *Assume that $\lambda > \lambda_*$. There exists $v \in \mathcal{K} := \{u \in \mathcal{H}_0^s(\Omega) : 0 \leqslant u \leqslant 1\}$ such that $v \neq 0$ and $v_{\lambda,m} \to v$ in $\mathcal{H}_0^s(\Omega)$ and in every Lebesgue space. Moreover, v satisfies*

$$\begin{cases} (-\Delta)^s v + \mathbf{h}_v(x) & = & \lambda v^{p-1} & \textit{in} & \Omega, \\ v & \geqslant & 0 & \textit{in} & \Omega, \\ v & = & 0 & \textit{in} & \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.78}$$

where $0 < \mathbf{h}_v(x) \leqslant \lambda$ and $\mathbf{h}_v(1-v) = 0$.

Proof. Let $(v_{\lambda,m})_m$ be a sequence of mountain pass solutions, for $\lambda > \lambda_*$ fixed. We argue in the same way as in the proof of Theorem 2.5.2, we easily obtain that there exists $v \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ such that

- $v_{\lambda,m} \rightharpoonup v$ weakly in $\mathcal{H}_0^s(\Omega)$.
- $v_{\lambda,m} \to v$ strongly in $L^q(\Omega)$ for all $q \geqslant 1$.
- $0 \leqslant v \leqslant 1$.

In order to show that $v \neq 0$, we use a different arguments since in this case we don't have a lower bound on $v_{\lambda,m}$ as in the estimate (2.62). We consider again the functional $\mathcal{I}_{\lambda_*}$ defined in (2.49). Notice that there exists $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ such that

$$\mathcal{I}_{\lambda_*}(u) < 0.$$

Let us define

$$\Gamma_* = \{\gamma : [0,1] \to \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega), \gamma \textit{ is continuous and } \gamma(0) = 0, \gamma(1) = u\},$$

and the mountain pass level

$$c_* := \inf_{\gamma \in \Gamma_*} \max_{[0,1]} \mathcal{I}_\lambda(\gamma(t)).$$

We claim that $\Gamma_m \subset \Gamma_*$. Indeed, let $\gamma \in \Gamma_m$ then

$$\gamma(1) = \mathcal{J}_{\lambda,m}(\kappa_m u) < 0.$$

Since $\mathcal{I}_\lambda(\kappa_m u_m) < \mathcal{J}_{\lambda,m}(\kappa_m u_m)$ then $\gamma \in \Gamma_*$. As consequence, we get

$$\max_{[0,1]} \mathcal{I}_\lambda(\gamma(t)) \leqslant \max_{[0,1]} \mathcal{J}_{\lambda,m}(\gamma(t))$$

and then

$$c_m = \inf_{\Gamma_m} \max_{[0,1]} \mathcal{J}_{\lambda,m}(\gamma(t)) \geqslant \inf_{\Gamma_*} \max_{[0,1]} \mathcal{I}_\lambda(\gamma(t)) = c_* \geqslant \rho_*,$$

where ρ_* is given by

$$\rho_* = \mathcal{I}_\lambda(\gamma(1)) = \frac{1}{2}\|u\|_s^2 - \frac{\lambda}{p}|u|_p^p > 0.$$

Therefore

$$\mathcal{J}_{\lambda,m}(v_{\lambda,m}) = c_m \geqslant \rho_* > 0,$$

yielding that $v_{\lambda,m} \neq 0$. The rest of the proof goes exactly as the proof of Theorem 2.5.2. ☐

2.7 Further discussions and concluding remarks

In this part of our study, we address the question: **What is the primary distinction between the two asymptotic behaviors?**

Observe that the difference appears on the functions $\mathbf{c}_u$ and $\mathbf{h}_v$. Indeed, while $\mathbf{c}_u \neq 0$ for every $p > 2$ and $\lambda > \lambda_*$, Thanks to the negativity of the energy functional which allows us to prove a lower bound, the difficulty seems in how to demonstrate that $\mathbf{h}_v \neq 0$ for p in $(2, 2_s^\star]$. Unfortunately, we cannot show that $\mathbf{h}_v \neq 0$, while we can only give some partial results depend on both the geometry, topology of the domain and the exponent p answering our question. Based on the results obtained by Dieb et.al in [46], we can state the following proposition.

Proposition 2.7.1. $\mathbf{h}_v \neq 0$ *under the following geometric conditions:*

① *If Ω is star-shaped domain and $p \geqslant 2_s^\star$, then $\mathbf{h}_v \neq 0$.*

② *Let $s \in (0,1)$ and $N \geqslant 2s$. If $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^2(\Omega)$. Then, there exists $p_0 := p_0(\Omega, s, \lambda) > 1$ such that $\mathbf{h}_v \neq 0$ for every $p \in (2, p_0)$.*

③ *If $\Omega \subset \mathbb{R}^2$ be a smooth convex domain and $p > 1$. Then, there exists p_0 large enough and $\sigma = \sigma(\Omega, p_0) \in (0,1)$, such that $\mathbf{h}_v \neq 0$ for $s \in (\sigma, 1]$.*

④ *If $\Omega \subset \mathbb{R}^N$ be a smooth bounded symmetric domain with respect to the hyperplanes $\{x_i = 0\}$ and convex in the direction x_i for all $1 \leqslant i \leqslant N$ with $N \geqslant 3$ and $p \in (2_s^\star - \varepsilon, 2_s^\star)$ for $\varepsilon > 0$ small enough. Then, there exists $\sigma = \sigma(\Omega, p) \in (0,1)$ such that for $s \in (\sigma, 1]$, $\mathbf{h}_v \neq 0$.*

To prove this proposition, we recall some useful tools that we will need in our discussion. We start by the well-known non-existence tool given in the nonlocal framework, the fractional Pohozaev identity by Ros-Oton and Serra in strictly star-shaped domains [96].

Theorem 2.7.1 (Pohozaev identity)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded star-shaped and $\mathcal{C}^{1,1}$-domain, $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function, and u a bounded solution to problem*

$$\begin{cases} (-\Delta)^s u & = f(u) \quad \text{in} \quad \Omega, \\ \qquad u & = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.79}$$

Then, $\dfrac{u}{\delta^s}$ has a continuous extension to Ω, that is, $\mathcal{C}_\delta^{0,\alpha}(\Omega)$, for some $\alpha \in (0,1)$. Moreover, the following identity holds:

$$(2s - N)\int_\Omega u f(u)\,dx + 2N\int_\Omega F(u)\,dx = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 \langle x, \nu\rangle\,d\sigma, \tag{2.80}$$

where ν is the unit outward normal to $\partial\Omega$ at x, $\delta(x)$ is the distance to the boundary, Γ is the Gamma function, and $F(u) = \displaystyle\int_0^u f(\tau)\,d\tau$.

Lemma 2.7.1. *Let $\Omega \subset \mathbb{R}^N$ be either a ball, a smooth convex domain, or a domain that is symmetric and convex in certain directions, and let $p \in (1, 2_s^\star - 1)$, $\lambda > -\lambda_{1,s}$. For $s \in (\sigma, 1]$ (with $\sigma \in (0,1)$), the fractional Dirichlet problem:*

$$\begin{cases} (-\Delta)^s u + \lambda u & = & u^p & \text{in} & \Omega, \\ u & > & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.81}$$

has a unique positive solution which is nondegenerate. This uniqueness and nondegeneracy also hold for least energy solutions in smooth convex domains, and for general solutions when p is sufficiently large in planar domains.

Lemma 2.7.2. *[46] Let $\Omega \subset \mathbb{R}^2$ be a smooth convex planar domain, $p > 1$. Then, there is $\sigma = \sigma(\Omega, p) \in (0,1)$ such that, for $s \in (\sigma, 1]$, the problem*

$$\begin{cases} (-\Delta)^s u & = & u^p & \text{in} & \Omega, \\ u & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega, \\ u & > & 0 & \text{in} & \Omega, \end{cases} \tag{2.82}$$

has a unique energy solution.

Lemma 2.7.3. *Let $s \in (0,1)$, $(p_n)_n \subset \left(1, \frac{N+2s}{N-2s}\right)$ be a sequence such that $p_n \to 1$. Let u_n be a solution to (2.81) with $p = p_n$, and let $M_n := |u_n|_\infty$. Then,*

$$M_n^{p_n-1} \to \lambda_{1,s} + \lambda$$

and

$$\frac{u_n}{M_n} \to \varphi_{1,s} \text{ uniformly in } \Omega \text{ and in } C^{2s+\alpha}_{loc}(\Omega) \text{ as } n \to \infty$$

for some $\alpha \in (0,1)$.

Lemma 2.7.4. *[46] Let $\Omega \subset \mathbb{R}^2$ be a smooth convex planar domain. Then, there exists $p_0 = p_0(\Omega) > 1$ such that, for any $p > p_0$, there is $\sigma = \sigma(\Omega, p) \in (0,1)$ such that, for $s \in (\sigma, 1]$, Problem (2.81) has a unique energy solution.*

Lemma 2.7.5. *[46] Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain symmetric with respect to the hyperplanes $\{x_i = 0\}$ and convex in the direction x_i for all $1 \leqslant i \leqslant N$. If either $N = 2$ and $p > 1$ or $N \geqslant 3$ and $p+1 \in (2^\star_s - \varepsilon, 2^\star_s)$ for $\varepsilon > 0$ small enough; then, there is $\sigma = \sigma(\Omega, p) \in (0,1)$ such that, for $s \in (\sigma, 1]$, Problem (2.81) has a unique solution.*

In the local case, we have the following result, see [[6], Theorem 1.2].

Lemma 2.7.6. *Let u_p be a solution to the local problem*

$$-\Delta u_p = \lambda u_p^{p-1} \text{ in } \Omega, \text{ and } u_p = 0 \text{ on } \partial\Omega. \tag{2.83}$$

Then

$$\lim_{p\to\infty} \|u_p\|_\infty = \sqrt{e}. \tag{2.84}$$

Now, we are ready to proceed with the demonstration.

Proof. ① In the first case we suppose that

- **Domain:** Ω star-shaped domain.

- **Exponent:** $p \geqslant 2_s^\star$.

Suppose by contradiction that $\mathbf{h}_v = 0$, then v is a positive solution to the problem

$$\begin{cases} (-\Delta)^s v &= \lambda v^{2_s^\star - 1} \quad \text{in} \quad \Omega, \\ v &> 0 \quad\quad\quad \text{in} \quad \Omega, \\ v &= 0 \quad\quad\quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

using Poholzaev identity (2.80) we have that

$$(2s - N)\int_\Omega v^{2_s^\star} dx + \frac{2N}{2_s^\star}\int_\Omega v^{2_s^\star} dx = \Gamma(1+s)^2 \int_{\partial\Omega} (\frac{v}{\delta^s})^2 (x, v) d\sigma.$$

since $(2s - N) + \dfrac{2N}{2_s^\star} = 0$, and Ω is a star shaped domain, we get that

$$\int_{\partial\Omega} \left(\frac{v}{\delta^s}\right)^2 (x, v) d\sigma = 0$$

then, $v = 0$ a.e. in Ω and this gives a contradiction since $v \neq 0$.

② In the second case, an analysis in the case when

- **Domain:** $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^2(\Omega)$.
- **Exponent:** p close to 2.

Assume that $\mathbf{h}_v = 0$, and v solves

$$\begin{cases} (-\Delta)^s v &= \lambda v^{p-1} \quad \text{in} \quad \Omega, \\ v &\geqslant 0 \quad\quad\quad \text{in} \quad \Omega, \\ v &= 0 \quad\quad\quad \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.85}$$

define $w_p = \lambda^{\frac{1}{p-2}} v$, the solution of following problem

$$\begin{cases} (-\Delta)^s w_p &= w_p^{p-1} \quad \text{in} \quad \Omega, \\ w_p &\geqslant 0 \quad\quad\quad \text{in} \quad \Omega, \\ w_p &= 0 \quad\quad\quad \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.86}$$

By Lemma 2.7.2, u_n is the unique solution of (2.86) for $p = p_n$ close to 2 and for $\lambda < \lambda_{1,s}$.

Furthermore, we set $M_n := |w_{p_n}|_\infty$. Therefore, by Lemma 2.7.3, we have

$$M_n^{p_n-2} \to \lambda_{1,s} \text{ as } n \to \infty\ .$$

Then

$$|v_{p_n}|_\infty^{p_n-2} = |\lambda^{\frac{-1}{p_n-2}} w_{p_n}|_\infty^{p_n-2} = \lambda^{\frac{-(p_n-2)}{p_n-2}} |w_{p_n}|_\infty^{p_n-2} = \frac{\lambda_{1,s}}{\lambda}$$

Hence

$$|v_{p_n}|_\infty \to \infty$$

This yields to a contradiction since $|v_{p_n}|_\infty \leqslant 1$.

③ In the third case, let us consider

- **Domain:** Ω is a convex domain of $\mathbb{R}^2$.
- **Exponent:** $p > 1$.

We use local arguments, for s is close to 1.

In the same way as before. Let $v_{s,p}$ be a positive solution of (2.85). Then by Lemma 2.7.4, $v_{s,p}$ is unique for p sufficiently large. Hence, $v_{\lambda,s,p} = \lambda^{\frac{1}{p-2}} v_{s,p}$ is the unique solution of (2.86) and we know from Lemma 2.7.6 and for s sufficiently close to 1 as mentioned in [Theorem 1.1, [46]] that

$$|v_{\lambda,s,p}|_\infty = |v_{\lambda,1,p}|_\infty + o_s(1), \text{ as } s \to 1. \tag{2.87}$$

Here $o_s(1) \to 0$ as $s \to 1$ and $v_{\lambda,1,p}$ is the unique positive solution of the local problem

$$-\Delta v = v^{p-1} \text{ in } \Omega, \text{ and } v = 0 \text{ on } \partial\Omega. \tag{2.88}$$

Notice that

$$|v_{\lambda,1,p}|_\infty = \lambda^{\frac{1}{p-2}} |v_{1,p}|_\infty$$

where $v_{1,p}$ satisfies (2.85) when $s = 1$. Thus, by Lemma 2.7.6, we obtain

$$|v_{\lambda,1,p}|_\infty = \lambda^{\frac{1}{p-2}} |v_{1,p}|_\infty = \sqrt{e}, \text{ as } p \to \infty.$$

Therefore,

$$|v_{\lambda,s,p}|_\infty = \sqrt{e} + o(1) > 1, \text{ as } p \to \infty. \tag{2.89}$$

So that $\mathbf{h}_v \neq 0$ in this case too.

④ In the fourth case, we assume that

- **Domain:** Ω is a bounded symmetric domain .
- **Exponent:** $p \in (2_s^\star - \varepsilon, 2_s^\star)$.

Let $v_{s,\varepsilon}$ be a positive solution of (2.85) with $p \in (2_s^\star - \varepsilon, 2_s^\star)$. Then by Lemma 2.7.5, $v_{s,\varepsilon}$ is unique for ε sufficiently small.

Hence, $v_{\lambda,s,\varepsilon} = \lambda^{\frac{1}{p-2s}} v_{s,\varepsilon}$ is the unique solution of (2.86). As shown in [46]

$$|v_{\lambda,s,\varepsilon}|_\infty = |v_{\lambda,1,\varepsilon}|_\infty + o_s(1), \text{ as } s \to 1. \tag{2.90}$$

Here $v_{\lambda,1,\varepsilon}$ is the unique positive solution of (2.88). Using [59] we have that $|v_{\lambda,1,\varepsilon}|_\infty \to \infty$, as $\varepsilon \to 0$. Therefore, $v_{\lambda,s,p} > 1$ also is a set of positive measure, then $\mathbf{h}_v \neq 0$.

□

3

A Nonlocal p-Logistic problem: existence, multiplicity and asymptotic behavior

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The aim of this chapter is to study a non-linear elliptic problem defined in a bounded domain involving the fractional p-Laplacian with a logistic type nonlinearity. We aim to characterize the range of parameter λ that allows the existence and multiplicity results, exhibiting three distinct cases, determined by the relationship between the exponent q and the condition on p. Additionally, we investigate the asymptotic behavior of the sequence of solutions, characterising their behavior as solution to a certain free boundary problem.

This chapter is an extended version of the work published in [16].

3.1 Introduction

This chapter is devoted to the study of the following nonlocal elliptic problem with logistic type reaction

$$\begin{cases} (-\Delta)_p^s u &= \lambda u^{q-1} - u^{m-1} \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

where Ω is bounded domain of $\mathbb{R}^N$, $N > ps$, $s \in (0,1)$, λ is a positive real parameter and m sufficiently large such that

$$2 \leqslant p < q < m.$$

In order to contextualize the research, we the review will begin by exploration of works relating to our study in the nonlocal framework.

In the nonlocal case:

Several studies focus on the following class of fractional elliptic problem

$$\begin{cases} (-\Delta)_p^s u &= f(u) \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2}$$

with various type of non-linearities [[21],[61], [68],[34]].

- Iannizoto et al. in [66] demonstrate global Hölder regularity result.
- When $\Omega = \mathbb{R}^N$, Torres Ledesma [105] established the existence of a nontrivial solution which is radially symmetric.
- For the reader's convenience, we refer to the most closely related work to our study, authored by Antonio Iannizzotto et al. in [61] where they proved existence and uniqueness of positive solutions within certain parameter ranges in cases of subdiffusion and equidiffusion of the following logistic problem

  $$(-\Delta)_p^s u = \lambda u^{q-1} - u^{r-1} \text{ in } \Omega, \text{ and } u = 0 \text{ in } \mathbb{R}^N \backslash \Omega,$$

  with $p \geqslant 2$ and $1 < q < r < p_s^\star$, where $p_s^\star$ is the critical Sobolev exponent. Additionally, in superdiffusive case they studied existence results according the value of λ.

In the local case:

Several authors have addressed nonlinear elliptic problem involving p-Laplacian such as [[54], [67], [71]], and a class of subcritical quasilinear elliptic equations with weight, variable parameter, and lack of compactness was studied in [88]. Also, in [29] the authors proved that the solution of p-Laplacian equation with poor summability convergences to the solution of variational inequality posed on a particular convex set.

Notice that when p = 2, Problem (3.1) reduces to the fractional elliptic problem that we addressed in Chapter 2.

As far as we are aware, The nonlocal p-Laplacian problem depending on supercritical nonlinearity and increasing power term is a novel contribution in the literature. Motivated by the above works, our main goal is to extend the results obtained in [22] to the p-Laplacian setting. Precisely, we discuss the existence results in terms of the parameters of λ, q, m even if $m > p_s^\star$ unlike [61] study, which deals only with the subcritical case. Additionally, we will study the asymptotic behavior of positive solutions as m tends to ∞, describing this behavior as free boundary limiting problem.

Notice that there are three distinct classifications for this problem type based on the principal exponent q with respect to p as in [50, 61]. In particular, we provide existence results for Problem (3.1) across various scenarios: superdiffusive, equidiffusive and subdiffusive case.

Plan of the chapter:

- ✓ In Section 3.2, we recall some preliminaries and we introduce some technical tools that will be used in this chapter.
- ✓ In Section 3.3 we establish a uniform bound for the solution of Problem (3.1).
- ✓ In Section 3.4, we prove the main existence results in the superdiffusive case, according to the value of λ.
- ✓ In Section 3.5, we perform an asymptotic analysis to the sequence of solutions obtained in Section 3.4.
- ✓ In Section 3.6, we treat the equidiffusive case, addressing existence, uniqueness and asymptotic analysis.
- ✓ In Section 3.7, we focus on the subdiffusive case, proving existence, uniqueness and asymptotic analysis.

3.2 Functional Framework and Variational Formulation

In this section, we will recall some useful tools that will be used in this chapter. Let Ω be a bounded domain in $\mathbb{R}^N$,

In order to obtain the existence result for the Problem (3.1), we consider the fractional Sobolev space defined by

$$X^s \overset{\text{def}}{=} \{u \in W_0^{s,p}(\Omega) : \int_\Omega u^m \, dx < +\infty\} = W_0^{s,p}(\Omega) \cap L^m(\Omega).$$

which is a reflexive Banach space and is renormed by setting

$$\|.\| = \|.\|_{s,p} + |.|_m.$$

Let us now make precise what we mean by a solution to Problem (3.1).

Definition 3.2.1. *We say that $u \in X^s$ is a weak solution of the Problem (3.1), if u satisfies*

$$\iint\limits_{D_\Omega} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y)) \, d\nu + \int\limits_\Omega u^{m-1} v \, dx = \lambda \int\limits_\Omega u^{q-1} v \, dx, \quad \forall v \in X^s. \tag{3.3}$$

where $d\nu = \frac{dxdy}{|x-y|^{N+ps}}$.

The fact that u is a weak solution of the problem (3.1) is equivalent to being a critical point of the energy functional

$$\mathcal{I}_{\lambda,m}(u) = \frac{1}{p} \iint_{D_\Omega} |u(x)-u(y)|^p \, d\nu + \frac{1}{m} \int_\Omega u_+^m \, dx - \frac{\lambda}{q} \int_\Omega u_+^q \, dx. \tag{3.4}$$

$\mathcal{I}_{\lambda,m}$ is well defined and of class $\mathcal{C}^1$. Furthermore, it's derivative is given by,

$$\langle \mathcal{I}'_{\lambda,m}(u), v\rangle = \iint\limits_{D_\Omega} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y)) \, d\nu + \int\limits_\Omega u_+^{m-1} v \, dx - \lambda \int\limits_\Omega u_+^{q-1} v \, dx, \forall v \in X^s. \tag{3.5}$$

Proposition 3.2.1. *Let $p > 1$ and $u, v \in W_0^{s,p}(\Omega)$, such that $(u-v)_+ \in W_0^{s,p}(\Omega)$ satisfies*

$$\langle (-\Delta)_p^s u - (-\Delta)_p^s v, (u-v)_+ \rangle_{s,p} \leqslant 0. \tag{3.6}$$

Then, $u \leqslant v$ in Ω.

Proof. See [[75], proof of Lemma 9]. We have that

$$\begin{aligned}&\langle(-\Delta)^s_p u-(-\Delta)^s_p v,(u-v)_+\rangle_{s,p}\\&=\iint\limits_{D_\Omega}|u(x)-u(y)|^{p-2}(u(x)-u(y)-|v(x)-v(y)|^{p-2}(v(x)-v(y))((u-v)_+(x)-(u-v)_+(y))\;d\nu\geqslant 0,\end{aligned}$$

we use the following identity

$$|a|^{p-2}a-|b|^{p-2}b=(p-1)(a-b)\int_0^1|b+t(a-b)|^{p-2}dt \tag{3.7}$$

with $a=u(x)-u(y)$ and $b=v(x)-v(y)$. hence

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))-|v(x)-v(y)|^{p-2}(v(x)-v(y))=(p-1)t[(u(x)-v(x))-(u(y)-v(y))]Q(x,y),$$

where

$$Q(x,y):=\int_0^1|u(x)-u(y)+t((v(x)-v(y))-(u(x)-u(y)))|^{p-2}\;dt.$$

Clearly, $Q(x,y)\geqslant 0$, and $Q(x,y)=0$ if and only if $u(x)=u(y)$ and $v(x)=v(y)$.

Now, we denote $\varphi=(u-v)$, and examine the integrand of the product:

$$\begin{aligned}[\varphi(x)-\varphi(y)][\varphi_+(x)-\varphi_+(y)]&=[\varphi_+(x)-\varphi_-(x)-\varphi_+(y)+\varphi_-(y)][\varphi_+(x)-\varphi_+(y)]\\&=[\varphi_+(x)-\varphi_+(y)]^2-[\varphi_-(x)+\varphi_-(y)][\varphi_+(x)-\varphi_+(y)].\end{aligned}$$

Since $\varphi_+(x)\varphi_-(x)=0$, the cross terms vanish and we obtain:

$$[\varphi(x)-\varphi(y)][\varphi_+(x)-\varphi_+(y)]=[\varphi_+(x)-\varphi_+(y)]^2+\varphi_-(x)\varphi_+(y)+\varphi_-(y)\varphi_+(x)\geqslant 0.$$

Hence

$$\langle(-\Delta)^s_p u-(-\Delta)^s_p v,(u-v)_+\rangle_{s,p}\leqslant 0,\ \text{ if and only if, } (u-v)_+=0,$$

which implies that $u(x)\leqslant v(x)$ a.e. Ω. ☐

The next two results can be found in Iannizzotto, Mosconi, and Squassina [[64], Theorems 1.1] for $p\geqslant 2$ and [[65], Theorem 1.1] for $1<p<2$. They will play a major role in the proof of existence results.

Proposition 3.2.2. *Let Ω be a smooth bounded domain, $p>1$. if $u\in W^{s,p}_0(\Omega)$ is a weak solution of* *the following problem*

$$\begin{cases} (-\Delta)_p^s u = f(x) & in \quad \Omega, \\ u = 0 & in \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.8}$$

where $f \in L^\infty(\Omega)$*. Then,* $\frac{u}{\delta^s}$ *admits a* α*-Hölder continuous extension to* $\bar{\Omega}$*, and there exist* $\alpha \in (0, s)$ *such that it satisfies the uniform bound*

$$\|\frac{u}{\delta^s}\|_{C^\alpha(\Omega)} \leqslant C|f|_\infty^{\frac{1}{p-1}},$$

where $C = C(N, p, s, \Omega) > 0$*. (See Section 1.2.2 for* $\alpha-$ *Holder definition).*

Proposition 3.2.3. *Let* $p > 1$ *and* $u \in W_0^{s,p}(\Omega)$*. Then,* u *is a local minimizer of* $\mathcal{I}_{\lambda,m}$ *in* $\mathcal{C}_0^s(\Omega)$*; there exists* $\rho_1 > 0$*, such that*

$$\mathcal{I}_{\lambda,m}(u) \leqslant \mathcal{I}_{\lambda,m}(u+v), \textit{ for all } v \textit{ in } \mathcal{C}_0^s(\Omega), \|v\|_{0,s} \leqslant \rho_1.$$

Then, u *is also a local minimizer of* $\mathcal{I}_{\lambda,m}$ *in* $W_0^{s,p}(\Omega)$*; there exists* $\rho_2 > 0$ *such that*

$$\mathcal{I}_{\lambda,m}(u) \leqslant \mathcal{I}_{\lambda,m}(u+v), \textit{ for all } v \textit{ in } W_0^{s,p}(\Omega), \|v\|_{s,p} \leqslant \rho_2.$$

Proof. For $p \geqslant 2$ see [[64], Theorem 1.1], and for $p \in (1, 2)$ see [[65], Theorem 1.1] . ☐

3.3 A priori estimate

In this section, we prove L^∞ a priori estimate, which will be crucial throughout our study.

Proposition 3.3.1. *Assume* $p < q < m$*. Let* $w \in X^s$ *is a weak positive solution of Problem* (3.1) *satisfies the following estimate*

$$|w|_\infty \leqslant \lambda^{\frac{1}{m-q}}. \tag{3.9}$$

Proof. For every $k, \varepsilon > 0$, we define the function $\Phi_\varepsilon : \mathbb{R} \to [0, 1]$

$$\Phi_\varepsilon(s) := \begin{cases} 0 & \text{if} \quad s < k, \\ \dfrac{s-k}{\varepsilon} & \text{if} \quad k < s < k + \varepsilon, \\ 1 & \text{if} \quad s \geqslant k + \varepsilon. \end{cases}$$

Let us denote the set

$$E_k = \{x \in \Omega : \; w(x) > k\}.$$

Taking $\varPhi_\varepsilon$ as a test function in the weak formulation of Problem (3.1)

$$\iint\limits_{D_\Omega} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\,d\nu + \int\limits_\Omega w_+^{m-1}\varPhi_\varepsilon(w)\ dx = \lambda\int\limits_\Omega w_+^{q-1}\varPhi_\varepsilon(w)\ dx.$$

Notice that $D_\Omega = (E_k \cup E_k^c)$, and $E_k = E_k^1 \cup E_k^2 = \{x \in \Omega :\ w(x) > k\}$, where

$$E_k^1 = \{x\in\Omega :\ k < w(x) < k+\varepsilon\},\ E_k^2 = \{x\in\Omega :\ w(x) > k+\varepsilon\}.$$

Let us split the set D_Ω in the following way

$$D_\Omega = D_1 \cup D_2 \cup D_3 \cup D_4$$

where

$$D_1 = \{(x,y)\in D_\Omega : w(x) > k, w(y) > k\}, D_2 = \{(x,y)\in D_\Omega : w(x)\leqslant k, w(y)\leqslant k\}$$

$$D_3 = \{(x,y)\in D_\Omega : w(x) > k,\ w(y)\leqslant k\}, D_4 = \{(x,y)\in D_\Omega : w(x)\leqslant k, w(y) > k\}$$

yielding that

$$\begin{aligned}
I_\Omega &= \iint\limits_{D_\Omega} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\ d\nu \\
&= \iint\limits_{D_1} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\ d\nu \\
&+ \iint\limits_{D_2} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\,d\nu \\
&+ \iint\limits_{D_3} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\ d\nu \\
&+ \iint\limits_{D_4} |w(x)-w(y)|^{p-2}(w(x)-w(y))(\varPhi_\varepsilon(w)(x)-\varPhi_\varepsilon(w)(y))\ d\nu \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}$$

To show that I_Ω is positive, we analyze each term separately.

$$\begin{aligned}
I_1 :\ &= \frac{1}{\varepsilon}\iint\limits_{E_k^1\times E_k^1} |w(x)-w(y)|^p\,dy\ dx + \frac{1}{\varepsilon}\iint\limits_{E_k^2\times E_k^1} (w(x)-w(y))^{p-1}(\varepsilon - w(y)+k)\,d\nu \\
&+ \frac{1}{\varepsilon}\iint\limits_{E_k^1\times E_k^2} (w(x)-w(y))^{p-1}(w(x)-k-\varepsilon)\,d\nu > 0.
\end{aligned}$$

Since in $E_k^c : \varPhi_\varepsilon(w)(x) = \varPhi_\varepsilon(w)(y) = 0$, then

$$\iint\limits_{D_2} |w(x) - w(y)|^{p-2}(w(x) - w(y))(\varPhi_\varepsilon(w)(x) - \varPhi_\varepsilon(w)(y))\ d\nu = 0$$

For I_3, since $x \in E_k^1$ then $w(x) \geqslant k$

$$\begin{aligned} I_3 : &= \frac{1}{\varepsilon}\iint\limits_{E_k^1 \times E_k^c} |w(x) - w(y)|^{p-2}(w(x) - w(y))(w(x) - k)\ d\nu + \iint\limits_{E_k^2 \times E_k^c} (w(x) - w(y))^{p-1}\ d\nu \\ &\geqslant \frac{1}{\varepsilon}\iint\limits_{E_k^1 \times E_k^c} |w(x) - k|^p\ d\nu + \iint\limits_{E_k^2 \times E_k^c} (w(x) - k)^{p-1}\ d\nu > 0 \end{aligned}$$

On the other hand, since $y \in E_k^2$ then $w(y) - k > 0$, and $x \in E_k^c$ then $w(x) < k$, we have that

$$\begin{aligned} I_4 : &= \frac{1}{\varepsilon}\iint\limits_{E_k^c \times E_k^1} |w(x) - w(y)|^{p-2}(w(x) - w(y))(-w(y) + k)\, d\nu - \iint\limits_{E_k^c \times E_k^2} (w(x) - w(y))^{p-1}\ d\nu \\ &\geqslant \frac{1}{\varepsilon}\iint\limits_{E_k^c \times E_k^1} |w(x) - w(y)|^p\ d\nu + \iint\limits_{E_k^c \times E_k^2} (w(y) - k)^{p-1}\ d\nu > 0. \end{aligned}$$

So that , we obtain

$$\begin{aligned} \int\limits_{E_k} w^{m-1}\varPhi_\varepsilon(w)\ dx &\leqslant \iint\limits_{D_\Omega} |w(x) - w(y)|^{p-2}(w(x) - w(y))(\varPhi_\varepsilon(w)(x) - \varPhi_\varepsilon(w)(y))\ d\nu + \int\limits_{E_k} w_+^{m-1}\varPhi_\varepsilon(w)\ dx \\ &\leqslant \lambda\int\limits_{E_k} w_+^{q-1}\ dx. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, by Fatou's Lemma and Hölder's inequality, we obtain

$$\begin{aligned} \int\limits_{E_k} w^{m-1}\ dx &\leqslant \lambda\int\limits_{E_k} w^{q-1}\ dx \\ &\leqslant \lambda\Big[\int\limits_{E_k} w^{m-1}\ dx\Big]^{\frac{q-1}{m-1}} |E_k|^{1-\frac{q-1}{m-1}}, \end{aligned}$$

that implies

$$k^{m-1}|E_k| \leqslant \int\limits_{E_k} w^{m-1}\ dx \leqslant \lambda^{\frac{m-1}{m-q}}|E_k|,$$

taking into account that $|E_k| \neq 0$, we deduce from the definition of k that

$$k \leqslant \lambda^{\frac{1}{m-q}},\ \forall t \in (0, |w|_\infty).$$

Specifically, taking $k = |w|_\infty$ yields the desired estimate (3.9). ☐

In the following result, the equidiffusive case $q = p$ and the subdiffusive case $q < p$, can be treated in a unified way. By following the same strategy developed for the superdiffusive regime as $p < q$, we derive an L^∞-a priori bound that holds for both cases as well.

Proposition 3.3.2. *Assume $q \leqslant p < m$. Let $w \in X^s$ be a weak positive solution of Problem (3.1). Then, w satisfies the following estimate*

$$|w|_\infty \leqslant \lambda^{\frac{1}{m-q}}. \tag{3.10}$$

Remark 3.3.1. *For the proof of Proposition 3.3.2, without altering the proof arguments of Proposition 3.3.1, the exponent of the reaction term q can be replaced by any exponent less to m, this yields (3.10).*

As a consequence of Propositions 3.3.1, 3.3.2, we provide the following a priori estimates.

Proposition 3.3.3. *Assume $p < q < m$ and let $w \in X^s$ be a weak solution of Problem (3.1). Then, w satisfies the following estimate*

$$|w|_m^m \leqslant \lambda^{\frac{m}{m-q}}|\Omega|. \tag{3.11}$$

$$\|w\|_{s,p}^p + |w|_m^m \leqslant \lambda^{\frac{m}{m-q}}|\Omega|. \tag{3.12}$$

Proof. Taking w as test function in (3.3)

$$\int_\Omega |w|^m \leqslant \iint_{D_\Omega} |w(x) - w(y)|^p \, d\nu + \int_\Omega w^m \, dx = \lambda \int_\Omega w^q \, dx,$$

we obtain by Hölder inequality

$$\int\limits_\Omega |w|^m \, dx \leqslant \lambda \int\limits_\Omega |w|^q \, dx \leqslant \lambda \Big[\int\limits_\Omega w^m \, dx \Big]^{\frac{q}{m}} |\Omega|^{1-\frac{q}{m}}.$$

Hence (3.11) is proved.

In the same way, we choose $v = w$ in (3.3), and by Hölder inequality and (3.11)

$$\begin{aligned} \iint\limits_{D_\Omega} |w(x) - w(y)|^p \, d\nu + \int\limits_\Omega w_+^m \, dx &= \lambda \int\limits_\Omega w_+^q \, dx \\ &\leqslant \lambda [\lambda^{\frac{m}{m-q}}]^{\frac{q}{m}} |\Omega|^{\frac{q}{m}} |\Omega|^{1-\frac{q}{m}} \\ &= \lambda^{\frac{m}{m-q}} |\Omega|. \end{aligned}$$

☐

3.4 Superdiffusive case: Existence and nonexistence

The principal existence result is presented in the following theorem, which improves and generalized the results obtained in Chapter 2, Theorem 2.4.1.

Theorem 3.4.1. *Let $s \in (0,1)$, $p > 1$ such that $N > ps$ and $p < q < m$. Then there exist positive constants $\Lambda(m)$, $\bar{\lambda}(m)$ with $0 < \Lambda(m) \leqslant \bar{\lambda}(m) < \infty$ such that:*

① *if $0 < \lambda < \Lambda(m)$, then Problem (3.1) does not have any solution in X^s.*

② *If $\lambda > \Lambda(m)$, Problem (3.1) has a maximal solution $z_{\lambda,m} \in X^s$.*

③ *if $\lambda > \Lambda(m)$, Problem (3.1) admits at least two positive solutions: a local minimum $u_{\lambda,m}$ in X^s such that*

- *If $\Lambda(m) < \lambda < \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) > 0$.*
- *If $\lambda = \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) = 0$.*
- *If $\lambda > \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_{\lambda,m}) < 0$.*

and a mountain pass solution $v_{\lambda,m}$ in X^s in every energy level.

④ *$(u_{\lambda,m})_\lambda$ is increasing with respect to λ.*

Outline of proof:

Before proceeding with the detailed demonstration, we outline the main steps of the proof.

① We begin by proving the existence of a largest value of λ, denoted $\Lambda(m)$ defined by

$$\Lambda(m) := \inf\{\lambda > 0 : \text{Problem (3.1) admits a non trivial solution}\}.$$

such that Problem (3.1) does not have any solution if $\lambda < \Lambda(m)$. This means that to assure the existence results we must take λ sufficiently large.

② For $\lambda > \Lambda(m)$, Problem (3.1) has a maximal solution.

③ We define $\bar{\lambda}(m)$ by

$$\bar{\lambda}(m) = \inf\{\lambda \geqslant \Lambda(m) : \mathcal{I}_{\lambda,m} \text{ has a local minimum } u_\lambda \in X^s \text{ such that } \mathcal{I}_{\lambda,m}(u_\lambda) < 0\}. \tag{3.13}$$

This critical value acts as a threshold that distinguishes different behaviors of the associated energy functional. In particular, Problem (3.1) admits at least two positive solutions for all $\lambda > \bar{\lambda}(m)$. More precisely

- We establish the existence of the first solution using a truncation method, yielding a local minimizer in $\mathcal{C}_0^s(\Omega)$. Following the approach of Iannizzotto et al. [68], we further show that this local minimizer is also a local minimizer in X^s-topology too.
    - We investigate the energy level of this solution as λ increases.
- From this local minimizer we obtain a second solution by applying Mountain Pass Theorem.
    - We analyze the energy behavior of this mountain pass solution with respect to the growth of λ, using a variant of Mountain Pass Theorem.

3.4.1 Non-existence result

Proposition 3.4.1. *If λ is small enough, then Problem (3.1) has only the trivial solution in X^s.*

Proof. Suppose that for λ small enough, Problem (3.1) admits a solution $u \in X^s$. Testing the equation by u, we obtain

$$\|u\|_{s,p}^p + |u|_m^m = \lambda |u|_q^q,$$

using estimate (3.9), we deduce that

$$\|u\|_{s,p}^p + |u|_m^m = \lambda |u|_q^q \quad \leqslant \quad \lambda |u|_\infty^{q-p} |u|_p^p \leqslant \frac{\lambda^{\frac{m-p}{m-q}}}{\lambda_{s,p}} \|u\|_{s,p}^p,$$

where $\lambda_{s,p}$ is the first eigenvalue of the fractional p-Laplacian defined in (1.13). Thus, we get

$$\|u\|_s^p \leqslant \frac{\lambda^{\frac{m-p}{m-q}}}{\lambda_{s,p}} \|u\|_{s,p}^p,$$

therefore

$$\lambda \geqslant \left(\lambda_{s,p}\right)^{\frac{m-q}{m-p}}.$$

This concludes the proposition. ☐

3.4.2 Existence results

Let us define

$$\Lambda(m) := \inf\{\lambda > 0 : \text{Problem (3.1) admits a non trivial solution in } X^s\}.$$

Proposition 3.4.2. *We have that* $0 < \Lambda(m) < \infty$.

Proof. From Proposition 3.4.1, we easily conclude that $\Lambda(m) > 0$. It remains to show that $\Lambda(m) < \infty$. To this end, we apply a direct minimization argument and construct a solution u_λ such that

$$\mathcal{I}_{\lambda,m}(u_\lambda) = \inf_{v\in X^s} \mathcal{I}_\lambda(v).$$

Thanks to the uniform bound givenin (3.9) of the solution u_λ, we obtain

$$\begin{aligned}\mathcal{I}_{\lambda,m}(u_\lambda) &\geqslant \frac{1}{p}\|u_\lambda\|_{s,p}^p + \frac{1}{m}|u_{\lambda,+}|_m^m - \lambda^{\frac{q}{m-q}}|\Omega| \\ &\geqslant \min\{\frac{1}{p}, \frac{1}{m}\}(\|u_\lambda\|_{s,p}^p + |u_{\lambda,+}|_m^m) - \lambda^{\frac{q}{m-q}}|\Omega| \to \infty \text{ as } \|u_\lambda\| \to \infty.\end{aligned}$$

Then, $\mathcal{I}_{\lambda,m}$ is coercive. Hence, we can easily show that $\mathcal{I}_{\lambda,m}$ weakly lower semi-continuous on X^s. Let ψ be the solution of the following problem

$$\begin{cases} (-\Delta)_p^s\psi + \psi^{m-1} = 1 & \text{in } \Omega, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{3.14}$$

We evaluate the energy functional of Problem (3.1), we have

$$\begin{aligned}\mathcal{I}_{\lambda,m}(\psi) &= \frac{1}{p}\|\psi\|_{s,p}^p + \frac{1}{m}|\psi|_m^m - \frac{\lambda}{q}|\psi|_q^q \\ &\leqslant \int_\Omega \psi\, dx - \frac{\lambda}{q}|\psi|_q^q \leqslant |\psi|_1(1 - \frac{\lambda}{q||\Omega|^{q-1}}|\psi|_1^{q-1}).\end{aligned}$$

We observe that, for λ sufficiently large $\mathcal{I}_{\lambda,m}(\psi) < 0$. Therefore,

$$\mathcal{I}_{\lambda,m}(u_\lambda) \leqslant \mathcal{I}_{\lambda,m}(\psi) < 0.$$

As a result, this yields that $u_\lambda \neq 0$. □

Existence of maximal solution

Proposition 3.4.3. *For all $\lambda > \Lambda(m)$, Problem* (3.1) *has a maximal solution $z_\lambda \in X^s$ in the sense: for every solution u to Problem* (3.1) *we have $z_\lambda \geqslant u$. Moreover, the family of solutions z_λ is increasing with respect to λ.*

Proof. Let us define $F(t) = Mt + f(t)$ where

$$f(t) = \begin{cases} \lambda t^{q-1} - t^{m-1} & \text{if} \qquad 0 \leqslant t \leqslant \bar{z}, \\ 0 & \text{otherwise} , \end{cases} \tag{3.15}$$

and $M > 0$ is fixed constant.

The function F is nondecreasing on $(0, \infty)$, since if $u \leqslant v$, the uniform bound (3.3.1) implies that $F(u) \leqslant F(v)$. We consider the iterative scheme for $n \geqslant 1$

$$\begin{cases} (-\Delta)^s_p u_{n+1} + M u_{n+1} & = M u_n + f(u_n) \quad \text{in} \quad \Omega \\ u_{n+1} & = 0 \qquad \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.16}$$

with initial data $u_0 = \bar{z}$ and $\bar{z} := \lambda^{\frac{1}{m-q}}$. By applying the maximum principle we obtain $u_{n+1} \leqslant u_n$, for all $n \geqslant 1$. Let $\underline{u}$ be a solution of (3.1), then using the estimate (3.9), we know $\underline{u} < \bar{z}$, then the classical monotone iteration argument implies the existence of z_λ such that $u_n \to z_\lambda$ and $\underline{u} \leqslant z_\lambda \leqslant \bar{z}$ a.e. in Ω.

It remains to prove that the monotonicity of the mapping $\lambda \mapsto z_\lambda$. Let $\lambda_1 < \lambda_2$, then z_{λ_1} is a subsolution to Problem (3.1) with $\lambda = \lambda_2$. Using the same argument as above we deduce that $z_{\lambda_1} < z_{\lambda_2}$. This ends the proof of the proposition. □

Existence of a local minimizer

To proceed with our analysis, we require a basic estimate which highlights how the nonlinearity behaves with respect to the parameter λ. This allows us to show that the sequence of solution u_λ is increasing with respect to λ.

Lemma 3.4.1. *Let $\lambda > 0$, $t > 0$ and $h_\lambda(t) = \lambda t^{q-1} - t^{m-1}$. Then, for any $\lambda > \mu > \Lambda(m)$, we have*

$$h_\lambda(t) \geqslant h_\mu(t).$$

Proof. Let $\lambda > \mu > \Lambda(m)$

$$\begin{aligned} h_\lambda(t) - h_\mu(t) &= \lambda t^{q-1} - t^{m-1} - (\mu t^{q-1} - t^{m-1}) \\ &\geqslant (\lambda - \mu) t^{q-1} > 0. \end{aligned}$$

Therefore, the result follows. □

To overcome the inability of applying the comparison principle directly, we use truncation method, which allows us to control h_λ to derive the increase of u_λ. Unfortunatly, we cannot establish the monotonicity of this mapping by using comparison principle, due to the fact that $t \mapsto \dfrac{f(t)}{t^{p-1}}$ is not decreasing for $t > 0$, since $q < m$.

Proposition 3.4.4. *Let $\lambda > \mu > \Lambda(m)$, and $w_\mu \in X^s$ is a maximal solution to Problem (3.1) for $\lambda > \mu > \Lambda(m)$. Then, the truncated problem*

$$\begin{cases} (-\Delta)_p^s u &= \hat{h}_\lambda(x,u) \quad \textit{in} \quad \Omega, \\ u &= 0 \qquad\qquad \textit{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.17}$$

where

$$\hat{h}_\lambda(x,t) = \begin{cases} \lambda {w_\mu}^{q-1} - {w_\mu}^{m-1} & \textit{if} \quad 0 \leqslant t \leqslant w_\mu \\ \lambda t^{q-1} - t^{m-1} & \textit{if} \quad t > w_\mu \end{cases}$$

admits a minimum solution $u_\lambda \in X^s$. Moreover,

1. *u_λ is increasing with respect to λ. Namely, if $\lambda > \mu > \Lambda(m)$ then $u_\lambda > u_\mu$.*
2. *u_λ is a positive solution to Problem (3.1).*

Proof. Consider the associated energy functional to the truncating problem (3.17)

$$\mathcal{E}_{\lambda,m}(u) = \frac{\|u\|_{s,p}^p}{p} - \int_\Omega H(x,u)\,dx$$

where $H(x,t) = \int_0^t \hat{h}_\lambda(x,s)\,dx.$

$\mathcal{E}_{\lambda,m}(u)$ is of class $\mathcal{C}^1(\Omega)$ and is well defined in X^s. To prove existence results to Problem (3.17), we use minimization method. The functional $\mathcal{E}_{\lambda,m}(u)$ is coercive.

Next, we prove the weak lower semi-continuity of $\mathcal{E}_{\lambda,m}$ on X^s;

Let $(v_k)_k \subset X^s$ be a minimizing sequence for $\mathcal{E}_{\lambda,m}$, due to the coercivity of $\mathcal{E}_{\lambda,m}$, we immediately see that $(v_k)_k$ is bounded in X^s, therefore there exists $v \in X^s$ such that, up to a subsequence still denoted

$(v_k)_k$

$$\begin{array}{llll} v_k & \rightharpoonup & v & \text{weakly in} \quad X^s, \\ v_k & \to & v & \text{strongly in} \quad L^q(\Omega) \text{ for all } q\in(1,+\infty), \ \cdot \\ v_k & \to & v & \text{a.e.} \quad \Omega, \end{array}$$

Besides, by the weak lower semi-continuity of the norm $\|.\|_{s,p}$ and $|.|_m$, we obtain

$$\begin{array}{rcl} \mathcal{E}_{\lambda,m}(u) & = & \dfrac{\|u\|_{s,p}^p}{p} - \dfrac{\lambda}{q}\|u\|_q^q + \dfrac{|u|_m^m}{m} - (1-\dfrac{1}{q})|w_\mu|_q^q + (1-\dfrac{1}{m})|w_\mu|_m^m \\ & \leqslant & \dfrac{1}{p}\lim_{k\to+\infty}\inf \|v_k\|_{s,p}^p + \dfrac{1}{m}\lim_{k\to+\infty}\inf |v_k|_m^m - \dfrac{\lambda}{p}\lim_{k\to+\infty}|v_k|_p^p - (1-\dfrac{1}{q})|w_\mu|_q^q + (1-\dfrac{1}{m})|w_\mu|_m^m \\ & \leqslant & \lim_{k\to+\infty}\inf(\dfrac{1}{p}\|v_k\|_{s,p}^p + \dfrac{1}{m}|v_k|_m^m - \dfrac{\lambda}{p}|v_k|_p^p - (1-\dfrac{1}{q})|w_\mu|_q^q + (1-\dfrac{1}{m})|w_\mu|_m^m) \\ & = & \lim_{k\to+\infty}\inf \mathcal{E}_{\lambda,m}(v_k) \end{array}$$

Therefore, there exits $u_\lambda \in X^s$ such that

$$\mathcal{E}_{\lambda,m}(u_\lambda) = \inf_{u\in X^s}\mathcal{E}_{\lambda,m}(u),$$

and u_λ is a solution of Problem (3.17).

Now, To prove that $u_\lambda > u_\mu$, if $\lambda > \mu > \Lambda$, where u_λ, u_μ denote the respective weak solutions of Problem (3.1) corresponding to λ and μ. From Lemma 3.4.1, we know that

$$(-\Delta)_p^s w_\mu = \mu {w_\mu}^{q-1} - {w_\mu}^{m-1} < \lambda {w_\mu}^{q-1} - {w_\mu}^{m-1}.$$

Taking $(w_\mu - u_\lambda)^+$ as test function

$$\begin{array}{rl} \langle(-\Delta)_p^s u_\lambda, (w_\mu - u_\lambda)^+\rangle_{s,p} & = \displaystyle\int_\Omega \hat{h}_\lambda(x,u_\lambda)(w_\mu - u_\lambda)^+\,dx \\ & = \displaystyle\int_\Omega (\lambda {w_\mu}^{q-1} - {w_\mu}^{m-1})(w_\mu - u_\lambda)^+\,dx \\ & = \displaystyle\int_\Omega (\lambda {w_\mu}^{q-1} - {w_\mu}^{m-1})(w_\mu - u_\lambda)^+\,dx + \int_\Omega (\mu {w_\mu}^{q-1} - {w_\mu}^{m-1})(w_\mu - u_\lambda)^+\,dx \\ & - \displaystyle\int_\Omega (\mu {w_\mu}^{q-1} - {w_\mu}^{m-1})(w_\mu - u_\lambda)^+\,dx. \end{array}$$

Hence

$$\langle(-\Delta)_p^s u_\lambda - (-\Delta)_p^s w_\mu, (w_\mu - u_\lambda)^+\rangle_{s,p} = \int_\Omega \Big((\lambda {w_\mu}^{q-1} - {w_\mu}^{m-1}) - (\mu {w_\mu}^{q-1} - {w_\mu}^{m-1})\Big)(w_\mu - u_\lambda)^+\,dx \geqslant 0$$

which implies that

$$\langle(-\Delta)_p^s w_\mu - (-\Delta)_p^s u_\lambda, (w_\mu - u_\lambda)^+\rangle_{s,p} \leqslant 0.$$

By using Lemma 3.2.1, we first obtain that $w_\mu - u_\lambda \leqslant 0$. Thus,

$$\hat{h}_\lambda(x, u_\lambda) = h_\lambda(u_\lambda),$$

which allows us to conclude that u_λ is a solution of Problem (3.1). In addition, since w_μ is a maximal solution of (3.1) for $\lambda = \mu > \Lambda(m)$, then we deduce that $u_\mu \leqslant w_\mu \leqslant u_\lambda$.

□

Before proceeding with our analysis, we recall the notion of local minimizers in both the $\mathcal{C}_0^s$ and X^s topologies.

Definition 3.4.1.

- *We say that $u \in X^s$ is a local minimizer of $\mathcal{I}_{\lambda,m}$ in $\mathcal{C}_0^s(\Omega)$, if there exists $\rho_1 > 0$ such that*

$$\mathcal{I}_{\lambda,m}(u) \leqslant \mathcal{I}_{\lambda,m}(u+v), \textit{ for all } v \textit{ in } \mathcal{C}_0^s(\Omega), \textit{ with } \|v\|_{0,s} \leqslant \rho_1.$$

- *We say that $u \in \mathcal{C}_0^s(\Omega)$ is a local minimizer of $\mathcal{I}_{\lambda,m}$ in X^s, if there exists $\rho_2 > 0$ such that*

$$\mathcal{I}_{\lambda,m}(u) \leqslant \mathcal{I}_{\lambda,m}(u+v), \textit{ for all } v \textit{ in } X^s, \textit{ with } \|v\| \leqslant \rho_2.$$

Proposition 3.4.5. *u_λ is a local minimizer of $\mathcal{I}_{\lambda,m}$ in the $\mathcal{C}_0^s(\Omega)$-topology. Moreover, u is also a local minimizer in the X^s-topology*

Proof. Let $\lambda > \mu > \Lambda(m)$ and let $\mathcal{V}$ be defined by

$$\mathcal{V} = \left\{ u_\mu + v \ \text{ s.t. } v \in \text{int}(\mathcal{C}_0^s(\Omega)^+) \right\}$$

which is an open subset of $\mathcal{C}_0^s(\Omega)$, and $u_\lambda > u_\mu$ we deduce that $u_\lambda \in \mathcal{V}$ and for all $u \in \mathcal{V}$ (i.e. $u > u_\mu$)

$$\begin{aligned}
\mathcal{E}_{\lambda,m}(u) &= \frac{\|u\|_{s,p}^p}{p} - \int_\Omega H(x,u)\,dx \\
&= \frac{\|u\|_{s,p}^p}{p} - \frac{\lambda}{q}|u|_q^q + \frac{|u|_m^m}{m} - (1-\frac{1}{q})|u_\mu|_q^q + (1-\frac{1}{m})|u_\mu|_m^m \qquad (3.18)\\
&= \mathcal{I}_{\lambda,m}(u) - C
\end{aligned}$$

such that $C \in \mathbb{R}$ and it is idependent of u. As u_λ minimizes $\mathcal{E}_{\lambda,m}(u)$, we can find $r > 0$ such that

$$\mathcal{E}_{\lambda,m}(u_\lambda) \leqslant \mathcal{E}_{\lambda,m}(u), \text{ for all } u \in \mathcal{B}_r(u_\lambda) \subset \mathcal{V}.$$

The relation between $\mathcal{E}_{\lambda,m}(u)$ and $\mathcal{I}_{\lambda,m}(u)$ given in (3.18), yields to

$$\mathcal{I}_{\lambda,m}(u_\lambda) - C \leqslant \mathcal{I}_{\lambda,m}(u) - C.$$

Proving that u_λ is indeed a local minimizer of $\mathcal{I}_{\lambda,m}$ in $\mathcal{C}_0^s$-topology. Then, by Proposition (3.2.3) is also a local minimizer in X^s-topology.

□

Now, we identify a value $\bar{\lambda}(m)$ that represents the threshold level for different behaviors of the energy functional

$$\bar{\lambda}(m) := \inf\{\lambda \geqslant \Lambda(m) : \mathcal{I}_{\lambda,m} \text{ has a local minimum } u_\lambda \text{ such that } \mathcal{I}_{\lambda,m}(u_\lambda) < 0\}. \tag{3.19}$$

Proposition 3.4.6. *Let $p < q < m$, if $\lambda = \bar{\lambda}(m)$ then Problem (3.1) admits a nontrivial weak solution $u_0 \in X^s$ with zero energy.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n > \bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$, as $n \to \infty$, denote $u_n = u_{\lambda_n}$ be a sequence of positive solutions of Problem (3.1) when $\lambda = \lambda_n$. Notice that $\mathcal{I}'_{\lambda,m}(u_n) = 0$, then from Proposition 3.3.3, there exists $C = C(\bar{\lambda}, m, q)$ such that

$$\|u_n\| \leqslant C. \tag{3.20}$$

Hence, $(u_n)_n$ is bounded in X^s, then there exists u_0 such that

- $u_n \rightharpoonup u_0$ in $W_0^{s,p}(\Omega)$,
- $u_n \to u_0$ in $L^r(\Omega)$, for all $r \geqslant 1$.
- $u_n \to u_0$ a.e in Ω.

So that, it remains to prove that $(u_n)_{n\in\mathbb{N}}$ converges strongly in $W_0^{s,p}(\Omega)$ to the nontrivial function u_0. Indeed, by taking $(u_n - u_0)$ as test function in (2.15) we get

$$\begin{aligned}\langle(-\Delta)_p^s u_n,\, u_n - u_0\rangle_{s,p} &= \lambda_n \int_\Omega u_n^{q-1}(u_n - u_0)\,dx - \int_\Omega u_n^{m-1}(u_n - u_0)\,dx\\ &\leqslant \lambda_n \|u_n\|_\infty^{q-1}\|u_n - u_0\|_1 \to 0, \text{ as } n \to \infty.\end{aligned}$$

By property 1 we can conclude that

$$u_n \to u_0 \textit{ strongly in } W_0^{s,p}(\Omega).$$

So, we can pass to the limit as $n \to \infty$ in $(3.57)_{\lambda=\lambda_n}$, we deduce that u_0 satisfies $(3.57)_{\lambda=\bar{\lambda}}$.

Claim. $u_0 \neq 0$.

Arguing by contradiction, assume that $u_n \to 0$ in $W_0^{s,p}(\Omega)$. Since $|u_n|_\infty \leqslant C$, such that $C = C(\lambda, m, q)$, using the regularity results in [66], we have

$$u_n \to 0 \text{ in } \mathcal{C}^{0,\beta}(\Omega), \text{ for some } \beta \in (0,1), \text{ and then } u_n \to 0 \text{ uniformly in } \Omega. \tag{3.21}$$

Let us set $v_n = \dfrac{u_n}{\|u_n\|_{s,p}}$ in $W_0^{s,p}(\Omega)$ such that v_n is bounded in $W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$, then up to a subsequence there exists v_0 such that $v_n \rightharpoonup v_0$ weakly in $W_0^{s,p}(\Omega)$ and strongly in $L^p(\Omega)$ for all $p > 1$. In addition, we have

$$(-\Delta)_p^s v_n = \frac{1}{\|u_n\|_{s,p}^{p-1}} (-\Delta)_p^s u_n$$

then v_n satisfies

$$(-\Delta)_p^s v_n = \lambda_n \|u_n\|_{s,p}^{q-p} v_n^{q-1} - \|u_n\|_{s,p}^{m-p} v_n^{m-1}. \tag{3.22}$$

We first prove that v_n converges strongly in $W_0^{s,p}(\Omega)$ to v. Indeed, using $v_n - v_0$ in $W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (3.22)

$$\begin{aligned} \langle (-\Delta)_p^s v_n, v_n - v_0 \rangle_{s,p} &= \lambda_n \|u_n\|_{s,p}^{q-p} \int_\Omega v_n^{q-1}(v_n - v_0)\,dx - \|u_n\|_{s,p}^{m-p} \int_\Omega v_n^{m-1}(v_n - v_0)\,dx \\ &\leqslant \lambda_n \frac{1}{\|u_n\|_{s,p}^{p-1}} \int_\Omega u_n^{q-1}(v_n - v_0)\,dx. \end{aligned} \tag{3.23}$$

In particular, $u_n \to 0$ uniformly in Ω, so for $n \in \mathbb{N}$ sufficiently large, we have $0 < u_n \leqslant 1$ which implies that

$$\begin{aligned} \langle (-\Delta)_p^s v_n, v_n - v_0 \rangle_{s,p} &\leqslant \lambda_n \frac{1}{\|u_n\|_{s,p}^{p-1}} \int_\Omega u_n^{p-1}(v_n - v_0)\,dx \\ &\leqslant \lambda_n \frac{1}{\|u_n\|_{s,p}^{p-1}} \|u_n\|_p^{p-1} |v_n - v_0|_p \\ &\leqslant \lambda_n \frac{C(\Omega)}{\|u_n\|_{s,p}^{p-1}} \|u_n\|_{s,p}^{p-1} |v_n - v_0|_p \\ &\leqslant \lambda_n C(\Omega) |v_n - v_0|_p \to 0. \end{aligned} \tag{3.24}$$

By property 1, we can conclude that $v_n \to v_0$ strongly in $W_0^{s,p}(\Omega)$, which implies that $\|v_n\|_{s,p} \to \|v_0\|_{s,p}$, hence $\|v_0\|_{s,p} = 1$. On the other hand, we take v_0 as test function in (3.22)

$$\begin{aligned} \langle (-\Delta)_p^s v_n, v_0 \rangle_{s,p} &= \lambda_n \|u_n\|_{s,p}^{q-p} \int_\Omega v_n^{q-1} v_0\,dx - \|u_n\|_{s,p}^{m-p} \int_\Omega v_n^{m-1} v_0\,dx \\ &\leqslant \lambda_n \|u_n\|_{s,p}^{q-p} |v_n|_\infty^{q-1} |v_0|_1 \to 0. \end{aligned} \tag{3.25}$$

We get $\|v_0\|_{s,p} = 0$, a contradiction.

☐

Corollary 3.4.1. *$\mathcal{I}_{\lambda,m}(u_\lambda)$ has different behaviors represented as follow:*

- *If $\Lambda(m) < \lambda < \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_\lambda) > 0$.*
- *If $\lambda = \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_\lambda) = 0$.*
- *If $\lambda > \bar{\lambda}(m)$, then $\mathcal{I}_{\lambda,m}(u_\lambda) < 0$.*

Proof. We proceed as in the proof in [22, Theorem 1.2]. The first and the last point are deduced from the definition of $\bar{\lambda}(m)$ and the second point is proved in Proposition 3.4.6. ☐

Existence of the second solution

In this section we prove the existence of a second solution from the local minimizer, obtained in Theorem 3.4.1, in each energy level by using a different version of the Mountain pass theorem.

➥ Positive/Negative energy level

Proposition 3.4.7. *Let $\lambda > \mu > \Lambda(m)$, and let $u_\lambda \in X^s$ be the local minimizer of $\mathcal{I}_{\lambda,m}$. Consider the truncated problem*

$$\begin{cases} (-\Delta)_p^s u &= g_\lambda(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.26}$$

where

$$g_\lambda(x,t) = \begin{cases} \lambda t^{q-1} - t^{m-1} & \text{if} \quad 0 \leqslant t \leqslant u_\lambda, \\ \lambda {u_\lambda}^{q-1} - t^{m-1} & \text{if} \quad t > u_\lambda. \end{cases}$$

Then, $u_\lambda \in X^s$ is a local minimizer of the truncated energy functional $\hat{\mathcal{I}}_{\lambda,m}$ in X^s. Moreover, 0 is also a strict local minimizer. As consequence, there exists $v_\lambda \in X^s \setminus \{0, u_\lambda\}$ a weak positive solution of Problem (3.26).

Proof. Let us consider the energy functional associated to Problem (3.26)

$$\begin{aligned} \hat{\mathcal{I}}_{\lambda,m}(u) &= \frac{\|u\|_{s,p}^p}{p} - \int_\Omega G(x,u)\,dx \\ &= \frac{\|u\|_{s,p}^p}{p} + \frac{|u|_m^m}{m} - \lambda \int_\Omega u_\lambda^{q-1} u\,dx - \lambda(1-\frac{1}{q})|u_\mu|_q^q \end{aligned}$$

where $G(x,t) = \displaystyle\int_0^t g_\lambda(x,s)ds$.

Step 1: We prove that the functional $\hat{\mathcal{I}}_{\lambda,m}$ admits a local minimizer u_λ.

Notice that, for $t \leqslant u_\lambda$, we have $g_\lambda(x,t) = \lambda t^{q-1} - t^{m-1}$, and if $t > u_\lambda$, $g_\lambda(x,t) = \lambda {u_\lambda}^{q-1} - t^{m-1}$. Therefore,

$$g_\lambda(x,t) \leqslant \lambda t^{q-1} - t^{m-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}_+,$$

by integration, we obtain

$$G_\lambda(x,t) \leqslant \lambda \frac{t^q}{q} - \frac{t^m}{m}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}_+,$$

then, we can conclude that

$$\hat{\mathcal{I}}_{\lambda,m}(u) \geqslant \mathcal{I}_{\lambda,m}(u), \text{ for all } u \text{ in } X^s. \tag{3.27}$$

Since u_λ is a local minimizer of $\mathcal{I}_{\lambda,m}(u)$, there exists $\rho > 0$ such that

$$\mathcal{I}_{\lambda,m}(u) \geqslant \mathcal{I}_{\lambda,m}(u_\lambda), \text{ for all } u \text{ in } X^s \text{ and } \|u - u_\lambda\| \leqslant \rho,$$

which implies that

$$\hat{\mathcal{I}}_{\lambda,m}(u) \geqslant \mathcal{I}_{\lambda,m}(u) \geqslant \mathcal{I}_{\lambda,m}(u_\lambda), \tag{3.28}$$

taking into account that

$$\mathcal{I}_{\lambda,m}(u_\lambda) = \hat{\mathcal{I}}_{\lambda,m}(u_\lambda),$$

hence

$$\hat{\mathcal{I}}_{\lambda,m}(u) \geqslant \hat{\mathcal{I}}_{\lambda,m}(u_\lambda), \text{ for all } u \text{ in } X^s \text{ and } \|u - u_\lambda\| \leqslant \rho.$$

So that, u_λ is a local minimizer of $\hat{\mathcal{I}}_{\lambda,m}$.

Step 2: we prove that 0 is a local minimizer of $\hat{\mathcal{I}}_{\lambda,m}$. Notice $f(t) = \lambda t^{q-1} - t^{m-1}$, for any $\varepsilon > 0$ there existes $\delta > 0$ s.t. $|t| < \delta$

$$f(t) \leqslant \varepsilon t^{p-1}. \tag{3.29}$$

by choosing $0 < \varepsilon < \lambda_{s,p}(\Omega)$. Thus, thanks to (3.9) we can find $\rho > 0$, such that for all $u \in \mathcal{C}_0^s(\Omega)$, $\|u\|_{0,s} \leqslant \rho$, we have

$$\hat{\mathcal{I}}_{\lambda,m}(u) \geqslant \frac{\|u\|_{s,p}^p}{p} - \frac{\varepsilon}{p} \int_\Omega u^p \, dx$$

by Poincaré's inequality we can conclude that

$$\hat{\mathcal{I}}_{\lambda,m}(u) \geqslant (\frac{\lambda_{s,p}}{p} - \frac{\varepsilon}{p})|u|_p^p > 0.$$

Therefore 0 is a local minimizer of $\hat{\mathcal{I}}_{\lambda,m}$ in $\mathcal{C}_0^s$-topology. Then, is also a local minimizer in X^s-topology.

Conclusion: u_λ and 0 are two local minimizers of $\hat{\mathcal{I}}_{\lambda,m}$ in X^s .

Step 3: We prove that the functional $\hat{\mathcal{I}}_{\lambda,m}$ is coercive.

From Proposition 3.4.2, we know that the energy functional $\mathcal{I}_{\lambda,m}$ is coercive in X^s. Moreover, using estimate (3.27), we conclude that $\hat{\mathcal{I}}_{\lambda,m}$ is also coercive in X^s.

Step 4: We need to check that the functional $\hat{\mathcal{I}}_{\lambda,m}$ satisfies the Palais-Smale condition.

Let $\{u_n\} \subset X^s$ is a sequence of $\hat{\mathcal{I}}_{\lambda,m}$ i.e.

$$\hat{\mathcal{I}}_{\lambda,m}(u_n) \to c$$

$$\hat{\mathcal{I}}'_{\lambda,m}(u_n) \to 0, in\ (X^s)'$$

This implies that, there exists a constant $C > 0$, such that

$$|\hat{\mathcal{I}}_{\lambda,m}(u_n)| \leqslant C,$$

We argue by contradiction that $u_n \to +\infty$, and since $\hat{\mathcal{I}}_{\lambda,m}$ is coercive, we get $\hat{\mathcal{I}}_{\lambda,m}(u_n) \to +\infty$ (contradiction). Then, u_n is bounded in X^s. In addition, X^s is a reflexive space, up to a subsequence, there exists $w \in X^s$, such that

$$\begin{array}{ccccc} u_n & \rightharpoonup & w & \text{weakly in} & X^s, n \to +\infty, \\ u_n & \to & w & \text{strongly in} & L^r(\Omega), \forall r \geqslant 1. \end{array}.$$

As $\hat{\mathcal{I}}'_{\lambda,m}(u_n) \to 0$ and $u_n \rightharpoonup w$ weakly in $W_0^{s,p}(\Omega)$ we also have $\hat{\mathcal{I}}'_{\lambda,m}(u_n)(u_n - w) \to 0$ and obviously $\hat{\mathcal{I}}'_{\lambda,m}(w)(u_n - w) \to 0$, then as $n \to +\infty$

$$\begin{array}{rcl} o(1) & = & \langle \hat{\mathcal{I}}'_{\lambda,m}(u_n) - \hat{\mathcal{I}}'_{\lambda,m}(w), (u_n - w) \rangle \\ & = & \langle (-\Delta)_p^s u_n - (-\Delta)_p^s w, u_n - w \rangle_{s,p} + \displaystyle\int_\Omega (|u_n|^{m-2} u_n - |w|^{m-2} w)(u_n - w)\, dx. \end{array}.$$

By Hölder inequality, L^∞-estimate and the strong convergence in every Lebesgue space we obtain through algebraic inequality (1.22) that

$$\begin{array}{rcl}\displaystyle\int_\Omega |u_n-w|^m\,dx & \leqslant & \displaystyle |\int_\Omega (|u_n|^{m-2}u_n-|w|^{m-2}w)(u_n-w)|dx \\ & \leqslant & |u_n|_m^{m-1}|u_n-w|_m+|w|_m^{m-1}|u_n-w|_m\to 0.\end{array}. \tag{3.30}$$

As consequence $\mathcal{L}_n\to 0$ as $n\to+\infty$ where

$$\begin{array}{rcl}\mathcal{L}_n & = & \langle(-\Delta)_p^s u_n-(-\Delta)_p^s w, u_n-w\rangle_{s,p} \\ & = & \|u_n\|_{s,p}^p+\|w\|_{s,p}^p-\langle(-\Delta)_p^s u_n, w\rangle-\langle(-\Delta)_p^s w, u_n\rangle.\end{array}$$

On the other hand, using that

$$\langle(-\Delta)_p^s u_n, w\rangle\leqslant\|u_n\|_{s,p}^{p-1}\|w\|_{s,p},$$

yields

$$\begin{array}{rcl}\mathcal{L}_n & \geqslant & \|u_n\|_{s,p}^p+\|w\|_{s,p}^p-\|u_n\|_{s,p}^{p-1}\|w\|_{s,p}-\|w\|_{s,p}^{p-1}\|u_n\|_{s,p} \\ & \geqslant & \|u_n\|_{s,p}^{p-1}(\|u_n\|_{s,p}-\|w\|_{s,p})+\|w\|_{s,p}^{p-1}(\|w\|_{s,p}-\|u_n\|_{s,p}) \\ & \geqslant & \Big(\|u_n\|_{s,p}-\|w\|_{s,p}\Big)\Big(\|u_n\|_{s,p}^{p-1}-\|w\|_{s,p}^{p-1}\Big).\end{array}$$

Since $\mathcal{L}_n\to 0$ and $u_n\rightharpoonup w$ in $W_0^{s,p}$, then $\|u_n-w\|_{s,p}\to 0$. As consequence of (3.30), we obtain

$$\|u_n-w\|_{s,p}+|u_n-w|_m\to 0\Rightarrow\|u_n-w\|\to 0, as\ n\to+\infty.$$

This shows that $u_n\to w$ strongly in X^s, and we conclude that $\hat{\mathcal{I}}_{\lambda,m}$ satisfies the compactness condition (Palais-Smale condition).

Step 5: Existence of the second solution to Problem (3.1).

We know from the mountain pass variant of Ghoussoub Preiss [55] that: If the Palais-Smale condition is satisfied and $\hat{\mathcal{I}}_{\lambda,m}$ has two distinct local minimum points, then $\hat{\mathcal{I}}_{\lambda,m}$ must have a third critical point. According to the above results and since 0 and u_λ are two distinct local minima. Since $\lambda\geqslant\bar{\lambda}(m)$, we have that $\hat{\mathcal{I}}_{\lambda,m}(u_\lambda)<0$. We define

$$c=\inf_{\gamma\in\Gamma_0^{u_\lambda}}\max_{t\in[0,1]}\hat{\mathcal{I}}_{\lambda,m}(\gamma(t)), \tag{3.31}$$

where

$$\Gamma_0^{u_\lambda}=\{\gamma:[0,1]\to X^s,\gamma \text{ is continuous and } \gamma(0)=0,\gamma(1)=u_\lambda\}. \tag{3.32}$$

such that $c\geqslant\max\{\hat{\mathcal{I}}_{\lambda,m}(0),\hat{\mathcal{I}}_{\lambda,m}(u_\lambda)\}$. Therefore, Mountain pass theorem [88] guarantees the existence of a critical point $v_\lambda \in X^s \setminus \{0, u_\lambda\}$, such that

$$\hat{\mathcal{I}}'_{\lambda,m}(v_\lambda) = 0.$$

Now, we aim to show that v_λ the solution of (3.26) is a second positive weak solution to Problem (3.1).

Taking $(v_\lambda - u_\lambda)^+$ as test function and using the fact that

$$\begin{aligned} \langle (-\Delta)_p^s v_\lambda, (v_\lambda - u_\lambda)^+ \rangle_{s,p} &= \int_\Omega g_\lambda(x, v_\lambda)(v_\lambda - u_\lambda)^+ \, dx \\ &\leqslant \int_\Omega (\lambda {u_\lambda}^{q-1} - {u_\lambda}^{m-1})(v_\lambda - u_\lambda)^+ \, dx \\ &= \langle (-\Delta)_p^s u_\lambda, (v_\lambda - u_\lambda)^+ \rangle_{s,p} \end{aligned}$$

Hence

$$\langle (-\Delta)_p^s v_\lambda - (-\Delta)_p^s u_\lambda, (v_\lambda - u_\lambda)^+ \rangle_{s,p} \leqslant 0.$$

This allows us to conclude, by using Proposition 3.2.1, that $v_\lambda \leqslant u_\lambda$. Thus, v_λ is a second solution of the main problem (3.1)

$$\begin{cases} (-\Delta)_p^s v_\lambda = \lambda {v_\lambda}^{q-1} - {v_\lambda}^{m-1} & \text{in } \Omega, \\ v_\lambda = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.33}$$

□

➥ Zero altitude case

In this direction we are looking for the second non trivial solution in the limiting case $\lambda = \bar{\lambda}(m)$ by using a different version of Mountain Pass Theorem.

Theorem 3.4.2. *For $\lambda = \bar{\lambda}(m)$ the problem admits a mountain pass solution $v^\star \in X^s$.*

Proof. we define

$$c^\star = \inf_{\gamma \in \Gamma^0_{u^\star}} \max_{t \in [0,1]} \mathcal{I}_{\lambda,m}(\gamma(t)), \tag{3.34}$$

where

$$\Gamma^0_{u^\star} = \{\gamma : [0,1] \to X^s, \gamma \text{ is continuous and } \gamma(0) = 0, \gamma(1) = u^\star\}. \tag{3.35}$$

$c^\star = 0$, then by Ghoussoub and Preiss mountain pass theorem [55], we deduce that there exists a mountain pass solution $v^\star$ of $\mathcal{I}_{\lambda,m}$ distinct from 0 and $u^\star$.

□

It remains to us to show the multiplicity result in the case when $\Lambda(m) < \lambda < \bar{\lambda}(m)$.

Theorem 3.4.3. *Assume that* $\Lambda(m) < \lambda < \bar{\lambda}(m)$ *Problem* (3.1) *admits a mountain pass solution* $v_2 \in X^s$.

Proof. From Corollary 3.4.1, we know that Problem (3.1) admits a local minimizer u_λ with $\mathcal{I}_{\lambda,m}(u_\lambda) > 0$. So, we define the path class

$$\Gamma^0_{u_\lambda} = \{\gamma : [0,1] \to X^s, \gamma \text{ is continuous and } \gamma(0) = u_\lambda, \gamma(1) = 0\}. \tag{3.36}$$

And the mountain pass level

$$c_1 = \inf_{\gamma \in \Gamma^0_{u_\lambda}} \max_{t \in [0,1]} \mathcal{I}_{\lambda,m}(\gamma(t)), \tag{3.37}$$

where $c_1 > \mathcal{I}_{\lambda,m}(u_\lambda) > 0$, then by Ambrostti Rabinowitz Theorem 1.3.2, we deduce that there exists a mountain pass solution v_1 of $\mathcal{I}_{\lambda,m}$ distinct from 0 and u_λ.

□

3.5 Asymptotic Analysis

The main goal of the present section is to perform an asymptotic analysis to the sequence of solutions found as m tends to ∞, investigating their different behaviors. In this study a crucial role will be played by the L^∞-a priori bound to prevent the lack of compactness even if $m > p^\star_s$.

Before studying the asymptotic behavior of the sequence of solutions, we first gain the asymptotic behavior of $\bar{\lambda}(m)$, the threshold for which $\mathcal{I}_{\lambda,m}$ has a negative energy, by using a comparison with the limit functional $\mathcal{I}_{\lambda,\infty}$, defined by

$$\mathcal{I}_{\lambda,\infty}(\xi) = \frac{1}{p}\|\xi\|^p_{s,p} - \frac{\lambda}{q}|\xi|^q_q, \forall \xi \in W^{s,p}_0(\Omega) \cap L^\infty(\Omega). \tag{3.38}$$

Let

$$Q(\phi) := \frac{q}{p}\frac{\|\phi\|^p_{s,p}|\phi|^{q-p}_\infty}{|\phi|^q_q} \quad \text{and} \quad \lambda^{\star\star} = \inf_{\phi \in W^{s,p}_0(\Omega)\cap L^\infty(\Omega)} Q(\phi). \tag{3.39}$$

The idea now is to see that $\lambda^{\star\star}$ is crucial asymptotic limit that describes the asymptotic behavior of $\bar{\lambda}(m)$ as $m \to \infty$. Furthermore, we will show that the infimum in $Q(\phi)$ is attained.

For simplicity, we denote $u_{\lambda,m}$ by u_m and $v_{\lambda,m}$ by v_m.

Theorem 3.5.1. *We have*

$$\lim_{m\to\infty} \bar{\lambda}(m) = \lambda^{\star\star}.$$

Proof. We first prove that $Q(\phi)$ is well defined and $\lambda^{\star\star} > 0$. For this purpose, we have $Q(\phi_{s,p}) < +\infty$, where $\phi_{s,p}$ is the first eigenfunction associated to the first eigenvalue $\lambda_{s,p}$ defined in (1.13). Furthermore, we have

$$\int_\Omega |\phi_{s,p}|^q = \int_\Omega |\phi_{s,p}|^{q-p}|\phi_{s,p}|^p \leqslant |\phi_{s,p}|_\infty^{q-p} \int_\Omega |\phi_{s,p}|^p \leqslant \frac{1}{\lambda_{s,p}}|\phi_{s,p}|_\infty^{q-p}|\phi_{s,p}|_{s,p}^p,$$

so that $\lambda^{\star\star} \geqslant \frac{q}{p}\lambda_{s,p} > 0$.

Next, we prove that

$$\bar{\lambda}(m) \leqslant \lambda^{\star\star} + o(1), \text{ as } m \to \infty.$$

Let $\psi \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and $t > 0$. Then, the functional $\mathcal{I}_{\lambda,m}$ is bounded

$$\mathcal{I}_{\lambda,m}(t\psi) \leqslant h_m(t),$$

where

$$h_m(t) = \frac{\alpha}{t^{q-p}} + \beta_m t^{m-q},$$

and

$$\gamma = \frac{1}{p}\frac{|\psi|_m^p}{|\psi|_q^q}, \quad \zeta_m = \frac{1}{m}\frac{|\psi|_m^m}{|\psi|_q^q}.$$

Let us analyze the function $h_m(t)$:

- Under the hypothesis $p < q < m$, we have $\lim\limits_{t\to+\infty} h_m(t) = \lim\limits_{t\to 0} h_m(t) = +\infty$. Consequently, h_m attains its global minimum at the point T_m given by

$$T_m = \Big(\frac{q-p}{m-q}\frac{\alpha}{\beta_m}\Big)^{\frac{1}{m-p}} = \Big(\frac{m(q-p)}{p(m-q)}|\psi|_m^p\Big)^{\frac{1}{m-p}}|\psi|_m^{\frac{-m}{m-p}}.$$

This allows us to obtain the condition under which $\mathcal{I}_{\lambda,m}(T_m\psi) < 0$

$$\mathcal{I}_{\lambda,m}(T_m\psi) < 0 \Leftrightarrow \lambda > Q_m(\psi) = qh_m(T_m), \tag{3.40}$$

where $h_m(T_m)$ is given by

$$h_m(T_m) = (\frac{1}{p})^{\frac{m-q}{m-p}}(\frac{q-p}{m-q})^{\frac{p-q}{m-p}}(m)^{\frac{p-q}{m-p}}(\frac{m-p}{m-q})|\psi|_m^{\frac{-m(p-q)}{m-p}}|\psi|_q^{-q}\|\psi\|_{s,p}^{\frac{p(m-q)}{m-p}}.$$

Observe that, as m tends to ∞ we obtain

$$\lim_{m\to\infty} T_m = \frac{1}{|\psi|_\infty}, \tag{3.41}$$

and

$$\lim_{m\to\infty} Q_m(\psi) = \frac{q}{p}\frac{\|\psi\|_{s,p}^p |\xi|_\infty^{q-p}}{|\psi|_q^q} = Q(\psi). \tag{3.42}$$

This indicates that

$$\lim_{m\to+\infty} \mathcal{I}_{\lambda,m}(T_m\psi) = \frac{1}{p}|\psi|_\infty^{-p}\|\psi\|_{s,p}^p - \frac{\lambda}{q}|\psi|_q^q|\psi|_\infty^{-q} = \mathcal{I}_{\lambda,\infty}(T_\infty\psi).$$

Hence, we can conclude from the key equivalence (3.40) that

$$\mathcal{I}_{\lambda,\infty}(T_\infty\psi) < 0 \Leftrightarrow \lambda > Q(\psi) \geqslant \lambda^{\star\star} \tag{3.43}$$

Thus, from (3.43), for every $\lambda > \lambda^{\star\star}$, we can fix $\psi_\infty \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ such that (3.43) is satisfied. Then, for all $\varepsilon > 0$ (sufficiently small) there exists m_0, such that

$$\mathcal{I}_{\lambda,m}(T_m\psi_\infty) \leqslant \mathcal{I}_{\lambda,\infty}(T_\infty\psi_\infty) + \varepsilon = \mathcal{I}_{\lambda,\infty}\Big(\frac{\psi_\infty}{|\psi|_\infty}\Big) + \varepsilon < 0$$

which implies that

$$\lambda^{\star\star} \geqslant \bar{\lambda}(m). \tag{3.44}$$

To complete the proof, we take $\lambda = \bar{\lambda}(m)$

$$\mathcal{I}_{\lambda,m}(u_m) = \frac{1}{p}\|u_m\|_{s,p}^p + \frac{1}{m}|u_m|_m^m - \frac{\bar{\lambda}(m)}{q}|u_m|_q^q = 0, \tag{3.45}$$

then

$$\frac{q}{p}\frac{\|u_m\|_s^2}{|u_m|_q^q} \leqslant \bar{\lambda}(m). \tag{3.46}$$

Multiplying both sides of the inequality by $|u_m|_\infty^{q-p}$ then

$$\lambda^{\star\star} \leqslant \bar{\lambda}(m) = \frac{q}{p}\frac{\|u_m\|_{s,p}^p|u_m|_\infty^{q-p}}{|u_m|_q^q} \leqslant \bar{\lambda}(m)|u_m|_\infty^{q-p}, \tag{3.47}$$

using (3.44), we get

$$\lambda^{\star\star} \leqslant \bar{\lambda}(m) \leqslant \lambda^{\star\star}|u_m|_\infty^{q-p}, \tag{3.48}$$

passing to the limit as $m \to \infty$ and using the fact that $|u|_\infty \leqslant 1$, we obtain

$$\lim_{m\to\infty} \bar{\lambda}(m) = \lambda^{\star\star}.$$

$\square$

3.5.1 Asymptotic behavior of minimum solution

The present subsection is devoted to the first main theorem, which investigates the asymptotic behavior of the sequence of local minimizers $u_{\lambda,m}$ (noted u_m). We establish the existence of a limiting profile by employing compactness and convergence techniques. Furthermore, we derive a uniform bound for the limiting profile as m tends to ∞.

Theorem 3.5.2. *Assume $p < q < m$. Then, there exists $\lambda^{\star\star} > 0$ such that for each $\lambda > \lambda^{\star\star} > \bar{\lambda}(m)$, there exists a sequence $(u_m)_m$ of minimum solutions and $u \in \mathcal{K} := \{w \in W_0^{s,p}(\Omega) : 0 \leqslant w(x) \leqslant 1\}$, such that $u \neq 0$ and*

$$u_m \to u \text{ strongly in } W_0^{s,p}(\Omega) \text{ and in evey Lebesgue space } L^r(\Omega), r \geqslant 1.$$

Moreover, u is a solution of the following problem

$$\begin{cases} (-\Delta)_p^s u + \mathbf{g}_u \chi_{\{u=1\}} & = & \lambda u^{q-1} & \text{in} & \Omega, \\ u & > & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.49}$$

where $0 < \mathbf{g}_u \leqslant \lambda, \mathbf{g}_u \neq 0,$ and $\mathbf{g}_u(x)[1 - u(x)] = 0$ a.e. in Ω.

The proof of Theorem 3.5.2 is divided in several propositions.

Proposition 3.5.1. *Let $(u_m)_{m>q} \subset X^s$ be the sequence of local minimizer of Problem (3.1). Then, there exists $u \in \mathcal{K} := \{w \in W_0^{s,p}(\Omega) : 0 \leqslant w(x) \leqslant 1\}$, such that $u \neq 0$ and*

$$u_m \to u \text{ strongly in } W_0^{s,p}(\Omega) \text{ and in every Lebesgue space}.$$

Proof. We take u_m as test function in (3.1), by (3.9) we obtain

$$\frac{1}{p}\|u_m\|_{s,p}^p + \frac{1}{m}|u_m|_m^m = \frac{\lambda}{q}|u_m|_q^q \leqslant \frac{\lambda|\Omega|}{q}|u_m|_\infty^q \leqslant C(\lambda, m, q, \Omega)$$

which implies that u_m is bounded in X^s. Since X^s is a reflexive space, up to a subsequence, there exists $u \in X^s$ such that

$$\begin{aligned} u_m &\rightharpoonup u \text{ weakly in } W_0^{s,p}(\Omega). \\ u_m &\to u \text{ strongly in } L^r(\Omega), \forall r \geqslant 1. \end{aligned}$$

Next, we prove the strong convergence in $W_0^{s,p}(\Omega)$, we take $u_m - u$ as test function in (3.1)

$$\langle(-\Delta)_p^s u_m, u_m - u\rangle + \int_\Omega {u_m}^{m-1}(u_m - u)\,dx - \lambda \int_\Omega {u_m}^{q-1}(u_m - u)\,dx = 0.$$

By (3.9) estimate, the second term tends to 0

$$|\int_\Omega {u_m}^{m-1}(u_m - u)|\,dx \leqslant |u_m|_\infty^{m-1}|u_m - u|_1 \leqslant \lambda^{\frac{m-1}{m-q}}|u_m - u|_1 \to 0, \tag{3.50}$$

and the same argument shows that

$$\int_\Omega {u_m}^{q-1}(u_m - u)\,dx \to 0.$$

So that

$$\langle(-\Delta)_p^s u_m, u_m - u\rangle_{s,p} \to 0,$$

by Property 1, we deduce that

$$u_m \to u \text{ strongly in } W_0^{s,p}(\Omega). \tag{3.51}$$

Now, we determine a convex set which contains the limit solution u. Notice that from Proposition 3.3.1, u_m is a solution in $\mathcal{K}_m$ where

$$\mathcal{K}_m := \{u_m \in X^s : 0 \leqslant |u_m|_\infty \leqslant \lambda^{\frac{1}{m-q}}\},$$

passing to the limit as m tends to ∞ implies that, there exists $u \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ such that:

$$0 \leqslant u(x) \leqslant 1.$$

Claim: $u \neq 0$**.**

Let us prove a lower bound on the $W_0^{s,p}$-norm of the sequence of minimum solution. Indeed, we have for $\lambda > \lambda^{\star\star}$, $\mathcal{I}_{\lambda,\infty}(u_m) < 0$. Hence,

$$\frac{1}{p}\|u_m\|_{s,p}^p \leqslant \frac{\lambda}{p}|u_m|_q^q.$$

Since $q \leqslant \frac{pN}{N-ps}$ then, by Hölder and Sobolev inequalities we get

$$\left(\frac{q}{pS\lambda|\Omega|^{1-\frac{p_s^\star}{q}}}\right)^{\frac{1}{q-p}} \leqslant \|u_m\|_{s,p},$$

letting $m \to \infty$, we obtain that $\|u\|_{s,p} \geqslant C$ and therefore $u \neq 0$. □

Proposition 3.5.2. *Let u be the limit of u_m, as m tends to ∞, is a solution to the following obstacle problem*

$$\langle(-\Delta)_p^s u,\, (v-u)\rangle_{s,p} \geqslant \lambda \int_\Omega u^{q-1}(v-u)\; dx, \quad \forall v \in \mathcal{K}. \tag{3.52}$$

Moreover, there exists $\mathbf{g}_u \in L^\infty(\Omega)$, such that

$$0 < \mathbf{g}_u \leqslant \lambda, \quad \mathbf{g}_u \neq 0, \quad \mathbf{g}_u(x)(1-u(x)) = 0 \textit{ a.e. in } \Omega \tag{3.53}$$

and u is a solution of the following equation

$$(-\Delta)_p^s u + \mathbf{g}_u = \lambda u^{q-1} \textit{ in } \Omega. \tag{3.54}$$

Proof. Let us take $\theta v - u_m$ as a test function in (3.1) such that $v \in \mathcal{K}$ and $0 < \theta < 1$

$$\langle(-\Delta)_p^s u_m, \theta v - u_m\rangle + \int\limits_\Omega {u_m}^{m-1}(\theta v - u_m)\, dx = \lambda \int\limits_\Omega {u_m}^{q-1}(\theta v - u_m)\, dx.$$

We analyze the second term

$$\int\limits_\Omega {u_m}^{m-1}(\theta v - u_m)\, dx = \int\limits_{\{x:\theta v - u_m \leqslant 0\}} {u_m}^{m-1}(\theta v - u_m)\, dx + \int\limits_{\{x:\theta v - u_m > 0\}} {u_m}^{m-1}(\theta v - u_m)\, dx.$$

Since $\theta < 1$ and $\{x : \theta v - u_m > 0\}$ we get

$$|{u_m}^{m-1}(\theta v - u_m)| \leqslant |\theta v|^{m-1}(\theta v + \theta v) \leqslant 2|\theta v|^m \leqslant 2\theta^m \xrightarrow[m\to+\infty]{} 0,$$

while

$${u_m}^{m-1}(\theta v - u_m) \leqslant 0, \text{ on the set } \{x : \theta v - u_m \leqslant 0\},$$

which implies that

$$\lim_{m\to\infty} \sup \int\limits_\Omega {u_m}^{m-1}(\theta v - u_m) dx \leqslant 0.$$

Then

$$\theta\langle(-\Delta)_p^s u_m, v\rangle \geqslant \lambda \int\limits_\Omega {u_m}^{q-1}(\theta v - u_m)dx + \|u_m\|_{s,p}^p.$$

Passing to the limit

- Thanks to the strong convergence of $u_m \to u$ in $W_0^{s,p}(\Omega)$, it follows that

$$\langle(-\Delta)_p^s u_m, v\rangle \to \langle(-\Delta)_p^s u, v\rangle.$$

- The lower semi-continuity of $W_0^{s,p}(\Omega)$-norm gives

$$\|u\|_{s,p} \leqslant \lim_{m\to\infty} \inf \|u_m\|_{s,p}$$

- The strong convergence $u_m \to u$ in $L^r(\Omega)$ for all $r \geqslant 1$, implies that

$$\int\limits_\Omega {u_m}^{q-1}(\theta v - u_m)\,dx \to \int\limits_\Omega u^{q-1}(\theta v - u)\,dx.$$

Then, $\forall v \in \mathcal{K}$ we obtain

$$\theta \iint\limits_{D_\Omega} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))\,d\nu \geqslant \int\limits_\Omega u^{q-1}(\theta v - u)\,dx + \iint_\Omega |u(x)-u(y)|^p\,d\nu.$$

Finally, as θ tends to 1 we get that u satisfies the following variational inequality

$$\langle(-\Delta)_p^s u, u - v\rangle \geqslant \lambda \int\limits_\Omega u^{q-1}(\theta v - u)\,dx, \ \forall v \in \mathcal{K}.$$

Now, we want to prove the second part of the theorem. From (3.3.1), we have

$$|u_m|_\infty^{m-1} \leqslant \lambda^{\frac{m-1}{m-p}}, \text{ for all } m > q,$$

then, there exists $\mathbf{g}_u \in L^\infty(\Omega)$ such that, up to a subsequence,

$$\left\{(u_m)^{m-1}\right\} \overset{\star}{\rightharpoonup} \mathbf{g}_u \; weakly - \star \; in \; L^\infty(\Omega),$$

which implies that $\mathbf{g}_u \geqslant 0$. On the other hand, to prove that $\mathbf{g}_u \leqslant \lambda$, we take the characteristic function $\mathbf{1}_E$ as test function, corresponding to the set

$$E := \{x \in \Omega : \ \mathbf{g}_u > \lambda\}.$$

Hence, by (3.9), we get

$$\int\limits_{\Omega} {u_m}^{m-1} \mathbf{1}_E \, dx = \int\limits_{E} {u_m}^{m-1} \, dx \leqslant \|u_m\|_\infty^{m-1} |E| \leqslant \lambda^{\frac{m-1}{m-q}} |E|,$$

passing to the limit as $m \to +\infty$, we obtain

$$\lambda |E| < \int_E \mathbf{g}_u \, dx \leqslant \lambda |E| \Rightarrow \ \mathbf{g}_u \leqslant \lambda.$$

Now, we take $\varphi \in W_0^{s,p}(\Omega)$ as test function in (3.3) then we pass to the limit as $m \to +\infty$, to conclude that u satisfies

$$\langle (-\Delta)_p^s u, v \rangle_{s,p} + \int_\Omega \mathbf{g}_u \varphi \, dx = \lambda \int_\Omega u^{q-1} \varphi \, dx, \forall \varphi \in W_0^{s,p}(\Omega). \tag{3.55}$$

Finally, let us take $\varphi = v - u$ in the last equality (3.55) with $v \in \mathcal{K}$

$$\iint_{D_\Omega} |u(x) - u(y)|^{p-2} (u(x) - u(y))((v-u)(x) - (v-u)(y)) \, d\nu + \int_\Omega \mathbf{g}_u (v-u) \, dx = \lambda \int_\Omega u^{q-1} (v-u) \, dx.$$

By (3.52) we deduce that

$$\int_\Omega \mathbf{g}_u (u - v) \, dx \geqslant 0, \ \forall v \in \mathcal{K}.$$

Thus, taking a sequence $v_j \in \mathcal{K}$ such that $v_j \xrightarrow[j \to +\infty]{} 1$ in $L^1(\Omega)$ to obtain

$$\int_\Omega \mathbf{g}_u (u-1) \geqslant 0 \Rightarrow \mathbf{g}_u (u-1) \, dx \equiv 0, \ as \ \mathbf{g}_u \geqslant 0 \ and \ u \leqslant 1.$$

To complete the proof,we prove that $\mathbf{g}_u \neq 0$, we take into account that

$$\mathcal{I}_{\lambda,\infty}(u) \leqslant -\eta < 0 \Rightarrow q\eta + \frac{q}{p} \iint_{D_\Omega} |u(x) - u(y)|^p \ d\nu \leqslant \lambda \int_\Omega u^q \, dx.$$

On the other hand taking $\varphi = u$ in (3.55)

$$\iint_{D_\Omega} (u(x) - u(y))^p \, d\nu - \int_\Omega \mathbf{g}_u u \, dx = \lambda \int_\Omega u^q \, dx,$$

So that

$$\int_\Omega \mathbf{g}_u u\,dx \geqslant q\eta + (\frac{q}{p}-1)\iint_{D_\Omega} |u(x)-u(y)|^p\,d\nu.$$

Since $\mathbf{g}_u \geqslant 0$ and $u \neq 0$, then $\mathbf{g}_u \neq 0$. $\square$

3.5.2 Asymptotic behavior of the mountain pass solution

In what follows we determine the asymptotic behavior of the sequence of mountain pass solutions $v_{\lambda,m}$ (noted v_m), as m tends to ∞.

Theorem 3.5.3. *Assume that $p<q<m$. Then, there exists $\lambda^{\star\star}>0$ such that for each $\lambda>\lambda^{\star\star}>\bar\lambda(m)$, there exists a sequence of solutions $(v_m)_m$ and $v\in\mathcal{K}$ such that $v\neq 0$ and*

$$v_m \to v \textit{ strongly in } W_0^{s,p}(\Omega) \textit{ and in evey Lebesgue space } L^r(\Omega)\,, r \geqslant 1.$$

Moreover, v is a solution of the following problem

$$\begin{cases} (-\Delta)_p^s v + \mathbf{g}_v\chi_{\{v=1\}} &= \lambda v^{q-1} \quad \textit{in} \quad \Omega,\\ v &> 0 \quad \textit{in} \quad \Omega,\\ v &= 0 \quad \textit{in} \quad \mathbb{R}^N\setminus\Omega, \end{cases} \tag{3.56}$$

where $0<\mathbf{g}_v\leqslant\lambda$ and $\mathbf{g}_v(x)[1-v(x)]=0$.

Proof. Let $\lambda>\lambda^{\star\star}$ and v_m be a sequence of mountain pass solutions. We argue in the same way as in the proof of Theorem 3.5.2, we easily obtain that there exists $v\in W_0^{s,p}(\Omega)\cap L^\infty(\Omega)$, such that

- $v_m \rightharpoonup v$ weakly in $W_0^{s,p}(\Omega)$.
- $v_m \to v$ strongly in $L^r(\Omega)$ for all $r\geqslant 1$.
- $0\leqslant v\leqslant 1$.

By Property (1) we conclude that $v_m\to v$ strongly in $W_0^{s,p}(\Omega)$.

Claim:$v\neq 0$.

We consider again the functional $\mathcal{I}_{\lambda,\infty}$ defined in (3.38). Notice that there exists $u\in W_0^{s,p}(\Omega)\cap L^\infty(\Omega)$ such that

$$\mathcal{I}_{\lambda,\infty}(u)<0.$$

Let us define

$$\Gamma_\infty = \{\gamma:[0,1]\to W_0^{s,p}(\Omega)\cap L^\infty(\Omega), \gamma \textit{ is continuous and } \gamma(0)=0, \gamma(1)=u\},$$

and the mountain pass level

$$c_\infty := \inf_{\gamma\in\Gamma_\infty} \max_{[0,1]} \mathcal{I}_{\lambda,\infty}(\gamma(t)).$$

Claim: $\Gamma_m \subset \Gamma_\infty$. Indeed, let $\gamma \in \Gamma_m$ then

$$\gamma(1) = \mathcal{I}_{\lambda,m}(T_m u) < 0,$$

since $\mathcal{I}_{\lambda,\infty}(u_m) < \mathcal{I}_{\lambda,m}(u_m)$ then $\gamma \in \Gamma_\infty$. As consequence, we get

$$\max_{[0,1]} \mathcal{I}_{\lambda,\infty}(\gamma(t)) \leqslant \max_{[0,1]} \mathcal{I}_{\lambda,m}(\gamma(t)),$$

and then

$$c_m = \inf_{\Gamma_m} \max_{[0,1]} \mathcal{I}_{\lambda,m}(\gamma(t)) \geqslant \inf_{\Gamma_\infty} \max_{[0,1]} \mathcal{I}_{\lambda,\infty}(\gamma(t)) = c_\infty \geqslant \rho_\infty,$$

where ρ_∞ is given by

$$\rho_\infty = \mathcal{I}_{\lambda,\infty}(\gamma(1)) = \frac{1}{p}\|u\|^p_{s,p} - \frac{\lambda}{p}|u|^q_q > 0.$$

Therefore

$$\mathcal{I}_{\lambda,m}(v_m) = c_m \geqslant \rho_\infty > 0,$$

yielding that $v_m \neq 0$. Taking into account the strong convergence of v_m to v in $W^{s,p}_0(\Omega)$ and in every Lebesgue space , we can pass to the limit to obtain that

$$\mathcal{I}_{\lambda,m}(v_m) \to \mathcal{I}_{\lambda,\infty}(v) \geqslant \rho_{\lambda,\infty} > 0,$$

then $v \neq 0$. The rest of the proof goes exactly as in the analysis of the asymptotic behavior of the sequence of local minimum in Theorem 3.5.2. □

3.6 Equidiffusive case $q = p$: Existence and asymptotic analysis

In the present section, we address the equidiffusive case as $p = q < m$. Precisely, we are interested by the following problem

$$\begin{cases} (-\Delta)^s_p u + u^{m-1} &= \lambda u^{p-1} \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.57}$$

The objective is to investigate the existence and uniqueness of solutions as a function of the parameter λ, and to demonstrate that the critical threshold for λ eventually is the first eigenvalue of fractional p-Laplacian problem with Direchlet boundary conditions $\lambda_{s,p}$, as defined in (1.13).

3.6.1 Existence and uniqueness results

Our first result in this subsection is the following Theorem.

Theorem 3.6.1. *Let* $p = q < m$.

1. *If* $0 < \lambda < \lambda_{s,p}$, *then Problem* (3.57) *does not have any positive solution in* X^s.
2. *If* $\lambda = \lambda_{s,p}$, *then Problem* (3.57) *has only the trivial solution in* X^s.
3. *If* $\lambda > \lambda_{s,p}$, *then Problem* (3.57) *has a unique positive weak solution* $w_{\lambda,m} \in X^s$.

Proof. The demonstration is based on two essential points: the first is the existence or nonexistence of a solution, and the second is its uniqueness. In particular, we also address the borderline case, where we show that the problem admits only the trivial solution.

- **Nonexistence results**

  Let us assume that $\lambda \in (0, \lambda_{s,p})$, and $u \in X^s$ satisfies (3.57). Taking u as test function in Problem (3.57), we obtain

  $$\|u\|_{s,p}^p \leqslant \|u\|_{s,p}^p + |u|_m^m = \lambda \|u\|_p^p,$$

  using the definition of $\lambda_{s,p}$ given in (1.13)

  $$\lambda_{s,p} \|u\|_p^p \leqslant \lambda \|u\|_p^p,$$

  then

  $$(\lambda_{s,p} - \lambda) \|u\|_p^p \leqslant 0,$$

  wich implies that $u = 0$. So that, Problem (3.57) admits only the trivial solution.

  Now, we treat the borderline case $\lambda = \lambda_{s,p}$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to $\lambda_{s,p}$, as $n \to \infty$, and let $(v_n)_n$ be the corresponding sequence of solutions of the following problem

$$\begin{cases} (-\Delta)^s_p v_n + v_n^{m-1} & = & \lambda_n v_n^{p-1} & \text{in} & \Omega, \\ v_n & > & 0 & \text{in} & \Omega, \\ v_n & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.58}$$

Testing (3.58) by $v_n \in X^s$, and by using Proposition 3.3.2, we get

$$\begin{aligned} \|v_n\|^p_{s,p} + |v_n|^m_m & = \lambda_n |v_n|^p_p \\ & \leqslant \lambda_n |v_n|^p_\infty |\Omega| \\ & \leqslant \lambda_n^{\frac{m}{m-p}} |\Omega|. \end{aligned}$$

Therefore, the sequence $(v_n)_n$ is bounded in X^s, then there exists $v_1 \in X^s$ such that

- $v_n \rightharpoonup v_1$ weakly in X^s.
- $v_n \to v_1$ strongly in $L^r(\Omega)$, for all $r \geqslant 1$.
- $v_n \to v_1$ a.e. in Ω.

Also, to prove the strong convergence of v_n to v_1, we follow the same arguments as in the proof of Theorem 3.4.6, we get that

$$v_n \to v_1, \text{ strongly in } W^{s,p}_0(\Omega).$$

Thus, v_n is a solution of Problem (3.58), then it satisfies

$$\langle (-\Delta)^s_p v_n, \phi \rangle_{s,p} + \int_\Omega v_n^{m-1} \phi \, dx = \lambda_n \int_\Omega v_n^{q-1} \phi \, dx, \text{ for all } \phi \in X^s. \tag{3.59}$$

Passing to the limit as $n \to \infty$, we deduce that v_1 solves

$$\begin{cases} (-\Delta)^s_p v_1 + {v_1}^{m-1} & = & \lambda_{s,p} {v_1}^{p-1} & \text{in} & \Omega, \\ v_1 & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.60}$$

Claim: $v_1 = 0$.

We take $v_1 \in X^s$ as a test function in (3.60)

$$\|v_1\|^p_{s,p} + |v_1|^m_m = \lambda_{s,p} |v_1|^p_p,$$

by the definition of the first eigenvalue (1.13), we get that

$$\|v_1\|_{s,p}^p + |v_1|_m^m = \inf \frac{\|v_1\|_{s,p}^p}{|v_1|_p^p} |v_1|_p^p.$$

Then, we conclude that $v_1 = 0$.

- **Existence result**

Let $\lambda > \lambda_{s,p}$, and define the energy functional associated to Problem (3.57)

$$\Phi_{\lambda,m}(u) = \frac{1}{p} \iint_{D_\Omega} |u(x) - u(y)|^p \, d\nu + \frac{1}{m} \int_\Omega u_+^m \, dx - \frac{\lambda}{p} \int_\Omega u_+^p \, dx. \tag{3.61}$$

$\Phi_{\lambda,m}$ is well defined and of class $\mathcal{C}^1$ on X^s.

To prove the existence result, we will use minimization method. The key steps are outlined below.

- **Step 1**: The functional $\Phi_{\lambda,m}$ is coercive. We use Hölder's inequality

$$\int_\Omega |u|^p \, dx \leqslant \left(\int_\Omega |u|^m \, dx \right)^{\frac{p}{m}} |\Omega|^{1-\frac{p}{m}}.$$

By a careful use of Young's inequality, we can control the term on the right hand. Indeed, we pose

$$a = \frac{|u|_m^p}{p^{\frac{p}{m}}}, b = \lambda p^{\frac{p}{m}} |\Omega|^{1-\frac{p}{m}},$$

then

$$ab \leqslant \frac{p}{pm} |u|_m^m + \frac{m-p}{m} \lambda^{\frac{m}{m-p}} p^{\frac{p}{m-p}} |\Omega|,$$

it results that

$$\Phi_{\lambda,m}(u) \geqslant \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{pm} |u|_m^m - C_{m,p},$$

where $C_{m,p} = \dfrac{m-p}{mp} \lambda^{\frac{m}{m-p}} p^{\frac{p}{m-p}} |\Omega|$. Then,

$$\Phi_{\lambda,m}(u) \geqslant \alpha_1 \|u\|^p - C_1,$$

where $\alpha_1 = \dfrac{1}{2^{p-1}} \min\{\frac{1}{p}, \frac{1}{pm}\}$.
As a consequence, as $\|u\| \to +\infty$, $\Phi_{\lambda,m}(u) \to +\infty$, which shows that $\Phi_{\lambda,m}$ is coercive.

- **Step 2**: $\Phi_{\lambda,m}$ is weakly lower semicontinuous in X^s.

Let $(v_k)_k \subset X^s$ be a minimizing sequence of $\Phi_{\lambda,m}$. From **Step 1**, we immediately see that $(v_k)_k$ is bounded in X^s, and therefore we can assume that there is a subsequence still denoted $(v_k)_k$, such that as $k \to +\infty$

$$\begin{array}{lcll} v_k & \rightharpoonup & v & \text{weakly in} \quad X^s, \\ v_k & \rightharpoonup & v & \text{weakly in} \quad W_0^{s,p}(\Omega), \\ v_k & \to & v & \text{a.e.} \quad \Omega. \end{array}$$

Thanks to L^∞-estimate, we conclude

$$v_k \to v \text{ strongly in } L^r(\Omega), \ \text{for all} \ r \geqslant 1.$$

Using the weak lower semi-continuity of the X^s-norm, we get

$$\Phi_{\lambda,m}(v) \leqslant \lim_{k\to+\infty} \inf \Phi_{\lambda,m}(v_k).$$

Therefore, there exists $w_{\lambda,m} \in X^s$ such that

$$\Phi_{\lambda,m}(w_{\lambda,m}) = \inf_{w\in X^s} \Phi_{\lambda,m}(w).$$

Conclusion: $w_{\lambda,m}$ is a global minimum for $\Phi_{\lambda,m}$, hence a weak solution of Problem (3.57).

- **Step 3**: $w_{\lambda,m}$ is nontrivial solution.

  Let $\varphi_{s,p} \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ be the first eigenfunction of the fractional p-Laplacian corresponding to $\lambda_{s,p}$. For any $\kappa > 0$, we have

$$\begin{aligned} \Phi_{\lambda,m}(\kappa\varphi_{s,p}) &= \frac{\kappa^p}{p}\|\varphi_{s,p}\|_{s,p}^p + \frac{\kappa^m}{m}|\varphi_{s,p}|_m^m - \lambda\frac{\kappa^p}{p}|\varphi_{s,p}|_p^p \\ &= \kappa^p\frac{(\lambda_{s,p}-\lambda)}{p}|\varphi_{s,p}|_p^p + \frac{\kappa^m}{m}|\varphi_{s,p}|_m^m. \end{aligned}$$

  Hence, for $\kappa > 0$ sufficiently small, we have $\Phi_{\lambda,m}(\kappa\varphi_{s,p}) < 0$, due to the fact that $p < m$ and $\lambda > \lambda_{s,p}$. Since $w_{\lambda,m}$ is a global minimum of $\Phi_{\lambda,m}$, it follows that

$$\Phi_{\lambda,m}(w_{\lambda,m}) \leqslant \Phi_{\lambda,m}(\kappa\varphi_{s,p}) < 0,$$

  this implies that $w_{\lambda,m} \neq 0$.

- **Uniqueness**

  To prove uniqueness, we employ Lemma 1.4.8.

$\square$

3.6.2 Asymptotic analysis

Now, we characterize the asymptotic behavior of the minimum solution $w_{\lambda,m}$ (noted w_m) of Problem (3.57) as a solution of a limiting problem.

Theorem 3.6.2. *Let $p = q < m$. Then, for $\lambda > \lambda_{s,p}$, there exists a sequence $(w_m)_m$ of minimum solutions to Problem (3.57) and $w \in \mathcal{K} := \{v \in W_0^{s,p}(\Omega) : 0 \leqslant v(x) \leqslant 1\}$. such that $w \neq 0$ and*

$$w_m \to w \text{ strongly in } W_0^{s,p}(\Omega) \text{ and in every Lebesgue space.}$$

Moreover, w solves

$$\begin{cases} (-\Delta)_p^s w + \mathbf{g}_w \chi_{\{w=1\}} & = & \lambda w^{q-1} & \text{in} & \Omega, \\ w & > & 0 & \text{in} & \Omega, \\ w & = & 0 & \text{in} & \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.62}$$

Such that

$0 < \mathbf{g}_w \leqslant \lambda, \mathbf{g}_w \neq 0, \mathbf{g}_w(x)[1 - w(x)] = 0$ *a.e. in* Ω *and* $|\{w = 1\}| > 0$.

Proof. The asymptotic behavior of w_m when m tends to ∞, is a subcase of Proposition 3.5.1.

For $\lambda > \lambda_{s,p}$, Problem (3.57) admits a sequence of minimum solutions w_m of $\Phi_{\lambda,m}$ such that

$$\Phi_{\lambda,m}(w_m) < 0$$

by Theorem (3.3.2), we get that $(w_m)_m$ is bounded in X^s. Then, up to a subsequence:

$$\begin{aligned} & w_m \rightharpoonup w \text{ weakly in } X^s, \\ & w_m \to w \text{ strongly in } W_0^{s,p}(\Omega), \qquad . \\ & w_m \to w \text{ strongly in } L^r(\Omega), \forall r \geqslant 1. \end{aligned}$$

In addition

$$0 \leqslant w(x) \leqslant 1.$$

Claim: $w \neq 0$.

Let us define

$$\Phi_{\lambda,\infty}(w_m) = \frac{1}{p}\iint_{D_\Omega} |w_m(x) - w_m(y)|^p d\mu - \frac{\lambda}{p}\int_\Omega w_m^p \, dx.$$

Then, by Fatou's Lemma we obtain that

$$\begin{aligned} &\Phi_{\lambda,\infty}(w) \leqslant \lim_{m\to+\infty} \inf \Phi_{\lambda,m}(w_m) < 0 \\ &\Rightarrow \Phi_{\lambda,\infty}(w) < 0 \\ &\Rightarrow w \neq 0. \end{aligned}$$

The same reasoning as in the proof of Proposition 3.5.2 applies here to complete the rest of the proof; yielding a limit solution w that satisfies the problem limit (3.62), with $\mathrm{g}_w \neq 0$, thanks to the negativity of the energy functional in this case. □

3.7 Subdiffusive case $q < p$: Existence and asymptotic analysis

In this section, we look to the Subdiffusive case when $1 < q < p$ and we deal with the following problem

$$\begin{cases} (-\Delta)_p^s u &= \lambda u^{q-1} - u^{m-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.63}$$

3.7.1 Existence result

Theorem 3.7.1. *Let $1 < q < p < m$. Then, for all $\lambda > 0$ Problem (3.63) has a unique solution $w_{\lambda,m} \in X^s$. Moreover, $w_{\lambda,m}$ is increasing with respect to λ. Furthermore, if $\lambda \to 0$, $w_{\lambda,m} \to 0$ in X^s.*

Proof. We fixe $\lambda > 0$, to prove the existence of positive solution we use minimization method. First, we define the energy functional associated to (3.63)

$$\Psi_{\lambda,m}(u) = \frac{1}{p}\iint_{D_\Omega} |u(x) - u(y)|^p \, d\nu + \frac{1}{m}\int_\Omega u_+^m \, dx - \frac{\lambda}{p}\int_\Omega u_+^p \, dx. \tag{3.64}$$

$\Psi_{\lambda,m}$ is well defined and of class $\mathcal{C}^1$ on X^s. We split the proof on the following steps.

- **Step 1:** $\Psi_{\lambda,m}$ is coercive.

  $$\Psi_{\lambda,m}(u) \quad \geqslant \quad \alpha\|u\|_{s,p} - C(\Omega).$$

  Then $\Psi_{\lambda,m}(u)$ tends to ∞ as $\|u\|_{s,p}$ to ∞.

- **Step 2:** The weak lower semicontinuity of $\Psi_{\lambda,m}(u)$.

  The coercivity of the energy functional implies the boundendness of the minimizer sequence $(u_n)_n$ and from (2.64) estimate, up to a subsequence, there exists u such that

  - $u_n \rightharpoonup u$ weakly in X^s.
  - $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(\Omega)$.
  - $u_n \to u$ strongly in $L^r(\Omega)$, forall $r \geqslant 1$.

  So that, there exists $w_{\lambda,m} \in X^s$ such that

$$\Psi_{\lambda,m}(w_{\lambda,m}) = \inf_{u \in X^s} \Psi_{\lambda,m}(u).$$

  Now, we prove that $v_{\lambda,m}$ is not trivial. Indeed, let us take $\psi \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and $t > 0$,

$$\Psi_{\lambda,m}(t\psi) = \frac{t^p}{p}\|\psi\|_{s,p}^p + \frac{t^m}{m}|\psi_+|_m^m - \lambda\frac{t^q}{q}|\psi_+|_q^q.$$

  Notice that, since $q < p < m$ and for $t > 0$ small enough we have $\Psi_{\lambda,m}(t\psi) < 0$, which allows us to conclude that $\Psi_{\lambda,m}(w_{\lambda,m}) < 0$, then $w_{\lambda,m} \neq 0$.

- **Step 2:** We address the question of uniqueness.

  Let v_0 be another solution of (3.63), we have for all $\sigma > 0$

$$\frac{f(\sigma)}{\sigma^{p-1}} = \lambda\sigma^{q-p} - \sigma^{m-p}.$$

  Since $q < p < m$, this mapping is decreasing in $(0, \infty)$. By applying the comparison principle (1.4.8), and following the same reasoning as in the equidiffusive case, we conclude that $w_{\lambda,m} = v_0$.

□

Proposition 3.7.1. *If $\lambda = 0$, the problem has only the trivial solution.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n \to 0$, as $n \to \infty$, denote $u_n = u_{\lambda_n}$ be a sequence of positive solutions of (3.1) with $q < p$. It is clear that $(u_n)_n$ is bounded in X^s. So that, as in Theorem 3.4.6, we obtain

$$u_n \to u_0 \ \text{ strongly in } W_0^{s,p}(\Omega) \text{ and in every Lebesgue space.}$$

$$u_n \to u_0 \text{ a.e. in } \Omega.$$

Passing to the limit as $n \to \infty$, we get weakly in Ω that u_0 satisfies

$$(-\Delta)_p^s u_0 + u_0^{m-1} = 0.$$

Testing by $u_0 \in W_0^{s,p}(\Omega)$, we conclude that $u_0 = 0$. □

3.7.2 Asymptotic Analysis

Theorem 3.7.2. *Let $q < p < m$. Then, for each $\lambda > 0$ there exists a sequence $(w_{\lambda,m})_m$ of minimum solutions and $v \in \mathcal{K} := \{v \in W_0^{s,p}(\Omega) : 0 \leqslant v(x) \leqslant 1\}$ such that $v \neq 0$*

$$w_{\lambda,m} \to v \textit{ strongly in } W_0^{s,p}(\Omega) \textit{ and in every Lebesgue space.}$$

Moreover, v is a solution of the following problem

$$\begin{cases} (-\Delta)_p^s v + \mathbf{g}_v \chi_{\{v=1\}} & = & \lambda v^{q-1} & \textit{in} & \Omega, \\ v & > & 0 & \textit{in} & \Omega, \\ v & = & 0 & \textit{in} & \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.65}$$

Such that

$0 < \mathbf{g}_v \leqslant \lambda, \mathbf{g}_v \neq 0, \mathbf{g}_v(x)[1 - v(x)] = 0$ *a.e. in* Ω *and* $|\{v = 1\}| > 0$.

Proof. The proof follows along the same lines as that of Theorem 3.6.2 in the equidiffusive case. □

Part II

Nonlocal Parabolic Problem

4

A class of fractional parabolic logistic problem

WE consider a nonlocal parabolic problem, with a logistic type nonlinearity

$$\begin{cases} \partial_t u + (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) &= u_0(x) \qquad\qquad\; \text{in} \quad \Omega, \\ u &= 0 \qquad\qquad\qquad\;\; \text{in} \quad \mathbb{R}^N \setminus \Omega \times (0,T), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N > 2s$, $s \in (0,1)$, $m > p > 2$ with $\lambda > 0$.

The purpose of this work is to prove the existence and uniqueness of a global weak positive solution. Furthermore, we study its asymptotic behavior, by determining it as a solution to limiting problem.

This chapter is an extended version of the work published [17].

4.1 Introduction

The present chapter is devoted to the study of the following nonlocal reaction-diffusion problem with logistic reaction

$$\begin{cases} \partial_t u + (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) &= u_0(x) \geqslant 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$, $0 < s < 1$, $2 < p < m$, λ is a positif parameter and u_0 is a nonnegative function in $L^1(\Omega)$.

The most related works to our study have extensively addressed the general questions of existence, and nonexistence of positive solutions. Particularly,

In the local case:

- In the linear case as $p = 2$, the authors in [93] studied the asymptotic behavior of positive solution as $m \to \infty$ of the following problem

  $$\begin{cases} \partial_t u - \Delta u &= \lambda u - b(x) u^{m-1} \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega \times (0,T), \end{cases} \tag{4.2}$$

  where $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\lambda > 0$, $m > 2$ and $b(x)$ is a nonnegative function in $L^\infty(\Omega)$. Inspired by the stationary version analyzed by [43], their study aims to extend the results obtained to the parabolic case. The objective is to demonstrate that the asymptotic behavior is determined by a limiting profile that solves a parabolic obstacle problem. Furthermore, they investigated existence, uniqueness under certain conditions, and its long time behavior of the solution.

- In [41] the authors considered the following parabolic problem with concave-convex nonlinearities

  $$\begin{cases} \partial_t u - \Delta u &= \lambda u^{m-1} + u^{p-1} \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega \times (0,T), \end{cases} \tag{4.3}$$

  with $u_0 \in L^\infty(\Omega)$, $1 < p < 2 < m$ and $\lambda > 0$. Their primary focus lies in the relations between the global (in time) solutions of the parabolic problem and the solutions of the corresponding stationary problem.

In the nonlocal case:

In recent years, significant attention has been devoted to the study of problems in the nonlocal framework.

- To the best of our knowledge, this type of logistic problem involving the fractional Laplacian has only been addressed by Klimsiak in [69], considering the linear case. Specifically, the author's focus is on the study of the following problem

$$\begin{cases} \partial_t u + (-\Delta)^s u = \lambda u - b u^m & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases} \tag{4.4}$$

  with u_0, b are bounded positive measurable functions on Ω and $\lambda > 0$. Klimsiak studied separately the asymptotic behaviors of positive solutions:

  - ➙ As t tends to infinity then as the exponent m tends to infinity.
  - ➙ As the exponent m tends to infinity then as t tends to infinity.

  The author concludes that, in the first scenario, the limiting configurations solve an elliptic free boundary problem. In the second scenario, they analyze the long-time behavior and show that the solution approaches the unique stationary solution of another specific free boundary problem. One of the main challenges they encountered was the nonlocal nature of the operator involved. To address this, they introduced a novel approach that blends techniques from classical Laplacian analysis with tools from probabilistic potential theory and stochastic analysis, allowing them to effectively study the asymptotic behavior in this nonlocal setting.

- Recently, in [42] the authors examined the following nonlocal parabolic logistic problem with harvesting in

$$\begin{cases} \partial_t u + (-\Delta)^s u = \lambda[a(x)u - b u^2 - h(x)] & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases} \tag{4.5}$$

where $u_0 \in L^2(\Omega)$, $\lambda, b > 0$ and a, h are nonnegative functions in $L^\infty(\Omega)$. Their primary focus is to establish existence and uniqueness of solutions using monotone iterations and sub- and supersolutions methods.

In the elliptic case

In chapter 2, we studied the stationary problem of (4.1), where we addressed the principal questions: existence, non-existence and multiplicity of solutions for the largest possible range of the parameters λ, p, m.

To be precise, let us recall the elliptic problem

$$\begin{cases} (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \mathbb{R}^N, \\ u &= 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.6}$$

and the associated energy functional.

$$\mathcal{J}_{\lambda,m}(u) = \frac{1}{2} \iint_{D_\Omega} |u(x) - u(y)|^2 \, d\mu + \frac{1}{m} \int_\Omega |u|^m \, dx - \frac{\lambda}{p} \int_\Omega |u|^p \, dx.$$

$\mathcal{J}_{\lambda,m}$ is well defined and $\mathcal{C}^1$ on $\mathcal{H}_0^s(\Omega) \cap L^m(\Omega)$.

The main goal of this chapter is to analyze the parabolic version of (4.6), answering the general questions about the global existence and uniqueness of weak positive solution to Problem (4.1). Then, we focus on the next interesting part, where we perform an asymptotic analysis to the sequence of solutions found when $m \to \infty$ such that the behavior will be determined by certain limit parabolic problem. Subsequently, we will determine the large-time asymptotics for the solution of this limiting problem and, at the same time, an existence result for a stationary limit problem. In this review, we focus on the critical thresholds λ_* defined in (2.46) by

$$\lambda_*(\Omega) = \inf_{v \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)} \frac{p}{2} \frac{\|v\|_s^2 |v|_\infty^{p-2}}{|v|_p^p}. \tag{4.7}$$

that governs the existence and multiplicity of solutions for Problem (4.6). Specifically, for $\lambda > \lambda_*$ the key results include:

1. If $2 < p < m$, existence of global minimum solution $u_{\lambda,m}$, such that $\mathcal{J}_{\lambda,m}(u_{\lambda,m}) < 0$.
2. If $2 < p \leqslant 2_s^* < m$, since $p < 2_s^*$ the functional $\mathcal{J}_{\lambda,m}$ satisfies the Palais Smale condition. This condition guarantees the existence of a second positive solution often referred to as mountain

pass solution $v_{\lambda,m}$. The energy of this solution can be characterized by the corresponding critical level, which is determined through the mountain pass method by

$$c_m = \inf_{\gamma \in \Gamma_m} \max_{t \in [0,1]} \mathcal{J}_{\lambda,m}(\gamma(t)), \tag{4.8}$$

where

$$\Gamma_m = \{\gamma : [0,1] \to \mathcal{H}^s_0(\Omega) \cap L^m(\Omega), \gamma \text{ is continuous and } \gamma(0) = 0, \gamma(1) = u_{\lambda,m}\}. \tag{4.9}$$

We focus on this case $\lambda \geqslant \lambda_*(\Omega)$, as it allows us to study the asymptotic behavior of the parabolic version of this problem.

Building on the insights from the aforementioned studies, the primary objective of this work is to investigate the parabolic counterpart of Problem (4.6). Our analysis begins by addressing the fundamental questions concerning the global existence and uniqueness of weak positive solutions to Problem (4.1). Following this, we explore further aspects, such as the asymptotic profile as $m \to \infty$, and identify the corresponding limit as the unique solution to a well-defined free boundary problem. Additionally, we turn our attention to the long-time dynamics of the solution as $t \to \infty$.

Plan of the chapter:

- ✓ In Section 4.2, we state the main results.
- ✓ In Section 4.3, we prove the existence of a unique global positive solution to Problem (4.1).
- ✓ In Section 4.4, we prove that the solution is uniformly bounded.
- ✓ In Section 4.5, we study the asymptotic behaviors, when $t \to \infty$, and then as $m \to \infty$.

4.2 Statement of the main result

The main objective of this section is to prove the existence and uniqueness of a global weak solution to Problem (4.1)

The primary objective of this section is to establish the existence and uniqueness of global solutions to the main nonlinear problem (4.1). To achieve this, we adopt a stepwise approach, beginning with the analysis of the following linear problem

$$\begin{cases} \partial_t u + (-\Delta)^s u &= h & \text{in } \Omega_T = \Omega \times (0,T), \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) &= u_0(x) & \text{in } \Omega, \end{cases} \tag{4.10}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain and $(h, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$ belong to suitable Lebesgue spaces. Specifically, if the data satisfy the conditions $(h, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$, this allows us to formulate the energy solution. To be more precise, we state the following definition.

Definition 4.2.1. *Assume that $(h, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$. We say that u is an energy solution to Problem (4.10) if $u \in L^2(0,T; H_0^s(\Omega)) \cap \mathcal{C}([0,T], L^2(\Omega))$, $\partial_t u \in L^2(0,T; H^{-s}(\Omega))$, and for all $v \in L^2(0,T; H_0^s(\Omega))$, we have*

$$\begin{aligned} &\int_\Omega u(.,T) v \, dx + \frac{1}{2} \int_0^T \iint_{D_\Omega} (u(x,t) - u(y,t))(v(x,t) - v(y,t)) \, d\mu \, dt \\ &= \iint_{\Omega_T} hv \, dx \, dt + \int_\Omega u_0 v \, dx \end{aligned}$$

The existence of an energy solution follows from classical arguments, see, for instance [74]. However, when the data lies in L^1, a more general concept of solution required. In order to state this definition precisely, we first introduce the dual problem.

$$(P_\varphi) \qquad \begin{cases} -\phi_t + (-\Delta)^s \phi &= \varphi & \text{in} & \Omega \times (0,T), \\ \phi(x,0) &= 0 & \text{in} & \Omega, \\ \phi(x,T) &= 0 & \text{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases}$$

with $\varphi \in L^\infty(\Omega_T)$.

It is important to note that the existence of a regular solution $\phi \in L^\infty(\Omega_T)$ is ensured by Felsinger and Kassmann in [48]. This allows us to introduce the set of test functions as

$$\mathcal{T} := \Big\{ \phi : \mathbb{R}^N \times (0,T) \to \mathbb{R} : \phi \text{ is solution to } (P_\varphi), \varphi \in L^\infty(\Omega_T) \Big\}. \tag{4.11}$$

We are now in a position to define the notion of a weak solution in this sense.

Definition 4.2.2. *Assume that $(h, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$. We say that $u \in \mathcal{C}([0,T]; L^1(\Omega))$ is a weak solution to Problem (4.10), if for any $\phi \in \mathcal{T}$, we have*

$$\iint_{\Omega_T} u \, (-\phi_t + (-\Delta)^s \phi) \, dx \, dt = \iint_{\Omega_T} h(u) \phi dx \, dt + \int_\Omega u_0(x) \phi(x,0) \, dx.$$

In order to establish the regularity of the solution to our problem, we first state the following existence result, established in [74], see also [2].

Theorem 4.2.1. *Assume that* $(h, u_0) \in L^r(\Omega_T) \times L^1(\Omega)$, *with* $1 \leqslant r < \dfrac{1}{s}$. *Then, there exists* $T > 0$ *such that Problem* (4.10) *admits a unique global weak solution* $u \in L^q(0, T; W_0^{s,q}(\Omega))$ *for all* $q < \frac{N+2s}{N+s}$ *and*

- $T_k(u) \in L^2(0, T; \mathcal{H}_0^s(\Omega))$ *for any* $k > 0$.
- $u \in L^\sigma(\Omega_T)$ *for any* $\sigma < \frac{N+2s}{N}$.

Moreover, $u \in L^p(0, T, W_0^{s,q}(\Omega))$ *for all* $1 \leqslant q < \frac{N+2s}{N+s}$. *In addition, we have*

$$||u||_{L^m(\Omega_T)} + ||u||_{L^p(0,T,W_0^{s,q}(\Omega))} \leqslant C(\Omega_T)\Bigg(||h||_{L^1(\Omega_T)} + |u_0|_1\Bigg)$$

Remark 4.2.1. *Notice that, according to Problem* (4.10), *we say that* $u \in \mathcal{C}([0, T]; L^1(\Omega))$ *is a weak solution of* (4.1), *if* $u_0 \in L^1(\Omega_T)$ *and* $h(u) = \lambda u^{p-1} - u^{m-1} \in L^1(\Omega_T)$ *for all* $1 < p < m$ *and* $T > 0$ *with* $\lambda > 0$.

4.3 Proof of the main result

In the present section, we aim to prove the fundamental existence result Theorem 4.2.1.

Outline of proof:

The proof of the result will be achieved in several steps structured as follow:

① *Existence result*: we use approximation method according to the following steps

- (a) Approximation scheme: we introduce an approximate problem of (4.10) with regular data, through which we address the principal questions regarding existence, regularity and uniqueness.
- (b) A priori estimates: we establish a priori estimates for the approximate solution.
- (c) Passing to the limit: we prove the main tool of the proof; the weak convergence of truncation, almost everywhere convergence and the equi-integrability of the nonlinear terms. Using compactness results, we show that the limit obtained is indeed a weak solution to Problem (4.1).

② *Globality of solution.*

③ *Uniqueness of solution.*

④ *A priori bound.*

Also in order to show the asymptotic behavior, we assume some additional conditions to initial data :

(H1)- $0 \leqslant u_0 \leqslant \lambda^{\frac{1}{m-p}}$.

(H2)- $E(u_0) < 0$, where

$$E(u_0) = \frac{1}{2}\|u_0\|_s^2 + \frac{1}{m}|u_0|_m^m - \frac{\lambda}{p}|u_0|_p^p.$$

4.3.1 Existence result

➢ Approximation scheme

• *Existence:*

Definition 4.3.1. *We say that $u_n \in L^2(0,T;H_0^s(\Omega)) \cap L^\infty(\Omega_T)$ is a solution to the following approximated problem, if for each n fixed, u_n satisfies*

$$\begin{cases} \partial_t u_n + (-\Delta)^s u_n &= h_n & \text{in } \Omega_T = \Omega \times (0,T), \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u_n(x,0) &= T_n(u_0(x)) & \text{in } \Omega, \end{cases} \tag{4.12}$$

where $(h_n)_n \subset L^\infty(\Omega_T)$ and $(u_{n,0})_n \subset L^\infty(\Omega)$ are the functions associated with truncated data.

Theorem 4.3.1. *Let u be the unique solution to Problem (4.10) obtained by approximation. If $h_n \to h$ in $L^1(\Omega_T)$ and $u_{n,0} \to u_0$ in $L^1(\Omega)$, then $u_n \to u$ in $\mathcal{C}([0,T];L^\sigma(\Omega))$ for all $\sigma < \frac{N+s}{N}$ and $u_n \to u$ in $L^q(0,T;W_0^{s,q}(\Omega))$ for all $q < \frac{N+2s}{N+s}$.*

Proof. Let $(h_n)_n \subset L^\infty(\Omega_T)$ be defined by

$$h_n = \lambda \frac{u_{n+}^{\ p}}{1 + \frac{1}{n}u_{n+}^{\ p}} - u_{n+}^{\ m},$$

where $h_n \to h$ in $L^1(\Omega_T)$. Assume also that $u_{n,0} \in L^\infty(\Omega)$ with $u_{n,0} \to u_0$ strongly in $L^1(\Omega)$, then by Theorem 4.2.1, we get the desired result. □

Proposition 4.3.1. *Let $(u_n)_{n\in\mathbb{N}}$ be solution of (4.12), such that $u_{n,0} \in L^\infty(\Omega)$ and $u_{n,0} \geqslant 0$, then $u_n \geqslant 0$.*

Proof. Testing Problem (4.12) with u_n^- and using standard arguments, we obtain $u_n^- = 0$. Therefore, the result follows.

□

- *Uniqueness:* To establish the uniqueness of the solution u_n, we require the following lemma.

Lemma 4.3.1 (Comparison principle). *If u, v in $L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ are solutions to the following problems*

$$\begin{cases} \partial_t u + (-\Delta)^s u &= \lambda u^{p-1} - u^{m-1} & \textit{in} & \Omega \times (0,T), \\ u(x,0) &= u_0(x) & \textit{in} & \Omega, \\ u &= 0 & \textit{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases}$$

and

$$\begin{cases} \partial_t v + (-\Delta)^s v &= \lambda v^{p-1} - v^{m-1} & \textit{in} & \Omega \times (0,T), \\ v(x,0) &= v_0(x) & \textit{in} & \Omega, \\ v &= 0 & \textit{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \end{cases}$$

and $u_0 \leqslant v_0$ in $(\mathbb{R}^N \setminus \Omega) \times (0,T)$. Then, $u \leqslant v$ a.e Ω_T.

Proof. Let $w \in L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ be the function defined by $w := u - v$, where w solves the following problem such that $u_0 \leqslant v_0$

$$\begin{cases} \partial_t w + (-\Delta)^s w &= \lambda(u^{p-1} - v^{p-1}) - (u^{m-1} - v^{m-1}) & \text{in} & \Omega \times (0,T), \\ w(x,0) &= (u_0 - v_0)(x) & \text{in} & \Omega, \\ w &= 0 & \text{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T). \end{cases}$$

By taking $(u-v)_+$ as test function and using inequality (1.26) for ε sufficiently small, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \Big((u-v)_+\Big)^2 dx \quad &+ \|(u-v)_+\|_s^2 + (1-\varepsilon\lambda)\int_\Omega (u^{m-1} - v^{m-1})(u-v)_+ dx \\ &\leqslant c_\varepsilon \lambda \int_\Omega (u-v)_+^2 dx. \end{aligned} \tag{4.13}$$

Since the terms in the left hand-side are non-negative, we conclude that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big((u-v)_+\Big)^2 dx \leqslant C_{\lambda,p}\int_\Omega \Big((u-v)_+\Big)^2 dx. \tag{4.14}$$

Therefore, by integrating (4.14) in time, we get

$$\int_{\Omega} \Big((u-v)_+\Big)^2 dx \leqslant e^{C_{\lambda,p}t} \int_{\Omega} \Big((u_0-v_0)_+\Big)^2 dx = 0. \tag{4.15}$$

Then, we deduce that $u \leqslant v$ a.e $\Omega \times (0,T)$. ☐

As consequence of the previous lemma, we can establish the monotony of u_n.

Lemma 4.3.2. *Let* $(u_n)_n$ *solution of* (4.12)*, then* u_n *is increasing in* n.

Proof. By applying Lemma 4.3.1, for $u_{n,0} \in L^\infty(\Omega)$, the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing with respect to n such that $u_n \leqslant u_{n+1}$. ☐

Now, we are able to prove the uniqueness of solution to Problem (4.12).

Proposition 4.3.2. *Assume that* $u_{n,0} \in L^\infty(\Omega)$. *Then, Problem* (4.12) *admits a unique positive solution* u_n *in* $L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$.

Proof. As consequence, of the above proposition (4.3.1), we obtain that u_n is a unique solution of (4.12) ☐

➢ A priori estimate

Now, we are going to prove some a priori estimates of approximate solution u_n which will allow us to pass to the limit in the approximating formulation.

Proposition 4.3.3. *Let* u_n *be a solution of Problem* (4.12)*, such that* $u_{n,0} \in L^\infty(\Omega)$. *Then,* $u_n \in L^\infty(\Omega_T)$.

Proof. We assume that $u_{n,0} \in L^\infty(\Omega)$ and, similarly, that the right-hand side of Problem (4.12) belongs to $L^\infty(\Omega_T)$. Using classical a priori estimate, it follows that $u_n \in L^\infty(\Omega_T)$. ☐

Proposition 4.3.4. *Let* u_n *be a solution of the approximated problem* (4.12)*. Then, for all* $n \in \mathbb{N}^*$ *we have*

$$\|T_k(u_n)\|_{L^2(0,T,\mathcal{H}_0^s(\Omega))} \leqslant C(\lambda, k, |\Omega_T|, |u_0|_1), \text{ for } k > 0. \tag{4.16}$$

$$\iint_{\Omega_T} u_n^{m-1} dx\, dt \leqslant C(\lambda, |\Omega_T|, |u_0|_1). \tag{4.17}$$

Proof. Let $k, n \in \mathbb{N}^*$ be fixed. Taking $\varphi = T_k(u_n)$ as test function in (4.12), we obtain by integration by part that

$$\begin{aligned}\int_\Omega \Theta_k(u_n(x,T))dx \quad &+\frac{1}{2}\int_0^T \iint_{D_\Omega} (u_n(x,t)-u_n(y,t))(T_k(u_n(x,t))-T_k(u_n(y,t)))\,d\mu\,dt \\ &= \lambda \iint_{\Omega_T} u_n^{p-1}T_k(u_n)\,dx\,dt + \int_\Omega \Theta_k(u_0(x))dx.\end{aligned} \tag{4.18}$$

where $\Theta_k(u) = \displaystyle\int_0^u T_k(r)dr$. This yields to

$$\begin{aligned}\int_\Omega \Theta_k(u_n(x,T))dx \quad &+\frac{1}{2}\int_0^T \iint_{D_\Omega} (u_n(x,t)-u_n(y,t))(T_k(u_n(x,t))-T_k(u_n(y,t)))\,d\mu\,dt \\ &+(1-\epsilon)\iint_{\Omega_T} u_n^{m-1}T_k(u_n)\,dx\,dt \leqslant C(\Omega_T, |u_0|_1) + \int_\Omega \Theta_k(u_0(x))dx,\end{aligned} \tag{4.19}$$

Using inequality (1.22) and by the fact that $u_n \geqslant 0$, $\Theta_k \geqslant 0$ and $\Theta_k(u_0) \leqslant k|u_0|$, one has that

$$\frac{1}{2}\int_0^T \iint_{D_\Omega} |T_k(u_n(x,t))-T_k(u_n(y,t))|^2\,d\mu\,dt \leqslant C(\lambda, k, |\Omega_T|, |u_0|_1).$$

Now, to prove estimate (4.17), we get by vertu of Young's inequality

$$\begin{aligned}\int_\Omega \Theta_k(u_n(x,T))dx \quad &+\int_0^T \iint_{D_\Omega} |T_k(u_n(x,t))-T_k(u_n(y,t))|^2\,d\mu\,dt \\ &+(1-\epsilon)\iint_{\Omega_T} u_n^{m-1}T_k(u_n)\,dx\,dt \qquad . \\ &\leqslant C(\lambda, |\Omega_T|, |u_0|_1).\end{aligned}$$

Drooping non-negative terms to deduce that

$$\iint_{\Omega_T} u_n^{m-1}T_k(u_n)\,dx\,dt \leqslant C(|\Omega_T|, |u_0|_1).$$

We deduce that, for all $k \geqslant 0$ (4.17)

$$\iint_{\Omega_T} u_n^{m-1}\,dx\,dt \leqslant C_2(\lambda, |\Omega_T|, |u_0|_1).$$

□

➢ Passing to the limit in the approximate problem

We are ready to prove the existence of weak solution u obtained as a limit of the unique sequence of solutions u_n. For that, we need the following Lemmas.

Remark 4.3.1.

- *Let $(u_n)_n$ be a sequence of measurable functions such that $T_k(u_n)$ is bounded in $L^2(0,T;\mathcal{H}_0^s(\Omega))$ for every $k>0$. Then, there exists a measurable function u such that $T_k(u)\in L^2(0,T;\mathcal{H}_0^s(\Omega))$. Moreover,*

$$u_n \to u \text{ a.e. in } \Omega_T \qquad \textit{and} \qquad T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } \ L^2(0,T;\mathcal{H}_0^s(\Omega)).$$

- *Since u_n is increasing sequence, we conclude that*

$$u_n^{m-1} \to u^{m-1} \quad \textit{in} \quad L^1(\Omega_T).$$

Lemma 4.3.3. *Let $(u_n)_n$ be a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega))$. Then, there exists a subsequence, still denoted $(u_n)_n$, and a function $u\in\mathcal{C}([0,T];L^1(\Omega))$ such that:*

$$u_n \to u \quad \textit{in} \quad \mathcal{C}((0,T);L^1(\Omega)).$$

Proof. We shall prove that $(u_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega))$.

Recalling that u_n is an increasing sequence, we fix $n>\ell>1$ and define $w_{n,\ell}=u_n-u_\ell\geqslant 0$. Consequently, $w_{n,\ell}$ satisfies the following problem

$$\left\{\begin{array}{rcll} \partial_t(w_{n,\ell})+(-\Delta)^s w_{n,\ell}+(u_n^{m-1}-u_\ell^{m-1}) & = & \lambda\Big(\frac{u_n^{p-1}}{1+\frac{1}{n}u_n^{p-1}}-\frac{u_\ell^{p-1}}{1+\frac{1}{\ell}u_\ell^{p-1}}\Big) & \text{in} \quad \Omega\times(0,T), \\ w_{n,\ell}(x,0) & = & u_{n,0}(x)-u_{l,0}(x) & \text{in} \quad \Omega, \\ w_{n,\ell} & = & 0 & \text{in} \quad (\mathbb{R}^N\setminus\Omega)\times(0,T). \end{array}\right. \tag{4.20}$$

Using $T_k(u_n-u_\ell)$ as test function in (4.20)

$$\begin{aligned} &\int_\Omega \Theta_k(u_n-u_\ell)(x,T)dx+\int_0^T\langle(u_n-u_\ell),T_k(u_n-u_\ell)\rangle_s dt \\ &+\iint_{\Omega_T}(u_n^{m-1}-u_\ell^{m-1})T_k(u_n-u_\ell)\,dx\,dt \\ &=\lambda\iint_{\Omega_T}\Big(\frac{u_n^{p-1}}{1+\frac{1}{n}u_n^{p-1}}-\frac{u_\ell^{p-1}}{1+\frac{1}{\ell}u_\ell^{p-1}}\Big)T_k(u_n-u_\ell)\,dx\,dt+\int_\Omega\Theta_k(u_n-u_\ell)(x,0)dx \\ &\leqslant k\lambda\int_0^T\int_{\{u_n-u_\ell\geqslant k\}}\Big|\frac{u_n^{p-1}}{1+\frac{1}{n}u_n^{p-1}}-\frac{u_\ell^{p-1}}{1+\frac{1}{\ell}u_\ell^{p-1}}\Big|\,dxdt \\ &+k\int_{\{u_{n,0}-u_{l,0}\geqslant k\}}|u_n(x,0)-u_\ell(x,0)|dx. \end{aligned} \tag{4.21}$$

dividing the inequality over k and using the fact that $\frac{\Theta(\sigma)}{k} \to \sigma$ as $k \to 0$. Moreover, we have

$$u_{n,0}, u_{l,0} \to u_0 \quad \text{in} \quad L^1(\Omega)$$

and

$$u_n^{p-1}, u_\ell^{p-1} \to u^{p-1} \quad \text{in } L^1(\Omega_T) \quad \text{as} \quad n, \ell \to \infty.$$

So, we can directly deduce that

$$\iint_{\Omega_T} |u_n - u_\ell| dx \leqslant \iint_{\Omega_T} \Big(\frac{u_n^{p-1}}{1+\frac{1}{n}u_n^{p-1}} - \frac{u_\ell^{p-1}}{1+\frac{1}{\ell}u_\ell^{p-1}}\Big) T_k(u_n - u_\ell)\, dxdt + \int_\Omega \Theta_k(u_n - u_\ell)(x,0) dx \to 0. \tag{4.22}$$

As $n, \ell \to \infty$. Therefore

$$\sup_{t\in[0,T]} \iint_{\Omega_T} |u_n - u_\ell| dx \leqslant \lambda \int_0^T \int_\Omega |u_n^{p-1} - u_\ell^{p-1}|\, dxdt + \int_\Omega |(u_n - u_\ell)(x,0)| dx \to 0.$$

Therefore, we conclude that $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega))$. Moreover, $u \in \mathcal{C}([0,T];L^1(\Omega))$ and there exists a sequence still denoted $(u_n)_n$ such that

$$u_n \to u \qquad \text{in} \quad \mathcal{C}([0,T];L^1(\Omega)).$$

□

Conclusion: u is a weak solution of (4.1)

Now, we can prove that u is a solution of (4.1). For that, we take $\varphi \in L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ as test function in (4.12), then

$$\begin{aligned} \int_0^T \int_\Omega \partial_t u_n \varphi dxdt &+ \int_0^T \iint_{D_\Omega} (u_n(x,t) - u_n(y,t))[\varphi(x,t) - \varphi(y,t)]\, d\mu\, dt \\ &+ \int_0^T \int_\Omega u_n^{m-1}\varphi dxdt = \lambda \int_0^T \int_\Omega u_n^{p-1}\varphi\, dxdt. \end{aligned} \tag{4.23}$$

By integration by part

$$\begin{aligned} -\int_0^T \int_\Omega u_n \partial_t \varphi dxdt &+ \int_0^T \iint_{D_\Omega} (u_n(x,t) - u_n(y,t))[\varphi(x,t) - \varphi(y,t)]\, d\mu\, dt \\ &+ \int_0^T \int_\Omega u_n^{m-1}\varphi dxdt = \lambda \int_0^T \int_\Omega u_n^{p-1}\varphi\, dxdt + \int_\Omega u_{n,0}\varphi(x,0) dx. \end{aligned}$$

For the first term we use the weak convergence of u_n to u in $L^2(0,T;\mathcal{H}_0^s(\Omega))$, with $\partial_t\varphi \in L^2(0,T;H^{-s}(\Omega))$,

$\varphi(x,T)=0$ and the strong convergence of $u_{n,0}$ to u_0 in $L^1(\Omega)$

$$\int_0^T\int_\Omega \partial_t u_n \varphi dxdt \quad = -\int_0^T\int_\Omega u_n\partial_t\varphi dxdt - \int\limits_\Omega u_{n,0}\varphi(x,0)dx.$$

We obtain that

$$\int_0^T\int_\Omega \partial_t u_n\varphi dxdt \to \int_0^T\int_\Omega \partial_t u\varphi dxdt = -\int_0^T\int_\Omega u\partial_t\varphi dxdt - \int\limits_\Omega u_0\varphi(x,0)dx.$$

On the other hand, the weak convergence in $L^2(0,T;\mathcal{H}_0^s(\Omega))$ implies that

$$\int_0^T\iint_{D_\Omega}(u_n(x,t)-u_n(y,t))[\varphi(x,t)-\varphi(y,t)]\,d\mu\,dt \to \int_0^T\iint_{D_\Omega}(u(x,t)-u(y,t))[\varphi(x,t)-\varphi(y,t)]\,d\mu\,dt.$$

Thanks to Lemma 4.3.1, since $u_n^{m-1}\to u^{m-1}$ in $L^1(\Omega_T)$, by Lebesgue dominated convergence Theorem, we get that

$$\int_0^T\int_\Omega u_n^{m-1}\varphi dxdt \to \int_0^T\int_\Omega u^{m-1}\varphi dxdt,$$

and

$$\int_0^T\int\limits_\Omega u_n^{p-1}\varphi\, dxdt \to \int_0^T\int\limits_\Omega u^{p-1}\varphi\, dxdt.$$

In conclusion, we deduce that u satisfies

$$\begin{aligned}-\int_0^T\int_\Omega u\partial_t\varphi dxdt \quad &+\int_0^T\iint_{D_\Omega}(u(x,t)-u(y,t))[\varphi(x,t)-\varphi(y,t)]\,d\mu\,dt\\ &+\int_0^T\int_\Omega u^{m-1}\varphi dxdt = \lambda\int_0^T\int\limits_\Omega u^{p-1}\varphi\,dxdt + \int\limits_\Omega u_0\varphi(x,0)dx.\end{aligned}$$

for all $\varphi\in L^2(0,T;\mathcal{H}_0^s(\Omega))\cap L^\infty(\Omega_T)$ with $\partial_t\varphi\in L^2(0,T;H^{-s}(\Omega))$, $\varphi(x,T)=0$.

4.3.2 Globality of solution

Before delving into the main result, we begin by defining the concept of globality.

Definition 4.3.2. *We say that u is a global solution to Problem (4.1), if u exists for all $t\geqslant 0$.*

Now, we prove the globality of solution in time.

Theorem 4.3.2. *Assume that $u_0\in L^1(\Omega)$, and for all $\lambda>0$ such that $\lambda>\lambda^*$. Then, the solution $u\in\mathcal{C}([0,T];L^1(\Omega))$ of Problem (4.1) exists globally and it is positive.*

Proof. Let us take the first eigenfunction $\varphi_{1,s}\in L^\infty(\Omega)$ as a test function in (4.1)

$$\frac{d}{dt}\int_\Omega u\varphi_{1,s}\,dx+\int_\Omega(-\Delta)^s u\varphi_{1,s}\,dx+\int_\Omega u^{m-1}\varphi_{1,s}\,dx=\lambda\int_\Omega u^{p-1}\varphi_{1,s}\,dx.$$

By virtue of the Definition of the first eigenvalue $\lambda_{1,s}$, given by (1.13), and the inequality (1.25) for ε sufficiently small, we obtain

$$\frac{d}{dt}\int_\Omega u\varphi_{1,s}\,dx+\lambda_{1,s}\int_\Omega u\varphi_{1,s}\,dx+(1-\lambda\varepsilon)\int_\Omega u^{m-1}\varphi_{1,s}\,dx\leqslant c_\varepsilon\int_\Omega u\varphi_{1,s}\,dx.$$

By dropping the non-negative terms, we get

$$\frac{d}{dt}\int_\Omega u\varphi_{1,s}\,dx\leqslant(\lambda c_\varepsilon-\lambda_{1,s})\int_\Omega u\varphi_{1,s}\,dx.$$

Putting $Y(t)=\int_\Omega u\varphi_{1,s}\,dx$, this reduces to

$$Y'(t)\leqslant(\lambda c_\varepsilon-\lambda_{1,s})Y(t),$$

Gronwall's inequality implies that

$$Y(t)=\int_\Omega u\varphi_{1,s}\,dx\leqslant Y(0)e^{(\lambda c_\varepsilon-\lambda_{1,s})t}. \tag{4.24}$$

This ensure the global existence. □

4.3.3 Uniqueness of the solution

Theorem 4.3.3. *Let* $u_0\in L^1(\Omega)$*. Then, Problem* (4.1) *admits a unique weak solution in* $\mathcal{C}([0,T];L^1(\Omega))$.

Proof. Suppose that u_1 is a minimal solution obtained as limit of approximation, and let define u_2 as another solution which satisfy $u_1\leqslant u_2$ with the same initial data. Taking $w=u_2-u_1$ satisfies

$$\begin{cases}\partial_t w+(-\Delta)^s w &= \lambda(u_2^{p-1}-u_1^{p-1})-(u_2^{m-1}-u_1^{m-1}) \quad \text{in } \Omega_T,\\ w(x,0) &= 0 \quad \text{in } \Omega,\\ w &= 0 \quad \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T).\end{cases} \tag{4.25}$$

From Proposition 1.3.2, since $\varphi_{1,s}$ is non-negative and belongs to $\mathcal{H}_0^s(\Omega)\cap L^\infty(\Omega)$, we choose it as a test function in (4.25).

$$\int_\Omega\partial_t w\varphi_{1,s}\,dx+\int_\Omega(-\Delta)^s w\varphi_{1,s}\,dx+\int_\Omega(u_2^{m-1}-u_1^{m-1})\varphi_{1,s}\,dx=\lambda\int_\Omega(u_2^{p-1}-u_1^{p-1})\varphi_{1,s}\,dx.$$

Hence, using the Definition of $\lambda_{1,s}$ and inequality (1.26) by choosing ε sufficiently small

$$\begin{aligned}\int_\Omega \partial_t(u_2-u_1)\varphi_{1,s}\,dx+\lambda_{1,s}\int_\Omega(u_2-u_1)\varphi_{1,s}\,dx \quad &+(1-\varepsilon\lambda)\int_\Omega(u_2^{m-1}-u_1^{m-1})\varphi_{1,s}\,dx\\ &=\lambda\int_\Omega(u_2^{p-1}-u_1^{p-1})\varphi_{1,s}\,dx\\ &\leqslant c_\varepsilon\int_\Omega(u_2-u_1)\varphi_{1,s}\,dx.\end{aligned}$$

Since $\varphi_{1,s}\geqslant 0$ and $u_2-u_1\geqslant 0$, it follows that

$$\int_\Omega(u_2-u_1)\varphi_{1,s}\,dx\geqslant 0.$$

Also, we have $\int_\Omega(u_2^{m-1}-u_1^{m-1})\varphi_{1,s}\,dx\geqslant 0$. This yields to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(u_2-u_1)\varphi_{1,s}\,dx\leqslant c_\varepsilon\int_\Omega(u_2-u_1)\varphi_{1,s}\,dx.$$

By putting $Y(t)=\int_\Omega(u_2-u_1)\varphi_{1,s}dx$, the following ordinary inequality is obtained

$$Y(t)'\leqslant c_\varepsilon Y(t).$$

We use Gronwall's inequality to deduce that $Y(t)=0$, for all $t\geqslant 0$, since $Y(0)=0$. Taking into account that $\varphi_{1,s}\geqslant 0$, $\varphi_{1,s}\neq 0$ and $u_2-u_1\geqslant 0$, to conclude that

$$(u_2-u_1)\varphi_{1,s}=0,\ \text{a.e. in }\Omega,\ \text{ for all }\varphi_{1,s}\in L^\infty(\Omega).$$

Then,

$$u_1=u_2 \qquad \text{a.e.} \qquad \Omega \tag{4.26}$$

□

4.3.4 Nonexistence result

Theorem 4.3.4. *Assume that λ small enough and $2<p<m$. Suppose that $u_0\in L^1(\Omega)$. Then, the problem (4.1) has no solution.*

Proof. As above taking as a test function $\varphi_{1,s}$ as a test function of (4.1), we follow closely the above result, and by using the inequality (1.25), we obtain

$$\frac{1}{2}\frac{d}{\partial t}\int_\Omega u\varphi_{1,s}\,dx+(\lambda_1-\lambda c_\varepsilon)\int_\Omega u\varphi_{1,s}\,dx+(1-\lambda_\epsilon\varepsilon)\int_\Omega u^{m-1}\varphi_{1,s}\,dx\leqslant 0.$$

Putting $Y(t) = \int_{\Omega} u\varphi_{1,s}\, dx$, by Jensen inequality and the fact that λ is small enough, this reduce to

$$Y'(t) + C_2 Y^{m-1}(t) \leqslant 0.$$

Thus, we integrate in t

$$\frac{1}{m-2}\left[\frac{1}{Y(0)^{m-2}} - \frac{1}{Y(t)^{m-2}}\right] \leqslant C_2 t.$$

We deduce that

$$Y(t) \leqslant \left[\frac{1}{Y(0)^{2-m} + (m-2)C_2 t}\right]^{\frac{1}{m-2}}.$$

Thus, as $t \to \infty$ the result follows. □

4.4 A priori bound

In this subsection, we first establish a uniform a priori bound, and then we provide a powerful estimate that allows us to determine the asymptotic behavior in the next section.

Theorem 4.4.1. *Assume that* $T, \lambda > 0$ *and* $2 < p < m$. *Let* $u_0 \in L^\infty(\Omega)$ *then* u *is a solution to Problem* (4.1) *such that* $u \in L^\infty(\Omega_T)$. *Furthermore, if* $|u_0|_\infty \leqslant \lambda^{\frac{1}{m-p}}$ *then*

$$|u|_{\infty,\Omega_T} \leqslant \lambda^{\frac{1}{m-p}}. \tag{4.27}$$

Proof. To prove the first point, we will use bootsrapping method:

- *Initialization:* We prove the initial bound in $L^m(\Omega_T)$. To this end, we take u as test function in (4.1)

$$\frac{1}{2}\int_{\Omega} u^2(x,T)\, dx + \int_0^T \|u(x,t)\|_s^2\, dt + |u|_m^m = \lambda |u|_p^p + \frac{1}{2}\int_{\Omega} u_0^2 dx.$$

By the inequality (1.24), we obtain

$$(1-\lambda\varepsilon)|u|_m^m \leqslant C_\varepsilon(|\Omega_T|) + \frac{1}{2}|u_0|_\infty^2 |\Omega|.$$

Thus, we have shown that

$$|u|_m^m \leqslant C(|\Omega_T|, |u_0|_\infty).$$

- *Bootstrap:* We prove that u is bounded in $L^\infty(\Omega_T)$. For that, we claim that for $\theta > 0$, if u is bounded in $L^{p+\theta}(\Omega_T)$ then u is bounded in $L^{m+\theta}(\Omega_T)$. We take u^θ as test function in (4.1)

$$\frac{1}{\theta+1}\int\limits_\Omega u^{\theta+1}(x,T)\,dx+C\int_0^T\|u^{\frac{\theta+1}{2}}(.,t)\|_s\,dt+\iint_{\Omega_T}|u|^{\theta+m}\,dx\,dt\leqslant\lambda\iint_{\Omega_T}|u|^{\theta+p}\,dx\,dt+\frac{1}{\theta+1}|u_0|_{\theta+1}.$$

Then, by the positivity of the the two first terms in the left hand side, we get

$$\iint_{\Omega_T}|u|^{\theta+m}\,dx\,dt\leqslant\lambda\iint_{\Omega_T}|u|^{\theta+p}\,dx\,dt+\frac{1}{\theta+1}|u_0|_\infty.$$

Which implies the claim.

We put $\theta_k+p=\theta_{k-1}+m$, such that $\theta_0=0$. Indeed, if we take u^{θ_1} as test function in (4.1) such that

$$p+\theta_1=m+\theta_0$$

we obtain that

$$\iint_{\Omega_T}|u|^{\theta_1+m}\,dx\,dt\leqslant\lambda\iint_{\Omega_T}|u|^{\theta_1+p}\,dx\,dt.$$

which implies that

$$\iint_{\Omega_T}|u|^{2m-p}\,dx\,dt\leqslant\lambda\iint_{\Omega_T}|u|^{m}\,dx\,dt.$$

since u is bounded in $L^m(\Omega_T)$, then it is bounded also in $L^{2m-p}(\Omega_T)$, then we choose θ_2 such that $p+\theta_2=m+\theta_1=2m-p$, and we take by the same way we take u^{θ_2} as test function

$$\iint_{\Omega_T}|u|^{3m-2p}\,dx\,dt\leqslant\lambda\iint_{\Omega_T}|u|^{2m-p}\,dx\,dt.$$

This implies that u is also bounded in $L^{3m-2p}(\Omega_T)$. by induction, we define

$$\theta_k=k(m-p),m>p,$$

therefore

$$\theta_k+p=\theta_{k-1}+m.$$

This implies that u is bounded in $L^{p+\theta_k}(\Omega_T)$ and in $L^{km-(k-1)p}(\Omega_T)$. As $k\to\infty$, $\theta_k\to\infty$, which yields that u is bounded in $L^\infty(\Omega_T)$.

To prove the second point, we take $\rho_+(x,t) = (u - \lambda^{\frac{1}{m-p}})_+$ as test function in (4.1)

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega} \rho_+(x,t)^2 dx + \iint\limits_{D_\Omega} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu = \int_{\Omega} (\lambda u^{p-1} - u^{m-1})\rho_+(x,t)\, dx. \tag{4.28}$$

We first analyze the second term, for that, let us split D_Ω into four subsets D_i, $1 \leqslant i \leqslant 4$, with

$$\begin{aligned}
D_1 &= \{(x,\, y, t) \in D_\Omega \times (0,T) : u(x,t) - \lambda^{\frac{1}{m-p}} > 0 \; and \; u(y,t) - \lambda^{\frac{1}{m-p}} > 0\},\\
D_2 &= \{(x,\, y, t) \in D_\Omega \times (0,T) : u(x,t) - \lambda^{\frac{1}{m-p}} \leqslant 0 \; and \; u(y,t) - \lambda^{\frac{1}{m-p}} \leqslant 0\},\\
D_3 &= \{(x,\, y, t) \in D_\Omega \times (0,T) : u(x,t) - \lambda^{\frac{1}{m-p}} > 0 \; and \; u(y,t) - \lambda^{\frac{1}{m-p}} \leqslant 0\},\\
D_4 &= \{(x,\, y, t) \in D_\Omega \times (0,T) : u(x,t) - \lambda^{\frac{1}{m-p}} \leqslant 0 \; and \; u(y,t) - \lambda^{\frac{1}{m-p}} > 0\}.
\end{aligned}$$

$$\iint\limits_{D_\Omega} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu = I_1 + I_2 + I_3 + I_4.$$

Where

$$I_i = \iint\limits_{D_i} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu, \text{ for } 1 \leqslant i \leqslant 4. \tag{4.29}$$

For the clarity, we analyze each term separately.

- In D_1, we have:

$$I_1 = \iint\limits_{D_1} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu = \iint\limits_{D_1} |u(x,t)-u(y,t)|^2\, d\mu \geqslant 0.$$

- If $(x,y,t) \in D_2 : \rho_+(x,t) = \rho_+(y,t) = 0$ implies that $I_2 = 0$.
- For I_3, we have $u(y,t) \leqslant \lambda^{\frac{1}{m-p}}$, then

$$\begin{aligned}
I_3 = \iint\limits_{D_3} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu &= \iint\limits_{D_3} (u(x,t)-u(y,t))(u(x,t)-\lambda^{\frac{1}{m-p}})\, d\mu\\
&\geqslant \iint\limits_{D_3} |u(x,t)-\lambda^{\frac{1}{m-p}}|^2 d\mu \geqslant 0.
\end{aligned}$$

- Also in D_4, we have $u(x,t) \leqslant \lambda^{\frac{1}{m-p}}$,

$$\begin{aligned}
I_4 = \iint\limits_{D_4} (u(x,t)-u(y,t))(\rho_+(x,t)-\rho_+(y,t))\, d\mu &= -\iint\limits_{D_4} (u(x,t)-u(y,t))(u(y,t)-\lambda^{\frac{1}{m-p}})\, d\mu\\
&\geqslant \iint\limits_{D_4} |u(y,t)-\lambda^{\frac{1}{m-p}}|^2 d\mu \geqslant 0.
\end{aligned}$$

This yields to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}((u-\lambda^{\frac{1}{m-p}})_+)^2dx &\leqslant \int_{\Omega}(\lambda u^{p-1}-u^{m-1})(u-\lambda^{\frac{1}{m-p}})_+\,dx\\ &= \int_{\{u-\lambda^{\frac{1}{m-p}}\geqslant 0\}}(\lambda u^{p-1}-u^{m-1})(u-\lambda^{\frac{1}{m-p}})\,dx.\end{aligned}$$

Notice that $\lambda u^{p-1}-u^{m-1}\leqslant 0$, if $u\geqslant \lambda^{\frac{1}{m-p}}$, yielding

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}((u-\lambda^{\frac{1}{m-p}})_+)^2dx \leqslant 0.$$

Putting $Z(t)=\int\limits_{\Omega}((u-\lambda^{\frac{1}{m-p}})_+)^2dx$, so, we get

$$\int\limits_{\Omega}((u-\lambda^{\frac{1}{m-p}})_+)^2dx \leqslant \int\limits_{\Omega}((u_0-\lambda^{\frac{1}{m-p}})_+)^2dx.$$

Since $u_0\leqslant \lambda^{\frac{1}{m-p}}$, we obtain that

$$(u-\lambda^{\frac{1}{m-p}})_+=0.$$

Thus, $0\leqslant u(x,t)\leqslant \lambda^{\frac{1}{m-p}}$. In particular,

$$|u|_{\infty,\Omega_T}\leqslant \lambda^{\frac{1}{m-p}}.$$

□

Remark 4.4.1. *Note that the solution to Problem* (4.1) *, denoted by u depends on the parameter m. Therefore, we will use the notation u_m to emphasize this dependency.*

4.5 Asymptotic analysis

In this section, we examine the asymptotic behavior of the unique solution u as $t\to\infty$, and subsequently as $m\to\infty$. Namely, we aim to determine the following limit

$$\lim_{t\to\infty}\left(\lim_{m\to\infty}u_m(x,t)\right)$$

More precisely, we determine the limit profiles of the solution in these two scenarios:

- *Asymptotic behavior as* $m \to \infty$ will be characterized by a limit profile, which solves a free boundary problem.

- *Asymptotic behavior as* $t \to \infty$ determines the limit of the limit profile obtained in the first scenario.

Before proceeding to the asymptotic analysis, it is essential to establish a number of preliminary results that will play a crucial role in the forthcoming arguments.

Proposition 4.5.1. *Let* $u_0 \in L^\infty(\Omega)$, $u \in L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ *be a weak solution Problem* (4.1). *Then*

$$\int_0^T \iint_{D_\Omega} |u_m(x,t) - u_m(y,t)|^2 d\mu \, dt \leqslant C_2(\lambda, |\Omega_T|, |u_0|_\infty). \tag{4.30}$$

$$\int_0^T \int_\Omega u_m^m \, dx \, dt \leqslant C_2(\lambda, |\Omega_T|, |u_0|_\infty). \tag{4.31}$$

$$\iint_{\Omega_T} |\partial_t(u_m)|^2 \, dx \, dt \leqslant M. \tag{4.32}$$

Proof. To prove the first and second estimates, we take u as test function in (4.1). As consequence of (4.27), we obtain

$$\begin{aligned} \frac{1}{2}\int\limits_\Omega |u_m(x,t)|^2 \, dx \quad &+ \quad \frac{1}{2}\int_0^T \iint_{D_\Omega} |u_m(x,t) - u_m(y,t)|^2 \, d\mu \, dt + \int_0^T \int_\Omega u_m^m \, dx \, dt \\ &= \quad \lambda \int_0^T \int_\Omega u_m^p \, dx \, dt + \frac{1}{2}\int\limits_\Omega |u_0|^2 dx \\ &\leqslant \quad \lambda^{\frac{p}{m-p}} |\Omega_T| + \frac{1}{2}|u_0|_\infty^2 := C_2(\lambda, m, |\Omega_T|, |u_0|_\infty). \end{aligned}$$

Then, we conclude (4.30) and (4.31).

Now, let us prove the last estimate (4.32). For a fixed $0 \leqslant t \leqslant T$,

$$\begin{aligned} \|\partial_t(u_m)(.,t)\|_{H^{-s}(\Omega)} &= \sup_{\|\phi\|_s \leqslant 1} \left|\partial_t(u_m)\phi dx\right| \\ &\leqslant \sup_{\|\phi\|_s \leqslant 1} \left|\iint_{D_\Omega} (u_m(x,t) - u_m(y,t))(\phi(x,t) - \phi(y,t) dx dy\right| d\mu \\ &+ \sup_{\|\phi\|_s \leqslant 1} \left|\lambda \int\limits_\Omega u_m^{p-1}\phi\right| + \sup_{\|\phi\|_s \leqslant 1} \left|\int\limits_\Omega u_m^{m-1}\phi\right| \\ &\leqslant \sup_{\|\phi\|_s \leqslant 1} \left\{\|u_m\|_s \|\phi\|_s + \lambda^{\frac{m-1}{m-p}} c(\Omega) + \lambda^{\frac{m-1}{m-p}} c(\Omega)\right\} \\ &\leqslant \|u_m\|_s + c_1(\Omega) \end{aligned}$$

such that $c_1(\Omega) = (\lambda^{\frac{m-1}{m-p}} + \lambda^{\frac{m-1}{m-p}})c(\Omega)$ then

$$\begin{aligned} \int_0^T \|\partial_t(u_m)(.,t)\|^2_{H^{-s}(\Omega)}dt &\leqslant \int_0^T \|u_m\|_s^2 dt + c_2(\lambda, m, \Omega_T) \\ &= C_2(\lambda, m, |\Omega_T|, |u_0|_\infty) + c_2(\lambda, m, \Omega_T) := M. \end{aligned}$$

□

Proposition 4.5.2. *Let $u_0 \in L^\infty(\Omega)$, that satisfies (H2), then*

$$|\partial_t u|_2 \to 0 \textit{ as } t \to \infty$$

Proof. We take the energy over time

$$E(u) = \mathcal{E}_{\lambda,m}(u(t)) = \frac{1}{2}\iint_{D_\Omega} |u(t) - u(t)|^2 d\mu + \frac{1}{m}\int_\Omega u^m dx - \frac{\lambda}{p}\int_\Omega u^p dx,$$

by derivation with respect to the time and substituting $\partial_t u$, we get

$$\frac{dE(u)}{dt} = \int_\Omega \partial_t u(-\Delta)^s u dx + \int_\Omega u^{m-1}\partial_t u dx - \int_\Omega \partial_t u u^{p-1} dx = -\langle \partial_t u, \partial_t u\rangle_2 \leqslant 0,$$

which implies that $E(u)$ is non-increasing in time. Therefore,

$$E(u) \leqslant E(u_0) \text{ for all } t > 0.$$

Moreover, under assumption $H2$, it follows that

$$E(u) < 0 \text{ for all } t > 0.$$

Since it is inferior bounded and negative for $\lambda > \lambda_*$, there exists a constant $\eta > 0$, such that for all $t > 0$

$$E(u) \geqslant -\eta.$$

By the Monotone Convergence Theorem, we conclude that $E(u)$ converges to a limit $E_\infty \geqslant -\eta$, when t tends to ∞. Therefore,

$$\frac{dE(u)}{dt} \to 0$$

as t tends to ∞. Then, we conclude that

$$|\partial_t u|_2^2 \to 0 \text{ as } t \to \infty$$

which implies that

$$\partial_t u \to 0 \qquad \text{a.e in } \Omega.$$

☐

Proposition 4.5.3. *Let $u_0 \in L^\infty(\Omega)$, that satisfies (H2), then $u \neq 0$*

Proof. We now prove that $u \neq 0$. To this end, we make use of the previous proposition, which ensures that $E(u) < 0$ when $\lambda > \Lambda(m)$, we have

$$\frac{1}{2}\|u(.,t)\|_s^2 \leqslant \frac{1}{2}\|u(.,t)\|_s^2 + \frac{1}{m}\int_\Omega u^m dx \leqslant \frac{\lambda}{p}|u(.,t)|_p^p. \tag{4.33}$$

We use u as a test function in (4.1)

$$\frac{1}{2}\frac{d}{dt}\|u(.,t)\|_2^2 + \|u(.,t)\|_s^2 + |u(.,t)|_m^m = \lambda|u(.,t)|_p^p,$$

We substitute in (4.33), we obtain

$$(\frac{1}{2} - \frac{1}{p})\|u(.,t)\|_s^2 \leqslant \frac{\lambda}{2p}\frac{d}{dt}|u(.,t)|_2^2 + \frac{1}{p}|u(.,t)|_m^m,$$

by Poincaré's inequality, and the uniform estimate, we get

$$C(\frac{1}{2} - \frac{1}{p})|u(.,t)|_2^2 \leqslant \frac{\lambda}{2p}\frac{d}{dt}|u(.,t)|_2^2 + \frac{1}{p}C_2(\Omega).$$

Putting now, $Y(t) = |u(.,t)|_2^2$, and denote $\beta = C(p-2)$, the above equation reduces to

$$\beta Y(t) \leqslant Y^{'}(t) + 2C_2(\Omega),$$

which has as solution

$$Y(t) \geqslant \frac{2C_2(\Omega)}{\beta} + |u_0|_2^2 e^{\beta t}.$$

Then, the result follows. ☐

4.5.1 Asymptotic behavior as $m \to \infty$

As in the elliptic case, we aim to perform an asymptotic analysis of the solutions found as $m \to \infty$. the asymptotic behavior of

Recalling that

$$\underline{\lambda}(m) := \inf\{\lambda : \mathcal{E}_{\lambda,m} \text{ has a local minimum } u_\lambda \in \mathcal{H}_0^s(\Omega) \cap L^m(\Omega) \text{ such that } \mathcal{E}_{\lambda,m}(u_\lambda) < 0\}. \quad (4.34)$$

and

$$E(u) = \mathcal{E}_{\lambda,m}(u(t)) = \frac{1}{2}\iint_{D_\Omega} |u(t) - u(t)|^2 d\mu + \frac{1}{m}\int_\Omega u^m dx - \frac{\lambda}{p}\int_\Omega u^p dx.$$

We will see that λ_* plays a crucial role in characterizing the asymptotic behavior of the solution as in the elliptic case. Moreover, from [Theorem 2.5.1, Chapter 2] we have that

$$\lim_{m\to\infty} \underline{\lambda}(m) = \lambda_*. \quad (4.35)$$

Where

$$\lambda_* = \inf_{\xi \in \mathcal{H}_0^s(\Omega)\cap L^\infty(\Omega)} \frac{p}{2}\frac{\|\xi\|_s^2 |\xi|_\infty^{p-2}}{|\xi|_p^p}, \quad (4.36)$$

is well-defined.

Theorem 4.5.1. *Let* $2 < p < m$, $u_0 \in L^\infty(\Omega)$ *that satisfies* $H1$ *and* $H2$. *For* $\lambda > \lambda_*$, *there exists* $u \in \mathcal{K} := \{v \in L^2(0, TH_0^s(\Omega)) : 0 \leqslant v(x,t) \leqslant 1\}$, *such that* $u \neq 0$ *and* u_m *converges strongly to* u *in* $L^2(0,T; H_0^s(\Omega))$ *and in every Lebesgue space as* $m \to \infty$. *Moreover,* u *is a solution of the bilateral obstacle problem*

$$\int_0^T \int\limits_\Omega \partial_t u(v-u)dx\, dt + \langle(-\Delta)^s u, v-u\rangle \geqslant \lambda\int_\Omega u^{p-1}(v-u)\, dx, \forall v \in \mathcal{K}. \quad (4.37)$$

In addition, there exists $\mathbf{c}_u \in L^\infty(\Omega_T)$, *such that* u *is a solution of the following equation :*

$$\partial_t u + (-\Delta)^s u + \mathbf{c}_u(x,t)\chi_{\{u=1\}} = \lambda u^{p-1} \qquad \text{in } \Omega_T \qquad u = 0 \text{ in } (\mathbb{R}^n \setminus \Omega_T) \quad (4.38)$$

where $0 < \mathbf{c}_u \leqslant \lambda$ *and* $\mathbf{c}_u(1-u) = 0$.

In order to prove this theorem, we need to establish several auxiliary lemmas first.

Lemma 4.5.1. *:*

1. $u_m \to u$ *strongly in* $L^2(0,T; H_0^s(\Omega))$ *as* $m \to \infty$.

2. *$u_m \to u$ strongly in every Lebesgue space.*

3. *$\partial_t u_m \to \partial_t u$ strongly in $L^2(\Omega_T)$.*

Proof. As consequence of Proposition 4.5.1, $(u_m)_m$ is bounded $L^2(0,T;\mathcal{H}_0^s(\Omega)) \cap L^m(\Omega_T)$, then up to a subsequence

- $u_m \rightharpoonup u$ weakly in $L^2(0,T;H_0^s(\Omega))$.
- $u_m \rightharpoonup u$ weakly in $L^m(\Omega_T)$.

As consequence of the uniform bound (4.27)

- $u_m \to u$ strongly in $L^q(\Omega_T)$ for all $q \geqslant 1$.
- Since u_m is a global solution, and $0 \leqslant u_m \leqslant \lambda^{\frac{1}{m-p}}$, we deduce that, when $m \to \infty$, u_m converge to u such that $0 \leqslant u \leqslant 1$.

Now, to prove the strong convergence in $L^2(0,T;H_0^s(\Omega))$, taking $(u_m - u)$ as test function in (4.1)

$$\int_0^T \int\limits_\Omega \partial_t u_m(u_m-u)dxdt + \int_0^T \int\limits_\Omega \langle u_m,\, u_m-u\rangle_s + \int_0^T \int_\Omega u_m^{m-1}(u_m-u)\,dx\,dt = \lambda \int_0^T \int_\Omega u_m^{p-1}(u_m-u)\,dxdt,$$

by the L^∞ a priori estimate (4.27), we get

$$\left| \int_{\Omega_T} |u_m|^{p-1}(u_m-u)\,dx\,dt \right| \leqslant \lambda^{\frac{p-1}{m-p}} |u_m-u|_{L^1(\Omega_T)}.$$

Passing to the limit as m tends to ∞

$$\left| \int_{\Omega_T} |u_m|^{p-1}(u_m-u)\,dx\,dt \right| \to 0 \text{ as } m \to \infty$$

which allows us to conclude that

$$\left| \int_{\Omega_T} |u_m|^{m-1}(u_m-u)\,dx\,dt \right| \to 0, \text{ as } m \to \infty.$$

By subtracting the terms which converge to 0

$$\int_0^T \int\limits_\Omega u_t(u_m-u)dx\,dt + \int_0^T \langle u, u_m-u\rangle_s dt = o(1),$$

we get

$$\int\limits_{\Omega} |u_m - u|^2 dx + \int_0^T \|u_m - u\|_s^2 dt = o(1).$$

Therefore, it follows that u_m converges to u strongly in $L^2(\Omega)$ and in $L^2(0,T;\mathcal{H}_0^s(\Omega))$ as $m \to \infty$. In addition, Proposition 4.5.3 implies that $u \neq 0$.

□

Lemma 4.5.2. $u(x,t) \in \mathcal{K} := \{v \in L^2(0,T,\mathcal{H}_0^s(\Omega)); 0 \leqslant u(x,t) \leqslant 1\}$*, for all* $t > 0$.

Proof. Let $k > 1$, we prove that $|\{(x,t) \in \Omega_T : u > k\}| = 0$ we have by (4.31)

$$k\Big|(x,t) \in \Omega_T : \{u_m(x,t) > k\}\Big| \leqslant \frac{1}{k^m} \iint_{\{u_m > k\}} u_m^m dx dt \leqslant \frac{C_1(\lambda, m, |\Omega_T|)}{k^m}. \tag{4.39}$$

Hence, by passing to the limit in m, we get

$$\Big|(x,t) \in \Omega_T : \{u > k\}\Big| = 0, \forall k > 1.$$

That is $u \in \mathcal{K}$. □

Lemma 4.5.3. *Let* u_m *converges to the unique solution* $u \in \mathcal{K}$ *of the bilateral obstacle problem*

$$\iint_{\Omega_T} \partial_t u(v-u) dx\, dt + \int_0^T \langle u,\, (v-u)\rangle_s\, dt \geqslant \lambda \iint_{\Omega_T} u^{p-1}(v-u)\, dx\, dt, \quad \forall v \in \mathcal{K}. \tag{4.40}$$

Proof. Let $v \in \mathcal{K}$, and let θ be any real number such that $0 < \theta < 1$. Using $\theta v - u_m$ as test function in (4.1)

$$\iint_{\Omega_T} \partial_t u_m(\theta v - u_m) dx\, dt + \int_0^T \langle u_m,\, \theta v - u_m\rangle_s\, dt + \int_0^T \int_\Omega u_m^{m-1} (\theta v - u_m) = \lambda \iint_{\Omega_T} u_m^{p-1} (\theta v - u_m). \tag{4.41}$$

We analyze the third term as

$$\begin{aligned} \int_0^T \int_\Omega u_m^{m-1} (\theta v - u_m) = &\int_0^T \int_{\{0 \leqslant u_m < \theta v\}} u_m^{m-1} (\theta v - u_m)\, dx\, dt \\ &+ \int_0^T \int_{\{u_m \geqslant \theta v\}} u_m^{m-1} (\theta v - u_m)\, dx\, dt. \end{aligned}$$

The first term of the right-hand side can be estimated as

$$u_m^{m-1} (\theta v - u_m) \leqslant [\theta v]^{m-1}(\theta v + \theta v) \leqslant 2\theta^m v = o(1) \text{ as } m \to \infty \textit{ since } \theta < 1. \tag{4.42}$$

For the second term, we have

$$u_m^{m-1}\left(\theta v-u_m\right)\leqslant 0 \text{ on } \{x: u_m(x)\geqslant \theta v\}$$

and this implies that

$$\limsup_{m\to\infty}\int_0^T\int_\Omega u_m^{m-1}\left(\theta v-u_m\right)\,dx\,dt\leqslant 0.$$

Then

$$\iint_{\Omega_T}\partial_t u_m(\theta v-u_m)dx\,dt+\theta\int_0^T\langle u_m,\,v\rangle_s\,dt\geqslant\lambda\iint_{\Omega_T}u_m^{p-1}\left(\theta v-u_m\right)\,dx\,dt+\int_0^T\|u_m\|_s^2.$$

We pass to the limit as $m\to\infty$, recalling that u_m converges strongly to u in $L^2(0,T;\mathcal{H}_0^s(\Omega))$ and $\partial_t u_m\to\partial_t u$ strongly in $L^2(\Omega)$, yielding that

$$\iint_{\Omega_T}u_t(\theta v-u)dx\,dt+\theta\langle u,\,v\rangle_s\geqslant\lambda\int_\Omega u^{p-1}\left(\theta v-u\right)\,dx+\|u\|_s^2, \tag{4.43}$$

For any v in $\mathcal{K}$ and any $\theta\in(0,1)$. Letting $\theta\to 1$, (4.40) follows.

To prove uniqueness, we assume that u_1,u_2 are two solutions of the variational inequality (4.40), i.e. for all $v\in\mathcal{K}$, they satisfy

$$\iint_{\Omega_T}\partial_t u_1(v-u_1)dx\,dt+\int_0^T\langle u_1,\,(v-u_1)\rangle_s\geqslant\lambda\int_0^T\int_\Omega u_1^{p-1}(v-u_1)\,dx\,dt,\quad\forall v\in\mathcal{K}.$$

and

$$\iint_{\Omega_T}\partial_t u_2(v-u_2)dx\,dt+\int_0^T\langle u_2,\,(v-u_2)\rangle_s\geqslant\lambda\int_0^T\int_\Omega u_2^{p-1}(v-u_2)\,dx\,dt,\quad\forall v\in\mathcal{K}.$$

Now, choose $v=u_2$ in the first inequality and $v=u_1$ in the second, we obtain by adding the two inequalities

$$\int_\Omega\partial_t(u_2-u_1)(u_1-u_2)dx+\langle u_2-u_1,\,(u_1-u_2)\rangle_s\geqslant\lambda\int_\Omega(u_2^{p-1}-u_1^{p-1})(u_2-u_1)\,dx,\quad\forall v\in\mathcal{K}.$$

Which is equivalent, and according to inequality (1.23), to the following

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(u_2-u_1)^2dx+\|u_2-u_1\|_s^2 &\leqslant \lambda\int_\Omega(u_2^{p-1}-u_1^{p-1})(u_1-u_2)\,dx\\ &\leqslant \lambda(p-2)\max\left\{|u_1|_\infty^{p-1},|u_2|_\infty^{p-1}\right\}\int_\Omega(u_2-u_1)^2\,dx\\ &\leqslant \lambda(p-2)\int_\Omega(u_2-u_1)^2\,dx.\end{aligned}$$

Putting $Y(t) = |u_2 - u_1|_2^2$, the above equation reduces to

$$Y(t) \leqslant 2\lambda(p-2)Y^{'}(t).$$

By the fact that $u_{1,0} = u_{2,0}$, we can deduce that

$$\int_\Omega (u_2 - u_1)^2(t)dx \leqslant e^{(2\lambda(p-2))t} \int_\Omega (u_0 - u_0)^2 \, dx = 0.$$

$\square$

Interestingly, the limiting profile that determines the asymptotic behavior turns out to be a solution of a free boundary problem with a nonlinear right-hand side. More precisely, we have

Proof of Theorem 4.5.1 According to the Lemmas 4.5.2, 4.5.1 and 4.5.3, the results follows, now, to finish our proof we need to prove the existence of $\mathbf{c}_u$. From (4.27), we have that

$$|u_m|_{\infty,\Omega_T}^{m-1} \leqslant \lambda^{\frac{m-1}{m-p}} \text{ for all } m > p,$$

Then, there exists $\mathbf{c}_u \in L^\infty(\Omega_T)$ such that, up to a subsequence,

$$\left\{(u_m)^{m-1}\right\} \overset{\star}{\rightharpoonup} \mathbf{c}_u \; weakly - \star \; in \; L^\infty(\Omega_T).$$

As a consequence, we obtain that $\mathbf{c}_u \geqslant 0$. Furthermore, u is a solution of

$$\partial_t u + (-\Delta)^s u = \lambda u^{p-1} - u^{m-1} \text{ a.e. } \Omega_T. \tag{4.44}$$

Moreover, as $m \to \infty$, we get that $0 \leqslant u(x,t) \leqslant 1$. In addition, considering χ_E the characteristic function of the set $E := \{(x,t) \in \Omega_T : \mathbf{c}_u > \lambda\}$ and using again (4.27), we get

$$\lambda^{\frac{m-1}{m-p}}|E| \geqslant \int_{\Omega_T} u_m^{m-1}\chi_E \, dxdt \xrightarrow{m\to\infty} \lambda|E| \geqslant \int_E \mathbf{c}_u \, dx\, dt > \lambda|E|.$$

Hence, $0 \leqslant \mathbf{c}_u \leqslant \lambda$. Taking $\varphi \in L^2(0,T;H_0^s(\Omega)) \cap L^m(\Omega_T)$ as a test function in (4.1) and passing to the limit it follows that u satisfies

$$\iint_{\Omega_T} \partial_t u\varphi dxdt + \int_0^T \langle u,\, \varphi\rangle_s \, dt + \iint_{\Omega_T} \mathbf{c}_u\varphi \, dx\, dt - \lambda \iint_{\Omega_T} u^{p-1}\varphi \, dx\, dt = 0, \quad \text{for all } v \in \mathcal{K}. \tag{4.45}$$

Finally, let us take $\varphi = (v-u)$, with $v \in \mathcal{K}$, in (4.45), then

$$\iint_{\Omega_T} \partial_t u(v-u)dxdt + \int_0^T \langle u,\, (v-u)\rangle_s dt + \iint_{\Omega_T} \mathbf{c}_u(v-u)\,dx\,dt - \lambda \iint_{\Omega_T} u^{p-1}(v-u)\,dx\,dt = 0, \text{ for all } v \in \mathcal{K}.$$

According to (4.43) we show that

$$\iint_{\Omega_T} \mathbf{c}_u(v-u)\,dx\,dt \leqslant 0, \quad \forall v \in \mathcal{K}.$$

Thus, we choose a sequence of $(v_j)_j \in \mathcal{K}$ such that $v_j \to 1$ in $L^1(\Omega)$, ve have

$$\iint_{\Omega_T} \mathbf{c}_u(1-u)\,dx\,dt \leqslant 0.$$

Hence, $\mathbf{c}_u \geqslant 0$ and $u \leqslant 1$, then

$$\mathbf{c}_u(x,t)\Big(u(x,t)-1\Big) = 0 \text{ a.e. in } \Omega_T. \tag{4.46}$$

By the fact that $E(u) \leqslant 0$, as proved in Proposition 4.5.2, we have

$$\lambda \int_0^T |u|_p^p\,dt > \frac{p}{2}\int_0^T \|u\|_s^2 dt.$$

By choosing u as test function in (4.44), we get

$$\lambda \int_0^T |u|_p^p\,dt = \iint_{\Omega_T} \mathbf{c}_u u\,dx\,dt + \int_0^T \|u\|_s^2 dt,$$

so that

$$\iint_{\Omega_T} \mathbf{c}_u u\,dx\,dt > \left(\frac{p}{2}-1\right)\int_0^T \|u\|_s^2\,dt > 0,$$

and since $\mathbf{c}_u \geqslant 0$, this shows that $\mathbf{c}_u \neq 0$, or equivalently, recalling (4.46),

$$\Big|\{x \in \Omega : u(x,t) = 1\}\Big| > 0.$$

This ends the proof of Theorem 4.5.1.

4.5.2 Asymptotic behavior as $t \to \infty$

Theorem 4.5.2. *Assume that $\lambda > \lambda_*$ and that $u_0 \in L^\infty(\Omega)$ satisfies the assumptions $H1$ and $H2$. There exists $\omega \in \mathcal{H}_0^s(\Omega)$ such that , as t $\to \infty$, the unique solution of Problem (4.38) u converges to*

ω in the following sense

$$u(.,t) \to \omega(.) \qquad \textit{strongly in } \mathcal{H}_0^s(\Omega).$$

$$\chi_{\{u=1\}} \to \chi_{\{\omega=1\}} \qquad \textit{strongly in } L^\alpha(\Omega) \textit{ for all } \alpha \geqslant 1.$$

Morevoer, ω is the unique solution of the following problem

$$\begin{cases} (-\Delta)^s\omega + \mathbf{c}_\omega(x)\chi_{\{\omega=1\}} &= \lambda\omega^{p-1} \quad \textit{in} \quad \Omega, \\ \omega &\geqslant 0 \qquad \textit{in} \quad \Omega, \\ \omega &= 0 \qquad \textit{in} \quad \mathbb{R}^N\backslash\Omega, \end{cases} \tag{4.47}$$

where $0 < \mathbf{c}_\omega(x) \leqslant \lambda$ and $\mathbf{c}_\omega(1-\omega) = 0$.

Proof. According to the above estimates, we obtain that $\|u(.,t)\|_s$ is bounded for $t > 0$, and up to subsequence, we obtain that

$$u(.,t) \rightharpoonup \omega \text{ weakly in } \mathcal{H}_0^s(\Omega) \text{ as } t \to \infty.$$

For t_n a subsequence $t_n \to \infty$ such that $u(.,t_n) \rightharpoonup \omega$, we obtain

$$\iint_{\Omega_T} \partial_t u(t_n)(v-u(t_n))dx\,dt + \int_0^T \langle u(t_n),\,(v-u(t_n))\rangle_s dt \geqslant \lambda \iint_{\Omega_T} u^{p-1}(t_n)(v-u(t_n))\,dx, \quad \forall v \in \mathcal{K}.$$

By Proposition 4.5.2, we have as $t_n \to \infty$

$$\langle \omega,\,(v-\omega)\rangle_s \geqslant \lambda \int_\Omega \omega^{p-1}(v-\omega)\,dx, \quad \forall v \in \mathcal{K}.$$

According to Proposition 4.5.3, for all $t > 0$, we have $u_m(x,t) > 0$, therefore, $u \neq 0$. And the result follows.

For the strong convergence, it is easy to show since by taking the difference and testing by $u - \omega$ and we obtain

$$u(.,t) \to \omega(.) \qquad \text{strongly in } \mathcal{H}_0^s(\Omega).$$

Following a standard arguments as in [93]. Since

$$0 \leqslant \chi_{\{u=1\}}(x,t) \leqslant 1 \quad \text{a.e. in } \Omega_T,$$

then, up to a subsequence, there exists a function $\chi^* \in L^\infty(\Omega)$, with $0 \leqslant \chi^*(x) \leqslant 1$, such that

$$\chi_{\{u=1\}}(x,t) \rightharpoonup^\star \chi^\star(x) \quad \text{weak-}\star \text{ in } L^\infty(\Omega), \quad \text{as } t \to +\infty.$$

implies that

$$\chi_{\{u=1\}}(x,t) \to \chi_{\{\omega=1\}}(x) \quad \text{strongly in } L^q(\Omega) \quad \text{for all } q \geqslant 1,$$

Moreover, since $0 < \mathbf{c}_u(x,t) \leqslant \lambda$ a.e. Ω_T and $u(x,t) \to \omega(x)$ a.e in Ω, then, up to a subsequence, there exists $\mathbf{c}_\omega$ such that

$$\mathbf{c}_u(x,t) \to \mathbf{c}_\omega(x), \text{ weakly-}\star \text{ in } L^\infty(\Omega)$$

In addition

$$\mathbf{c}_u(x,t)(1-u(x,t)) = 0 \quad \text{a.e. in } \Omega_T,$$

we pass to the limit and obtain

$$\mathbf{c}_\omega(x)(1-\omega(x)) = 0 \quad \text{a.e. in } \Omega.$$

This implies that $\mathbf{c}_\omega(x) > 0$ only on the set $\{\omega = 1\}$, and vanishes elsewhere and $\mathbf{c}_\omega(x) \leqslant \lambda$.

Furthermore, since

$$\chi_{\{u=1\}}(x,t) \to \chi_{\{\omega=1\}}(x) \quad \text{strongly in } L^q(\Omega) \quad \text{for all } q \geqslant 1,$$

and $0 < \mathbf{c}_u(x,t) \leqslant \lambda$, we deduce by the Dominated Convergence Theorem that

$$\mathbf{c}_u(x,t)\chi_{\{u=1\}}(x,t) \to \mathbf{c}_\omega(x)\chi_{\{\omega=1\}}(x) \quad \text{strongly in } L^1(\Omega).$$

To conclude, we take φ as a test function in (4.38) and pass to the limit, thereby obtaining that ω solves Problem (4.47). □

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Abstract :

This study investigates a class of nonlocal scalar field problems with logistic type nonlinearity set in a bounded domain, emphazing the role of increasing power in the equation. Initialy, we introduce a linear operator the fractional Laplacian, addressing the existence, non-existence, and multiplicity of solutions. Subsequently, we perform an asymptotic analysis as an nonlinear exponent increases indefinitely. Building on this, we extend the study to the nonlinear framework by introducing the nonlinear fractional p-Laplacian and establishing foundational results for the convex, linear, and concave cases. Finally, we explore the evolutionary version of the initial problem with L^1 data, addressing the main questions of existence, globality, uniqueness, and large-time behavior, thereby characterizing the solution dynamics under exponent growth.

Key words : Fractional Laplacian, fractional p-Laplacian, nonlocal scalar field problem, logistic problems, increasing power, asymptotic analysis, non-local elliptic problems, variational methods, nonlocal parabolic problem.

Résumé

Cette étude examine une classe de problèmes de champs scalaires non-local avec une non-linéarité de type logistique dans un domaine borné. Dans un premier temps, nous analysons le rôle de la puissance croissante dans l'équation en introduisant le Laplacien fractionnaire comme opérateur linéaire, en abordant l'existence, la non-existence et la multiplicité des solutions. Ensuite, nous effectuons une analyse asymptotique lorsque certains exposants non linéaires augmentent indéfiniment. Par la suite, nous étendons l'étude au cadre non linéaire en introduisant le Laplacien fractionnaire non linéaire et en établissant des résultats fondamentaux pour les cas: convexe, linéaire et concave. Enfin, nous explorons la version évolutive du problème initial avec des données L^1, en abordant les questions clés d'existence, de globalité, d'unicité et de comportement à long terme, caractérisant ainsi la dynamique de la solution sous la croissance des exposants.

Mots-clés : Laplacien fractionnaire, p-Laplacien fractionnaire, problème scalaire non local, problèmes logistiques, puissance croissante, analyse asymptotique, problèmes elliptiques non locaux, méthodes variationnelles, problème parabolique non local.

ملخص

تتناول هذه الدراسة فئة من مشاكل الحقول العددية ذات اللاخطية من النوع اللوجستيكي غير المحلي في نطاق محدود. في البداية، نقوم بدراسة دور القوى المتزايدة في المعادلة عن طريق تقديم لابلاسيان الكسري كمؤثر خطي، مع معالجة وجود، وعدم وجود، وتعدد الحلول. بعد ذلك، نقوم بإجراء تحليل غير متماثل عندما تزداد احدى الأسس اللاخطية بلا حدود. استنادًا إلى هذا، نوسّع الدراسة إلى الإطار اللاخطي من خلال اضافة ب لابلاسيان كسري غير خطي وتأسيس نتائج أساسية للحالات المحدبة، الخطية، المقعرة . أخيرًا، ندرس نسخة القطع المكافئ للمشكلة الأولية بالتطرق الى أسئلة الوجود، الشمولية و التفرد ، وسلوك الحل عند الزمن اللانهائي ، وبالتالي تحديد ديناميكيات الحل تحت تأثير نمو الأسس.

الكلمات المفتاحية

لابلاسيان كسري، ب لابلاسيان كسري، مشكلة الحقل العددي غير المحلي، المشاكل اللوجستية، القوة المتزايدة، التحليل التقاربي، المشاكل الإهليلجية غير المحلية، طرق التباين، مشكل القطع المكافئ غير المحلي.