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Thème

CONTRIBUTION TO QUASILINEAR ELLIPTIC PROBLEMS

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Contents

Introduction

In recent years considerable interest has been focused on the problem of type:

$$-\Delta_p u = f(\lambda, x, u, \nabla u) \tag{1}$$

with the boundary conditions

$$u = 0 \text{ in } \partial\Omega \tag{2}$$

or

$$\lim_{x \to \partial\Omega} u(x) = \infty \tag{3}$$

or

$$u|_{\partial\Omega} = \int_{\Omega} \Phi(x, y)\, u(y)\, dy \tag{4}$$

where Ω is a bounded region in $\mathbf{R}^N$, $N \geq 1$ with smooth boundary $\partial\Omega$, $\Delta_p u = div\left(|\nabla u|^{p-2} \nabla u\right)$ is the $p-$Laplace operator , $p > 1$, λ is a real parameter, $\Phi : \partial\Omega \times \Omega \to \mathbf{R}_+$ is a positive kernel and $f : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ is a given function.

The problems of the type (1-2) appears in the study of non-Newtonian fluids. The quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo plastics. If $p = 2$, they are Newtonian fluids (see for example Diaz [81] and its bibliography).

The problems of the type (1-3) comes originally from differential geometry and electrohydrodynamics and the problem of the type (1-4) are motivated by a model arising from quasi-static thermoelasticity.

In this work, we are interested with the existence and the multiplicity of the solutions of the type (1-2) and (1-3) in the one dimensional case and (1-4) in the higher dimension.

In order to prove our results we use a variety of techniques that is:

(i) Quadrature method for the $p-$Laplace operator.

(ii) Angular function technique.

(iii) Method of upper and lower solutions.

Chapter 2 is concerned by the question of existence, multiplicity and the zeros of solutions for the problem:

$$\begin{cases} \left(|u_k'|^{p-2} u_k'\right)' = f_k\left(t, u_1, u_2, ..., u_n, u_1', u_2', ..., u_n'\right) \text{ in } (a,b) \\ u_k(a)\sin\alpha_k - u_k'(a)\cos\alpha_k = 0 \\ u_k(b)\sin\beta_k - u_k'(b)\cos\beta_k = 0 \end{cases} \tag{5}$$

where $k = 1, 2, ..., n.$ and $p_k > 1$ for all $k = 1, 2, ..., n$. The functions $f_k : [a,b] \times \mathbf{R}^{2n} \to \mathbf{R}$ are continuous, $\alpha_k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\beta_k \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. The results of this chapter are generalizations to the case $p \neq 2$ of a result of Shekhter [177]. The results of this chapter are done in [49].

In chapter 3, we study the existence and multiplicity results of positive solutions for the boundary value problem:

$$\begin{cases} -\left(|u'|^{p-2} u'\right)' = \lambda f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{6}$$

where $p > 1$, λ is a positive real parameter and f is a $p-$concave-$p-$convex function. The problem (6) was studied for the case $p = 2$ in [203]. In this chapter we show, using a quadrature method the existence of a constant $\lambda_* > 0$ such that:

i) If $\lambda < \lambda_*$, problem (6) admits exactly two positive solutions;

ii) If $\lambda = \lambda_*$, problem (6) admits exactly one positive solution;

iii) If $\lambda > \lambda_*$, problem (6) has no positive solution.

The results of this chapter are a generalizations of those published in [14] and [77].

In chapter 4, we study the existence and multiplicity results of solutions for the problem:

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda g(t)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{7}$$

where $p > 1$, λ is a positive real parameter, $0 < \mu < p-1 < \nu$ and $g : [a,b] \to \mathbf{R}_+^*$ is of class C^1. Using the angular function technique we show that the results obtained in [75] holds for any $p > 1$. The results of this chapter are published in [76].

In chapter 5, we study the existence and multiplicity results of positive radially symmetric solutions for the problem:

$$\begin{cases} -\Delta_p u = \lambda\left(u^{\alpha-1} + u^{q-1}\right) \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{8}$$

where Ω denotes the unit ball in $\mathbf{R}^N$, Δ_p is the $p-$Laplace operator, λ is a positive real parameter, $\alpha = p^* = \dfrac{Np}{N-p}$, $N > p$ and $1 < q < p$.

In this chapter we show, using a shooting method (see [117] for the case $p = 2$) the existence of a constant $\lambda_* > 0$ such that:

i) If $\lambda < \lambda_*$, problem (6) admits at least positive radially symmetric solutions.

ii) If $\lambda = \lambda_*$, problem (6) admits at least one positive radially symmetric solution.

iii) If $\lambda > \lambda_*$, problem (6) has no positive radially symmetric solutions.

The novelty here is that we do not assume (as in the case in [27]) the condition

$$\frac{2N}{N+2} < p < 3 \text{ or } p \geq 3 \text{ and } p > q > \alpha - \frac{2}{p-1}$$

is satisfied.

The results of this chapter are done in [73].

Chapter 6 is concerened with the necessary and sufficient conditions for the existence and the multiplicity of boundary blow-up positive solutions for the quasilinear boundary value problems

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda f(u) \text{ in } (0,1) \\ \lim\limits_{x \to 0^+} u(x) = +\infty = \lim\limits_{x \to 1^-} u(x) \end{cases} \tag{9}$$

where $p > 1$, λ is a positive real parameter and $f : \mathbf{R} \to \mathbf{R}$ is a continuous function. The aim of this work is to give a generalization of the results obtained by Anuradha & al [29] and Shin-Hwa Wang [207] for the case $p > 1$. The results of this chapter are done in [79].

In chapter 7, we investigate the existence of solutions of the multipoint boundary value problem

$$\begin{cases} -\Delta_p u = f(x,u) \text{ in } \Omega \\ u(x) = \int\limits_{\Omega} \Phi(x,y)\, u(y)\, dy \text{ on } \partial\Omega \end{cases} \tag{10}$$

where Δ_p is the p–Laplace operator with $p > 1$, Ω is a bounded domain of class $C^{1,\alpha}$, $0 < \alpha < 1$, with smooth boundary $\partial\Omega$ $f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and $\Phi : \partial\Omega \times \bar{\Omega} \to \mathbf{R}_+$ is a smooth function. We rely on the upper and lower solutions to provide a constructive method for obtaining at least one solution. At our Knowledge, this is the first time that the method of upper and lower solutions is being used in the context of quasilinear elliptic problem with nonlocal boundary conditions. The results of this chapter are done in [74].

Chapter 1

Quadrature method and angular function technique

1.1 Introduction

This chapter is essentially divided into two parts in which the shooting method serves as the underlying technique.

The first part is concerned with the description of the quadrature method for the $p-$Laplace operator.

The second part is devoted to an important method know as the "angular function technique". This method is enable us to solve an important problems concerning the existence, multiplicity and the zeros of solutions for the nonautonomous boundary value problems.

1.2 Quadrature method

1.2.1 Notation

For any integer $k \geq 1$, let

$$S_k^+ = \left\{ \begin{array}{c} u \in C^1([a,b]) : \ u \text{ admits exactly } \ (k-1) \ \text{ zeros in } \ (a,b) \\ \text{are all simple, } u(a) = u(b) = 0 \text{ and } u'(a) > 0 \end{array} \right\}$$

Definition Let $u \in C([a,b])$ be a real valued function with two consecutive zeros $x_1 < x_2$. We call a hump of u the restriction of u to the open interval (x_1, x_2).

Let A_k^+ $(k \geq 1)$ be the subset of S_k^+ consisting of the functions u satisfying:

(i) Every hump of u is symmetrical about the center of the interval of its definition.

(ii) Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.

(iii) The derivative of each hump of u vanishes once and only once.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$.

1.2.2 Description of the method

Denote by g a nonlinearity and by p a real parameter, and we assume

$$g \in C(\mathbf{R}, \mathbf{R}) \text{ and } 1 < p < +\infty \tag{1.1}$$

and consider the boundary value problem,

$$\begin{cases} -(\varphi_p(u'))' = g(u) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{1.2}$$

where $\varphi_p(x) = |x|^{p-2}x$, $x \in \mathbf{R}$. Denote by $p' = \dfrac{p}{p-1}$ the conjugate exponent of p. Define $G(s) = \int_0^s g(t)\,dt$. For any $E > 0$ and $\kappa = +, -$, let,

$$X_\kappa(E) = \{s \in \mathbf{R} : \kappa s \geq 0 \text{ and } E - p'G(\xi) > 0,\ \forall \xi,\ 0 \leq \kappa\xi \leq \kappa s\} \text{ and,}$$

$$r_\kappa(E) = \begin{cases} 0 \text{ if } X_\kappa(E) = \emptyset \\ \kappa \sup(\kappa X_\kappa(E)) \text{ otherwise} \end{cases}$$

Note that r_κ may be infinite. We shall also make use of the following sets,

$$\tilde{D}_+ = \left\{ E > 0 : \ 0 < |r_\kappa(E)| < +\infty \text{ and } \kappa \int_0^{r_\kappa(E)} \left[E - p'G(\xi)\right]^{-\frac{1}{p}} d\xi < +\infty \right\}$$

$$\tilde{D} = \tilde{D}_+ \cap \tilde{D}_-$$

Also, let $\tilde{D}_k^\kappa := \tilde{D}$ if $k \geq 2$, and $\tilde{D}_1^\kappa = \tilde{D}_k$. Define the following time-maps

$$T_\kappa(E) = \kappa \int_0^{r_\kappa(E)} \left[E - p'G(\xi)\right]^{-\frac{1}{p}} d\xi, E \in \tilde{D}_\kappa$$

$$T_{2n}^\kappa(E) = n\left(T_+(E) + T_-(E)\right), \, n \in \mathbf{N}, \, E \in \tilde{D}$$

$$T_{2n+1}^\kappa(E) = T_{2n}^\kappa(E) + T_\kappa(E), \, n \in \mathbf{N}, \, E \in \tilde{D}.$$

Theorem 1 *Assume that (1.1) holds. Let $E \geq 0$, $k \geq 1$ and $\kappa = +, -$. Then:*

Problem (1.2) admits a solution $u_k^\kappa \in A_k^\kappa$ satisfying $\left[(u_k^\kappa)'(a)\right]^p = \kappa E$ if and only if $E \in \tilde{D}_k^\kappa \cap (0, +\infty)$ and $T_k^\kappa(E) = \dfrac{(b-a)}{2}$ and in this case the solution is unique.

Proof. See the proof in [2].

1.3 Angular function technique

Consider the boundary value problem

$$\left(\varphi_p(u')\right)' = f(t, u, u') \tag{1.3}$$

$$u(a)\left|\sin\alpha\right|^{\frac{2-p}{p-1}} \sin\alpha - u'(a)\left|\cos\alpha\right|^{\frac{2-p}{p-1}} \cos\alpha = 0 \tag{1.4}$$

$$u(b)\,|\sin\beta|^{\frac{2-p}{p-1}}\sin\beta - u'(b)\,|\cos\beta|^{\frac{2-p}{p-1}}\cos\beta = 0 \tag{1.5}$$

where $\varphi_p(x) = |x|^{p-2}x$, $x \in \mathbf{R}$ with $p > 1$, $f : [a,b] \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function and $\alpha, \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definition 2.1: A function $u \in C^1[a,b]$ such that $\varphi_p(u') \in C^1(a,b)$ is called a solution of problem (1.3), (1.4)and (1.5) if:

(i) u satisfies (1.3) for each $t \in (a,b)$.

(ii) u satisfies (1.4) and (1.5).

Definition 2.2: By a nondegenerate solution of problem (1.3), (1.4)and (1.5) we mean a function u such that $u^2(t) + (u')^2(t) \neq 0$ for all $t \in [a,b]$.

We distinguish two cases:

1.3.1 The case $1 < p < 2$

Let u be a solution of (1.3) such that

$$u^{2(p-1)}(t) + (u')^{2(p-1)}(t) \neq 0, \ \forall t \in [a,b]$$

We define the angular function associated to u by

$$\tan\varphi(t) = \frac{|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)}$$

and set,

$$\begin{cases} |u(t)|^{p-2}u(t) = r(t)\cos\varphi(t) \\ |u'(t)|^{p-2}u'(t) = r(t)\sin\varphi(t) \end{cases}$$

where $r(t) = \sqrt{u^{2(p-1)}(t) + (u')^{2(p-1)}(t)}$ for all $t \in [a,b]$.

If u is a solution of (1.3), then $(r(t), \varphi(t))$ is a solution of the following system

$$r'(t) = f(t, u, u') \sin\varphi(t) + (p-1) r(t) \sin\varphi(t) |\sin\varphi(t)|^{\frac{2-p}{p-1}} \cos\varphi(t) |\cos\varphi(t)|^{\frac{p-2}{p-1}} \quad (1.6)$$

$$\varphi'(t) = \frac{f\left(t, \widetilde{u}(r(t), \varphi(t)), \widetilde{v}(r(t), \varphi(t))\right)}{r(t)} \cos\varphi(t) - (p-1) |\sin\varphi(t)|^{\frac{p}{p-1}} |\cos\varphi(t)|^{\frac{p-2}{p-1}} \quad (1.7)$$

where

$$\widetilde{u}(r(t), \varphi(t)) = |r(t) \cos\varphi(t)|^{\frac{2-p}{p-1}} r(t) \cos\varphi(t)$$

and

$$\widetilde{v}(r(t), \varphi(t)) = |r(t) \sin\varphi(t)|^{\frac{2-p}{p-1}} r(t) \sin\varphi(t)$$

Remarks

(i) The set of angular functions corresponding to a given u is infinite, each of these functions can be uniquely specified by indicating its value in a.

(ii) If t_0 is a simple zero of u then $u(t_0) = 0$ and $u'(t_0) \neq 0$. Consequently $r(t_0) \neq 0$ and $\varphi(t_0) = \frac{\pi}{2} \pm k\pi$ with $k \in \mathbf{N}$. This shows that the simple zeros of a solution u of (1.3) are obtained by studying the equation

$$\varphi(t) = \frac{\pi}{2} \pm k\pi, \ k \in \mathbf{N}$$

where φ is a solution of the equation (1.7).

(iii) u is a solution of (1.3), (1.4) and (1.5) if and only if its angular function $\varphi(t)$ satisfies

$$\varphi(a) = \alpha, \varphi(b) = \beta + k\pi$$

for some integer k.

1.3.2 The case $2 \leq p < +\infty$

Let u be a nondegenerate solution of (1.3). In this case we define the angular function φ associated to u by letting

$$\tan\varphi(t) = \frac{u'(t)}{u(t)}$$

and set,

$$\begin{cases} u(t) = r(t)\cos\varphi(t) \\ u'(t) = r(t)\sin\varphi(t) \end{cases}$$

where $r(t) = \sqrt{u^2(t) + (u')^2(t)}$ for all $t \in [a,b]$.

If u is a solution of (1.3), then $(r(t), \varphi(t))$ is a solution of the following system

$$\varphi'(t) = \frac{f(t, r(t)\cos\varphi(t), r(t)\sin\varphi(t))\cos\varphi(t)}{(p-1)|r(t)\sin\varphi(t)|^{p-2}\, r(t)} - \sin^2\varphi(t)$$

$$r'(t) = \left[r(t)\cos\varphi(t) + \frac{f(t, r(t)\cos\varphi(t), r(t)\sin\varphi(t))}{(p-1)|r(t)\sin\varphi(t)|^{p-2}}\right]\sin\varphi(t)$$

Chapter 2

Oscillations results of a weakly coupled system

ABSTRACT: Weakly-coupled systems are studied using phase plane analysis. Infinitely many solutions are obtained for an equation of the Emden-Fowler type.

2.1 INTRODUCTION

In this chapter we study the existence and oscillations of solutions of the system,

$$\left(|u_k'(t)|^{p_k-2}u_k'(t)\right)' = f_k\left(t,\ u_1,\cdots,\ u_n,\ u_1',\ \cdots u_n'\right), \tag{2.1}$$

$$u_k(a)\sin\alpha_k\left|\sin\alpha_k\right|^{\frac{2-p_k}{p_k-1}} - u_k'(a)\cos\alpha_k\left|\cos\alpha_k\right|^{\frac{2-p_k}{p_k-1}} = 0, \tag{2.2}$$

$$u_k(b)\sin\beta_k\left|\sin\beta_k\right|^{\frac{2-p_k}{p_k-1}} - u_k'(b)\cos\beta_k\left|\cos\beta_k\right|^{\frac{2-p_k}{p_k-1}} = 0, \tag{2.3}$$

for $k = 1,\ 2,\ \cdots,\ n$ and $p_k > 1$.

The functions $f_k : [a,\ b] \times \mathbf{R}^{2n} \to \mathbf{R}$ are continuous, $\alpha_k \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right[$ and $\beta_k \in \left]-\frac{\pi}{2},\ \frac{\pi}{2}\right]$. The boundary conditions considered in (2.2) and (2.3) are a generalization of Dirichlet conditions $(\alpha_k = -\frac{\pi}{2}, \beta_k = \frac{\pi}{2})$and of Neumann conditions $(\alpha_k = \beta_k = 0)$.

If $u = (u_1,\ \cdots,\ u_n)$ is a solution of (2.1) with u_k non degenerate for each k, i.e.,

$$u_k^{2(p_k-1)}(t) + \left(u_k'\right)^{2(p_k-1)}(t) \neq 0,\ \forall t \in [a,\ b],$$

so, we may define the angular function associated to u_k by,

$$\operatorname{tg}(\varphi_k(t)) = \frac{|u_k'(t)|^{p_k-2}u_k'(t)}{|u_k(t)|^{p_k-2}u_k(t)}$$

and set,

$$\begin{cases} |u_k(t)|^{p_k-2}u_k(t) &= r_k(t)\cos\varphi_k(t) \\ |u_k'(t)|^{p-2}u_k'(t) &= r_k(t)\sin\varphi_k(t). \end{cases}$$

If $u = (u_1,\ \cdots,\ u_n)$ is a solution of (2.1) then $(r_k(t),\ \varphi_k(t))_{1\leq k\leq n}$ is a solution of the differential system,

$$\begin{aligned} \varphi_k'(t) = {} & \frac{f_k(t,\ u_1,\ \cdots,\ u_n,\ u_1',\ \cdots,\ u_n')}{r_k(t)}\cos\varphi_k(t) - \\ & (p_k-1)\left|\sin\varphi_k(t)\right|^{\frac{p_k}{p_k-1}}\left|\cos\varphi_k(t)\right|^{\frac{p_k-2}{p_k-1}} \end{aligned} \tag{2.4}$$

$$r_k'(t) = f_k(t, u_1, \cdots, u_n, u_1', \cdots, u_n') \sin\varphi_k(t) - r_k(t) \sin\varphi_k(t) \times \\ \times |\sin\varphi_k(t)|^{\frac{2-p_k}{p_k-1}} \cos\varphi_k(t) |\cos\varphi_k(t)|^{\frac{p_k-2}{p_k-1}} \tag{2.5}$$

for $k = 1, 2, \cdots, n$.

The set of angular functions corresponding to the same component u_k is infinite. However, to each component u_k one may associate uniquely an angular function φ_k by fixing its value at a to be α_k.

From $r_k^2(t) = u_k^{2(p_k-1)}(t) + (u_k')^{2(p_k-1)}(t)$, we deduce that if t_0 is a simple zero of u_k $(u_k'(t_0) \neq 0)$ then $r_k(t_0) \neq 0$. Also, one has $\varphi_k(t_0) = \frac{\pi}{2} \pm j_k\pi$, $j_k \in \mathbf{N}$. So, the study of a simple zero of u_k is reduced to study of φ_k satisfying $\varphi_k(t_0) = \frac{\pi}{2} \pm j_k\pi$, $j_k \in \mathbf{N}$.

Even for single equation, the results presented here are generalizations to the case $p \neq 2$ of a result of Shekhter [177, lemma 3]. Weakly-coupled systems have been studied by several authors see for instance A. Capietto ,J.Mawhin and F.Zanolin [56] for periodic boundary conditions and Henrard [113] for Sturm-Liouville boundary conditions where $p_k = 2$.

The paper is organized as follows. In section 2 we state and prove our main result. In section 3 we give two applications.The first application deals with Emden-Fowler equation and the second with weakly coupled systems.Note that Emden-Fowler's equation was studied by R. Emden in an astrophysical investigation and involved in the study of the electronic distribution in a heavy atom [135]. The estimates presented in the sequel are similar to the ones presented in [177]. In this work we show that the case $1 < p \leq 2$ is slightly different from the case $p > 2$.

2.2 MAIN RESULT

Definition 2 *By a solution of problem (2.1),(2.2) and (2.3),we mean a function* $u = (u_1, \cdots, u_n) \in C^1(]a, b[, \mathbf{R}^n)$ *such that for each* $k = 1, 2, \cdots, n$, $|u_k'(t)|^{p_k-2} u_k'(t) \in C^1(]a, b[, \mathbf{R})$ *and (2.1),(2.2) and (2.3) are satisfied.*

Remarks:

Let $\varphi_p(s) = |s|^{p-2} s$, with $p > 1$.Since $\lim_{s\to 0^+} \varphi_p(s) = \lim_{s\to 0-} \varphi_p(s) = 0$, we can extend $\varphi_p(s)$ to $s = 0$ by taking $\varphi_p(0) = 0$.

2.2.1 Case where $p_k \in \left]1,\ 2\right]$ for each $k = 1, \cdots, n$

Consider the problem

$$\begin{aligned}
&\left(\left|u_k'(t)\right|^{p_k-2} u_k'(t)\right)' = f_k\left(t,\ u_1, \cdots,\ u_n,\ u_1',\ \cdots u_n'\right) \\
&u_k(a) = z_k \cos\alpha_k \left|z_k \cos\alpha_k\right|^{\frac{2-p_k}{p_k-1}} \\
&u_k'(a) = z_k \sin\alpha_k \left|z_k \sin\alpha_k\right|^{\frac{2-p_k}{p_k-1}},\quad z_k \in \mathbf{R}_*^+, k = 1, ..., n.
\end{aligned} \tag{C1}$$

Theorem 3 *Let $k \in \{1, ..., n\}$.Assume that $\rho_k : I \rightarrow \mathbf{R}_+$ and $q_k : I \longrightarrow \mathbf{R}_+$ be continuous such that $\rho_k(t) > 0$ for each $t \in I$.Let $h_k : \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$ be a continuous function satisfying for any sufficiently small $\varepsilon_k > 0$:*

$$\lim_{r_k \to 0^+} \frac{1}{r_k} h_k\left(r_1 \left|\cos\varphi_1\right|, \cdots, r_n \left|\sin\varphi_n\right|\right) = +\infty$$

(or

$$\lim_{r_k \to +\infty} \frac{1}{r_k} h_k\left(r_1 \left|\cos\varphi_1\right|, \cdots, r_n \left|\sin\varphi_n\right|\right) = +\infty)$$

uniformly with respect to $\varphi_k \in \left[0, \frac{\pi}{2} - \varepsilon_k\right]$, φ_j and r_j for all $j \neq k$. Then, for any positive integer j_k and any positive real number ν_k, there exists a constant $\mu_{j_k}(\nu_k)$ such that if $u = (u_k)_{1 \leq k \leq n}$ is a solution of (C1) with u_k non degenerate for each $k = 1, 2, ...n$, defined for all $t \in [a, b]$ and satisfy the inequality

$$\begin{aligned}
&f_k\left(t,\ u_1,\ \cdots,\ u_n,\ u_1',\ \cdots,\ u_n'\right) \operatorname{sign} u_k(t) \leq \\
&\quad -\rho_k(t)\, h_k\left(\left|u_1\right|^{p_1-1},\ \cdots,\ \left|u_n\right|^{p_n-1},\ \left|u_1'\right|^{p_1-1},\ \cdots \left|u_n'\right|^{p_n-1}\right) \\
&\quad + (p_k - 1)\, q_k(t) \left(\left|u_k\right|^{p_k-1} + \left|u_k'\right|^{p_k-1}\right)
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&0 < \sqrt{u_k^{2(p_k-1)}(t) + \left(u_k'\right)^{2(p_k-1)}(t)} \leq \mu_{j_k}(\nu_k) \\
&\qquad \left(\textit{resp. } \sqrt{u_k^{2(p_k-1)}(t) + \left(u_k'\right)^{2(p_k-1)}(t)} \geq \mu_j(\nu_k)\,\right)
\end{aligned} \tag{2.7}$$

for all $t \in [a,\ b] \subset I$, with,

$$\int_a^b \rho_k(t)\, dt \geq \nu_k > 0 \tag{2.8}$$

Then u_k admits at least j_k zeros in $[a,\ b]$.

Proof. The proof will be given in two steps.

step.1 Using differential inequalities, we will control the angular function φ_k of equation (2.4) by another angular function ψ_k associated to a simpler problem.In fact,

for any given j_k, there exists a positive δ_k such that if $\tau_2 - \tau_1 \leq \delta_k$ $(\tau_1, \tau_2 \in I)$ then, since ρ_k is continuous with respect to all its arguments,

$$2\sqrt{2}\int_{\tau_1}^{\tau_2} q_k(s)\,ds \leq 1 \text{ and } 4j_k\int_{\tau_1}^{\tau_2} \rho_k(s)\,ds < \nu_k. \tag{2.9}$$

Let $Q_k = \int_a^b q_k(t)\,dt$, $\sigma_k = \frac{1}{4}\min\left\{\frac{\pi}{2}, \frac{\delta_k}{1+Q_k}\right\}$ and let us choose $\mu_{j_k}(\nu_k) > 0$ such that,

$$\frac{1}{r_k}h_k(r_1\cos\varphi_1, \cdots, r_n\cos\varphi_n, r_1\sin\varphi_1, \cdots, r_n\sin\varphi_n) \geq \frac{2j_k}{\nu_k}\left[\frac{\pi - 2(p_k-1)\sigma_k + Q_k(p_k-1)\sqrt{2}}{\sin(\sigma_k(p_k-1)/2)}\right]$$

for $0 < r \leq \mu_{j_k}(\nu_k)$ (resp. $r_k \geq \mu_{j_k}(\nu_k)$), $0 \leq \varphi_k \leq \frac{\pi}{2} - \frac{\sigma_k}{2}$. Let $[a, b] \subset I$ and $u_k \in C^1(]a, b[)$ satisfying (5.11) and (2.7). Let $r_k(t) = \sqrt{u_k^{2(p_k-1)}(t) + (u_k')^{2(p_k-1)}(t)}$ and consider a maximal solution ψ_k of the problem,

$$\begin{cases} \psi_k'(t) = [\sqrt{2}(p_k-1)q_k(t) - (\rho_k(t)/r_k(t)) \times \\ \qquad \times h_k(r_1(t)|\cos\varphi_1(t)|, \cdots, r_n(t)|\sin\varphi_n(t)|)] \times \\ \qquad \times |\cos\psi_k(t)| - (p_k-1)|\sin\psi_k(t)|^{\frac{p_k}{p_k-1}}|\cos\psi_k(t)|^{\frac{p_k-2}{p_k-1}} \\ \psi_k(a) = \frac{\pi}{2}. \end{cases} \tag{2.10}$$

If $\varphi_k : [a, b] \longrightarrow \mathbf{R}$ is an angular function associated to u_k and $\varphi_k(a) = \alpha_k$ then (2.4), (2.10) and the theorem of differential inequalities[[138] Theorem 14.2] imply that $\varphi_k(t) \leq \psi_k(t)$ in $[a, b]$.

The inequality (2.8) implies that there exist $t_i \in [a, b]$ $(i = 0, \cdots, j)$ such that,

$$\int_a^{t_i} \rho_k(s)\,ds = \left(\frac{i_k}{j_k}\right)\nu_k. \tag{2.11}$$

step.2 Let us show by induction that the inequality,

$$\psi_k(t_{i_k}) \leq \frac{\pi}{2} - \pi i_k, \tag{2.12}$$

is valid for any $i_k \in \{0, \cdots, j_k\}$. One has: If $2\psi_k(t_k^*) \leq \pi - 2\pi i_k$ for some $t_k^* \in [a, b]$ and some integer i_k then $2\psi_k(t) < \pi - 2\pi i_k$ in $]t_k^*, b]$. So, (5.3) is valid for $i_k = 0$. Suppose that it is valid for some $i_k = m_k \in \{0, \cdots, j_k - 1\}$ then $2\psi_k(t) < \pi - 2\pi m_k$ in the interval $]t_{m_k}, b]$.

claim for each $k = 1, 2, \cdots, n$ we have the following inequality

$$\psi_k(t_{m_k} + 4\sigma_k) \leq \frac{\pi}{2} - \pi m_k - (p_k - 1)\sigma_k \tag{2.13}$$

Proof. Suppose that (5.4) is not true. One has two possibilities:

(i) For $t_{m_k} \leq t_k \leq t_{m_k} + 4\sigma_k$, one has,

$$\psi_k(t_k) > \frac{\pi}{2} - \pi m_k - 2(p_k - 1)\sigma_k. \tag{2.14}$$

(ii) There exists $s_k \in [t_{m_k}, t_{m_k} + 4\sigma_k[$ such that,

$$\psi_k(s_k) = \frac{\pi}{2} - \pi m_k - 2(p_k - 1)\sigma_k$$

and,

$$\psi_k(t_k) > \frac{\pi}{2} - \pi m_k - 2(p_k - 1)\sigma_k, \quad \text{in }]s_k, t_{m_k} + 4\sigma_k[.$$

In the case **(i)**,

$$\begin{aligned}\psi_k(t_{m_k}+4\sigma_k)-\psi_k(t_{m_k}) &\le \sqrt{2}(p_k-1)\sin 2\sigma_k\int_{t_{m_k}}^{t_{m_k}+4\sigma_k} q_k(t)\,dt- \\ &\quad -4\sigma_k(p_k-1)(\cos 2\sigma_k)^{\frac{p_k}{p_k-1}}2^{\frac{2-p_k}{2(p_k-1)}} \\ &\le (p_k-1)2\sqrt{2}\sigma_k\int_{t_{m_k}}^{t_{m_k}+4\sigma_k} q_k(t)\,dt- \\ &\quad -4\sigma_k(p_k-1)(\cos 2\sigma_k)^{\frac{p_k}{p_k-1}}2^{\frac{2-p_k}{2(p_k-1)}} \\ &\le (p_k-1)\sigma_k-4\sigma_k(p_k-1)2^{\frac{-p_k}{2(p_k-1)}}2^{\frac{2-p_k}{2(p_k-1)}} \\ &= -(p_k-1)\sigma_k,\end{aligned}$$

that is to say,

$$\begin{aligned}\psi_k(t_{m_k}+4\sigma_k) &\le \psi_k(t_{m_k})-(p_k-1)\sigma_k \\ &\le \frac{\pi}{2}-\pi m_k-(p_k-1)\sigma_k.\end{aligned}$$

Suppose that **(ii)** is true, one has,

$$\begin{aligned}\psi_k(t_{m,k}+4\sigma_k)-\psi_k(s_k) &\le (p_k-1)\sqrt{2}\sin 2\sigma_k\int_{s_k}^{t_{m,k}+4\sigma_k} q_k(t)\,dt \\ &\le (p_k-1)\sigma_k,\end{aligned}$$

that is to say,

$$\begin{aligned}\psi_k(t_{m_k}+4\sigma_k) &\le \psi_k(s_k)+(p_k-1)\sigma_k \\ &= \frac{\pi}{2}-\pi m_k-(p_k-1)\sigma_k.\end{aligned}$$

So, in all cases one has,

$$\psi_k(t_{m_k}+4\sigma_k)\le\frac{\pi}{2}-\pi m_k-(p_k-1)\sigma_k.$$

Assume that,

$$\psi_k(t_{m_k+1}-4\sigma_k)>-\frac{\pi}{2}-\pi m_k+(p_k-1)\sigma_k.$$

So, one has,

$$-\frac{\pi}{2}-\pi m_k+\frac{1}{2}(p_k-1)\sigma_k\le\psi_k(t_k)\le\frac{\pi}{2}-\pi m_k-\frac{1}{2}(p_k-1)\sigma_k,$$

in $[t_{m_k} + 4\sigma_k,\ t_{m_k+1} - 4\sigma_k]$. In the contrary case, there exists,

$$[s_{1,k},\ s_{2,k}] \subset [t_{m_k} + 4\sigma_k,\ t_{m_k+1} - 4\sigma_k]$$

such that $\psi_k(s_{1,k}) \le \psi_k(t_k) \le \psi_k(s_{2,k})$ for $s_{1,k} \le t_k \le s_{2,k}$ and $\psi_k(s_{1,k}) = \frac{\pi}{2} - \pi m_k + \frac{1}{2}(p_k - 1)\sigma_k$, $\psi_k(s_{2,k}) = \frac{\pi}{2} - \pi m_k + (p_k - 1)\sigma_k$ which is impossible, because one has,

$$\begin{aligned}\psi_k(s_{2,k}) - \psi_k(s_{1,k}) = \tfrac{1}{2}(p_k - 1)\sigma_k \le \sqrt{2}\sigma_k(p_k - 1)\int_{s_{1,k}}^{s_{2,k}} q_k(t)\,dt \\ -\tfrac{3}{4}(p_k - 1)(s_{2,k} - s_{1,k}),\end{aligned}$$

which is a contradiction with the choice of σ_k.

Using inequality (5.4) we will show that (5.3) is true for $i_k = m_k + 1$.

From (2.9) and (5.2) one has,

$$\begin{aligned}\int_{t_{m_k}+4\sigma_k}^{t_{m_k+1}-4\sigma_k} \rho_k(s)\,ds &= \int_{t_{m_k}}^{t_{m_k+1}} \rho_k(s)\,ds - \\ &- \int_{t_{m_k}}^{t_{m_k}+4\sigma_k} \rho_k(s)\,ds - \int_{t_{m_k+1}-4\sigma_k}^{t_{m_k+1}} \rho_k(s)\,ds \\ &\ge \frac{\nu_k}{j_k} - \frac{\nu_k}{4j_k} - \frac{\nu_k}{4j_k} = \frac{\nu_k}{2j_k},\end{aligned} \tag{2.15}$$

and then : $t_{m_k+1} - t_{m_k} - 8\sigma_k > \delta_k$. From the relations (2.7), (2.8), (2.10), and (2.15) one gets, $\psi_k(t_{m_k+1} - 4\sigma_k) - \psi_k(t_{m_k} + 4\sigma_k) \le$

$$\begin{aligned}&\le \sqrt{2}(p_k - 1)\int_{t_{m_k}+4\sigma_k}^{t_{m_k+1}-4\sigma_k} q_k(t)\,|\cos\psi_k(t)|\,dt - \\ &\quad - \int_{t_{m_k+1}+4\sigma_k}^{t_{m_k+1}-4\sigma_k} \frac{\rho_k(t)}{r_k(t)}|\cos\psi_k(t)| \times \\ &\qquad h_k(r_1(t)|\cos\psi_1(t)|,\ \cdots,\ r_n(t)|\cos\psi_n(t)|,\ \cdots,\ r_n(t)|\sin\psi_n(t)|)\,dt \\ &\le \sqrt{2}(p_k - 1)Q_k - \left[\frac{\pi - 2(p_k - 1)\sigma_k + Q_k(p_k - 1)\sqrt{2}}{\sin(\sigma_k(p_k - 1)/2)}\right]\sin\tfrac{1}{2}\sigma_k(p_k - 1) \\ &= -\pi + 2(p_k - 1)\sigma_k\end{aligned}$$

that is to say,

$$\begin{aligned}\psi_k(t_{m_k+1}-4\sigma_k) &\leq \psi_k(t_{m_k}+4\sigma_k)-\pi+2(p_k-1)\sigma_k \\ &\leq \frac{\pi}{2}-\pi m_k-(p_k-1)\sigma_k-\pi+2(p_k-1)\sigma_k \\ &= \frac{\pi}{2}-\pi(m_k+1)+(p_k-1)\sigma_k,\end{aligned}$$

and then,

$$\psi_k(t_{m_k+1}) \leq \frac{\pi}{2}-\pi(m_k+1)$$

.

Taking $i_k \leq m_k+1$. we have proved (5.3).So with $i_{k=j_k}$, one gets

$$\varphi_k(t_{j_k}) \leq \frac{\pi}{2}-\pi j_k,$$

which shows that u_k admits at least j_k zeros in $[a,\ b]$.

2.2.2 Case where $p_k \in]2,\ +\infty[$ for each $k=1,\cdots,n$

Consider, in this case, the problem,

$$\left(|u_k'(t)|^{p_k-2}u_k'(t)\right)' = f_k(t,\ u_1,\cdots,u_n,\ u_1',\ \cdots,\ u_n') \tag{2.16}$$

$$\begin{aligned}&u_k(a)\sin\alpha_k-u_k'(a)\cos\alpha_k=0 \\ &u_k(b)\sin\beta_k-u_k'(b)\cos\beta_k=0,\ k=1,\ \cdots,\ n,\end{aligned} \tag{2.17}$$

where $f_k \in C([a,\ b]\times\mathbf{R}^{2n},\ \mathbf{R})$, $\alpha_k \in \left]-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ and $\beta_k \in \left]-\frac{\pi}{2},\ \frac{\pi}{2}\right]$.

Let $u_k \in C^1([a,\ b],\ \mathbf{R})$ be a non degenerate component of the solution u of the equation (2.16). One may define the angular function associated to u_k by,

$$\operatorname{tg}\varphi_k(t) = \frac{u_k'(t)}{u_k(t)}.$$

Set

$$\begin{cases} u_k(t) &= r_k(t)\cos\varphi_k(t) \\ u'_k(t) &= r_k(t)\sin\varphi_k(t). \end{cases}$$

It is not difficult to see that $(r_k(t), \varphi_k(t))_{1\le k\le n}$ is a solution of the differential system,

$$\varphi'_k(t) = \frac{f_k(t, u_1, \cdots, u_n, u'_1, \cdots, u'_n)\cos\varphi_k(t)}{(p_k-1)\,|r_k(t)\sin\varphi_k(t)|^{p_k-2}\,r_k(t)} - \sin^2\varphi_k(t) \tag{2.18}$$

$$r'_k(t) = \left[(p_k-1)\,r_k(t)\cos\varphi_k(t) + f_k\left(t, u_1, \cdots, u_n, u'_1, \cdots, u'_n\right)\right]\sin\varphi_k(t). \tag{2.19}$$

In this case we consider the auxiliary problem

$$\begin{aligned} &\left(|u'_k(t)|^{p_k-2}\,u'_k(t)\right)' = f_k(t, u_1, \cdots, u_n, u'_1, \cdots, u'_n) \\ &u_k(a) = z_k\cos\alpha_k \\ &u'_k(a) = z_k\sin\alpha_k, \quad z_k\in\mathbf{R}^+_*, k=1,...,n. \end{aligned} \tag{C2}$$

Theorem 4 *Let $k\in\{1,...,n\}$. Assume that $\rho_k : I \longrightarrow \mathbf{R}_+$ and $q_k : I \to \mathbf{R}_+$ be continuous functions and assume that $\rho_k(t) > 0$ for each $t\in I$ and $u\in\mathbf{R}^n$.*

Also assume that $h_k : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$ is continuous and for any $\varepsilon_k > 0$,

$$\lim_{r_k\to+0^+} \frac{h_k\left(|r_1\cos\varphi_1|^{p_1-1}, \cdots, |r_n\sin\varphi_n|^{p_n-1}\right)}{r_k^{p_k-1}} = +\infty$$

(or

$$\lim_{r_k\to+\infty} \frac{h_k\left(|r_1\cos\varphi_1|^{p_1-1}, \cdots, |r_n\sin\varphi_n|^{p_n-1}\right)}{r_k^{p_k-1}} = +\infty)$$

uniformly with respect to $\varphi_k\in\left[0, \frac{\pi}{2}-\varepsilon_k\right]$, φ_j and r_j for all $j\neq k$ Then, for any positive integer j_k and any positive real number ν_k, there exists a constant $\mu_{l_k}(\nu_k)$ such that if $u = (u_k)_{1\le k\le n}$ is a solution of (C2) with u_k non degenerate for each $k=1,2,...n$, defined for all $t\in[a,b]$ and satisfy the inequality

$$\begin{aligned} &f_k(t, u_1, \cdots, u_n, u'_1, \cdots, u'_n)\operatorname{sign} u_k(t) \le \\ &-\rho_k(t)\,h_k\left(|u_1|^{p_1-1}, \cdots |u_n|^{p_n-1}, |u'_1|^{p_1-1}, \ldots, |u'_n(t)|^{p_n-1}\right) + \\ &+q_k(t)\left(|u_k(t)|^{p_k-1} + |u'_k(t)|^{p_k-1}\right) \end{aligned}$$

$$0 < \sqrt{u_k^2(t) + \left(u_k'\right)^2(t)} \leq \mu_{j_k}(\nu_k) \quad (resp.\ \sqrt{u_k^2(t) + \left(u_k'\right)^2(t)} \geq \mu_{j_k}(\nu_k)\,)$$

in $[a,\ b]$, *with,*

$$[a,\ b] \subset I \quad and \quad \int_a^b \rho_k(s)\, ds \geq \nu_k,$$

Then u_k *has at least* j_k *zeros in* $[a,\ b]$.

Proof. Similar to that of theorem 3. ∎

2.3 APPLICATIONS

2.3.1 Emden Fowler equation:

We consider the problem,

$$\left(\left|u'(t)\right|^{p-2} u'(t)\right)' = -q(t)\left|u(t)\right|^{\lambda(p-1)} \operatorname{sign} u \tag{2.20}$$

$$\begin{aligned} u(a)\sin\alpha\left|\sin\alpha\right|^{\frac{2-p}{p-1}} - u'(a)\cos\alpha\left|\cos\alpha\right|^{\frac{2-p}{p-1}} &= 0 \\ u(b)\sin\beta\left|\sin\beta\right|^{\frac{2-p}{p-1}} - u'(b)\cos\beta\left|\cos\beta\right|^{\frac{2-p}{p-1}} &= 0, \end{aligned} \tag{2.21}$$

$q \in C^1\left([a,\ b],\ \mathbf{R}_*^+\right)$, $p \in \left]1,\ 2\right]$, $\lambda(p-1) > 1$,$\alpha \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right[$and $\beta \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right[$.

Theorem 5 . *The problem (2.20)-(31) admits infinitely many solutions.*

For each $z \in \mathbf{R}_*^+$, let $u(t,z)$ be a solution of the following problem:

$$\begin{cases} \left(\left|u'(t)\right|^{p-2} u'(t)\right)' = -q(t)\left|u(t)\right|^{\lambda(p-1)} \operatorname{sign} u \\ u(a) = z\cos\alpha\left|z\cos\alpha\right|^{\frac{2-p}{p-1}} \\ u'(a) = z\sin\alpha\left|z\sin\alpha\right|^{\frac{2-p}{p-1}}, \quad z \in \mathbf{R}_*^+, \end{cases} \tag{C3}$$

Lemma 6 (i) *The solution* $u(t,z)$ *of problem* $(C3)$ *is defined for all* $t \in [a,b]$.

(ii) *Let* $r(t,z) = \left(u^{2(p-1)}(t,z) + (u')^{2(p-1)}(t,z)\right)^{\frac{1}{2}}, \forall t \in [a,\ b]$.

Then $\lim\limits_{z \to +\infty} r(t,z) = +\infty$.

Proof. **(i)** Suppose there exist a sequence $(t_n)_{n\in\mathbb{N}}$ which tends to $t_* \in [a,b]$ such that

$$\lim_{n\to+\infty} \left(|u(t_n,z)| + |u'(t_n,z)|\right) = +\infty$$

The mean value theorem shows that

$$\lim_{n\to+\infty} \left(|u'(t_n,z)| = |u'(t_*,z)| = +\infty\right.$$

Let

$$F(t,u) = \int_0^u q(t)f(s)ds \text{ with } f(s) = |s|^{\lambda(p-1)} sgn(s)$$

and

$$E(t,z) = \frac{p-1}{p}|u'(t,z)|^p + F(t,u(t,z))$$

Then

$$\begin{aligned} E'(t,z) &= (p-1)|u'(t,z)|^{p-2} u'(t,z)u''(t,z) + \frac{\partial F}{\partial t}(t,u(t,z)) + \frac{\partial F}{\partial u}(t,u(t,z))\,u'(t,z) \\ &= \int_0^{u(t,z)} q'(t)f(u(s))ds \le M \int_0^{u(t,z)} f(u(s))ds \text{ where } M = \sup_{t\in[a,b]} |q'(t)|. \end{aligned}$$

Let

$$\tilde{F}(u) = \int_0^u f(s)ds$$

we obtain $E'(t,z) \le M\tilde{F}(u(t,z))$.On the other hand, $F(t,u(t,z)) \ge m\tilde{F}(u(t,z))$ where $m = \inf_{t\in[a,b]} q(t)$.As $m > 0$, it follows that $E'(t,z) \le \frac{M}{m}E(t,z)$.This implies that

$$E(t_*,z) \le \exp\left(\frac{M}{m}(b-a)\right) E(t,a) \quad \forall t \in [a,b]$$

which is impossible.

(ii) It is a consequence of lemma 2.6.3. in [42].

Proof of theorem 3: Let $\varphi(t,\ z)$ be the angular function corresponding to $u(t,\ z)$. By theorem 3, for any $n \in \mathbf{N}^*$, there exists a positive real z_1 such that,

$$\varphi(b,z) \le \beta - \pi n, \ \text{ for } \ z \ge z_1,$$

so that , $\lim_{z\to+\infty}\varphi(b,\ z)=-\infty$. Since φ is continuous with respect to z (See [42] lemma 2.6.2 pp. 118) Hence there exists infinitely many positive real numbers z such that,

$$\varphi(b,z)=\beta-k\pi,\ k\in\mathbf{N}^{*},$$

and then problem (2.20)-(31) admits infinitely many solutions.

Remark 1 *When $q\equiv 1$, similar results were obtained in [93] ,where the authors consider the case of an autonomous superlineair term using the time map of the superlinear term which is not defined when the weight $q(t)$ is not a constant function.*

2.3.2 Weakly coupled systems:

Now, let us consider the problem

$$\left(\left|u_k'(t)\right|^{p_k-2}u_k'(t)\right)'+g_k(u_k)=p_k(t,u,u') \tag{2.22}$$

$$\begin{aligned}&u_k(a)\sin\alpha_k-u_k'(a)\cos\alpha_k=0\\&u_k(b)\sin\beta_k-u_k'(b)\cos\beta_k=0,k=1,...,n.\end{aligned} \tag{2.23}$$

where $u=(u_1,u_2,...,u_n)$, $\alpha_k,\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right[$, $\beta_k\in\left]-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ and $p_k\geq 2$ for all $k=1,2,...,n$.The fuctions g_k and p_k satisfies respectively the following hypothesis:

(g) The function $g_k:\mathbf{R}\to\mathbf{R}$ is continuous and odd with

$$\lim_{|u_k|\to+\infty}\frac{g_k(u_k)}{|u_k|^{p_k-2}u_k}=+\infty$$

(p) The function $p_k:\mathbf{R}\times\mathbf{R}^{2n}\to\mathbf{R}$ is continuous with

$$p_k(t,u,u')=0 \text{ if } u_k'\leq 0$$

Assume that there exists a continuous function $q_k:[a,b]\to\mathbf{R}^+$ such that

$$p_k(t,u,u')\leq q_k(t)\left|u_k'\right|^{p_k-1}$$

For $k = 1, ..., n$.

Theorem 7 *The problem (2.22),(2.23) admits infinitely many solutions.*

For each $z = (z_1, z_2, ..., z_n)$,let $u(t, z)$ be a solution of the following problem:

$$\begin{cases} \left(\left|u_k'(t)\right|^{p-2} u_k'(t)\right)' + g_k(u_k) = p_k(t, u, u') \\ u_k(a) = z_k \cos\alpha_k \\ u_k'(a) = z_k \sin\alpha_k, \ z_k \in \mathbf{R}_*^+ \end{cases} \qquad (C4)$$

We need a preliminary result.

Lemma 8

i) *For each $k = 1, ..., n$ $u_k(t, z)$ is defined for all $t \in [a, b]$.*

ii) *For any $\gamma_k > 0$, there exist $\delta_k > 0$ such that if $\sqrt{(u_k(a,z))^2 + (u_k'(a,z))^2} \le \gamma_k$ then $\sqrt{(u_k(t,z))^2 + (u_k'(t,z))^2} \le \delta_k$ for all $t \in [a, b]$ and $k = 1, 2, ..., n$.*

Proof.

i) Assume by contradiction that there exists a sequence $(t_l)_{l\in\mathbb{N}}$ converging to $t_* \in [a, b]$ such that

$$\lim_{l\to+\infty} \sum_{k=1}^{n} \left(\left|u_k(t_l, z)\right| + \left|u_k'(t_l, z)\right|\right) = +\infty$$

then by the mean value theorem, we have

$$\lim_{l\to+\infty} \sum_{k=1}^{n} \left|u_k'(t_l, z)\right| = \sum_{k=1}^{n} \left|u_k'(t_*, z)\right| = +\infty$$

Consider

$$E(t, z) := \sum_{k=1}^{n} \left(\frac{p_k - 1}{p_k} \left|u_k'(t, z_k)\right|^{p_k} + G(u_k(t, z))\right)$$

where $G_k(u) = \int_0^u g_k(s)ds$

we have

$$\begin{aligned} E'(t,z) &= \sum_{k=1}^{n} p_k\left(t,u,u'\right)) u'_k(t,z) \\ &\leq \sum_{k=1}^{n} q_k(t)\left|u'_k(t,z)\right|^{p_k} \\ &\leq 2M\sum_{k=1}^{n}\frac{p_k-1}{p_k}\left|u'_k(t,z_k)\right|^{p_k} \end{aligned}$$

where $M = \max\limits_{1\leq i\leq n}\left(\sup\limits_{t\in[a,b]} q_i(t)\right)$
It follows that

$$E'(t,z) \leq 2ME(t,z)$$

Hence

$$E(t_*,z) \leq E(a,z)\, e^{2M(t_*-a)}$$

ii For each $k = 1,2,...,n$, let $E_k(t,z) = \dfrac{p_k-1}{p_k}\left|u'_k(t,z)\right|^{p_k} + G_k(u_k(t,z))$

The hypothesis **(g)** implies that there exist $M_k > 0$ such that

$$G_k(u_k) \geq -M_k$$

Hence $E_k(t,z) \geq \dfrac{p_k-1}{p_k}\left|u'_k(t,z)\right|^{p_k} - M_k$
it follows that

$$\frac{p_k-1}{p_k}\left|u'_k(t,z)\right|^{p_k} - M_k \leq E_k(a,z)\, e^{2M(b-a)}$$

This last inequality shows that

$$u_k(t,z) - u_k(a,z) \leq \left[\tfrac{p_k}{p_k-1}\left(\left(M_k + L_k(\gamma_k)\, e^{2M(b-a)}\right)\right)\right]^{\frac{1}{p_k}}(b-a)$$

where

$$L_k(\gamma_k) = \frac{p_k-1}{p_k}\left|\gamma_k\right|^{p_k} + \max_{|u_k|\leq\gamma_k} G_k(u_k)$$

In other words, we have

$$
\begin{aligned}
r_k(t,z) &= \sqrt{(u_k(t,z))^2 + (u_k'(t,z))^2} \\
&\le \sqrt{\begin{array}{c}\left(\gamma_k + \left[\frac{p_k}{p_k-1}\left(\left(M_k + L_k(\gamma_k)\, e^{2M(b-a)}\right)\right)\right]^{\frac{1}{p_k}}(b-a)\right)^2 \\ + \left(\left[\frac{p_k}{p_k-1}\left(\left(M_k + L_k(\gamma_k)\, e^{2M(b-a)}\right)\right)\right]^{\frac{1}{p_k}}\right)^2\end{array}} = \delta_k
\end{aligned}
$$

for all $t \in [a,b]$.

Lemma 9 *Let $k \in \{1,2,...,n\}$ be fixed.Given any $r > 0$, there exists a number $R_k(r)$ such that if $z_k \ge R_k(r)$ and $z_1, z_{2,...,} z_{k-1}, z_{k+1}, ..., z_n$ are arbitary then $r_k(t,z) \ge r$ for all $t \in [a,b]$.*

Proof. Let $k \in \{1,2,...,n\}$ be fixed and let $r > 0$ be given. Consider a solution $u_k(t,z)$ passing through the set $S_k = \{(t,z_k) : t \in [a,b], z_k \le r\}$. By (**ii**) there exist $R_k(r)$ such that $r_k(t,z) \le R_k(r)$ where $u_k(t,z)$ passing through S_k. Hence if $z_k \ge R_k(r)$ then $r_k(t,z) > r$ for all $t \in [a,b]$.

Proof of theorem 4 : By the preceding lemma we have

$$
\lim_{z_k \to +\infty} r_k(t,z) = \left(|u_k(t,z)|^2 + |u_k'(t,z)|^2\right)^{\frac{1}{2}} = +\infty
$$

The rest of the proof is similar to that of theorem 3.

Remark 2 ***a)-*** *In this last theorem, we suppose only one side condition on the perturbation $p_k(t,u,u')$ .Similar results can be found in[113] and [93] .*

b)-This last theorem can be generalized for the case $p_k \in \left]1,2\right]$ for each $k = 1,2,...,n$.

Chapter 3

Exact number of positive solutions for a class of quasilinear boundary value problems

Abstract Using time-mapping approach, we study the exact multiplicity of positive solutions of the boundary value problem:

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

where $p > 1$, $\lambda > 0$ and f is a $p-$concave- $p-$convex function.

3.1 Introduction

The purpose of this chapter is to study the exact multiplicity of positive for the quasilinear problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.2}$$

where $p > 1$, $\lambda > 0$ are real parameters and f is a $p-$concave-$p-$convex function.

Our study is motivated by some recent works on elliptic problems with concave-convex nonlinearities. This question was studied by several authors, in particular T.Bartsch et al [41], Ambrosetti et al [25] and I.Peral & al [26]. In [25], the authors investigate the following problem:

$$\begin{cases} -\Delta u = u^{\alpha} + \lambda u^{\beta} \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.3}$$

with $0 < \beta < 1 < \alpha \leq 2^*$, $2^* = \dfrac{N+2}{N-2}$ for $N > 2$ and $2^* = +\infty$ for $N = 2$. The authors prove the existence of a constant $\gamma \in \mathbb{R}$ such that for all $\lambda \in (0, \gamma)$, the problem(3.3) admits at least two solutions. One solution, denoted u_λ, is obtained using lower and upper solution method, when the concave term u^β is essential, and the other solution, denoted v_λ, is obtained using a variatonel technics, when the essential term is the convex term u^α. In [34], the authors showed the existence of additional pair of solutions which can change sign for all $0 < \lambda < \lambda_*$, with λ_* possibly smaller then γ. Their method relies as a critical point of a functional I_λ constrained on a suitable manifold M_λ.

In [27], the problem:

$$\begin{cases} -\Delta_p u = \lambda h(u) + g(r,u) \text{ in } |x| = r < 1 \\ u = 0 \text{ in } |x| = r = 1 \end{cases} \tag{3.4}$$

is studied where Δ_p denotes the $p-$Laplace operator in $\mathbb{R}^N$, $\lambda \geq 0$

$h(u) = |u|^{q-2} u$, $1 < q < p$ near $u = 0$ and g is of higher order with respect to h at $u = 0$. The authors showed the existence of infinitely many continua of radial solutions branching of

$\lambda = 0$ from the trivial solution, each continuum being characterized by nodal properties. As a consequence (3.4) posses infinitely many radial solutions for $\lambda > 0$ and small. This bound is the counterpart, for radial solutions that change sign of the classical result by Gidas and Spurck [95]. The problem (3.2) was studied for the case $p = 2$ in [203] . The aim of this work is to show that the same result holds for any $p > 1$.

3.2 Main result and the method used

We consider the boundary value problem

$$\begin{cases} -\left(|u'|^{p-2} u'\right)' = \lambda f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.5}$$

where $p > 1$, $\lambda > 0$ are real parameters. Assume that $f \in C^2(0,+\infty) \cap C[0,+\infty)$ and $F(u)$ satisfies:

(H1) $f(u) > 0$ for $u > 0$ and $f(0) \geq 0$.

(H2) $\lim_{u \to 0^+} \dfrac{f(u)}{|u|^{p-2} u} = \lim_{u \to +\infty} \dfrac{f(u)}{|u|^{p-2} u} = +\infty$

(H3) $\lim_{u \to +\infty} (pF(u) - uf(u)) = -\infty$ and $\lim_{u \to +\infty} ((p-1) f(u) - uf'(u)) = -\infty$

(H4) $(p-2) f'(u) - uf''(u) > 0$ for $0 < u < c$ and

$(p-2) f'(u) - uf''(u) < 0$ for $u > c$ for some number $c > 0$.

To state our result, define

$S_1^+ = \{u \in C^1([0,1]) ;\ u > 0 \text{ in } (0,1),\ u(0) = u(1) = 0 \text{ and } u'(0) > 0\}$ and let A_1^+ the subset of S_1^+ composed by those the functions u satisfying:

i) u is symmetrical about $\frac{1}{2}$.

ii) The derivative of u vanishes once and only once.

The main result of this work is:

Theorem 10 *There exists a number $\lambda_* > 0$ such that*

i) *If $\lambda < \lambda_*$, problem (3.5) admits exactly two solutions and they belong to A_1^+.*

ii) *If $\lambda = \lambda_*$, problem (3.5) admits a unique solution and it belongs to A_1^+.*

iii) *If $\lambda = \lambda_*$, problem (3.5) admits no positive solution.*

In order to state the method, consider the boundary value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = g(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.6}$$

where $g \in C^1(\mathbb{R}^+, \mathbb{R})$ and $p > 1$.

Define $G(s) = \int_0^s g(t)\,dt$

For any $E \in (0, \infty)$, let

$$X_+(E) = \left\{ s > 0;\ E - \frac{p}{p-1} G(\zeta) > 0,\ \forall \zeta,\ 0 < \zeta < s \right\}$$

and

$$S_+(E) = \begin{cases} 0 \text{ if } X_+(E) = \emptyset \\ \sup X_+(E) \text{ otherwise} \end{cases}$$

Let

$$D = \{E > 0; 0 < S_+(E) < +\infty \text{ and } g(S_+(E)) > 0\}$$

and define the following time-map

$$T_+(E) = \int_0^{S_+(E)} \left[E - \frac{p}{p-1} G(u)\right]^{-\frac{1}{p}} du, E \in D.$$

Theorem 11 *Let $E \in (0, +\infty)$ and let $p > 1$, problem (3.6) admits a positive solution $u \in A_1^+$ satisfying $(u'(0))^p = E$ if and only if $E \in D$ and $T_+(E) = \frac{1}{2}$, and in this case the solution is unique and it sup-norm is equal to $S_+(E)$.*

3.3 Proof of main result

As usual, in order to define the time map, we need the following technical lemma.

Lemma 12 *Consider the equation in* $s \in \mathbb{R}$ *:*

$$E - \frac{p}{p-1} F(s) = 0 \tag{3.7}$$

where $p > 1$, $\lambda > 0$ *and* $E > 0$ *are real parameters. Then for any* $E > 0$, *equation (3.7) admits a unique positive zero* $r_+ = r_+(p, \lambda, E)$. *Moreover*

i) *The function* $E \mapsto r_+(p, \lambda, E)$ *is* C^1 *in* $(0, +\infty)$ *and*

$$\frac{\partial r_+(p, \lambda, E)}{\partial E} = \frac{(p-1)E}{\lambda p f(r_+(p, \lambda, E))} > 0, \ \forall p > 1, \ \forall \lambda > 0 \text{ and } \forall E > 0$$

ii) $\lim_{E \to 0^+} r_+(p, \lambda, E) = 0^+$.

iii) $\lim_{E \to +\infty} r_+(p, \lambda, E) = +\infty$.

Proof. For any fixed $p > 1$, $\lambda > 0$ and $E > 0$, consider the function

$$s \mapsto H(p, \lambda, E, s) := E - \frac{p}{p-1} \lambda F(s)$$

defined in $\mathbb{R}$ which is strictly decreasing with $H(p, \lambda, E, 0) = E$ and $\lim_{s \to +\infty} H(p, \lambda, E, s) = -\infty$. So, it is clear that equation (3.7) admits for any $E > 0$, a unique positive zero, $r_+ = r_+(p, \lambda, E)$.

Now, for any $p > 1$ and $\lambda > 0$, consider the real valued function

$$(E, s) \mapsto H_+(E, s) := E - \frac{p}{p-1} \lambda F(s)$$

defined in $\Omega_+ = (0, +\infty)^2$. One has $H_+ \in C^1(\Omega_+)$ and

$$\frac{\partial H_+(E, s)}{\partial s} = -\frac{p}{p-1} \lambda f(s) \text{ in } \Omega_+$$

hence

$$\frac{\partial H_{+}(E,s)}{\partial s} < 0 \text{ in } \Omega_{+}$$

and one may observe that $r_{+}(p,\lambda,E)$ belongs to the open interval $(0,+\infty)$ and satisfies from its definition

$$H_{+}(E, r_{+}(p,\lambda,E)) = 0 \tag{3.8}$$

So, one can makes use of the implicit function to show that the function $E \mapsto r_{+}(p,\lambda,E)$ is $C^{1}((0,+\infty),\mathbb{R})$ and to obtain the expression of $\dfrac{\partial r_{+}(p,\lambda,E)}{\partial E}$ given in **i)**. Hence, for any fixed $p > 1$ and $\lambda > 0$, the function defined in $(0,+\infty)$ by $E \mapsto r_{+}(p,\lambda,E)$ is strictly increasing and bounded from below by 0 and from above by $+\infty$. Then, the limit $\lim_{E\to 0^{+}} r_{+}(p,\lambda,E) = l_{0^{+}}$ and the limit $\lim_{E\to+\infty} r_{+}(p,\lambda,E) = l_{+\infty}$ exist and belong to $(0,+\infty]$. Moreover

$$0 \leq l_{0^{+}} < l_{+\infty} \leq +\infty$$

We observe that, for any fixed $p > 1$ and $\lambda > 0$, the function

$$(E,s) \mapsto H_{+}(E,s)$$

is continuous in $[0,+\infty)^{2}$ and the function $E \mapsto r_{+}(p,\lambda,E)$ is continuous in $(0,+\infty)$ and satisfies (3.8) as E tends to 0^{+}, we get

$$0 = \lim_{E\to 0^{+}} H_{+}(E, r_{+}(p,\lambda,E))$$

Hence, $l_{0^{+}}$ is a zero belonging to $[0,+\infty)$, to the equation in s

$$H_{+}(0,s) = 0$$

By resolving this equation in $[0,+\infty)$, we obtain

$$l_{0^+} = 0$$

Assume that $l_{+\infty} < +\infty$, then by passing to the limit in (3.8) as E tends to $+\infty$, we get

$$+\infty = \lambda \frac{p}{p-1} F\left(l_{+\infty}\right) < +\infty$$

which is impossible. So $l_{+\infty} = +\infty$. ■

Now we are ready, for any $p > 1$, $\lambda > 0$ and $E > 0$ to compute $X_+ (p, \lambda, E)$ as defined in section 2. In fact $X_+ (p, \lambda, E) =]0, r_+ (p, \lambda, E)[$. Then

$$s_+ (p, \lambda, E) := \sup X_+ (p, \lambda, E) = r_+ (p, \lambda, E)$$

and since $f (s) > 0$, $\forall s \in \mathbb{R}^*$ then

$$\begin{aligned} D &:= \{E > 0; 0 < s_+ (p, \lambda, E) < +\infty \text{ and } f (s_+ (p, \lambda, E)) > 0\} \\ &= (0, +\infty) \end{aligned}$$

By lemma 12, we have

$$\lim_{E \to 0^+} s_+ (p, \lambda, E) = 0 and \lim_{E \to +\infty} s_+ (p, \lambda, E) = +\infty$$

$$\frac{\partial s_+ (p, \lambda, E)}{\partial E} = \frac{(p-1) E}{p \lambda f (s_+ (p, \lambda, E))} > 0, \forall E \in D.$$

At present we define, for any $p > 1$, $\lambda > 0$ and $E \in D$ the time map

$$T_+ (p, \lambda, E) := \int_0^{s_+(p,\lambda,E)} \left[E - \frac{\lambda p}{p-1} F(u)\right]^{-\frac{1}{p}} du$$

and a simple change of variables shows that

$$\begin{aligned} T_{+}(p,\lambda,E) &= s_{+}(p,\lambda,E)\int_0^1 \left[E-\frac{\lambda p}{p-1}F(s_{+}(p,\lambda,E)\,u)\right]^{-\frac{1}{p}} du \\ &= s_{+}(p,\lambda,E)\left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}}\int_0^1 \left[F(s_{+}(p,\lambda,E)) - F(s_{+}(p,\lambda,E)\,u)\right]^{-\frac{1}{p}} du \end{aligned}$$

Lemma 13 *We have*

i) $\lim\limits_{E\to 0^+} T_{+}(p,\lambda,E) = 0^+$.

ii) $\lim\limits_{E\to +\infty} T_{+}(p,\lambda,E) = 0^+$.

Proof.

i) We have

$$\begin{aligned} \lim_{E\to 0^+} T_{+}(p,\lambda,E) &= \lim_{E\to 0^+} s_{+}(p,\lambda,E)\left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}}\int_0^1 \left[F(s_{+}(p,\lambda,E)) - F(s_{+}(p,\lambda,E)\,u)\right]^{-\frac{1}{p}} du \\ &= \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}}\int_0^1 \lim_{s_{+}(p,\lambda,E)\to 0^+}\left[\frac{F(s_{+}(p,\lambda,E)) - F(s_{+}(p,\lambda,E)\,u)}{(s_{+}(p,\lambda,E))^{p}}\right]^{-\frac{1}{p}} du \end{aligned}$$

On the other hand, one has

$F(s_{+}(p,\lambda,E)) - F(s_{+}(p,\lambda,E)\,u) = f(s_{+}(p,\lambda,E)\,c)\,s_{+}(p,\lambda,E)\,(1-u)$ with $c\in(u,1)$

Using the last formula, we obtain

$$\begin{aligned} &\int_0^1 \lim_{s_{+}(p,\lambda,E)\to 0^+}\left[\frac{F(s_{+}(p,\lambda,E)) - F(s_{+}(p,\lambda,E)\,u)}{(s_{+}(p,\lambda,E))^{p}}\right]^{-\frac{1}{p}} du \\ &= \int_0^1 \lim_{s_{+}(p,\lambda,E)\to 0^+}\left[\frac{f(s_{+}(p,\lambda,E)\,c)\,(1-u)}{(s_{+}(p,\lambda,E))^{p-1}}\right]^{-\frac{1}{p}} du \\ &= 0^+ \end{aligned}$$

So,

$$\lim_{E\to 0^+} T_{+}(p,\lambda,E) = \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \times 0^+ = 0^+$$

ii) In a similar manner as in **i)**, we prove that $\lim_{E \to +\infty} T_+(p, \lambda, E) = 0^+$. ■

The previous lemma shows that $T_+(p, \lambda, .)$ admits at least one critical point.

We observe that

$$T_+(p, \lambda, E) = \tilde{T}(p, \lambda, s(p, \lambda, E))$$

where $s(p, \lambda, E) = s_+(p, \lambda, E)$ and

$$\tilde{T}(p, \lambda, s) = \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \int_0^s [F(s) - F(u)]^{-\frac{1}{p}} du$$

Since for each $\lambda > 0$, the function $E \mapsto s_+(p, \lambda, E)$ is an increasing C^1-diffeomorphism from $(0, +\infty)$ onto itself, and

$$\frac{\partial T_+(p, \lambda, E)}{\partial E} = \frac{\partial \tilde{T}(p, \lambda, s(p, \lambda, E))}{\partial s} \frac{\partial s(p, \lambda, E)}{\partial E}$$

So, to study the variations of $E \mapsto T_+(p, \lambda, E)$, it suffices to study of $\tilde{T}$. That is, $\tilde{T}$ attains a local maximum (resp.minimum) value at s_* if and only if $T_+(p, \lambda, .)$ do so at $s_{p,\lambda}^{-1}(s_*)$ where $s_{p,\lambda}^{-1}$ is the inverse function of $s(p, \lambda, .)$. So, from the previous lemma, it follows that $\lim_{s \to 0^+} \tilde{T}(p, \lambda, s) = \lim_{s \to +\infty} \tilde{T}(p, \lambda, s) = 0$.

Then $\tilde{T}$ admits at least a maximum value.

We have

$$\frac{\partial \tilde{T}(p, \lambda, s)}{\partial s} = \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \int_0^s \frac{(K(p, s) - K(p, u))}{s[F(s) - F(u)]^{\frac{p+1}{p}}} du$$

where

$$K(p, u) = pF(u) - uf(u)$$

We have

$$\frac{\partial K(p,u)}{\partial u} = (p-1) f(u) - uf'(u) \tag{3.9}$$

$$\frac{\partial^2 K(p,u)}{\partial u^2} = (p-2) f'(u) - uf''(u) \tag{3.10}$$

Since f satisfies (**H1**)-(**H4**), we see that

$$K(p,0) = 0, \frac{\partial K(p,0)}{\partial u} \geq 0, \lim_{u\to+\infty} K(p,u) < 0$$

In addition, by (3.9) and (3.10), there exists a real numbers A and B with $0 < c < A < B$ such that

$$\frac{\partial K(p,u)}{\partial u} > 0 \text{ on } (0,A)$$

$$\frac{\partial K(p,A)}{\partial u} = 0$$

$$\frac{\partial K(p,u)}{\partial u} < 0 \text{ on } (A,+\infty)$$

and

$$K(p,u) > 0 \text{ on } (0,B)$$

$$K(p,B) = 0$$

$$K(p,u) < 0 \text{ on } (B,+\infty)$$

So, it follows that

$$\frac{\partial \tilde{T}(p,\lambda,s)}{\partial s} > 0,\ \forall s \in (0,A)$$

$$\frac{\partial \tilde{T}(p,\lambda,s)}{\partial s} < 0,\ \forall s \in (B,+\infty)$$

Under, one additional resonable hypothesis on the function $\frac{uf'(u)}{f(u)}$ on $(0,B)$, we show that $\tilde{T}$ admits exactly one critical point, a maximum on $(0,+\infty)$.

In addition to **(H1)**-**(H4)**, suppose that f satisfies

(H5) $\frac{uf'(u)}{f(u)} \geq -\frac{1}{p+1}$ on $(0,A)$ and $\frac{uf'(u)}{f(u)}$ is increasing on (A,B).

Lemma 14 $\tilde{T}(p,\lambda,.)$ *admits a unique critical point,* $s^*(p,\lambda)$, *at which it attains it global maximum value.*

We observe that

$$\frac{\partial^2 \tilde{T}(p,\lambda,s)}{\partial s^2} = \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \left(\frac{p+1}{p^2}\right) \int_0^s \frac{(K(p,s)-K(p,u))^2}{s^2\left[F(s)-F(u)\right]^{\frac{2p+1}{p}}}du + \frac{1}{p}\int_0^s \frac{(\Psi(p,s)-\Psi(p,u))}{s^2\left[F(s)-F(u)\right]^{\frac{p+1}{p}}}du$$

where

$$\Psi(p,u) = -p(p+1)F(u) + 2puf(u) - u^2 f'(u)$$

A simple computations shows that

$$\frac{\partial^2 \tilde{T}(p,\lambda,s)}{\partial s^2} = \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \frac{1}{ps^2} \int_0^s \frac{-\frac{p+1}{p}(\Delta K)\left(\Delta \tilde{f}\right) + \left(\Delta \tilde{K}'\right)(\Delta F)}{[\Delta F]^{\frac{2p+1}{p}}}du$$

where

$$\Delta K = K(p,s) - K(p,u)$$

$$\Delta F = F(s) - F(u)$$

$$\Delta\tilde{f} = sf(s) - uf(u)$$

$$\Delta\tilde{k}' = s\frac{\partial K(p,s)}{\partial u} - u\frac{\partial K(p,u)}{\partial u}$$

To prove lemma 14, we state and prove the following lemmas 15 and 16.

Lemma 15 *Suppose that f satisfies **(H1)**-**(H5)**, then it follows that the maximum of* $\frac{\Delta\tilde{f}}{\Delta F}$ *on* $[0,s]$ *attains at* $u = s$ *for* $s \in (A,B)$, *and* $\max\limits_{0\le u\le s}\frac{\Delta\tilde{f}}{\Delta F} = \frac{f(s)+sf'(s)}{f(s)}$ *for* $s \in (A,B)$.

Proof. For fixed $s \in (A,B)$, we have

$$K(p,s) = pF(s) - sf(s) > 0$$

and

$$\frac{\partial K(p,s)}{\partial u} = (p-1)f(s) - sf'(s) < 0$$

So,

$$\frac{\Delta\tilde{f}}{\Delta F}\Big|_{u=0} = \frac{sf(s)-uf(u)}{F(s)-F(u)}\Big|_{u=0} = \frac{sf(s)}{F(s)} < p$$

$$\frac{\Delta\tilde{f}}{\Delta F}\Big|_{u=s} = \frac{f(s)+sf'(s)}{f(s)} > \frac{f(s)+(p-1)f(s)}{f(s)} = p$$

Then for fixed $s \in (A,B)$, the maximum of $\frac{\Delta\tilde{f}}{\Delta F}$ attains at $u = s$ or at some point interior on $(0,s)$.

Put

$$h(u) = 1 + \frac{uf'(u)}{f(u)}$$

We have

$$h(u) < p \text{ for } 0 < u < A$$

$$h(A) = p$$

$$h(u) \text{ is increasing on } (A, B)$$

This yields

$$h(s) - h(u) \geq 0 \text{ for } 0 < u < s \text{ and } s \in (A, B)$$

For $s \in (A, B)$, suppose that the maximum of $\dfrac{\Delta \tilde{f}}{\Delta F}$ attains at $u_0 \in (0, s)$.

Then

$$\left(\frac{\Delta \tilde{f}}{\Delta F}\right)' |_{u=u_0} = 0$$

which is equivalently to:

$$f(u_0)[sf(s) - u_0 f(u_0)] - [F(s) - F(u_0)]\left[u_0 f'(u_0) + f(u_0)\right] = 0$$

This yields

$$\frac{sf(s) - u_0 f(u_0)}{F(s) - F(u_0)} = \frac{u_0 f'(u_0) + f(u_0)}{f(u_0)}$$

So,

$$\frac{\Delta \tilde{f}}{\Delta F} |_{u=u_0} = h(u_0)$$

Hence

$$\frac{\Delta \tilde{f}}{\Delta F}|_{u=u_0} = h(u_0) \leq h(s) = \frac{\Delta \tilde{f}}{\Delta F}|_{u=s}$$

This contradicts the assumption that the maximum of $\frac{\Delta \tilde{f}}{\Delta F}$ does not occurs at $u = s$ for $s \in (A, B)$. ■

We define

$$\Delta \hat{f}' = s^2 f'(s) - u^2 f'(u)$$

We have the following lemma

Lemma 16 *Suppose that f satisfies **(H1)**-**(H5)**, then it follows that the minimum of $\frac{\Delta \hat{f}'}{\Delta \tilde{f}}$ on $[0, s]$ attains at $u = 0$ for $s \in (A, B)$, and $\min_{0 \leq u \leq s} \frac{\Delta \hat{f}'}{\Delta \tilde{f}} = \frac{sf'(s)}{f(s)}$ for $s \in (A, B)$.*

Proof. By (**H5**), we have

$$\begin{aligned}
(xf(x))' &= f(x) + xf'(x) \\
&\geq f(x) - \frac{1}{p+1} f(x) \\
&= \frac{p}{p+1} f(x) \geq 0
\end{aligned}$$

Then

$$(xf(x))' > 0 \text{ on } (0, +\infty)$$

So,

$$sf(s) - uf(u) > 0 \text{ for } 0 < u < s < B$$

This yields

$$\Delta \tilde{f} > 0 \text{ for } 0 < u < s < B$$

For $s \in (A, B)$ and $0 < u < s$, we have

$$\frac{\Delta \hat{f}'}{\Delta \tilde{f}} - \frac{\Delta \hat{f}'}{\Delta \tilde{f}}|_{u=0} = \frac{uf(u)\left[\frac{sf'(s)}{f(s)} - \frac{uf'(u)}{f(u)}\right]}{sf(s)-uf(u)} = \frac{uf(u)\left[h(s)-h(u)\right]}{sf(s)-uf(u)} \geq 0$$

On the other hand, we have

$$\frac{\Delta \hat{f}'}{\Delta \tilde{f}}|_{u=s} = \lim_{u \to s} \frac{s^2 f'(s) - u^2 f'(u)}{sf(s) - uf(u)} = \frac{2sf'(s) + s^2 f''(s)}{f'(s) + sf''(s)}$$

Since $f \in C^2(0, +\infty)$ and h is increasing in (A, B), we obtain

$$\frac{\Delta \hat{f}'}{\Delta \tilde{f}}|_{u=s} - \frac{\Delta \hat{f}'}{\Delta \tilde{f}}|_{u=0} = \frac{uh'(u)f(u)}{f(u) + uf'(u)}|_{u=s} \geq 0$$

Hence the minimum of $\frac{\Delta \hat{f}'}{\Delta \tilde{f}}$ on $[0, s]$ attains at $u = 0$ for $s \in (A, B)$ and $\min\limits_{0 \leq u \leq s} \frac{\Delta \hat{f}'}{\Delta \tilde{f}} = \frac{sf'(s)}{f(s)}$

Proof. of lemma(14)

Let $M := \max\limits_{0 \leq u \leq s} \frac{\Delta \tilde{f}}{\Delta F}$ and $m := \min\limits_{0 \leq u \leq s} \frac{\Delta \hat{f}'}{\Delta \tilde{f}}$

We have

$$\frac{\partial^2 \tilde{T}(p, \lambda, s)}{\partial s^2} + \frac{M}{p} s \frac{\partial \tilde{T}(p, \lambda, s)}{\partial s}$$
$$= \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \int_0^s \frac{\frac{M}{p}\left[p(\Delta F)^2 - (\Delta \tilde{f})(\Delta F)\right] + \frac{p+1}{p}(\Delta \tilde{f})^2 - (p+1)(\Delta \tilde{f})(\Delta F) - (\Delta \hat{f}')(\Delta F) + (p-1)(\Delta \tilde{f})(\Delta F)}{s^2 (\Delta F)^{\frac{2p+1}{p}}} du$$
$$= \left(\frac{p-1}{\lambda p}\right)^{\frac{1}{p}} \int_0^s \frac{\frac{M}{p}\left[p(\Delta F)^2 - (\Delta \tilde{f})(\Delta F)\right] + \frac{p+1}{p}(\Delta \tilde{f})^2 - 2(\Delta \tilde{f})(\Delta F) - (\Delta \hat{f}')(\Delta F)}{s^2 (\Delta F)^{\frac{2p+1}{p}}} du$$

Let

$$Q = \frac{M}{p}\left[p(\Delta F)^2 - \left(\Delta \tilde{f}\right)(\Delta F)\right] + \frac{p+1}{p}\left(\Delta \tilde{f}\right)^2 - 2\left(\Delta \tilde{f}\right)(\Delta F) - \left(\Delta \hat{f}'\right)(\Delta F)$$

and define the function $\tilde{h}$ by

$$\tilde{h}(x) := xf(x) - \frac{p}{p+1}F(x)$$

We have

$$\begin{aligned}\tilde{h}'(x) &= \frac{1}{p+1}f(x) + xf'(x) \\ &= f(x)\left[\frac{1}{p+1} + \frac{xf'(x)}{f(x)}\right] \geq 0\end{aligned}$$

Then, for $0 < u < s < B$

$$\frac{sf(s) - uf(u)}{F(s) - F(u)} \geq \frac{p}{p+1}$$

So,

$$L := \frac{\Delta\tilde{f}}{\Delta F} \geq \frac{p}{p+1}$$

which implies that

$$\begin{aligned}Q &\leq \frac{p+1}{p}\left(\Delta\tilde{f}\right)^2 - \left(2 + \frac{M}{p} + m\right)\left(\Delta\tilde{f}\right)(\Delta F) + M(\Delta F)^2 \\ &= (\Delta F)^2\left[\frac{p+1}{p}L^2 - \left(2 + \frac{M}{p} + m\right)L + M\right]\end{aligned}$$

Put

$$P(L) := \frac{p+1}{p}L^2 - \left(2 + \frac{M}{p} + m\right)L + M$$

We observe that

$$P(L) = \frac{p+1}{p}(L - M)\left(L - \frac{p}{p+1}\right)$$

Since $\frac{p}{p+1} \leq L \leq M$, we obtain

$$P(L) \leq 0$$

and consequently

$$Q \leq 0$$

We can show that Q is not identically zero for fixed $s \in (A, B)$, $0 < u < s$.

Hence

$$\frac{\partial^2 \tilde{T}(p, \lambda, s)}{\partial s^2} + \frac{M}{p} s \frac{\partial \tilde{T}(p, \lambda, s)}{\partial s} < 0 \text{ for } s \in (A, B)$$

which proves the uniqueness of the critical point of $\tilde{T}(p, \lambda, .)$. ■

Proof. of theorem(10)

From the preceding lemmas one has the following picture of the function $s \mapsto \tilde{T}(p, \lambda, s)$ which is defined on $(0, +\infty)$:

$\lim_{s \to 0^+} \tilde{T}(p, \lambda, s) = \lim_{s \to +\infty} \tilde{T}(p, \lambda, s) = 0^+$, and $\tilde{T}(p, \lambda, .)$ admits a unique maximum value $\tilde{T}(p, \lambda, s_*)$. So,

i) If $\tilde{T}(p, \lambda, s_*) < \frac{1}{2}$, problem (3.5) admits no positive solution.

ii) If $\tilde{T}(p, \lambda, s_*) = \frac{1}{2}$, problem (3.5) admits a unique positive solution.

iii) If $\tilde{T}(p, \lambda, s_*) > \frac{1}{2}$, problem (3.5) admits exactly two positive solutions.

Then, if one put $\lambda_* = \frac{p-1}{p} \left(2s_*(p, \lambda, E) \int_0^1 \left[F(s_*(p, \lambda, E)) - F(s_*(p, \lambda, E) u) \right]^{-\frac{1}{p}} du \right)^p$, theorem (10) follows. ■

Chapter 4

Multiplicity results for quasilinear boundary-value problems with concave-convex nonlinearities

Abstract This chapter is concerned with multiplicity results for the problem

$$\begin{cases} -\left(\left|u'\right|^{p-2}u'\right)' = \lambda g\left(t\right)\left(\left|u\right|^{\nu-1}u + \left|u\right|^{\mu-1}u\right), t \in (a,b) \\ u\left(a\right) = u\left(b\right) = 0 \end{cases}$$

where $p > 1$, λ is a strictly positive real parameter, $0 < \mu < p-1 < \nu$ and $g : [a,b] \to \mathbf{R}_+^*$ is of class C^1. We use the angular function technique for showing the existence of solutions.

4.1 Introduction

The purpose of this chapter is to study the existence and multiplicity of solutions to the problem:

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda g(t)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right), \ t \in (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{4.1}$$

where $p > 1$, λ is a strictly positive real parameter, $0 < \mu < p-1 < \nu$ and $g : [a,b] \to \mathbf{R}_+^*$ is of class C^1. We investigate the influence of the combined concave-convex nonlinearities on the multiplicity of the solutions. This question was studied by several authors (see, for instance, [14], [25]-[27], [41], [76], [92], [162], [175], [198], [199], [202], [203], and [213]).In [25], the authors investigated the following problem:

$$\begin{cases} -\Delta u &= u^{\gamma_1} + \lambda u^{\gamma_2} & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

with $0 < \gamma_1 < 1 < \gamma_2$, $\lambda > 0$ and the set Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. The authors prove that (4.2) has a minimal solution u_λ for $\lambda \in (0,\Lambda)$ with some $\Lambda > 0$, and there exists $A > 0$ such that for all $\lambda \in (0,\Lambda)$, (4.2) has at most one solution u_λ such that $\|u_\lambda\|_\infty \leq A$. Moreover, if the condition $\gamma_1 \leq 2^*$ holds then for all $\lambda \in (0,\Lambda)$ problem (4.2) has a second solution $v_\lambda > u_\lambda$, where $2^* = \frac{N+2}{N-2}$ for $N > 2$ and $2^* = +\infty$ for $N = 1, 2$.

At the end of the paper [25], the authors indicated that it could be interesting to study in detail the problem (4.2) in the special case when $N = 1$ and $\Omega = (a,b)$, see [[25], section 6, (d)]. This study was done by S.Villegas [199] by means of a quadrature method. He shows that there exist two monotone divergent sequences (ε_n) and (L_n); $\varepsilon_n \leq L_n$ satisfying:

(i) If $\lambda \in (0, \varepsilon_n)$, then problem (4.2) has exactly two pairs of opposite solutions with $(n+1)$ zeros.

(ii) If $\lambda \in [\varepsilon_n, L_n)$, then problem (4.2) has at least two pairs of opposite solutions with $(n+1)$ zeros.

(iii) If $\lambda = L_n$, then problem (4.2) has at least one pair of opposite solutions with $(n+1)$ zeros.

(iv) If $\lambda > L_n$, then problem (4.2) has no pair of opposite solutions with $(n+1)$ zeros.

In[76], the authors shows that the results obtained by S.Villegas remain true for problem (4.1) in the case $p = 2$.

The aim of this work is to show that the same results obtained in [76] holds for any $p > 1$.

The paper is organized as follows. In section 2 we give some definitions and present the method used to prove the results of this paper. Some preliminary lemmas are the aim of section 3. Next, in section 4 we state and prove our main result. Finally, we conclude the paper with some remarks in section 5.

4.2 Definitions and the method used

Consider the boundary value problem

$$\left(|u'|^{p-2} u'\right)' = f\left(t, u, u'\right) \tag{4.3}$$

$$|u\left(a\right)|^{\frac{2-p}{p-1}} u\left(a\right) \sin\alpha - |u'\left(a\right)|^{\frac{2-p}{p-1}} u'\left(a\right) \cos\alpha = 0 \tag{4.4}$$

$$|u\left(b\right)|^{\frac{2-p}{p-1}} u\left(b\right) \sin\beta - |u'\left(b\right)|^{\frac{2-p}{p-1}} u'\left(b\right) \cos\beta = 0 \tag{4.5}$$

where $p > 1$, $f : [a, b] \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function and $\alpha, \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definition 1: A function $u \in C^1[a, b]$ such that $|u'|^{p-2} u' \in C^1(a, b)$ is called a solution of problem (4.3), (4.4)and (4.5) if:

(i) u satisfies (4.3) for each $t \in (a, b)$.

(ii) u satisfies (4.4) and (4.5).

Definition 2: By a nondegenerate solution of problem (4.3), (4.4)and (4.5) we mean a function u such that $u^2(t) + (u')^2(t) \neq 0$ for all $t \in [a, b]$.

4.2.1 Angular function technique

To obtain our results, we use the well-know angular function technique (see for instance, [42], [88], [138] and [177]).

We distinguish two cases:

The case $1 < p < 2$

Let u be a solution of (4.3) such that

$$u^{2(p-1)}(t) + (u')^{2(p-1)}(t) \neq 0, \ \forall t \in [a, b]$$

We define the angular function associated to u by

$$\tan \varphi(t) = \frac{|u'(t)|^{p-2} u'(t)}{|u(t)|^{p-2} u(t)}$$

and set,

$$\begin{cases} |u(t)|^{p-2} u(t) = r(t) \cos \varphi(t) \\ |u'(t)|^{p-2} u'(t) = r(t) \sin \varphi(t) \end{cases}$$

where $r(t) = \sqrt{u^{2(p-1)}(t) + (u')^{2(p-1)}(t)}$ for all $t \in [a, b]$.

If u is a solution of (4.3), then $(r(t), \varphi(t))$ is a solution of the following system

$$\varphi'(t) = \frac{f(t, u, u')}{r(t)} \cos \varphi(t) - (p-1) |\sin \varphi(t)|^{\frac{p}{p-1}} |\cos \varphi(t)|^{\frac{p-2}{p-1}} \tag{4.6}$$

$$r'(t) = f(t, u, u') \sin \varphi(t) - r(t) \sin \varphi(t) |\sin \varphi(t)|^{\frac{2-p}{p-1}} \cos \varphi(t) |\cos \varphi(t)|^{\frac{p-2}{p-1}} \tag{4.7}$$

Remarks

(i) The set of angular functions corresponding to a given u is infinite, each of these functions can be uniquely specified by indicating its value in a.

(ii) If t_0 is a simple zero of u then $u(t_0) = 0$ and $u'(t_0) \neq 0$. Consequently $r(t_0) \neq 0$ and $\varphi(t_0) = \frac{\pi}{2} \pm k\pi$ with $k \in \mathbf{N}$. This shows that the simple zeros of a solution u of (4.3) are obtained by studying the equation

$$\varphi(t) = \frac{\pi}{2} \pm k\pi, \ k \in \mathbf{N}$$

where φ is a solution of the equation (4.6).

(iii) u is a solution of (4.3), (4.4) and (4.5) if and only if its angular function $\varphi(t)$ satisfies

$$\varphi(a) = \alpha, \ \varphi(b) = \beta + k\pi$$

for some integer k.

The case $2 \leq p < +\infty$

Let u be a nondegenerate solution of (4.3). In this case we define the angular function φ associated to u by letting

$$\tan\varphi(t) = \frac{u'(t)}{u(t)}$$

and set,

$$\begin{cases} u(t) = r(t)\cos\varphi(t) \\ u'(t) = r(t)\sin\varphi(t) \end{cases}$$

where $r(t) = \sqrt{u^2(t) + (u')^2(t)}$ for all $t \in [a, b]$.

If u is a solution of (4.3), then $(r(t), \varphi(t))$ is a solution of the following system

$$\varphi'(t) = \frac{f(t, u, u')\cos\varphi(t)}{(p-1)\,|r(t)\sin\varphi(t)|^{p-2}\, r(t)} - \sin^2\varphi(t)$$

$$r'(t) = \left[(p-1)\,r(t)\cos\varphi(t) + f(t,u,u')\right]\sin\varphi(t)$$

4.3 Preliminary Lemmas

Consider the Cauchy problem

$$\begin{cases} \left(|u'|^{p-2}u'\right)' = f(t,u,u') \\ u(a) = c\cos\alpha \\ u'(a) = c\sin\alpha \end{cases} \tag{4.8}$$

where $c \in \mathbf{R}_+^*$.

Lemma 17 *Let $1 < p < 2$. Assume $q_1, q_2 : [a,b] \to \mathbf{R}_+$ are continuous function satisfying for all $\varepsilon > 0$ small enough $\lim\limits_{r\to 0^+} \dfrac{h(r\cos\varphi, r\sin\varphi)}{r} = +\infty$ (*
resp $\lim\limits_{r\to+\infty} \dfrac{h(r\cos\varphi, r\sin\varphi)}{r} = +\infty$) uniformly with respect to $\varphi \in \left[0, \frac{\pi}{2} - \varepsilon\right]$. Then for all integer n and positive real number $\tilde{\nu}$, there exists $\tilde{\mu}_n(\tilde{\nu})$ such that if u is a nondegenerate solution of (4.8) defined for all $t \in [a,b]$ and satisfy the inequalities

$$\begin{aligned} f(t,u,u')\, signu(t) \;&\leq\; -q_1(t)\, h\left(|u(t)|^{p-1}, |u'(t)|^{p-1}\right) \\ &\quad + q_2(t)\left(|u(t)|^{p-1} + |u'(t)|^{p-1}\right) \end{aligned} \tag{4.9}$$

with

$$0 < \sqrt{u^{2(p-1)}(t) + (u')^{2(p-1)}(t)} \leq \tilde{\mu}_n(\tilde{\nu}) \tag{4.10}$$

(resp $\sqrt{u^{2(p-1)}(t) + (u')^{2(p-1)}(t)} \geq \tilde{\mu}_n(\tilde{\nu})$) for all $t \in [a,b]$ where

$$\int_a^b q_1(t)\,dt \geq \tilde{\nu} > 0 \tag{4.11}$$

Then u admits at least n zeros in $[a,b]$.

Proof. The proof will be given in several steps

Step1: Using differential inequalities, we will control the angular function φ of equation (4.6) by another angular function ψ associated to a simpler problem. In fact, for any given $n \in \mathbf{N}$, there exists a positive constant δ such that if $\tau_2 - \tau_1 \leq \delta$ ($\tau_1,\ \tau_2 \in [a, b]$) then

$$2\sqrt{2}\int_{\tau_1}^{\tau_2} q_2(s)\,ds \leq 1 \text{ and } 4n\int_{\tau_1}^{\tau_2} q_1(s)\,ds < \tilde{\nu} \tag{4.12}$$

Let $Q = \int_a^b q_2(s)\,ds$, $\sigma = \frac{1}{4}\min\left\{\frac{\pi}{2}, \frac{\delta}{1+Q}\right\}$ and let us choose $\tilde{\mu}(\tilde{\nu}) > 0$ such that

$$\frac{h(r\cos\varphi, r\sin\varphi)}{r} \geq \frac{2n}{\tilde{\nu}}\left[\frac{\pi - 2(p-1)\sigma + Q(p-1)\sqrt{2}}{\sin\frac{\sigma(p-1)}{2}}\right]$$

for $0 < r \leq \tilde{\mu}(\tilde{\nu})$ (resp. $r \geq \tilde{\mu}(\tilde{\nu})$), $0 \leq \varphi \leq \frac{\pi}{2} - \frac{\sigma}{2}$. Let u be a nondegenerate solution of (4.8) defined for all $t \in [a, b]$ and satisfy (4.9)-(4.11). Set $r(t) = \sqrt{u^{2(p-1)}(t) + (u')^{2(p-1)}(t)}$ and consider a maximal solution of the problem

$$\begin{cases} \psi'(t) = \left[\sqrt{2}(p-1)\,q_2(t) - \dfrac{q_1(t)}{r(t)}h\left(r(t)\left|\cos\psi(t)\right|, r(t)\left|\sin\psi(t)\right|\right)\right] \times \\ \qquad \times \left|\cos\psi(t)\right| - (p-1)\left|\sin\psi(t)\right|^{\frac{p}{p-1}}\left|\cos\psi(t)\right|^{\frac{p-2}{p-1}} \\ \psi(a) = \dfrac{\pi}{2} \end{cases} \tag{4.13}$$

If $\varphi : [a, b] \to \mathbf{R}$ is an angular function associated to u and $\varphi(a) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then by (4.9) the theorem on differential inequalities (see[[138], Theorem 15.2]) implies that

$$\varphi(t) \leq \psi(t) \text{ for } a \leq t \leq b$$

The inequality (4.11) implies that there exists $t_i \in [a, b]$ such that

$$\int_a^{t_i} q_1(s)ds = \left(\frac{i}{n}\right)\tilde{\nu} \tag{4.14}$$

Step 2: Let us show by induction that the inequality

$$\psi(t_i) \leq \frac{\pi}{2} - \pi i \tag{4.15}$$

is valid for any $i \in \{0, ..., n\}$.

If $2\psi(t^*) \leq \pi - 2\pi i$ for some $t^* \in [a, b]$ and some integer i then $2\psi(t) < \pi - 2\pi i$ in $]t^*, b]$. So, (4.15) is valid for $i = 0$. Suppose that (4.15) is valid for some $i = m \in \{0, ..., n-1\}$ then $2\psi(t) < \pi - 2\pi m$ in the interval $]t_m, b]$.

Claim: We have the following inequality

$$\psi(t_m + 4\sigma) \leq \frac{\pi}{2} - \pi m - (p-1)\sigma \tag{4.16}$$

Proof. Suppose that (4.16) is not true. One has two possibilities

(i) For $t_m \leq t \leq t_m + 4\sigma$, one has

$$\psi(t) > \frac{\pi}{2} - \pi m - 2(p-1)\sigma$$

(ii) There exists $s \in [t_m, t_m + 4\sigma)$ such that

$$\psi(s) = \frac{\pi}{2} - \pi m - 2(p-1)\sigma$$

and

$$\psi(t) > \frac{\pi}{2} - \pi m - 2(p-1)\sigma \text{ in } (s, t_m + 4\sigma)$$

In the case **(i)**,

$$\begin{aligned}\psi(t_m+4\sigma)-\psi(t_m) &\leq \sqrt{2}(p-1)\sin 2\sigma \int_{t_m}^{t_m+4\sigma} q_2(s)\,ds - 4\sigma(p-1)\times \\ &\times(\cos 2\sigma)^{\frac{p}{p-1}}\, 2^{\frac{2-p}{2(p-1)}} \\ &\leq (p-1)2\sqrt{2}\sigma \int_{t_m}^{t_m+4\sigma} q_2(s)\,ds - 4\sigma(p-1)\times \\ &\times(\cos 2\sigma)^{\frac{p}{p-1}}\, 2^{\frac{2-p}{2(p-1)}} \\ &\leq (p-1)\sigma - 4\sigma(p-1)2^{-\frac{p}{p-1}}2^{\frac{2-p}{2(p-1)}} \\ &= -(p-1)\sigma\end{aligned}$$

Hence,

$$\begin{aligned}\psi(t_m+4\sigma) &\leq \psi(t_m)-(p-1)\sigma \\ &\leq \tfrac{\pi}{2}-\pi m-(p-1)\sigma\end{aligned}$$

Suppose that (**ii**) is true, one has

$$\begin{aligned}\psi(t_m+4\sigma)-\psi(s) &\leq (p-1)\sqrt{2}\sin 2\sigma \int_{s}^{t_m+4\sigma} q_2(t)\,dt \\ &\leq (p-1)\sigma\end{aligned}$$

Thus,

$$\begin{aligned}\psi(t_m+4\sigma) &\leq \psi(s)+(p-1)\sigma \\ &\leq \tfrac{\pi}{2}-\pi m-(p-1)\sigma\end{aligned}$$

So, in all cases one has

$$\psi(t_m+4\sigma)\leq \frac{\pi}{2}-\pi m-(p-1)\sigma$$

Assume that,

$$\psi(t_{m+1}-4\sigma) > -\frac{\pi}{2}-\pi m+(p-1)\sigma$$

So, one has

$$-\frac{\pi}{2}-\pi m+\frac{(p-1)\sigma}{2}\leq\psi(t)\leq\frac{\pi}{2}-\pi m-\frac{(p-1)\sigma}{2}$$

in $[t_m+4\sigma,t_{m+1}-4\sigma]$. In the contrary case, there exists $[s_1,s_2]\subset[t_m+4\sigma,t_{m+1}-4\sigma]$ such that $\psi(s_1)\leq\psi(t)\leq\psi(s_2)$ for $s_1\leq t\leq s_2$ and $\psi(s_1)=\frac{\pi}{2}-\pi m+\dfrac{(p-1)\sigma}{2}$, $\psi(s_2)=\frac{\pi}{2}-\pi m+(p-1)\sigma$ which is impossible because one has

$$\psi(s_2)-\psi(s_1)=\frac{(p-1)\sigma}{2}\leq\sqrt{2}\sigma(p-1)\int_{s_1}^{s_2}q_2(t)\,dt-\frac{3}{4}(p-1)(s_2-s_1)$$

which is a contradiction with the choice of σ.

Using inequality (4.16) we will show that (4.15) is true for $i=m+1$.

From (4.12) and (4.14) one has

$$\begin{aligned}\int_{t_m+4\sigma}^{t_{m+1}-4\sigma}q_1(s)\,ds &= \int_{t_m}^{t_{m+1}}q_1(s)\,ds-\int_{t_m}^{t_m+4\sigma}q_1(s)\,ds-\int_{t_{m+1}-4\sigma}^{t_{m+1}}q_1(s)\,ds \\ &\geq \frac{\tilde{\nu}}{n}-\frac{\tilde{\nu}}{4n}-\frac{\tilde{\nu}}{4n}=\frac{\tilde{\nu}}{2n}\end{aligned}\tag{4.17}$$

Then $t_{m+1}-t_m-8\sigma>\delta$. From the relations(4.10), (4.11), (4.13) and (4.17) one gets

$$\begin{aligned}\psi(t_{m+1}-4\sigma)-\psi(t_m+4\sigma) &\leq \sqrt{2}(p-1)\int_{t_m+4\sigma}^{t_{m+1}-4\sigma}q_2(t)\left|\cos\psi(t)\right|dt \\ &\quad -\int_{t_m+4\sigma}^{t_{m+1}-4\sigma}\frac{q_1(t)\,h\left(r(t)\left|\cos\psi(t)\right|,r(t)\left|\sin\psi(t)\right|\right)}{r(t)}\left|\cos\psi(t)\right|dt \\ &\leq \sqrt{2}(p-1)Q-\left[\frac{\pi-2(p-1)\sigma+\sqrt{2}(p-1)Q}{\sin\frac{\sigma(p-1)}{2}}\right]\sin\frac{\sigma(p-1)}{2} \\ &= -\pi+2(p-1)\sigma\end{aligned}$$

which implies

$$\begin{aligned}\psi(t_{m+1}-4\sigma) &\leq \psi(t_m+4\sigma)-\pi+2(p-1)\sigma \\ &\leq \tfrac{\pi}{2}-\pi m-(p-1)\sigma-\pi+2(p-1)\sigma \\ &= \tfrac{\pi}{2}-\pi(m+1)+(p-1)\sigma\end{aligned}$$

and then

$$\psi(t_{m+1}) \leq \frac{\pi}{2} - \pi(m+1)$$

The inequality (4.15) is valid for $i = m+1$. Then, we obtain

$$\varphi(t_n) \leq \frac{\pi}{2} - \pi n$$

which shows that u admits at least n zeros in $[a,b]$. ■

Lemma 18 *Let $2 \leq p < +\infty$. Assume $q_1, q_2 : [a,b] \to \mathbf{R}_+$ are continuous function satisfying for all $\varepsilon > 0$ small enough* $\lim_{r \to 0^+} \frac{h\left(|r\sin\varphi|^{p-1}, |r\sin\varphi|^{p-1}\right)}{r^{p-1}} = +\infty$ *(resp* $\lim_{r\to+\infty} \frac{h\left(|r\sin\varphi|^{p-1}, |r\sin\varphi|^{p-1}\right)}{r^{p-1}} = +\infty$*) uniformly with respect to $\varphi \in \left[0, \frac{\pi}{2} - \varepsilon\right]$. Then for all integer n and positive real number $\tilde{\nu}$, there exists $\tilde{\mu}_n(\tilde{\nu})$ such that if u is a nondegenerate solution of (4.8) defined for all $t \in [a,b]$ and satisfying the inequalities*

$$\begin{aligned} f(t,u,u')\, signu(t) \leq & -q_1(t)\, h\left(|u(t)|^{p-1}, |u'(t)|^{p-1}\right) \\ & + q_2(t)\left(|u(t)|^{p-1} + |u'(t)|^{p-1}\right) \end{aligned}$$

with

$$0 < \sqrt{u^2(t) + (u')^2(t)} \leq \tilde{\mu}_n(\tilde{\nu})$$

(resp $\sqrt{u^2(t) + (u')^2(t)} \geq \tilde{\mu}_n(\tilde{\nu})$*) for all $t \in [a,b]$ where*

$$\int_a^b q_1(t)\, dt \geq \tilde{\nu} > 0$$

Then u admits at least n zeros in $[a,b]$.

Proof. Similar to that of lemma 17.

Remark: Lemmas 17 and 18 are generalizations to the case $p \neq 2$ of a result of Shekhter [[177],lemma3].

4.4 Main result

In this section, we state and prove our main result

Theorem 19 *There exists a decreasing sequence* $(\lambda_n)_{n\geq 1}$ *such that*

(i) *If* $\lambda > \lambda_n$*, then problem (4.1) has no solution with n zeros in* (a,b).

(ii) *If* $\lambda = \lambda_n$*, then problem (4.1) has at least one pair of opposite solutions with n zeros in* (a,b).

(iii) *If* $\lambda < \lambda_n$*, then problem (4.1) has at least two pairs of opposite solutions with n zeros in* (a,b).

Remark: We will give the proof for the case where $p \in (1,2)$. The adaptation of the other case may be handled similarly.

Proof. Consider the Cauchy problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right) = \lambda g(t)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right) \\ u(a) = 0 \\ u'(a) = z, \;\; z \in \mathbf{R}_+^* \end{cases} \tag{4.18}$$

Lemma 20 *Each solution u of (4.18) is defined for all* $t \in [a,b]$.

Proof. Suppose that there exist a sequence $(t_n)_{n\geq 1}$ converging to $t_* \in [a,b]$ such that $\lim\limits_{n\to+\infty} (|u(t_n,\lambda,z)| + |u'(t_n,\lambda,z)|) = +\infty$. The mean value theorem shows that $\lim\limits_{n\to+\infty} |u'(t_n,\lambda,z)| = |u'(t_*,\lambda,z)| = +\infty$

Let

$$E(t,\lambda,z) = \frac{p-1}{p}|u'(t,\lambda,z)|^p + F(t,u(t,\lambda,z))$$

where

$$F(t,u)=\int_0^u \lambda g(t)\left(|s|^{\nu-1}s+|s|^{\mu-1}s\right)ds$$

We have

$$\begin{aligned}\frac{\partial E(t,\lambda,z)}{\partial t} &= \int_0^{u(t,\lambda,z)} \lambda g'(t)\left(|s|^{\nu-1}s+|s|^{\mu-1}s\right)ds \\ &\leq \lambda\|g'\|_0 \int_0^{u(t,\lambda,z)} \left(|s|^{\nu-1}s+\lambda|s|^{\mu-1}s\right)ds=\lambda\|g'\|_0 G(u)\end{aligned}$$

where $\|g'\|_0=\sup_{t\in[a,b]}|g'(t)|$ and $G(u)=\int_0^u\left(|s|^{\nu-1}s+\lambda|s|^{\mu-1}s\right)ds$

Let $\tilde{m}=\inf_{t\in[a,b]} g(t)$. Then

$$\frac{\partial E(t,\lambda,z)}{\partial t}\leq\frac{\|g'\|_0}{\tilde{m}}F(t,u(t,\lambda,z))\leq\frac{\|g'\|_0}{\tilde{m}}E(t,\lambda,z)$$

which implies that

$$E(t_*,\lambda,z)\leq E(a,\lambda,z)\exp\left(\frac{\|g'\|_0}{\tilde{m}}(b-a)\right)\tag{4.19}$$

which is a contradiction. ■

Lemma 21 *The problem (4.18) has nonzero solution satisfying zero initial condition.*

Proof. Suppose that u is a nontrivial solution of (4.18) with $u(a)=u'(a)=0$.

By the inequality (4.19), we have

$$E(t,\lambda,0)\leq E(a,\lambda,0)\exp\left(\frac{\|g'\|_0}{\tilde{m}}(b-a)\right)$$

Then $E(t,\lambda,0)=0$ for all $t\in(a,b]$ since $E(a,\lambda,0)=0$. This implies that $u(t,\lambda,0)\equiv 0$ for all $t\in[a,b]$ which is a contradiction. ■

Lemma 22 *For any $\gamma > 0$, there exist $\delta > 0$ such that if $z \leq \gamma$ then* $\sqrt{u^{2(p-1)}(t,\lambda,z) + (u')^{2(p-1)}(t,\lambda,z)}$
δ for all $\lambda > 0$ and $t \in [a,b]$.

Proof. There exist a constant $M_1 > 0$ such that

$$F(t, u(t,\lambda,z)) \geq -M_1$$

Hence

$$E(t,\lambda,z) \geq \frac{p-1}{p} |u'(t,\lambda,z)|^p - M_1$$

It follows that

$$\frac{p-1}{p} |u'(t,\lambda,z)|^p - M_1 \leq E(a,\lambda,z) \exp\left(\frac{\|g'\|_0}{\tilde{m}}(b-a)\right)$$

Then

$$\begin{aligned} \tfrac{p-1}{p} |u'(t,\lambda,z)|^p &\leq M_1 + \left[\tfrac{p-1}{p} |u'(a,\lambda,z)|^p + F(a, u(a,\lambda,z))\right] \times \\ &\quad \times \exp\left(\tfrac{\|g'\|_0}{\tilde{m}}(b-a)\right) \\ &= M_1 + \tfrac{p-1}{p} z^p \exp\left(\tfrac{\|g'\|_0}{\tilde{m}}(b-a)\right) \end{aligned}$$

The last inequality shows that

$$u(t,\lambda,z) - u(a,\lambda,z) \leq \left[\tfrac{p}{p-1}\left(M_1 + \tfrac{p-1}{p} z^p \exp\left(\tfrac{\|g'\|_0}{\tilde{m}}(b-a)\right)\right)\right]^{\frac{1}{p}} (b-a)$$

In other words, we have

$$\begin{aligned} r(t,\lambda,z) &= \sqrt{u^{2(p-1)}(t,\lambda,z) + (u')^{2(p-1)}(t,\lambda,z)} \\ &\leq \delta \end{aligned}$$

with

$$\begin{aligned} \delta = \Bigg[&\left[\left[\tfrac{p}{p-1}\left(M_1 + \tfrac{p-1}{p} z^p \exp\left(\tfrac{\|g'\|_0}{\tilde{m}}(b-a)\right)\right)\right]^{\frac{1}{p}} (b-a)\right]^{2(p-1)} + \\ &+ \left[\tfrac{p}{p-1}\left(M_1 + \tfrac{p-1}{p} z^p \exp\left(\tfrac{\|g'\|_0}{\tilde{m}}(b-a)\right)\right)\right]^{\frac{2(p-1)}{p}} \end{aligned}$$

Lemma 23 *Given $r > 0$, there exist a number $R(r)$ such that if $z \geq R(r)$ then* $\sqrt{u^{2(p-1)}(t,\lambda,z) + (u')^{2(p}}$
r for all $t \in [a,b]$.

Proof. Let $r > 0$ be given. Consider a solution $u(t,\lambda,z)$ passing through the set $S = \{(t,\lambda,z) : t \in [a,b], \lambda > 0, z \leq r\}$. By the preceding lemma there exists $R(r)$ such that $r(t,\lambda,z) \leq$

$R(r)$ where $u(t,\lambda,z)$ passing through S. Hence if $z \geq R(r)$ then $\sqrt{u^{2(p-1)}(t,\lambda,z)+(u')^{2(p-1)}(t,\lambda,z)} > r$ for all $t \in [a,b]$. ■

Now, let $\varphi(t,\lambda,z)$ be the angular function associated to a solution $u(t,\lambda,z)$ of problem (4.18) such that $\varphi(a,\lambda,z)=\frac{\pi}{2}$. By lemma 23, we will have that $\lim_{z\to+\infty} r(t,\lambda,z) = \lim_{z\to+\infty} \sqrt{u^{2(p-1)}(t,\lambda,z)+(u')^{2(p-1)}(t,\lambda,z)} = +\infty$. Hence, lemma 17 implies that for all $m \in \mathbf{N}$, there exists $z_1>0$ such that for all $z \geq z_1$, we have $\varphi(b,\lambda,z) \leq \frac{\pi}{2} - \pi m$; which means that $\lim_{z\to+\infty} \varphi(b,\lambda,z) = -\infty$. Similarly, lemma 17 implies that $\lim_{z\to 0^+} \varphi(b,\lambda,z) = -\infty$ since $\lim_{z\to 0^+} r(t,\lambda,z) = 0^+$ by lemma 21.

The angular function φ satisfies

$$\varphi'(t) = F(t,r,\cos\varphi,\sin\varphi) \tag{4.20}$$

where

$$\begin{aligned} F(t,r,\cos\varphi,\sin\varphi) &= -\lambda g(t)\left(r^{\frac{\nu+1-p}{p-1}}(t)\left|\cos\varphi\right|^{\frac{\nu+p-1}{p-1}} + r^{\frac{\mu+1-p}{p-1}}(t)\left|\cos\varphi\right|^{\frac{\mu+p-1}{p-1}}\right) \\ &\quad -(p-1)\left|\sin\varphi(t)\right|^{\frac{p}{p-1}}\left|\cos\varphi(t)\right|^{\frac{p-2}{p-1}} \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial F(t,r,\cos\varphi,\sin\varphi)}{\partial r} &= -\lambda g(t)\, r^{\frac{\mu+2-2p}{p-1}}\left|\cos\varphi\right|^{\frac{\mu+p-1}{p-1}} \times \\ &\quad \times \left[\tfrac{\nu+1-p}{p-1} r^{\frac{\nu-\mu}{p-1}}(t)\left|\cos\varphi\right|^{\frac{\nu-\mu}{p-1}} + \tfrac{\mu+1-p}{p-1}\right] \end{aligned}$$

The last equality shows that $\frac{\partial F(t,r,\cos\varphi,\sin\varphi)}{\partial r} > 0$ for r sufficiently small and $\frac{\partial F(t,r,\cos\varphi,\sin\varphi)}{\partial r} < 0$ for r sufficiently large. Applying theorems 15.3 and 15.4 in [138], we obtain that $\varphi(b,\lambda,z)$ is strictly decreasing for z sufficiently large and $\varphi(b,\lambda,z)$ is strictly increasing for z sufficiently small. Now, it is clear from the previous study that φ is bounded from above.

Let $\hat{\varphi}(b,\lambda) = \sup_{z\in(0,+\infty)} \varphi(b,\lambda,z)$. Then, we have

(i) If $\hat{\varphi}(b,\lambda) > \frac{\pi}{2} - \pi m$, then problem (4.1) has at least two pairs of opposite solutions with exactly m zeros in (a,b).

(ii) If $\hat{\varphi}(b,\lambda) = \frac{\pi}{2} - \pi m$, then problem (4.1) has at least one pair of opposite solution with exactly m zeros in (a,b).

(iii) If $\hat{\varphi}(b,\lambda) < \frac{\pi}{2} - \pi m$, then problem (4.1) has no solution with exactly m zeros in (a,b).

Now, we will show that the equation

$$\hat{\varphi}(b,\lambda) = \frac{\pi}{2} - \pi m$$

has a unique solution λ_m for each m.

Lemma 24 *φ is decreasing with respect to λ.*

Proof. Since

$$\begin{aligned} \tilde{g}(t,\lambda) \quad := \quad & -\lambda g(t)\left(r^{\frac{\nu+1-p}{p-1}}(t)\left|\cos\varphi\right|^{\frac{\nu+p-1}{p-1}} + r^{\frac{\mu+1-p}{p-1}}(t)\left|\cos\varphi\right|^{\frac{\mu+p-1}{p-1}}\right) \\ & -(p-1)\left|\sin\varphi(t)\right|^{\frac{p}{p-1}}\left|\cos\varphi(t)\right|^{\frac{p-2}{p-1}} \end{aligned}$$

is strictly decreasing with respect to λ and $\varphi(a)$ is independent of λ, then the theorem of differential inequalities shows that $\varphi(t,z,\lambda)$ is decreasing with respect to λ for a given value $t \in (a,b]$. ∎

Let φ_1 and φ_2 be respectively the solutions of the following problems

$$\begin{cases} \varphi_1'(t) & = -\lambda M\left(r^{\frac{\nu+1-p}{p-1}}(t)\left|\cos\varphi_1\right|^{\frac{\nu+p-1}{p-1}} + r^{\frac{\mu+1-p}{p-1}}(t)\left|\cos\varphi_1\right|^{\frac{\mu+p-1}{p-1}}\right) \\ & \quad -(p-1)\left|\sin\varphi_1(t)\right|^{\frac{p}{p-1}}\left|\cos\varphi_1(t)\right|^{\frac{p-2}{p-1}} \\ \varphi_1(a) & = \frac{\pi}{2} \end{cases}$$

and

$$\begin{cases} \varphi_2'(t) & = -\lambda\tilde{m}\left(r^{\frac{\nu+1-p}{p-1}}(t)\left|\cos\varphi_2\right|^{\frac{\nu+p-1}{p-1}} + r^{\frac{\mu+1-p}{p-1}}(t)\left|\cos\varphi_2\right|^{\frac{\mu+p-1}{p-1}}\right) \\ & \quad -(p-1)\left|\sin\varphi_2(t)\right|^{\frac{p}{p-1}}\left|\cos\varphi_2(t)\right|^{\frac{p-2}{p-1}} \\ \varphi_2(a) & = \frac{\pi}{2} \end{cases}$$

where $M = \|g\|_0$ and $\tilde{m}$ is the constant defined in the proof of lemma 20.

Lemma 25 *One has $\varphi_1(t,\lambda) \leq \varphi(t,\lambda,z) \leq \varphi_2(t,\lambda)$ for all $t \in [a,b]$.*

Proof. It is a consequence of the theorem of differential inequalities.

Now consider the problem

$$\begin{cases} -\left(|u'(t)|^{p-2}u'(t)\right) = \lambda M\left(|u(t)|^{\nu-1}u(t) + |u(t)|^{\mu-1}u(t)\right) \\ u(a) = u(b) = 0 \end{cases} \tag{4.21}$$

Let

$$T_1^+(p,\lambda,E) := \int_0^{S_+(p,\lambda,E)} \left[E - \frac{p}{p-1}\lambda M\left(\frac{|u|^{\nu+1}}{\nu+1} + \frac{|u|^{\mu+1}}{\mu+1}\right)\right]^{\frac{-1}{p}} du \tag{4.22}$$

where $S_+(p,\lambda,E)$ is the first positive zero of the equation

$$E - \frac{p}{p-1}\lambda M\left(\frac{|u|^{\nu+1}}{\nu+1} + \frac{|u|^{\mu+1}}{\mu+1}\right) = 0$$

We note that $T_1^+(p,\lambda,E)$ is the half-time between two consecutive zeros of the solution u of (4.21).

Lemma 26 *Consider the equation in $s \in \mathbf{R}$*

$$E - \frac{p}{p-1}\lambda M\left(\frac{|s|^{\nu+1}}{\nu+1} + \frac{|s|^{\mu+1}}{\mu+1}\right) = 0 \tag{4.23}$$

where $p > 1$, $\lambda > 0$ and $E > 0$ are real parameters. Then for any $E > 0$, equation (4.23) admits a unique positive zero $S_+ = S_+(p,\lambda,E)$. Moreover

(i) *The function $E \longmapsto S_+(p,\lambda,E)$ is C^1 in $(0,+\infty)$ and*

$$\frac{\partial S_+(p,\lambda,E)}{\partial E} = \frac{p-1}{p\lambda M\left((S_+(p,\lambda,E))^{\nu} + (S_+(p,\lambda,E))^{\mu}\right)} > 0, \ \forall p > 1,\ \forall \lambda > 0 \text{ and } \forall E > 0.$$

(ii) $\lim_{E\to 0^+} S_+(p,\lambda,E) = 0^+.$

(iii) $\lim_{E\to+\infty} S_+(p,\lambda,E) = +\infty.$

Proof. For any fixed $p > 1$, $\lambda > 0$ and $E > 0$, consider the function

$$s \mapsto G(p, \lambda, E, s) := E - \frac{p}{p-1} \lambda M \left(\frac{|s|^{\nu+1}}{\nu+1} + \frac{|s|^{\mu+1}}{\mu+1} \right)$$

defined in $\mathbf{R}$ which is strictly decreasing with $G(p, \lambda, E, 0) = E$ and $\lim_{s \to +\infty} G(p, \lambda, E, s) = -\infty$. So, it is clear that equation (4.23) admits for any $E > 0$, a unique positive zero, $S_+ = S_+(p, \lambda, E)$.

Now, for any $p > 1$ and $\lambda > 0$, consider the real valued function

$$(E, s) \mapsto G_+(E, s) := E - \frac{p}{p-1} \lambda M \left(\frac{|s|^{\nu+1}}{\nu+1} + \frac{|s|^{\mu+1}}{\mu+1} \right)$$

defined in $\Omega_+ = (0, +\infty)^2$. We have $G_+ \in C^1(\Omega_+)$ and

$$\frac{\partial G_+(E, s)}{\partial s} = -\frac{p}{p-1} \lambda M \left(|s|^{\nu} + |s|^{\mu} \right) \text{ in } \Omega_+$$

Hence

$$\frac{\partial G_+(E, s)}{\partial s} < 0 \text{ in } \Omega_+$$

and one may observe that $S_+(p, \lambda, E)$ belongs to the open interval $(0, +\infty)$ and satisfies from its definition

$$G_+(E, S_+(p, \lambda, E)) = 0 \tag{4.24}$$

So, one can use of the implicit function theorem to show that the function $E \mapsto S_+(p, \lambda, E)$ is $C^1((0, +\infty), \mathbf{R}_+)$ and to obtain the expression of $\dfrac{\partial S_+(p, \lambda, E)}{\partial E}$ given in **(i)**. Hence for any $p > 1$ and $\lambda > 0$, the function defined in $(0, +\infty)$ by $E \mapsto S_+(p, \lambda, E)$ is strictly increasing and bounded from below by 0 and from above by $+\infty$. Then the limit $\lim_{E \to 0^+} S_+(p, \lambda, E) = l_0$ exists and the limit $\lim_{E \to +\infty} S_+(p, \lambda, E) = l_{+\infty}$ exists and belong to $(0, +\infty]$. Moreover

$$0 \leq l_0 \leq l_{+\infty} \leq +\infty$$

We observe that, for any $p > 1$ and $\lambda > 0$, the function

$$(E, s) \mapsto G_+(E, S)$$

is continuous in Ω_+ and the function $E \mapsto S_+(p, \lambda, E)$ is continuous in $(0, +\infty)$ and satisfies (4.24). So, by passing to the limit in (4.24) as E tends to 0^+, one gets

$$0 = \lim_{E \to 0^+} G_+(E, S_+(p, \lambda, E))$$

Hence, l_0 is a zero, belonging to $[0, +\infty)$, to the equation in s :

$$G_+(0, s) = 0$$

By resolving this equation in $[0, +\infty)$, we obtain $l_0 = 0$. The point **(ii)** is proved.

Assume that $l_{+\infty}$ is finite, then by passing to the limit in (4.24) as E tends to $+\infty$, we will obtain that

$$\frac{p}{p-1} \lambda M \left(\frac{l_{+\infty}^{\nu+1}}{\nu+1} + \frac{l_{+\infty}^{\mu+1}}{\mu+1} \right) = +\infty$$

which is impossible. Thus, we deduce that $l_{+\infty} = +\infty$. ■

Lemma 27 *We have for all $\lambda > 0$ and $0 < \mu < p - 1 < \nu$*

(i) $\lim_{E \to 0^+} T_1^+(p, \lambda, E) = 0^+$

(ii) $\lim_{E \to +\infty} T_1^+(p, \lambda, E) = 0^+$

Proof.

(i) Letting $u = S_+(p, \lambda, E)\, t$ in (4.22), we will obtain

$$T_1^+(p, \lambda, E) = \int_0^1 \left[\tfrac{p}{p-1} \lambda M \left[\tfrac{(S_+(p,\lambda,E))^{\nu+1-p}}{\nu+1} \left(1 - t^{\nu+1}\right) + \right. \right.$$
$$\left. + \tfrac{(S_+(p,\lambda,E))^{\mu+1-p}}{\mu+1} \left(1 - t^{\mu+1}\right) \right]^{\frac{-1}{p}} dt$$

Then

$$0 < T_1^+(p,\lambda,E) \leq \frac{(S_+(p,\lambda,E))^{\frac{\mu+1-p}{p}}}{\left[\frac{pM\lambda}{(p-1)(\mu+1)}\right]^{\frac{1}{p}}} \int_0^1 \frac{dt}{[1-t^{\mu+1}]^{\frac{1}{p}}}$$

So, by passing to the limit as E tends to 0^+, one gets

$$0 \leq \lim_{E\to 0^+} T_1^+(p,\lambda,E) \leq \lim_{E\to 0^+} \frac{(S_+(p,\lambda,E))^{\frac{\mu+1-p}{p}}}{\left[\frac{pM\lambda}{(p-1)(\mu+1)}\right]^{\frac{1}{p}}} \int_0^1 \frac{dt}{[1-t^{\mu+1}]^{\frac{1}{p}}} = 0.$$

(ii) We have

$$0 < T_1^+(p,\lambda,E) \leq \frac{(S_+(p,\lambda,E))^{\frac{\nu+1-p}{p}}}{\left[\frac{pM}{(p-1)(\mu+1)}\right]^{\frac{1}{p}}} \int_0^1 \frac{dt}{[1-t^{\nu+1}]^{\frac{1}{p}}}$$

So, by passing to the limit as E tends to $+\infty$, one gets

$$0 \leq \lim_{E\to +\infty} T_1^+(p,\lambda,E) \leq \lim_{E\to +\infty} \frac{(S_+(p,\lambda,E))^{\frac{\nu+1-p}{p}}}{\left[\frac{pM}{(p-1)(\mu+1)}\right]^{\frac{1}{p}}} \int_0^1 \frac{dt}{[1-t^{\nu+1}]^{\frac{1}{p}}} = 0.$$

Lemma 28 *We have for all $E > 0$ and $0 < \mu < p-1 < \nu$*

(i) $\lim_{\lambda\to 0^+} S_+(p,\lambda,E) = +\infty$

(ii) $\lim_{\lambda\to +\infty} S_+(p,\lambda,E) = 0^+$

Proof. **(i)** By lemma 26, we have

$$E - \frac{p}{p-1}\lambda M\left(\frac{(S_+(p,\lambda,E))^{\nu+1}}{\nu+1} + \frac{(S_+(p,\lambda,E))^{\mu+1}}{\mu+1}\right) = 0$$

or

$$\frac{(p-1)E}{p\lambda M} = \frac{(S_+(p,\lambda,E))^{\nu+1}}{\nu+1} + \frac{(S_+(p,\lambda,E))^{\mu+1}}{\mu+1}$$

Letting $\lambda \to 0^+$, we get **(i)**.

(ii) In a similar manner as in **(i)**, we prove **(ii)**.

Lemma 29 *We have for all* $E > 0$ *and* $0 < \mu < p - 1 < \nu$

(i) $\lim_{\lambda \to 0^+} T_1^+ (p, \lambda, E) = +\infty$

(ii) $\lim_{\lambda \to +\infty} T_1^+ (p, \lambda, E) = 0^+$

Proof. **(i)** We observe that

$$\begin{aligned} T_1^+ (p, \lambda, E) &= \int_0^{S_+(p,\lambda,E)} \left[E - \tfrac{p}{p-1} \lambda M \left(\frac{|u|^{\nu+1}}{\nu+1} + \frac{|u|^{\mu+1}}{\mu+1} \right) \right]^{\frac{-1}{p}} du \\ &\geq \int_0^{S_+(p,\lambda,E)} \frac{du}{E^{\frac{1}{p}}} \\ &= \frac{S_+(p,\lambda,E)}{E^{\frac{1}{p}}} \end{aligned}$$

One has

$$\lim_{\lambda \to 0^+} S_+ (p, \lambda, E) = +\infty$$

So,

$$\lim_{\lambda \to 0^+} T_1^+ (p, \lambda, E) \geq \lim_{\lambda \to 0^+} \frac{S_+ (p, \lambda, E)}{E^{\frac{1}{p}}} = +\infty$$

Hence

$$\lim_{\lambda \to 0^+} T_1^+ (p, \lambda, E) = +\infty$$

(ii) We have

$$\begin{aligned} T_1^+ (p, \lambda, E) &= \int_0^{S_+(p,\lambda,E)} \left[E - \tfrac{p}{p-1} \lambda M \left(\frac{|u|^{\nu+1}}{\nu+1} + \frac{|u|^{\mu+1}}{\mu+1} \right) \right]^{\frac{-1}{p}} du \\ &= \left(\frac{p-1}{\lambda p M} \right) \int_0^1 \left[\tfrac{(S_+(p,\lambda,E))^{\nu+1-p}}{\nu+1} \left(1 - t^{\nu+1}\right) + \right. \end{aligned}$$

$$+\frac{(S_+(p,\lambda,E))^{\mu+1-p}}{\mu+1}\left(1-t^{\mu+1}\right)\Big]^{\frac{-1}{p}}dt$$

One has

$$\lim_{\lambda\to+\infty} S_+(p,\lambda,E)=0^+$$

Hence

$$\lim_{\lambda\to+\infty} T_1^+(p,\lambda,E)=0^+$$

The previous lemma, shows that the angular function $\varphi_2(b,\lambda)$ has respectively the limits $\frac{\pi}{2}$ when λ tends to zero and $-\infty$ when λ tends to $+\infty$. Similarly, we obtain the same limits for φ_1. Consequently, we obtain the existence of a decreasing sequence $(\lambda_n)_{n\geq 1}$ such that:

(i) If $\lambda>\lambda_n$, then problem (4.1) has no solution with n zeros in (a,b).

(ii) If $\lambda=\lambda_n$, then problem (4.1) has at least one pair of opposite solutions with n zeros in (a,b).

(iii) If $\lambda<\lambda_n$, then problem (4.1) has at least two pairs of opposite solutions with n zeros in (a,b).

4.5 Remarks

(i) Since $\varphi(b,\lambda,z)$ is strictly monotone for z small enough and z large, we can deduce that the exact number of solutions is exactly two when λ is small enough.

(ii) One of the motivations for studying problems of the type (4.1) are the existence and multiplicity of radially symmetric solutions to the boundary value problems of the form

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right)=\lambda\rho(|x|)\left(|u|^{\nu-1}u+|u|^{\mu-1}u\right) \text{ in } B(R_1,R_2) \\ u=0 \text{ on } \partial B(R_1,R_2) \end{cases} \tag{4.25}$$

where $B(R_1,R_2)=\left\{x\in\mathbf{R}^N,\ R_1<|x|<R_2\right\}$ and $\rho:(R_1,R_2)\to\mathbf{R}_+^*$ is of class C^1.

The radially symmetric solutions of (4.25) satisfies

$$\begin{cases} -\left(\left|u'(r)\right|^{p-2}u'(r)\right)' - \frac{N-1}{r}\left|u'(r)\right|^{p-2}u'(r) = \lambda\rho(r)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right) \text{ in } (R_1, R_2) \\ u(R_1) = u(R_2) = 0 \end{cases} \tag{4.26}$$

If $N = p$, we make the change of variable $r = e^t$, then (4.26) becomes

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \lambda e^{-pt}\rho\left(e^t\right)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right) \text{ in } (LogR_1, LogR_2) \\ u(LogR_1) = u(LogR_2) = 0 \end{cases}$$

If $N \neq p$, let $t = \left(\frac{|N-p|}{(p-1)r}\right)^{\frac{N-p}{p-1}}$, then (4.26) becomes

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \lambda t^{\frac{p(1-N)}{p-1}}\rho\left(\frac{|N-p|}{p-1}t^{-\frac{p-1}{N-p}}\right)\left(|u|^{\nu-1}u + |u|^{\mu-1}u\right) \text{ in } (t_1, t_2) \\ u(t_1) = u(t_2) = 0 \end{cases}$$

where $t_1 = \left(\frac{|N-p|}{(p-1)R_1}\right)^{\frac{N-p}{p-1}}$ and $t_2 = \left(\frac{|N-p|}{(p-1)R_2}\right)^{\frac{N-p}{p-1}}$

Applying theorem 19, we have the following corollary

Corollary 30 *There exists a decreasing sequence (λ_n) such that*

i) *If $\lambda > \lambda_n$, then problem (4.25) has no radially symmetric solutions with n zeros in $B(R_1, R_2)$.*

ii) *If $\lambda = \lambda_n$, then problem (4.25) has at least one pair of opposite radially symmetric solutions with n zeros in $B(R_1, R_2)$.*

iii) *If $\lambda < \lambda_n$, then problem (4.25) has at least two pairs of opposite radially symmetric solutions with n zeros in $B(R_1, R_2)$.*

Chapter 5

On the number of positive radially symmetric solutions for a quasilinear Dirichlet problem on a ball

Abstract: Using a shooting method,we study the existence and multiplicity of positive radially symmetric solutions for the quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = \lambda \left(u^{\alpha-1} + u^{q-1}\right) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where Ω denotes the unit ball in $\mathbb{R}^N$, Δ_p is the $p-$Laplace operator, $\lambda > 0$, $\alpha = p^* = \dfrac{Np}{N-p}$, $N > p$ and $1 < q < p$

5.1 Introduction

The purpose of this chapter is to study the existence and multiplicity of positive radially symmetric solutions for the quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = \lambda\left(u^{\alpha-1}+u^{q-1}\right) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

where Ω denotes the unit ball in $\mathbb{R}^N$, Δ_p is the $p-$Laplace operator, $\lambda > 0$, $\alpha = p^* = \dfrac{Np}{N-p}$, $N > p$ and $1 < q < p$. Our study is motivated by some recent works on elliptic problems with concave-convex nonlinearities.

In [25], the authors investigate the following problem:

$$\begin{cases} -\Delta u = u^{\gamma}+\lambda u^{\beta} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.3}$$

with $0 < \gamma < 1 < \alpha \leq 2^*$, $2^* = \frac{N+2}{N-2}$ for $N > 2$ and $2^* = +\infty$ for $N = 2$. The authors prove the existence of a constant $\Lambda \in \mathbb{R}$ such that for all $\lambda \in (0, \Lambda)$ the problem (5.3) admits at least two solutions. One solution denoted u_λ, is obtained using lower and upper solution method, when the concave term u^β is essential and the other solution denoted v_λ, is obtained using variational technics, when the essential term is the convex term u^γ. In [26], the authors showed the existence of an additional pair of solutions which can change sign for all $0 < \lambda < \lambda_*$, with λ_* possibly smaller then Λ. Their method relies on a critical point theory. In fact, these solutions arise as critical points of a functional I_λ constrained on a suitable manifold M_λ. In [27], the problem:

$$\begin{cases} -\Delta_p u = \lambda h(u) + g(r,u) & \text{in } |x| = r < 1 \\ u = 0 & \text{in } |x| = r = 1 \end{cases} \tag{5.4}$$

is studied where λ is a positive real parameter, $h(u) = |u|^{q-2}u$, $1 < q < p$ near $u = 0$ and g is of higher order with respect to h at $u = 0$.The authors showed the existence of infinitely many continuum of radial solutions branching at $\lambda = 0$, from the trivial solution, each continuum being characterized by nodal properties. As a consequence (5.4) possess infinitely many radial solutions for λ strictly positive and small. The main ingredient of the proof is an

a priori bound. This bound is the counterpart for radial solutions that change sign of the classical result by Gidas and Spurck [95].

In the present paper we show, using a shooting method (see [117] for the case $p = 2$) the existence of a constant $\lambda_* > 0$ such that:

i) If $\lambda < \lambda_*$, then problem (5.2) admits at least two positive radially symmetric solution.

ii) If $\lambda = \lambda_*$, then problem (5.2) admits at least one positive radially symmetric solution.

iii) If $\lambda > \lambda_*$, then problem (5.2) has no positive radially symmetric solution.

The novelty here is that we do not assume (as is the case in [27])the condition

$$\frac{2N}{N+2} < p < 3 \text{ or } p \geq 3 \text{ and } p > q > \alpha - \frac{2}{p-1}$$

is satisfied.

The chapter is organized as follows. In section 2, we state our main result. In section 3, we present some preliminary lemmas. In section 4, we prove our main result.

5.2 Statement of the result

Theorem 31 *There exists a continuous function $F : (0, +\infty) \to (0, +\infty)$ such that u is a positive radially symmetric solution of (5.2) if and only if*

$\lambda = F(u(0))$. *Moreover* $\lim_{d \to 0^+} F(d) = \lim_{d \to +\infty} F(d) = 0$ *and there exist* $\lambda_* > 0$ *such that:*

i) If $\lambda < \lambda_*$ then problem (5.2) admits at least two positive radially symmetric solutions,

ii) If $\lambda = \lambda_*$ then problem (5.2) admits at least one positive radially symmetric solution,

iii) If $\lambda > \lambda_*$ then problem (5.2) has no positive radially symmetric solution.

5.3 Preliminaries

We first note that radial solutions to (5.2) correspond to solutions of the problem

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \lambda\left(u^{\alpha-1} + u^{q-1}\right) = 0, \text{ in } 0 < r < 1 \tag{5.5}$$

$$u(r) > 0, \text{ in } 0 < r < 1 \tag{5.6}$$

$$u'(0) = 0 \tag{5.7}$$

$$u(1) = 0 \tag{5.8}$$

For $d > 0$, we define $u(., \lambda, d) := u(.)$ as the solution to the initial value problem

$$\begin{cases} \left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \lambda\left(u^{\alpha-1} + u^{q-1}\right) = 0, \ r \in]0,1] \\ u(0) = d, \ u'(0) = 0 \end{cases} \tag{5.9}$$

It can be shown using the contraction mapping principle that for every $(\lambda, d) \in \mathbf{R}_*^+ \times \mathbf{R}_*^+$ problem (5.9) has a unique positive solution $u(r) := u(r, \lambda, d)$ on the interval $[0, 1[$.

Set $E(r) := E(r, \lambda, d) = \frac{p-1}{p}|u'(r)|^p + \lambda\left(\frac{u^\alpha(r)}{\alpha} + \frac{u^q(r)}{q}\right)$

Let $G(r) := rE(r) + \frac{N-p}{p}|u'(r)|^{p-2}u'(r)u(r)$

Lemma 32 *Let u be a positive solution of (5.9). Then*

$$r^{N-1}G(r) - r_1^{N-1}G(r_1) = \lambda\frac{p+N(p-1)}{p}\int_{r_1}^{r} s^{N-1}\left(\frac{\alpha+1}{\alpha}u^\alpha(s) + \frac{q+1}{q}u^q(s)\right)ds$$

for all $r_1, r \in [0, 1]$.

Proof. Multiplying the equation in (5.9) by $r^N u'(r)$ and integrating the resulting equation over $[r_1, r]$, we obtain

$$\int_{r_1}^{r}\left[\begin{array}{c} s^N\left(|u'(s)|^{p-2}u'(s)\right)'u'(s) + (N-1)s^{N-1}|u'(s)|^p \\ +\lambda s^N\left(u^{\alpha-1}(s) + u^{q-1}(s)\right)u'(s) \end{array}\right]ds = 0 \tag{5.10}$$

We have

$$\int_{r_1}^{r} s^N \left(\left|u'(s)\right|^{p-2} u'(s) \right)' u'(s)\, ds = \tfrac{p-1}{p} r^N \left|u'(r)\right|^p - \tfrac{p-1}{p} r_1^N \left|u'(r_1)\right|^p - \tfrac{N(p-1)}{p} \int_{r_1}^{r} s^{N-1} \left|u'(s)\right|^p ds \tag{5.11}$$

On the other hand, we have:

$$\begin{aligned} & \int_{r_1}^{r} s^N \left(\alpha u^{\alpha-1}(s) + q u^{q-1}(s) \right) u'(s)\, ds \\ = \; & r^N \left(u^\alpha(r) + u^q(r) \right) - r_1^N \left(u^\alpha(r_1) + u^q(r_1) \right) \\ & -N \int_{r_1}^{r} s^{N-1} \left(u^\alpha(s) + u^q(s) \right) ds \end{aligned} \tag{5.12}$$

Now from (5.10), (5.11) and (5.12), we obtain:

$$\begin{aligned} & \int_{r_1}^{r} \left[\begin{array}{c} s^N \left(|u'(s)|^{p-2} u'(s) \right)' u'(s) + (N-1) s^{N-1} |u'(s)|^p \\ +\lambda s^N \left(u^{\alpha-1}(s) + u^{q-1}(s) \right) u'(s) \end{array} \right] ds \\ = \; & \tfrac{p-1}{p} r^N |u'(r)|^p - \tfrac{p-1}{p} r_1^N |u'(r_1)|^p - \tfrac{N(p-1)}{p} \int_{r_1}^{r} s^{N-1} |u'(s)|^p ds \\ & + (N-1) \int_{r_1}^{r} s^{N-1} |u'(s)|^p ds + \lambda r^N \left(\frac{u^\alpha(r)}{\alpha} + \frac{u^q(r)}{q} \right) \\ & -\lambda r_1^N \left(\frac{u^\alpha(r_1)}{\alpha} + \frac{u^q(r_1)}{q} \right) - \lambda N \int_{r_1}^{r} s^{N-1} \left(\frac{u^\alpha(s)}{\alpha} + \frac{u^q(s)}{q} \right) ds \end{aligned}$$

This yields

$$\begin{aligned} & r^N E(r) - r_1^N E(r_1) \\ = \; & \tfrac{p-N}{p} \int_{r_1}^{r} s^{N-1} |u'(s)|^p ds + \lambda N \int_{r_1}^{r} s^{N-1} \left(\frac{u^\alpha(s)}{\alpha} + \frac{u^q(s)}{q} \right) ds \end{aligned} \tag{5.13}$$

Multiplying the equation (5.5) by $r^{N-1} u(r)$, we obtain:

$$r^{N-1}\left(\left|u'(r)\right|^{p-2}u'(r)\right)'u(r)+(N-1)\,r^{N-2}\left|u'(r)\right|^{p-2}u'(r)\,u(r)+\lambda r^{N-1}\left(u^{\alpha}(r)+u^{q}(r)\right)=0 \tag{5.14}$$

We have:

$$\begin{aligned}
&\int_{r_1}^{r} s^{N-1}\left(\left|u'(s)\right|^{p-2}u'(s)\right)'u(s)\,ds \\
&= r^{N-1}\left|u'(r)\right|^{p-2}u'(r)\,u(r)-r_1^{N-1}\left|u'(r_1)\right|^{p-2}u'(r_1)\,u(r_1) \\
&-(N-1)\int_{r_1}^{r} s^{N-2}\left|u'(s)\right|^{p-2}u'(s)\,u(s)\,ds-\int_{r_1}^{r} s^{N-1}\left|u'(s)\right|^{p}ds
\end{aligned} \tag{5.15}$$

It follows from (5.14) and (5.15) that

$$\begin{aligned}
&\int_{r_1}^{r} s^{N-1}\left|u'(s)\right|^{p}ds \\
&= r^{N-1}\left|u'(r)\right|^{p-2}u'(r)\,u(r)-r_1^{N-1}\left|u'(r_1)\right|^{p-2}u'(r_1)\,u(r_1) \\
&+\lambda\int_{r_1}^{r} s^{N-1}\left(u^{\alpha}(s)+u^{q}(s)\right)ds
\end{aligned} \tag{5.16}$$

Substituting (5.16) in (5.13), we get:

$$\begin{aligned}
&r^{N}E(r)-r_1^{N}E(r_1) \\
&=\frac{p-N}{p}r^{N-1}\left|u'(r)\right|^{p-2}u'(r)\,u(r)-\frac{p-N}{p}r_1^{N-1}\left|u'(r_1)\right|^{p-2}u'(r_1)\,u(r_1) \\
&+\lambda\frac{p+N(p-1)}{p}\int_{r_1}^{r} s^{N-1}\left(\frac{\alpha+1}{\alpha}u^{\alpha}(s)+\frac{q+1}{q}u^{q}(s)\right)ds
\end{aligned}$$

Therefore

$$\begin{aligned}
&r^{N-1}G(r)-r_1^{N-1}G(r_1) \\
&=\lambda\frac{p+N(p-1)}{p}\int_{r_1}^{r} s^{N-1}\left(\frac{\alpha+1}{\alpha}u^{\alpha}(s)+\frac{q+1}{q}u^{q}(s)\right)ds
\end{aligned}$$

Lemma 33 *Let u be a positive solution of (5.9). Then* $\frac{\partial u}{\partial \lambda}(1,\lambda,d) < 0$.

Proof. For any $s > 0$, we define

$$v(r) := u(rs,\lambda,d)$$

Then v satisfies

$$\begin{cases} \left(|v'(r)|^{p-2} v'(r)\right)' + \frac{N-1}{r} |v'(r)|^{p-2} v'(r) + \lambda s^p \left(v^{\alpha-1}(r) + v^{q-1}(r)\right) = 0 \\ v(0) = d, \ v'(0) = 0 \end{cases}$$

By the uniqueness of the solution to the initial value problem (5.9), this implies that

$$u(rs,\lambda,d) = u(r,\lambda s^p,d)$$

Differentiating this equality with respect to s, we obtain:

$$ru'(rs,\lambda,d) = p\lambda s^{p-1} \frac{\partial u}{\partial \lambda}(r,\lambda s^p,d)$$

Taking $s = 1$, we get:

$$ru'(r,\lambda,d) = p\lambda \frac{\partial u}{\partial \lambda}(r,\lambda,d)$$

Then $\frac{\partial u}{\partial \lambda}(r,\lambda,d) < 0$, for all $r \in]0,1]$.
Now, we define the function h by

$$h(r) = -\frac{r^{p-1} |u'(r)|^{p-2} u'(r)}{|u(r)|^{p-2} u(r)}$$

Claim: h is a strictly increasing function.

Proof. Simple computation and lemma (32) give:

$$
\begin{aligned}
h'(r) &= \frac{\begin{array}{c} pr^{p-2}\left|u(r)\right|^{p-2}\left[\lambda r\left(\frac{u^{\alpha}(r)}{\alpha}+\frac{u^{q}(r)}{q}\right)+\lambda\frac{N(p-1)+p}{p}r^{1-N}\int_0^r s^N\left(\begin{array}{c}\frac{\alpha+1}{\alpha}u^{\alpha}(s)\\ +\frac{q+1}{q}u^{q}(s)\end{array}\right)ds\right] \\ -\lambda r^{p-1}\left(u^{\alpha+p-2}(r)+u^{q+p-2}(r)\right)\end{array}}{|u(r)|^{2(p-1)}} \\
&\geq \frac{\lambda r^{p-1}\left[\left(\frac{p}{\alpha}+\left[(N(p-1)+p)\left(\frac{\alpha+1}{\alpha}\right)-1\right]\right)u^{\alpha+p-2}(r)+\left(\frac{p}{q}+\left[(N(p-1)+p)\left(\frac{q+1}{q}\right)-1\right]\right)u^{q+p-2}(r)\right]}{|u(r)|^{2(p-1)}} \\
&> \lambda r^{p-1}\frac{p-\alpha+(N(p-1)+p)(\alpha+1)}{\alpha}\left(u(r)\right)^{\alpha-p}>0
\end{aligned}
$$

which implies that h is a strictly increasing function.

Lemma 34 *Let u be a positive solution of (5.9). Then there exist a constant $M_0>0$ and a unique $\tilde{r}\in(0,1)$ such that $u(\tilde{r})=M_0\tilde{r}^{-\frac{p}{\alpha-p}}$. Moreover if $0<M<M_0$, then there exist exactly two positive numbers r_1 and $r_2\in(0,1)$ such that $u(r_i)=Mr_i^{-\frac{p}{\alpha-p}}, i=1,2$.*

Proof. Let $\overline{r}\in[0,1]$ be such that $M_0=\max\{u(\overline{r})\overline{r}^{\gamma},\ \overline{r}\in[0,1]\}$ with $\gamma=\dfrac{p}{\alpha-p}$. Then the graph of u is tangent to the graph of $u(\overline{r})=M_0\overline{r}^{-\gamma}$ in $\overline{r}$, and $u(r)\leq M_0r^{-\gamma}$ for all $M<M_0$. Now if $M<M_0$, the graph of u intersects the graph of $M\overline{r}^{-\gamma}$ exactly in two points.

Suppose $0<r_1<r_2<r_3<1$ are the three first points such that $u(r_i)=Mr_i^{-\gamma}$, $i=1,2,3$, then

$u(r_1)<u(r_2)<u(r_3)$.

Define Z by

$$Z(r)=Mr^{-\gamma}$$

Then, we have

$Z(r_2)=u(r_2)$, $Z'(r_2)>u'(r_2)$, $Z(r_3)=u(r_3)$ and $Z'(r_3)<u'(r_3)$.

We have

$$
\begin{aligned}
h(r_3) &= -\frac{r_3^{p-1}\left|u'(r_3)\right|^{p-2}u'(r_3)}{\left|u(r_3)\right|^{p-2}u(r_3)} \\
&< -\frac{r_3^{p-1}\left|Z'(r_3)\right|^{p-2}Z'(r_3)}{\left|Z(r_3)\right|^{p-2}Z(r_3)} \\
&= \gamma^{p-1}
\end{aligned}
$$

and

$$\begin{aligned} h(r_2) &= -\frac{r_2^{p-1}|u'(r_2)|^{p-2}u'(r_2)}{|u(r_2)|^{p-2}u(r_2)} \\ &> -\frac{r_2^{p-1}|Z'(r_2)|^{p-2}Z'(r_2)}{|Z(r_2)|^{p-2}Z(r_2)} \\ &= \gamma^{p-1} \end{aligned}$$

Therefore $h(r_3) < h(r_2)$, with $r_2 < r_3$ which is a contradiction.

Suppose now, $u(\overline{r})\overline{r}^{\gamma} = u(\widetilde{r})\widetilde{r}^{\gamma} = M_0$, we obtain $h(\overline{r}) = h(\widetilde{r}) = \gamma^{p-1}$, which is contradiction. Then $\widetilde{r}$ is unique.

It is not difficult to prove that, $\widetilde{r}$ is the unique element in $(0,1)$ such that $h(\widetilde{r}) = \left(\dfrac{N-p}{p}\right)^{p-1}$.

Estimation of M_0

We have

$$h'(r) \geq \lambda c r^{p-1}(u(r))^{\alpha-p}$$

with $c = \dfrac{(p-\alpha)+(N(p-1)+p)(\alpha+1)}{\alpha}$

Integrating this inequality over $[0,\widetilde{r}]$, we obtain:

$$\begin{aligned} h(\widetilde{r}) - h(0) &\geq \lambda c \int_0^{\widetilde{r}} s^{p-1}(u(s))^{\alpha-p}\,ds \\ &\geq \lambda c (u(\widetilde{r}))^{\alpha-p}\widetilde{r}^{p} \\ &= \frac{\lambda c}{p} M_0^{\alpha-p} \end{aligned}$$

Since $h(\widetilde{r}) = \left(\dfrac{N-p}{p}\right)^{p-1}$, we have:

$$M_0 \leq \left[\frac{p}{\lambda c}\left(\frac{N-p}{p}\right)^{p-1}\right]^{\frac{1}{p-1}}$$

Lemma 35 *We have* $\widetilde{r} = O\left(d^{-\frac{\alpha-p}{p}}\right)$

Proof. Let $r_0 = d^{-\frac{\alpha - p}{p}}$ and put $K_0 = r_0^p (u(r_0))^{\alpha - p}$

We have:

$$\begin{aligned} r^{N-1} |u'(r)|^{p-2} u'(r) &= -\lambda \int_0^r s^{N-1} \left(u^{\alpha-1}(s) + u^{q-1}(s)\right) ds \\ &\geq -\lambda \frac{\left(d^{\alpha-1} + d^{q-1}\right)}{N} r^N \end{aligned}$$

Then

$u'(r) \geq -\frac{\lambda^{\frac{1}{p-1}} d^{\frac{q-1}{p-1}} \left(1 + d^{\alpha-q}\right)^{\frac{1}{p-1}}}{N} r^{\frac{1}{p-1}}$

Integrating this inequality over $[0, r_0]$, we get:

$$\begin{aligned} u(r_0) - u(0) &\geq -\left(\frac{p-1}{Np}\right) \lambda^{\frac{1}{p-1}} d^{\frac{q-1}{p-1}} \left(1 + d^{\alpha-q}\right)^{\frac{1}{p-1}} r_0^{\frac{p}{p-1}} \\ &= -\left(\frac{p-1}{Np}\right) \lambda^{\frac{1}{p-1}} d^{\frac{q-1}{p-1}} \left(1 + d^{\alpha-q}\right)^{\frac{1}{p-1}} d^{-\frac{\alpha-p}{p-1}} \end{aligned}$$

which leads to

$$\begin{aligned} u(r_0) &\geq d - \left(\frac{p-1}{Np}\right) \lambda^{\frac{1}{p-1}} d^{\frac{q-1}{p-1}} \left(1 + d^{\alpha-q}\right)^{\frac{1}{p-1}} r_0^{\frac{p}{p-1}} \\ &\geq d - \left(\frac{p-1}{Np}\right) \lambda^{\frac{1}{p-1}} 2^{\frac{1}{p-1}} d^{-\frac{\alpha-q}{p-1}} \text{ if } d \geq 1 \\ &= \left[1 - \frac{(2\lambda)^{\frac{1}{p-1}} (p-1)}{Np}\right] d \end{aligned}$$

Since $u(r_0) = K_0^{\frac{1}{\alpha - p}} r_0^{-\frac{p}{\alpha - p}}$, we obtain

$$\begin{aligned} K_0^{\frac{1}{\alpha-p}} &\geq r_0^{\frac{p}{\alpha-p}} \left[1 - \frac{(2\lambda)^{\frac{1}{p-1}}(p-1)}{Np}\right] d \\ &= \left(1 - \frac{(2\lambda)^{\frac{1}{p-1}}(p-1)}{Np}\right) \\ &\geq \left(\frac{Np - 2^{\frac{1}{p-1}}(p-1)}{Np}\right) \text{ if } \lambda \leq 1. \end{aligned}$$

On the other hand:

$$h'(r) \geq \lambda c r^{p-1} (u(r))^{\alpha-p}$$

Integrating this inequality over $[0, \widetilde{r}]$, we obtain:

$$\begin{aligned}\int_{r_0}^{\widetilde{r}} h'(r)\,dr &\geq \lambda c \int_{r_0}^{\widetilde{r}} r^{p-1}(u(r))^{\alpha-p}\,dr \\ &\geq \lambda c \int_{r_0}^{\widetilde{r}} r^{p-1} K_0 r^{-p} dr \\ &= \lambda c K_0 \log \frac{\widetilde{r}}{r_0}\end{aligned}$$

Since $h(\widetilde{r}) = \left(\frac{N-p}{p}\right)^{p-1}$, we obtain:

$$\frac{\widetilde{r}}{r_0} \leq \mu, \text{ where } \mu \text{ is a constant}$$

Then $\widetilde{r} = O\left(d^{-\frac{\alpha-p}{p}}\right)$.

Lemma 36 *If u is a positive solution of (5.9), then we have*
$\lim\limits_{d\to+\infty} \int_0^1 r^{N-1} u(r,\lambda,d)\,dr = 0.$

Proof. Let $\varepsilon > 0$ be sufficiently small such that for $r > \widetilde{r} + \delta$ where $\delta > 0$, we have:

$$\left(\frac{p}{N-p}\right)^{p-1} h(r) \geq 1 + \varepsilon$$

That is

$$-\left(\frac{p}{N-p}\right)^{p-1} \frac{r^{p-1}|u'(r)|^{p-2}u'(r)}{|u(r)|^{p-2}u(r)} \geq 1 + \varepsilon$$

Then, we have

$$-\frac{u'(r)}{u(r)} \geq (1+\varepsilon)^{\frac{1}{p-1}} \left(\frac{N-p}{p}\right) \frac{1}{r}$$

Integrating this inequality over $[\widetilde{r}, r]$, we obtain:

$$\log\frac{u(\widetilde{r})}{u(r)} \geq (1+\varepsilon)^{\frac{1}{p-1}}\frac{N-p}{p}\log\frac{r}{\widetilde{r}}$$

Hence

$$\begin{aligned} u(r) &\leq u(\widetilde{r})\,\widetilde{r}^{(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} r^{-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} \\ &= c_1\widetilde{r}^{-\frac{N-p}{p}}\widetilde{r}^{(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} r^{-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} \\ &= c_1\widetilde{r}^{\left(\frac{N-p}{p}\right)\left((1+\varepsilon)^{\frac{1}{p-1}}-1\right)} r^{-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} \end{aligned}$$

This implies that

$$\begin{aligned} \int_{\widetilde{r}}^{1} r^{N-1}u(r)\,dr &\leq c_1\widetilde{r}^{\frac{N-p}{p}\left((1+\varepsilon)^{\frac{1}{p-1}}-1\right)}\left[\frac{1-\widetilde{r}^{N-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)}}{N-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)}\right] \\ &= c_1\left[\frac{\widetilde{r}^{\frac{N-p}{p}\left((1+\varepsilon)^{\frac{1}{p-1}}-1\right)}-\widetilde{r}^{\frac{p-N}{p}+N}}{N-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)}\right] \\ &\leq \frac{c_1\widetilde{r}^{\frac{N-p}{p}\left((1+\varepsilon)^{\frac{1}{p-1}}-1\right)}}{N-(1+\varepsilon)^{\frac{1}{p-1}}\left(\frac{N-p}{p}\right)} \longrightarrow 0 \text{ as } d\to+\infty \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \int_{0}^{\widetilde{r}} r^{N-1}u(r)\,dr &\leq \tfrac{d}{N}\widetilde{r}^{N} \\ &= \tfrac{d}{N}\left(d^{-\frac{p}{N-p}}\right)^{N} \\ &= \frac{d^{-\left[\frac{N(p-1)+p}{N-p}\right]}}{N} \longrightarrow 0 \text{ as } d\to+\infty \end{aligned}$$

Finally, we obtain

$$\lim_{d\to+\infty}\int_{0}^{1} r^{N-1}u(r,\lambda,d)\,dr = 0.$$

Lemma 37 *If u is a positive solution of (5.9), then we have*

$$\lim_{d\to0^{+}}\int_{0}^{1} r^{N-1}u(r,\lambda,d)\,dr = 0.$$

Proof. We have

$$\int_0^1 r^{N-1}u(r,\lambda,d)\,dr \leq d\int_0^1 r^{N-1}dr = \frac{d}{N}$$

This implies that

$$\lim_{d\to 0^+}\int_0^1 r^{N-1}u(r,\lambda,d)\,dr \leq \lim_{d\to 0^+}\frac{d}{N}=0^+$$

5.4 Proof of theorem (31)

As $\frac{\partial u}{\partial \lambda}(1,\lambda,d)<0$, the implicit function theorem shows that if

$S=\{(\lambda,d)\,;u(1,\lambda,d)=0$ and $u(r,\lambda,d)>0$ for all $r\in[0,1[\}$ then there exists a differentiable function F such that $S=\{(F(d),d)\,;d\in(0,+\infty)\}$.

By lemma (37) there exists a constant $a\in\,]0,1[$ such that

$\lim_{d\to+\infty}u(a,F(d),d)=0$. Then if $\overline{\lim_{d\to 0^+}}F(d)>0$, we have, for a some sequence (d_n) such that $d_n\to 0^+$

$$\frac{F(d_n)}{|u(a,F(d_n),d_n)|^{p-2}u(a,F(d_n),d_n)}\to+\infty$$

Since $u(r,F(d_n),d_n)$ satisfy

$$\left(|u'|^{p-2}u'\right)'+\frac{N-1}{r}|u'|^{p-2}u'+\frac{F(d_n)}{|u|^{p-2}u}\left(u^{\alpha-1}+u^{q-1}\right)|u|^{p-2}u=0$$

The comparison theorem of Sturm (see [86]) shows that $u(r,F(d_n),d_n)$ must have a zero in $[a,1)$, which contradicts the fact that $u(r,F(d_n),d_n)$ is positive in $(0,1)$.

By lemma (36) there exists a constant $c\in\,]0,1[$ such that

$\lim_{d\to+\infty}u(c,F(d),d)=0$. Then if $\overline{\lim_{d\to+\infty}}F(d)>0$, we have, for a some sequence (d_n) such that $d_n\to+\infty$

$$\frac{F(d_n)}{|u(c,F(d_n),d_n)|^{p-2}u(c,F(d_n),d_n)}\to+\infty$$

Also, the same argument shows that $u(r,F(d_n),d_n)$ must have a zero in $[c,1)$, which contradicts the fact that $u(r,F(d_n),d_n)$ is positive in $(0,1)$.

Let $d_* \in (0, +\infty)$ be such that $F(d_*) = \max_{d \in (0,+\infty)} F(d) = \lambda_*$. Then we have:

i) If $\lambda < \lambda_*$ then problem (5.2) admits at least two positive radially symmetric solutions,

ii) If $\lambda = \lambda_*$ then problem (5.2) admits at least one positive radially symmetric solution,

iii) If $\lambda > \lambda_*$ then problem (5.2) has no positive radially symmetric solution.

Chapter 6

Existence and multiplicity results for quasilinear boundary value problems with blow-up boundary conditions

Abstract This chapter is concerned with the necessary, sufficient conditions for the existence and the multiplicity of boundary blow-up nonnegative solutions of the quasilinear boundary value problem:

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = \lambda f\left(u\right) \text{ in } (0,1) \\ \lim\limits_{x\to 0^+} u\left(x\right) = +\infty = \lim\limits_{x\to 1^-} u\left(x\right) \end{cases}$$

where $p > 1$, $\varphi_p\left(y\right) = \left|y\right|^{p-2} y$, $\left(\varphi_p\left(u'\right)\right)'$ is the one dimensional p–Laplacian, λ is a strictly positive real parameter and f is a continuous function. We use the quadrature method for showing the existence of solutions.

6.1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 1$. A solution $u \in C^2(\Omega)$ of the following boundary value problem

$$-\Delta u = f(u) \text{ in } \Omega \tag{6.1}$$

$$u(x) \to +\infty \text{ as } x \to \partial\Omega \tag{6.2}$$

is called a boundary blow-up solution.

This type of problems has been extensively studied. The first striking result in that case is due to Bieberbach [44] who proved that if Ω is a bounded domain in $\mathbf{R}^2$ with C^2 submanifold $\partial\Omega$ and $f(u) = -\exp(u)$, then (6.1)-(6.2) admits a unique solution $u \in C^2(\Omega)$ such that $u(x) - \ln\left(d(x)^{-2}\right)$ is bounded on Ω, where $d(x)$ denotes the distance from the point x to the boundary $\partial\Omega$.

This result, was extend by Rademacher [173] to smooth bounded domain in $\mathbf{R}^3$. Later Lazer and Mckenna [142] extend the results of Bieberbach for the case when Ω is a smooth bounded domain in $\mathbf{R}^N$. In this case the problem plays an important role, when $N = 2$ in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, according to [173], in the study of electrical potential in a glowing hollow metal body.

Problems of type (6.1)-(6.2) has been discussed under aspects of existence of solutions, uniqueness and asymptotic behavior near the boundary.

The question of existence of blow-up solutions at least for monotone f, was studied by Keller [134] and Osserman [159]. They gave a sufficient conditions on f for the existence of positive solutions:

$$f \text{ is locally Lipschitz continuous and nondecreasing on } [0, +\infty)$$

$$f(0) = 0 \text{ and } \int^{+\infty} [F(u)]^{\frac{-1}{2}} du < \infty \text{ where } f = F'$$

Keller applied the results to electrohydrodynamics, namely to the problem of the equilibrium of a charged gas in a conducting container, see [133].

The question of the uniqueness and the asymptotic behavior near the boundary has been discussed by many others.

For the special case where $f(u) = -u^{\frac{N+2}{N-2}}$ and $N > 2$, Loewner and Nirenberg [145] prove that if $\partial\Omega$ consists of the disjoint union of finitely compact C^∞ manifolds, each having codimension less than $\frac{N}{2} + 1$, then there exists a unique solution of problem (6.1)-(6.2). Later, Bandle and Marcus ([37] and [38]) and Lazer Mckenna [141] extended the results of [145] to a much larger class of nonlinearities including $f(u) = -u^a$ with $a > 1$. For smooth domain, they obtained the asymptotic behavior of the blow-up solutions near the boundary and under the monotonicity assumption on f, they could deduce the uniqueness of the positive blow-up solutions. The uniqueness was established also by Kondrat'ev and Nikishkin [137]. In fact they have showed that for $f(u) = -u^a$ with $a > 1$, $\partial\Omega$ is a C^2-manifold and the Laplace operator is replaced by a more general second order elliptic operator. The case where the Laplace operator is replaced by the $p-$Laplace operator has been discussed by Diaz and Letelier [80].

Note that for $f(u) = -u^a$ with $a > 1$, problem (6.1)-(6.2) is of interest in the study of the subsonic motion of a gas when $a = 2$ (see [171]) and is related to a problem involving superdiffusion, particularly for $1 < a \leq 2$ (see [84] and [85]).

The first result of nonuniqueness was obtained by McKenna, Reichel and Walter, in the special case when the domain Ω is a ball and $f(u) = -|u|^a$. More precisely, they proved that for $1 < a < N^*$ (note that $N^* = \frac{N+2}{N-2}$ for $N \geq 3$ and $N^* = \infty$ for $N = 1, 2$) there are exactly two blow-up solutions: one positive and one sign-changing. For $a \geq N^*$, there is a unique blow-up solution and it is positive. They first proved the radial blow-up solutions for the $p-$Laplace operator using the monotonicity and shooting methods and the use of variational techniques and Pohozaev's identity.

Subsequently, Aftalion and Reichel [16] extended the existence of at least two blow-up solutions to convex, bounded C^1 domains for general nonlinearities f including

$$f(u) = \begin{cases} -u^a \text{ with } a > 1 \text{ if } u > 0 \\ -(-u)^b \text{ with } 1 < b < N^*, \text{ if } u < 0 \end{cases}$$

For general nonlinearities f and in the one dimensional case Anuradha, Brown and Shivaji [29] and Shin-Hwa Wang [207] considered problem(6.1)-(6.2). Using quadrature method they have studied the existence and the multiplicity of boundary blow-up nonnegative solutions.

During, the last decedade, the $p-$Laplacian operator $\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$ $p > 1$ has been widely investigated. This operator is linear if and only if $p = 2$. So, several authors are interested by questions as, does such a result, know for $p = 2$, steel hold for $p \neq 2$? If not, what minimal informations can one gets such a result for $p > 1$?

In this chapter, we will discus the necessary, sufficient conditions for the existence and the multiplicity of boundary blow-up nonnegative solutions of the quasilinear boundary value problem:

$$-\left(\varphi_p\left(u'\right)\right)' = \lambda f\left(u\right) \text{ in } (0,1) \tag{6.3}$$

$$\lim_{x\to 0^+} u\left(x\right) = +\infty = \lim_{x\to 1^-} u\left(x\right) \tag{6.4}$$

where $p > 1$, $\varphi_p\left(y\right) = |y|^{p-2}\,y$, $\left(\varphi_p\left(u'\right)\right)'$ is the one dimensional $p-$Laplacian, $\lambda > 0$ and $f : \mathbf{R}^+ \to \mathbf{R}$ is a continuous function.

The aim of this work is to give a generalization of the results obtained by Anuradha, Brown and Shivaji [29] and Shin-Hwa Wang [207] for the case $p > 1$.

The chapter is organized as follows. In section 2, we present the method used for proving the main results of this paper. In section 3 we state and prove our main results. In section 4 we give some examples to illustrate our results. Finally in section 5 we give an appendix.

6.2 Quadrature method

To obtain our results, we use of the well know quadrature method. This method enable us to look for nonnegative solutions of (6.3)-(6.4) in a prescribed subset of $C^1\left(0,1\right)$.

By a nonnegative solution to problem (6.3)-(6.4), we mean a nonnegative function $u \in C^1\left(0,1\right)$ with $\varphi_p\left(u'\right) \in C^1\left(0,1\right)$ satisfying (6.3)-(6.4).

Let A^+ the subset of $C^1\left(0,1\right)$ composed by the functions u satisfying:

(i) $u(x) \geq 0$, $\forall x \in (0,1)$, and $\lim_{x \to 0^+} u(x) = \lim_{x \to 1^-} u(x) = +\infty$

(ii) u is symmetrical about $\frac{1}{2}$.

(iii) The derivative of u vanishes once and only once in $(0,1)$.

Consider the boundary value problem (6.3)-(6.4) and assume that f, p and λ satisfy the following condition:

$$f \in C\left(\mathbf{R}^+, \mathbf{R}\right), \ p > 1 \text{ and } \lambda > 0 \tag{6.5}$$

Let

$$F(u) := \int_0^u f(t)\, dt$$

and

$$I := \{s \geq 0 : f(s) < 0 \text{ and } F(s) > F(u) \ \forall u > s\}$$

We have the following result

Theorem 38 *Assume that (6.5) holds. The problem (6.3)-(6.4) admits a unique nonnegative solution* $u \in A^+$ *with* $\rho = \inf_{x \in (0,1)} u(x)$ *if and only if* $\rho \in I$ *and* $G(\rho) = \lambda^{\frac{1}{p}}$, *where* G *is defined by*

$$G(\rho) := 2 \int_{\rho}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\left(F(\rho) - F(\xi)\right)\right]^{\frac{1}{p}}}$$

One may observe that this result doesn't give informations about solutions to (6.3)-(6.4) outside A^+. The following proposition give some useful informations.

$$f \in C\left(\mathbf{R}^+, \mathbf{R}\right), \ f(u) < 0, \ \forall u \geq 0, \ p > 1 \text{ and } \lambda > 0 \tag{6.6}$$

$$f \text{ is locally Lipschitzian in } \mathbf{R}^+, \ 1 < p \leq 2 \text{ and } \lambda > 0 \tag{6.7}$$

Proposition 39 *Denote by S the nonnegative solution set of (6.3)-(6.4).*

(i) *If (6.6) holds then $S \subset A^+$.*

(ii) *If (6.7) holds then $S \subset A^+$.*

Proof. The proof of proposition39 is established in the Appendix.

Proof. of theorem38

Let u be a solution of problem (6.3)-(6.4) belonging to A^+. Thus u takes its minimum at $\frac{1}{2}$, u is symmetric with respect to $\frac{1}{2}$, $u' < 0$ in $\left(0, \frac{1}{2}\right)$ and $u' > 0$ in $\left(\frac{1}{2}, 1\right)$. Hence (6.3)-(6.4) is equivalent to the following problem defined on $\left(0, \frac{1}{2}\right)$:

$$-\left(\varphi_p\left(u'\right)\right)' = \lambda f\left(u\right) \text{ in } \left(0, \frac{1}{2}\right) \tag{6.8}$$

$$\lim_{x \to 0^+} u\left(x\right) = +\infty \tag{6.9}$$

$$u'\left(\frac{1}{2}\right) = 0 \tag{6.10}$$

Multiplying the equation (6.8) by $u'$and integrating the resulting equation over $\left(x, \frac{1}{2}\right)$, we obtain

$$\left|u'\right|^p\left(x\right) + \frac{p}{p-1}\lambda F\left(u\left(x\right)\right) = \frac{p}{p-1}\lambda F\left(\rho\right) \tag{6.11}$$

where $\rho = u\left(\frac{1}{2}\right)$

Since $u' < 0$ in $\left(0, \frac{1}{2}\right]$, we obtain

$$u'\left(x\right) = -\left[\frac{p}{p-1}\lambda\left(F\left(\rho\right) - F\left(u\left(x\right)\right)\right)\right]^{\frac{1}{p}}, \text{ for all } x \in \left(0, \frac{1}{2}\right]$$

and thus

$$\frac{-u'\left(x\right)}{\left[\frac{p}{p-1}\left(F\left(\rho\right) - F\left(u\left(x\right)\right)\right)\right]^{\frac{1}{p}}} = \lambda^{\frac{1}{p}}, \text{ for all } x \in \left(0, \frac{1}{2}\right)$$

Integrating the last equality on $(0,x)$, we obtain

$$\int_{u(x)}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\left(F(\rho)-F(\xi)\right)\right]^{\frac{1}{p}}} = \lambda^{\frac{1}{p}} x, \text{ for all } x \in \left(0, \frac{1}{2}\right) \tag{6.12}$$

Letting $x \longrightarrow \frac{1}{2}$ in (6.12) gives

$$G(\rho) := 2\int_{\rho}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\left(F(\rho)-F(\xi)\right)\right]^{\frac{1}{p}}} = \lambda^{\frac{1}{p}}$$

It follows that if u is a nonnegative solution of problem (6.3)-(6.4) belonging to A^{+}, there exists $\rho_{*} \in \tilde{I}$ where

$$\tilde{I} := \left\{ s \geq 0 : F(s) > F(u) \ \forall u > s \text{ and } \int_{s}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\left(F(\rho)-F(\xi)\right)\right]^{\frac{1}{p}}} < +\infty \right\}$$

such that $u\left(\frac{1}{2}\right) = \rho_{*}$ and $G(\rho_{*}) = \lambda_{*}^{\frac{1}{p}}$.

Conversely, given $\lambda_{*} > 0$, if there $\rho_{*} \in I$ is such that $G(\rho_{*}) = \lambda_{*}^{\frac{1}{p}}$, then we can obtain a nonnegative solution of problem (6.3)-(6.4) belonging to A^{+} as follows. Define the function h_{+} on $(\rho_{*}, +\infty)$ by

$$h_{+}(u) := 2\int_{u}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\left(F(\rho)-F(\xi)\right)\right]^{\frac{1}{p}}}$$

Notice that $h_{+}(\rho_{*}) = G(\rho_{*}) = \lambda_{*}^{\frac{1}{p}}$ and

$$0 \leq h_{+}(u) \leq G(\rho_{*}) \text{ for all } u \in [\rho_{*}, +\infty)$$

Thus, h_{+} is well defined on $[\rho_{*}, +\infty)$. Moreover, it is a decreasing diffeomorphism from $(\rho_{*}, +\infty)$ onto $(0, G(\rho_{*}))$,

$$h'_+(u) = \frac{-2}{\left[\frac{p}{p-1}(F(\rho) - F(u))\right]^{\frac{1}{p}}} < 0 \text{ for all } u \in (\rho_*, +\infty)$$

Let u_+ be the inverse of h_+ defined by

$$u_+(x) = h_+^{-1}\left(\lambda_*^{\frac{1}{p}} x\right) \in [\rho_*, +\infty), \text{ for all } x \in \left(0, \frac{1}{2}\right]$$

and let u be defined on $(0, 1)$ by

$$u(x) = \begin{cases} u_+(x) & \text{if } x \in \left(0, \frac{1}{2}\right] \\ u_+(1-x) & \text{if } x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

It is easy to show that this function u is a solution of problem (6.3)-(6.4) belonging to A^+ and satisfies $\inf_{x \in (0,1)} u(x) = u\left(\frac{1}{2}\right) = \rho_*$. Let us prove it uniqueness. Assume that v is also a solution of problem (6.3)-(6.4) belonging to A^+ and satisfies

$$\inf_{x \in (0,1)} v(x) = v\left(\frac{1}{2}\right) = \rho_*$$

By (6.12) it follows that

$$\lambda_*^{\frac{1}{p}} x = \int_{u(x)}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}(F(\rho) - F(\xi))\right]^{\frac{1}{p}}} = \int_{v(x)}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}(F(\rho) - F(\xi))\right]^{\frac{1}{p}}} \text{ for all } x \in \left(0, \frac{1}{2}\right]$$

Thus,

$$\int_{u(x)}^{v(x)} \frac{d\xi}{\left[\frac{p}{p-1}(F(\rho) - F(\xi))\right]^{\frac{1}{p}}} = 0, \text{ for all } x \in \left(0, \frac{1}{2}\right]$$

Thus,

$$u = v \text{ on } \left(0, \frac{1}{2}\right]$$

and by symmetry it follows that

$$u = v \text{ on } (0,1)$$

Therefore, by $I \subset \tilde{I}$, theorem38 is proved.

Remark: The proof of theorem 38 is similar to that of theorem 5 in [3].

6.3 Main results

6.3.1 Existence results

Consider the sequence of functions $(g_n)_{n\in\mathbf{N}^*}$ defined by:

$$g_1(t) = \ln t := \ln_1 t$$

$$g_2(t) = \ln(\ln t) := \ln_2 t$$

$$g_n(t) = \ln(g_{n-1}(t)) := \ln_n t$$

We have the following results

Theorem 40 *Let $p > 1$ and $n \in \mathbf{N}^*$. If there exists any solution to (6.3)-(6.4) for any $\lambda > 0$, then*

$$\lim_{u\to+\infty} \sup \frac{-f(u)}{u^{p-1}\ln_1^p u \ln_2^p u \ldots \ln_n^p u} = +\infty \tag{6.13}$$

Proof. Assume that $\lim\limits_{u\to+\infty} \sup \dfrac{-f(u)}{u^{p-1}\ln_1^p u \ln_2^p u \ldots \ln_n^p u} \neq +\infty$. Then there exists constants $K > 0$ and $M_1 > 0$ such that

$$
\begin{aligned}
-f(u) &\le Ku^{p-1}\prod_{j=1}^{n}\ln_j^p u \text{ for all } u > M_1 \\
&< Ku^{p-1}\prod_{j=1}^{n}\ln_j^{p-1} u\left(1+\sum_{k=1}^{n}\prod_{i=k}^{n}\ln_i u\right) \text{ for all } u > M_1
\end{aligned}
$$

This implies that

$$
\begin{aligned}
-F(u) &= \int_0^u -f(t)\,dt \\
&= -F(M_1) + \int_{M_1}^{u} -f(t)\,dt \\
&\le -F(M_1) + t^{p-1}\int_{M_1}^{u} K\prod_{j=1}^{n}\ln_j^{p-1} t\left(1+\sum_{k=1}^{n}\prod_{j=k}^{n}\ln_j t\right)dt \\
&= -F(M_1) + \tfrac{K}{p}\left(u^p\prod_{j=1}^{n}\ln_j^p u - M_1^p\prod_{j=1}^{n}\ln_j^p M_1\right)
\end{aligned}
$$

Let $\rho \in I$, then

$$
F(\rho) - F(u) < F(\rho) - F(M_1) + \frac{K}{p}\left(u^p\prod_{j=1}^{n}\ln_j^p u - M_1^p\prod_{j=1}^{n}\ln_j^p M_1\right)
$$

If we put $K_1 = F(\rho) - F(M_1) - \frac{K}{p}M_1^p\prod_{j=1}^{n}\ln_j^p M_1$, we obtain

$$
F(\rho) - F(u) < K_1 + \frac{K}{p}u^p\prod_{j=1}^{n}\ln_j^p u \tag{6.14}
$$

On other hand, there exists $M_2 > 0$ such that:

$$
\frac{p-1}{p}Ku^p\prod_{j=1}^{n}\ln_j^p u > K_1 \text{ for all } u > M_2 \tag{6.15}
$$

Let $M = \max(M_1, M_2)$, then combining (6.14) and (6.15) one gets:

$$F(\rho) - F(u) < Ku^p \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M \tag{6.16}$$

without loss of generality, we may assume that $M > \max(\rho, M_1)$ and obtain from (6.16) that

$$\begin{aligned}
G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{+\infty} \frac{du}{[F(\rho)-F(u)]^{\frac{1}{p}}} \\
&\geq 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{M}^{+\infty} \frac{du}{[F(\rho)-F(u)]^{\frac{1}{p}}} \\
&= \frac{2}{K^{\frac{1}{p}}}\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{M}^{+\infty} \frac{du}{\left[u^p \prod_{j=1}^{n} \ln_j^p u\right]^{\frac{1}{p}}} \\
&= \frac{2}{K^{\frac{1}{p}}}\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \lim_{y \to +\infty} [\ln_{n+1} y]_M^y = +\infty
\end{aligned}$$

So, we have that $G(\rho)$ does not exist if $\lim\limits_{u\to+\infty} \sup \dfrac{-f(u)}{u^{p-1}\ln_1^p u \ln_2^p u \ldots \ln_n^p u} \neq +\infty$ and theorem40 follows from theorem38.

Theorem 41 *Let $p > 1$ and $n \in \mathbf{N}^*$. If f satisfy*

$$\lim_{u\to+\infty} \inf \frac{-f(u)}{u^{p-1}\ln_1^p u \ln_2^p u \ldots \ln_{n-1}^p u \ln_n^{p+1} u} = L \tag{6.17}$$

then there exists solutions to (6.3)-(6.4) for some $\lambda > 0$. Furthermore, $G(\rho)$ is well defined and continuous for all $\rho \in I$.

To prove theorem41, we need a technical lemma which is similar to [[29], Lemma 4.1].

Lemma 42 *Let f satisfy (6.17) and $\rho \in [\rho_1, \rho_2] \subset I$. Then there exists $C > 0$ and $M > 0$ such that*

$$F(\rho) - F(u) \geq Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M \tag{6.18}$$

where

$$C = \begin{cases} \frac{L}{6p} & if \quad 0 < L < +\infty \\ \frac{1}{3p} & if \qquad L = +\infty \end{cases} \tag{6.19}$$

Proof. If f satisfies (6.17), then there exists a constant $M_3 > 0$ such that

$$\begin{aligned} -f(u) &> 3pCu^{p-1} \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M_3 \\ &> h(u) := 2Cu^{p-1} \prod_{j=1}^{n} \ln_j^{p-1} u \left[p \left(1 + \sum_{k=1}^{n} \prod_{j=k}^{n} \ln_j u \right) \ln_n u + 1 \right] \end{aligned} \tag{6.20}$$

where C is the constant defined in (6.19). Then, for all $u > M_3$, one has

$$\begin{aligned} -F(u) &= -F(M_3) + \int_{M_3}^{u} -f(t)\, dt \\ &\geq -F(M_3) + \int_{M_3}^{u} h(t)\, dt \\ &= -F(M_3) + 2C \left[u^p \ln_n u \prod_{j=1}^{n} \ln_j^p u - M_3^p \ln_n M_3 \prod_{j=1}^{n} \ln_j^p M_3 \right] \end{aligned}$$

If we put $K = -F(M_3) - 2CM_3^p \ln_n M_3 \prod_{j=1}^{n} \ln_j^p M_3 + \inf_{\rho \in [\rho_1, \rho_2]} F(\rho)$, we obtain

$$F(\rho) - F(u) > K + 2Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M_3,\ \rho \in [\rho_1, \rho_2] \tag{6.21}$$

On other hand there exist $M_4 > 0$ such that

$$Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \geq -K \text{ for all } u > M_4 \tag{6.22}$$

and then combining (6.21) and (6.22) one gets:

$$F(\rho) - F(u) \geq Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M,\ \rho \in [\rho_1, \rho_2]$$

where $M = \max(M_3, M_4)$.

This completes the proof of lemma35.

Proof. of theorem41 Let $\rho \in I$. Since I is open, there exists ρ_1, $\rho_2 \in I$ such that $\rho \in (\rho_1, \rho_2)$ and $[\rho_1, \rho_2] \subset I$

Let f satisfy (6.17), by lemma35 there exists a constants $M > 0$ and $C > 0$ such that

$$F(\rho) - F(u) \geq Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M,\ \rho \in [\rho_1, \rho_2] \tag{6.23}$$

where C is the constant defined in (6.19).

Note that

$G(\rho) = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{+\infty} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} < +\infty$ if and only if there exists $\delta > 0$ such that

$$\int_{\rho}^{\rho+\delta} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} < +\infty,\ 0 < \delta < \rho_2 - \rho$$

and

$$\int_{M}^{+\infty} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} < +\infty$$

where we suppose without loss of generality that $M > \rho_2$.

Since $[\rho_1, \rho_2] \subset I$, we have

$$\bar{L} := \inf_{z \in [\rho_1, \rho_2]} (-f(z)) > 0 \tag{6.24}$$

Using (6.24) and the mean value theorem, one gets

$$F(\rho) - F(u) = -f(z)(u - \rho) \geq L(u - \rho) \text{ for all } u \text{ in } [\rho_1, \rho_2]$$

Since $\rho + \delta < \rho_2$, one has

$$\int_{\rho}^{\rho+\delta} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} \leq \frac{1}{L^{\frac{1}{p}}} \int_{\rho}^{\rho+\delta} \frac{du}{[u - \rho]^{\frac{1}{p}}}$$

$$= \frac{p}{(p-1)\,L^{\frac{1}{p}}} \delta^{\frac{p-1}{p}} < +\infty \tag{6.25}$$

Also, from (6.23) it follows that

$$\int_M^{+\infty} \frac{du}{\left[F(\rho) - F(u)\right]^{\frac{1}{p}}} \leq \frac{1}{C^{\frac{1}{p}}} \int_M^{+\infty} \frac{du}{u \ln_n^{\frac{1}{p}} u \prod_{j=1}^{n} \ln_j u}$$

$$= \frac{p}{C^{\frac{1}{p}}} \left(\ln_{n+1} M\right)^{-\frac{1}{p}} < +\infty \tag{6.26}$$

Thus, from (6.25) and (6.26) it follows that

$$G(\rho) < +\infty \text{ for all } \rho \text{ in } I.$$

Hence G is well defined on I, and by theorem38 there exists a solution to (6.3)-(6.4) for $\lambda = [G(\rho)]^p$ given any ρ in I.

Also, G is continuous at ρ. This can be shown by defining

$$G_n(\rho) = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho+\frac{1}{n}}^{\rho+n} \frac{du}{\left[F(\rho) - F(u)\right]^{\frac{1}{p}}}, \; \forall n \in \mathbf{N}^*, \; \forall \rho \in [\rho_1, \rho_2]$$

Since G_n is a proper integral of a continuous integrand for each $n \in \mathbf{N}^*$, G_n is continuous on $[\rho_1, \rho_2]$. We will have that G is continuous on $[\rho_1, \rho_2]$, and thus at ρ, if we can shown that $G_n \to G$ uniformly as $n \to +\infty$.

Since I is open, there exists $\delta > 0$ such that $[\rho_1, \rho_2 + \delta] \subset I$ which implies that

$$\tilde{L} := \inf_{z \in [\rho_1, \rho_2 + \delta]} (-f(z)) > 0$$

Choose N_1 big enough so that $\frac{1}{n} \leq \delta$, for all $n \geq N_1$. Thus, we obtain by the mean value theorem

$$2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_{\rho}^{\rho+\frac{1}{n}}\frac{du}{\left[F(\rho)-F(u)\right]^{\frac{1}{p}}} \le 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_{\rho}^{\rho+\frac{1}{n}}\frac{du}{\left[\tilde{L}(u-\rho)\right]^{\frac{1}{p}}} = \left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{n^{\frac{p-1}{p}}\tilde{L}^{\frac{1}{p}}},\ \forall n\ge N_1,\ \forall\rho\in[\rho_1,\rho_2] \tag{6.27}$$

Also, choosing N_2 big enough so that $\rho+n\ge\rho_1+n\ge M$ for all $n\ge N_2$, where $M>0$ as in lemma35, we have

$$\begin{aligned}2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_{\rho+n}^{\infty}\frac{du}{\left[F(\rho)-F(u)\right]^{\frac{1}{p}}} &\le \left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{C^{\frac{1}{p}}}\int_{\rho+n}^{\infty}\frac{du}{u\ln_n^{\frac{1}{p}}u\prod_{j=1}^{n}\ln_j u}\\ &\le \left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{C^{\frac{1}{p}}}\int_{\rho_1+n}^{\infty}\frac{du}{u\ln_n^{\frac{1}{p}}u\prod_{j=1}^{n}\ln_j u}\\ &= \left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{C^{\frac{1}{p}}(\ln_{n+1}(\rho_1+n))^{\frac{1}{p}}},\forall n\ge N_2 \text{ and } \forall\rho\in[\rho_1,\rho_2]\end{aligned}$$

Letting $N:=\max(N_1,N_2)$, we have from the previous inequality and (6.27) that $|(G-G_n)(\rho)|\le\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{n^{\frac{p-1}{p}}\tilde{L}^{\frac{1}{p}}}+\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\frac{2}{C^{\frac{1}{p}}(\ln_{n+1}(\rho_1+n))^{\frac{1}{p}}}$ $\forall\, n\ge N$ and $\forall\,\rho\in[\rho_1,\rho_2]$ which implies $\sup_{\rho\in[\rho_1,\rho_2]}|(G-G_n)(\rho)|\to 0$ as $n\to+\infty$. Thus $G_n\to G$ uniformly on $[\rho_1,\rho_2]$ as $n\to+\infty$. Therefore G is continuous on all of I. The proof of theorem41 is complete.

Theorem 43 *Let f satisfy (6.17). Then $G(\rho)\to 0^+$ as $\rho\to+\infty$.*

Proof. Let f satisfy (6.17). By theorem41 we have that G exists and is continuous on I. Also, note that (6.20) implies that there exists a constants $M_3>0$ and $C>0$ such that

$$\begin{aligned}-f(u) &> 3pCu^p\ln_n u\prod_{j=1}^{n}\ln_j^p u \text{ for all } u>M_3\\ &> h(u)=2Cu^{p-1}\ln_n u\left[\prod_{j=1}^{n}\ln_j^{p-1}u\left(p\left[\ln_n u+\sum_{i=1}^{j}\prod_{j=1}^{n}\ln_j u\right]+p+1\right)\right]\end{aligned} \tag{6.28}$$

For $u>\rho\ge M_3$, one has

$$
\begin{aligned}
F(\rho) - F(u) &= \int_{\rho}^{u} -f(t)\,dt \\
&\geq \int_{\rho}^{u} h(t)\,dt \\
&= 2C\left[u^p \ln_n u \prod_{j=1}^{n} \ln_j^p u - \rho^p \ln_n \rho \prod_{j=1}^{n} \ln_j^p \rho\right]
\end{aligned}
$$

If we let $\rho^* = 2\rho$, then for $u \geq \rho^* = 2\rho$, we obtain

$$
\begin{aligned}
F(\rho) - F(u) &\geq 2C\left[u^p \ln_n u \prod_{j=1}^{n} \ln_j^p u - \rho^p \ln_n \rho \prod_{j=1}^{n} \ln_j^p \rho\right] \\
&\geq \tfrac{2^p-1}{2^p} C u^p \ln_n u \prod_{j=1}^{n} \ln_j^p u
\end{aligned}
\tag{6.29}
$$

because

$$
u^p \ln_n u \prod_{j=1}^{n} \ln_j^p u > 2^p \rho^p \ln_n \rho \prod_{j=1}^{n} \ln_j^p \rho
$$

By (6.29), we have for $\rho \geq M_3$

$$
\begin{aligned}
G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{\infty} \frac{du}{\left[F(\rho) - F(u)\right]^{\frac{1}{p}}} \\
&\leq 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \left\{ \int_{\rho}^{\rho_*} \frac{du}{\left[F(\rho) - F(u)\right]^{\frac{1}{p}}} + \frac{2^{\frac{p-1}{p}}}{[(2^p-1)C]^{\frac{1}{p}}} \int_{\rho_*}^{\infty} \frac{du}{u \ln_n^{\frac{1}{p}} u \prod_{j=1}^{n} \ln_j u} \right\} \\
&= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \left\{ \int_{\rho}^{\rho_*} \frac{du}{\sqrt{-f(z)}\,(u-\rho)^{\frac{1}{p}}} + \frac{2^{\frac{p-1}{p}}}{[(2^p-1)C]^{\frac{1}{p}} \ln_{n+1}^{\frac{1}{p}} \rho_*} \right\}
\end{aligned}
$$

where $z = z(u) \in (\rho, u)$ for each $u \in (\rho, \rho_*)$ exists by the mean value theorem. However, by (6.28), we have

$$\begin{aligned} -f(z) &> 2pCz^{p-1}\ln_n z\prod_{j=1}^{n}\ln_j^p z \\ &> 2pC\rho^{p-1}\ln_n \rho\prod_{j=1}^{n}\ln_j^p \rho \text{ for } z>\rho\geq M_3 \end{aligned}$$

This implies that

$$\begin{aligned} G(\rho) &\leq 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}}\left\{\int_{\rho}^{\rho_*}\frac{du}{\sqrt{-f(z)}\,(u-\rho)^{\frac{1}{p}}}+\frac{2^{\frac{p-1}{p}}}{[(2^p-1)C]^{\frac{1}{p}}\ln_{n+1}^{\frac{1}{p}}\rho_*}\right\} \\ &\leq 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}}\left\{\frac{p}{(p-1)\left[2pC\rho^{p-1}\ln_n\rho\prod_{j=1}^{n}\ln_j^p\rho\right]^{\frac{1}{p}}}\rho^{p-1}+\frac{2^{\frac{p-1}{p}}}{[(2^p-1)C]^{\frac{1}{p}}\ln_{n+1}^{\frac{1}{p}}\rho_*}\right\}\to 0 \text{ as } \rho\to+\infty \end{aligned}$$

The proof of theorem43 is complete.

Theorem 44 *Let f satisfy (6.17) and $0\in I$, then*

$$G(0)<\infty \tag{6.30}$$

Proof. Let $0\in I$, then there exists $\rho_2\in I$ such that $[0,\rho_2]\subset I$.

It is well know that $G(0)=2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_0^{\infty}\frac{du}{[-F(u)]^{\frac{1}{p}}}<+\infty$ if and only if there exists numbers $0<\delta<M$ such that

$$\int_0^{\delta}\frac{du}{[-F(u)]^{\frac{1}{p}}}<+\infty,\ 0<\delta<\rho_2$$

and

$$\int_M^{+\infty}\frac{du}{[-F(u)]^{\frac{1}{p}}}<+\infty,\ M>2$$

Now $[0,\rho_2]\subset I$ implies $\hat{L}:=\inf_{z\in[0,\delta]}(-f(z))>0$. For $u\in[0,\delta]$, by the mean value theorem, we have

$$\begin{aligned} -F(u) &= F(0) - F(u) \\ &\geq \inf_{z\in[0,\delta]} (-f(z))(u-0) \\ &= \hat{L}u \end{aligned}$$

Thus

$$\begin{aligned} \int_0^\delta \frac{du}{[-F(u)]^{\frac{1}{p}}} &\leq \frac{1}{\hat{L}^{\frac{1}{p}}} \int_0^\delta \frac{du}{u^{\frac{1}{p}}} \\ &= \frac{p}{(p-1)\hat{L}^{\frac{1}{p}}} \delta^{\frac{p-1}{p}} < +\infty \end{aligned} \tag{6.31}$$

Also, if f satisfies (6.17), it is know that there exists $M > 0$ such that

$$-F(u) \geq Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u \text{ for all } u > M \tag{6.32}$$

Using the inequality (6.32), we obtain

$$\begin{aligned} \int_M^{+\infty} \frac{du}{[-F(u)]^{\frac{1}{p}}} &\leq \frac{1}{C^{\frac{1}{p}}} \int_M^{+\infty} \frac{du}{\left[Cu^p \ln_n u \prod_{j=1}^{n} \ln_j^p u\right]^{\frac{1}{p}}} \\ &= \frac{p}{C^{\frac{1}{p}} (\ln_n M)^{\frac{1}{p}}} < +\infty \end{aligned} \tag{6.33}$$

The proof of theorem34 is complete.

6.3.2 Multiplicity results

Theorem 45 : *Let f satisfy (6.17). If there exists $s \geq 0$ such that $s \in I$ and f is nonincreasing in $[s, +\infty)$, then G is strictly decreasing in $[s, +\infty)$.*

Proof. : Suppose that there exist $s \geq 0$ such that $s \in I$ and f is nonincreasing in $[s, +\infty)$. Let $\rho_1, \rho_2 \in I$ such that $\rho_2 > \rho_1 \geq s$ and $\delta := \rho_2 - \rho_1$

Let

$$Y_1(u) := F(\rho_1) - F(u)$$

$$Y_2(u) := F(\rho_1 + \delta) - F(u + \delta)$$

Since

$$-f(u) \le -f(u+\delta) \text{ for all } u \ge s$$

it follows that

$$Y_1(u) = \int_{\rho_1}^{u} -f(\omega)\, d\omega \le \int_{\rho_1}^{u} -f(\omega+\delta)\, d\omega = Y_2(u) \text{ for all } u \ge \rho_1 \tag{6.34}$$

Also, condition (6.17) implies that f is not eventually constant and so there exists a $u^* > \rho_1$ such that

$$-f(u^*) < -f(u^* + \delta) \tag{6.35}$$

Since f is continuous, it follows from (6.34)and (6.35) that

$$Y_1(u) < Y_2(u) \text{ for all } u \ge u^* \tag{6.36}$$

Which implies that

$$G(\rho_1) = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho_1}^{+\infty} \frac{du}{[Y_1(u)]^{\frac{1}{p}}} > 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho_2}^{+\infty} \frac{du}{[Y_2(u)]^{\frac{1}{p}}} = G(\rho_2)$$

Thus, G is strictly decreasing on $[s, +\infty)$. Therefore, theorem45 is proved.

Case where the nonlinearity is strictly negative

This subsection is devoted to the study of problem (6.3)-(6.4) where the nonlinearity f is strictly negative in $[0, +\infty)$.

Theorem 46 *Let f satisfy (6.17). If $f(0) < 0$ and f is nonincreasing on $[0, +\infty)$, then there exists $\lambda^* > 0$ such that:*

(i) *If $\lambda > \lambda^*$, problem (6.3)-(6.4) admits no nonnegative solution.*

(ii) *If $\lambda \leq \lambda^*$, problem (6.3)-(6.4) admits a unique nonnegative solution and this solution is in A^+.*

Proof. Since $f(0) < 0$ and f is nonincreasing on $[0, +\infty)$, $I = [0, +\infty)$. Thus, G is well defined and continuous in I (theorem41), G is strictly decreasing (theorem45) and $\lim_{\rho \to +\infty} G(\rho) \to 0^+$ (theorem43). So, the equation $G(\rho) = \lambda^{\frac{1}{p}}$ admits a unique solution in I if and only if $\lambda^{\frac{1}{p}} \leq G(0)$, which is equivalent to $\lambda \leq \lambda^*$ with $\lambda^* := [G(0)]^p$. Therefore, theorem46 is proved.

Case where the nonlinearity vanishes exactly once or at least once

This subsection is devoted to the study of problem (6.3)-(6.4) where the nonlinearity f vanishes at least once or exactly once in $[0, +\infty)$.

Main results The main results of this subsection are:

Theorem 47 *Assume that $p \in (1, 2]$, $f \in C^1(\mathbf{R}^+, \mathbf{R})$ satisfying (6.17). If there exists $\beta \geq 0$ such that $f(\beta) = 0$, $I = (\beta, +\infty)$, and f is nonincreasing on $[\beta, +\infty)$. Then, for every $\lambda > 0$, problem (6.3)-(6.4) admits a unique nonnegative solution and this solution is in A^+. Furthermore, for any solution u of (6.3)-(6.4), $u\left(\frac{1}{2}\right) > \beta$.*

Theorem 48 *Let $p \in (1, 2]$ and assume that f satisfies*

(i) $f \in C^1(\mathbf{R}^+, \mathbf{R})$.

(ii) *f satisfies (6.17).*

(iii) *There exist $\beta > 0$ such that*

$$f(u) < 0 \ \forall u > 0 \text{ with } u \neq \beta \text{ and } f(\beta) = 0$$

Then, for every $\lambda > 0$, problem (6.3)-(6.4) admits a nonnegative solution and this solution is in A^+. Furthermore, there exists $\lambda_* > 0$ such that problem (6.3)-(6.4) admits at least two nonnegative solutions for all $\lambda \geq \lambda_*$.

In order to prove our main results we need the following lemma

Lemma 49 *Assume that $p \in (1, 2]$, $f \in C^1(\mathbf{R}^+, \mathbf{R})$ satisfying (6.17) and there exists $s \in [0, +\infty)$ such that $f(s) = 0$. If there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \subset I$, then $G(\rho) \to +\infty$ as $\rho \to s^+$. Furthermore, if there exists $\varepsilon > 0$ such that $(s - \epsilon, s) \subset I$, then $G(\rho) \to +\infty$ as $\rho \to s^-$.*

Proof. Assume that there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \subset I$. Then there exists a sequence $(s_n)_{n\in\mathbf{N}}$ in I such that $s_n \to s^+$ as $n \to +\infty$. Since f is of class C^1 in $\mathbf{R}^+$, then there exists $\delta > 0$ and $M > 0$ such that

$$-f(u) \leq M(u - s) \text{ for all } u \text{ in } (s, s+\delta) \tag{6.37}$$

If we choose N large enough that $s_n + \frac{\delta}{2} < s + \delta$ for each $n \geq N$, then we have by (6.37) that

$$F(s_n) - F(u) = \int_{s_n}^{u} -f(w)\,dw \leq \frac{M(u-s)^2}{2} \text{ for all } u \text{ in } \left(s_n, s_n + \frac{\delta}{2}\right)$$

For each $n \geq N$. This estimates give us the lower bound

$$
\begin{aligned}
G(s_n) &> 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{s_n}^{s_n+\frac{\delta}{2}} \frac{du}{[F(s_n)-F(u)]^{\frac{1}{p}}} \\
&\geq 2^{p+1}\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{s_n}^{s_n+\frac{\delta}{2}} \frac{du}{\left[M(u-s)^2\right]^{\frac{1}{p}}} \\
&= \begin{cases} \frac{2^{p+1}p}{(p-2)M^{\frac{1}{p}}}\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\left[\left(s_n-s+\frac{\delta}{2}\right)^{\frac{p-2}{p}}-(s_n-s)^{\frac{p-2}{p}}\right] & \text{if } p\neq 2 \\ \frac{2^{\frac{5}{2}}}{\sqrt{M}}\left[\ln\left(s_n-s+\frac{\delta}{2}\right)-\ln(s_n-s)\right] \text{ if } p=2 & \end{cases}
\end{aligned}
\tag{6.38}
$$

If $p \in (1,2]$, it is clear that $G(s_n) \to +\infty$ as $n \to +\infty$.

Similarly, we can prove that if there exist $\varepsilon > 0$ such that $(s-\varepsilon, s) \subset I$ then $G(\rho) \to +\infty$ as $\rho \to s^-$. The proof of lemma 49 is complete.

Proof. of theorem47 From the preceding results one has the following picture of the function $\rho \mapsto G(\rho)$ which is defined in $(\beta, +\infty)$: $\lim_{\rho\to\beta^+} G(\rho) = +\infty$, $\lim_{\rho\to+\infty} G(\rho) = 0^+$ and G is strictly decreasing in $(\beta, +\infty)$. So, the equation $G(\rho) = \lambda^{\frac{1}{p}}$, admits a unique solution for any $\lambda > 0$. Lastly, since $u\left(\frac{1}{2}\right) = \rho$, where $G(\rho) = \lambda^{\frac{1}{p}}$, it is clear that $u\left(\frac{1}{2}\right) > \beta$ for any solution u of (6.3)-(6.4). Theorem47 is proved.

Proof. of theorem 48 By theorem 43 and lemma 49, we have that G maps $(\beta, +\infty)$ onto $(0, +\infty)$. This guarantees existence of solutions $\forall\lambda > 0$. Also, by lemma49 we have $G(\rho) \to +\infty$ as $\rho \to \beta^-$. Let $\lambda^* := \left[\inf_{\rho\in[0,\beta)} G(\rho)\right]^p$ which exists since G is positive and well defined on $[0, \beta)$. For $\lambda \geq \lambda^*$, there exists $\rho_1 \in [0, \beta)$, $\rho_2 \in (\beta, +\infty)$ such that $G(\rho_1) = G(\rho_2) = \lambda^{\frac{1}{p}}$. Theorem48 is proved.

Case where the nonlinearity vanishes exactly $2n$ times with $n \in \mathbf{N}^*$

This subsection is devoted to the study of problem (6.3)-(6.4) where the nonlinearity f vanishes exactly $2n$ times with in $[0, +\infty)$.

Main results The main results of this subsection are

Theorem 50 *Let $p \in (1,2]$ and assume that f satisfies:*

(i) $f \in C^1(\mathbf{R}^+, \mathbf{R})$.

(ii) *f satisfies (6.17).*

(iii) *There exists* $(a,b) \in [0,+\infty)^2 : 0 < a < b$ *such that*

1) $f(a) = 0 = f(b)$

2) $f < 0$ *in* $[0,a) \cup (b,+\infty)$ *and* $f > 0$ *in* (a,b)

3) $\int_0^b f(t)\,dt < 0$

Then, for every $\lambda > 0$, *problem (6.3)-(6.4) admits a nonnegative solution and this solution is in* A^+. *Furthermore, there exists* $\lambda_* > 0$ *such that problem (6.3)-(6.4) admits at least two nonnegative solutions for all* $\lambda \geq \lambda_*$ *one of which is such that* $u\left(\frac{1}{2}\right) > b$ *and one of which is that* $u\left(\frac{1}{2}\right) < \nu$, *where* $\nu \in (0,a)$ *is such that* $F(\nu) = F(b)$.

Theorem 51 *Let* $p \in (1,2]$ *and assume that f satisfies:*

(i) $f \in C^1(\mathbf{R}^+, \mathbf{R})$.

(ii) *f satisfies (6.17).*

(iii) *There exists* $a_1, b_1, a_2, b_2, ..., a_n, b_n$ *such that*

1) $0 < a_1 \leq b_1 < a_2 \leq b_2 < ... < a_n \leq b_n < +\infty$

2) $f(a_i) = f(b_i) = 0$ *for all* $i \in \{1,2,...,n\}$.

3) $f < 0$ *in* (b_i, a_{i+1}) *for all* $i \in \{1,2,...,n\}$ *and* $f > 0$ *in* (a_i, b_i) *for all* $i \in \{1,2,...,n\}$.

4) $F(b_{i+1}) < F(b_i)$ *for all* $i \in \{1,2,...,n\}$ *and* $F(b_1) < 0$.

Then, for every $\lambda > 0$, *problem (6.3)-(6.4) admits a nonnegative solution and this solution is in* A^+. *Furthermore, there exists* $\bar{\lambda} > 0$ *such that problem (6.3)-(6.4) admits at least* $2n$ *nonnegative solutions for all* $\lambda \geq \bar{\lambda}$.

In order to prove our results, we need the following lemma

Lemma 52 *Let $p \in (1,2]$ and assume that f satisfies the conditions of theorem50. Then there exists a unique $\nu \in (0,a)$ such that $I = [0,\nu) \cup (b,+\infty)$ and* $\lim_{\rho \to \nu^-} G(\rho) = +\infty$

Proof. of lemma52 Since $F(b) < 0$ and f is strictly positive in (a,b), then $F(a) < F(b) < 0$. On other hand since F is continuous and strictly decreasing in $(0,a)$ and $F(0) = 0$, then there exists a unique $\nu \in (0,a)$ such that $F(\nu) = F(b)$. So, $F(\nu) > F(u)$ for all $u \in (\nu, b)$ and since F is strictly decreasing in $(b,+\infty)$ then $F(\nu) > F(u)$ for all $u \in (b,+\infty)$. It follows that

$$F(\nu) \geq F(u), \ \forall u \in (\nu,+\infty) \tag{6.39}$$

On other hand since F is strictly decreasing in $[0,\nu]$, then

$$\forall \rho \in [0,\nu): \ F(\rho) > F(u), \ \forall u \in (\rho,\nu] \tag{6.40}$$

From (6.39) and (6.40) it follows that

$$\forall \rho \in [0,\nu): \ F(\rho) > F(u), \ \forall u \in (\rho,+\infty)$$

then,

$$[0,\nu) \subset I \tag{6.41}$$

On other hand, since F is strictly decreasing in $[\nu,a)$, then

$$\forall \rho \in [\nu,a): \ F(\rho) \leq F(\nu) = F(b)$$

then,

$$[\nu,a) \cap I = \emptyset$$

and since $f(a) = f(b) = 0$ and $f > 0$ in (a,b), then

$$[\nu,b] \cap I = \emptyset \tag{6.42}$$

On other hand since F is strictly decreasing in $(b, +\infty)$, then

$$\forall \rho \in (b, +\infty): \; F(\rho) > F(u), \; \forall u \in (\rho, +\infty)$$

This implies that

$$(b, +\infty) \subset I \tag{6.43}$$

From (6.41), (6.42) and (6.43) it follows that

$$I = [0, \nu) \cup (b, +\infty)$$

Now we are going to show that $\lim\limits_{\rho \to \nu^-} G(\rho) = +\infty$

Since f is of class C^1 in $\mathbf{R}^+$, then there exists $\varepsilon > 0$ and $k_1 > 0$ such that

$$|f(u) - f(b)| \leq k_1 |u - b| \text{ for all } u \text{ in } (b, b + \varepsilon)$$

Hence if $0 \leq \rho < \nu$ and $\rho < u < +\infty$, one has

$$\begin{aligned} 0 < F(\rho) - F(u) &= F(\rho) - F(\nu) + F(\nu) - F(u) \\ &= F(\rho) - F(\nu) + F(b) - F(u) \\ &\leq \max_{0 \leq \rho \leq \nu} |f(\rho)| \, |\rho - \nu| + F(b) - F(u) \\ &= k_2 |\rho - \nu| + F(b) - F(u) \end{aligned}$$

where $k_2 := \max\limits_{0 \leq \rho \leq \nu} |f(\rho)|$

Then,

$$
\begin{aligned}
G(\rho) &= \int_{\rho}^{+\infty} \frac{du}{[F(\rho)-F(u)]^{\frac{1}{p}}} \\
&\geq \int_{\rho}^{+\infty} \frac{du}{[k_2|\rho-\nu|+F(b)-F(u)]^{\frac{1}{p}}} \\
&\geq \int_{b}^{b+\varepsilon} \frac{du}{[k_2|\rho-\nu|+F(b)-F(u)]^{\frac{1}{p}}} \\
&\geq \int_{b}^{b+\varepsilon} \frac{du}{\left[k_2|\rho-\nu|+k_1(u-b)^2\right]^{\frac{1}{p}}} \\
&= \int_{b}^{b+\varepsilon} H_\rho(u)\,du
\end{aligned}
$$

where $H_\rho(u) := \left[k_2|\rho-\nu|+k_1(u-b)^2\right]^{\frac{-1}{p}}$. But as $\rho \to \nu^-$, H_ρ is a nondecreasing sequence of measurable functions. Hence by the monotone convergence theorem,

$$
\begin{aligned}
\lim_{\rho\to\nu^-} G(\rho) &\geq \lim_{\rho\to\nu^-} \int_{b}^{b+\varepsilon} H_\rho(u)\,du \\
&= k_1^{\frac{-1}{p}} \int_{b}^{b+\varepsilon} |u-b|^{\frac{-2}{p}}\,du = +\infty \text{ if } 1<p\leq 2
\end{aligned}
$$

Therefore,

$$
\lim_{\rho\to\nu^-} G(\rho) = +\infty
$$

The proof of lemma52 is complete.

Proof. of theorem 50 By theorem 43 and lemma 49 we have G maps $(b,+\infty)$ onto $(0,+\infty)$. This guarantees the existence of solutions for all $\lambda > 0$. Also, by lemma52 we have $\lim_{\rho\to\nu^-} G(\rho) = +\infty$. Let $\lambda^* := \left[\inf_{\rho\in[0,\nu)} G(\rho)\right]^p$ which exists since G is positive and well defined in $[0,\nu)$. For $\lambda \geq \lambda^*$, then there exists $\rho_1 \in [0,\nu)$ and $\rho_2 \in (b,+\infty)$ such that $G(\rho_1) = \lambda^{\frac{1}{p}} = G(\rho_2)$ and thus (6.3)-(6.4) admits at least two nonnegative solutions.The proof of theorem50 is complete.

Proof. . of theorem 51 Here, if $a_i = b_i$ for some $i = 1, ..., n$ then choose $\nu_i := a_i = b_i$. If $a_i < b_i$, then choose $\nu_i \in (0, a_i)$ as in the proof of theorem50 so that it satisfies the hypothesis of lemma52. Note that enforcing $F(b_{i-1}) < F(b_i)$ (where b_0 is defined to be zero) and choosing ν_i such that $F(u) < F(\nu_i)$ $\forall u \in (\nu_i, b_i)$ forces $\nu_i > b_{i-1}$. Thus, $\nu_i \in (b_{i-1}, a_i]$. Now, using lemma52, when necessary, we have that $I = (0, \nu_1) \cup (b_1, \nu_2) \cup (b_2, \nu_3) \cup ... \cup (b_{n-1}, \nu_n) \cup (b_n, +\infty)$, and $([\nu_1, b_1] \cup [\nu_2, b_2] \cup ... [\nu_n, b_n]) \cap I = \emptyset$. Also, using lemma49 when $\nu_i = a_i = b_i$ and lemma52 when $\nu_i < a_i < b_i$, we have $\lim_{\rho \to \nu_i^-} G(\rho) = +\infty$. Using lemma49 we have $\lim_{\rho \to b_i^+} G(\rho) = +\infty$ for $i = 1, 2, ..., n$.

Now, let $\lambda_i^* = \inf_{\rho \in (b_{i-1}, \nu_i)} [G(\rho)]^p$ for $i = 1, 2, ..., n$ and $\bar{\lambda} := \max_{i=1,2,...,n} \lambda_i^*$. Then, for $\lambda > \bar{\lambda}$ there exists at least two nonnegative solutions with $u\left(\frac{1}{2}\right) = \rho \in [b_{i-1}, \nu_i); i = 2, ..., n$ while at least one nonnegative solution exists with $u\left(\frac{1}{2}\right) \in [0, \nu_1)$ and at least one nonnegative solution with $u\left(\frac{1}{2}\right) \in (b_n, +\infty)$ exists. Therefore, theorem 51 is proved.

6.4 Examples

6.4.1 Example 1

Consider the following problem:

$$\begin{cases} -(\varphi_p(u'))' = -\lambda \exp(u) \text{ in } (0,1) \\ \lim_{x \to 0^+} u(x) = \lim_{x \to 1^-} u(x) = +\infty \end{cases} \tag{6.44}$$

where $\varphi_p(y) = |y|^{p-2} y$, $p > 1$ and $\lambda > 0$ are real parameters.

The main result of this example is

Theorem 53 *There exists a real number $\hat{\lambda} > 0$ such that:*

(i) *If $\lambda > \hat{\lambda}$, problem (6.44) admits no nonnegative solution.*

(ii) *If $\lambda \leq \hat{\lambda}$, problem (6.44) admits a unique nonnegative solution and this solution is in A^+.*

Proof. In this example, one has

$$f(u) = -\exp(u) \text{ and } F(u) = \int_0^u f(s)\, ds = 1 - \exp(u)$$

We have

$$I := \{s \geq 0 : f(s) < 0 \text{ and } F(s) > F(u), \forall u > s\} = [0, +\infty)$$

The function G is defined on $I = [0, +\infty)$ by

$$\begin{aligned} G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{+\infty} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} \\ &= 2e^{-\frac{\rho}{p}} \left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{1}^{+\infty} \frac{dv}{v[v-1]^{\frac{1}{p}}} \end{aligned}$$

Using the change of variables $v = \dfrac{1}{\cos^2\theta}$, one gets

$$\begin{aligned} G(\rho) &= 4\exp\left(-\frac{\rho}{p}\right)\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{0}^{\frac{\pi}{2}} \cos^{\frac{2}{p}-1}\theta \sin^{1-\frac{2}{p}}\theta d\theta \\ &= 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} B\left(\frac{1}{p}, \frac{p-1}{p}\right) \exp\left(-\frac{\rho}{p}\right) \end{aligned}$$

where $B(k, l)$ is the Euler beta function defined by

$$B(k, l) = 2\int_{0}^{\frac{\pi}{2}} \sin^{2k-1}\theta \cos^{2l-1}\theta d\theta, \; k > 0 \text{ and } l > 0$$

We observe that

$\lim_{\rho \to 0^+} G(\rho) = G(0)$, $\lim_{\rho \to +\infty} G(\rho) = 0^+$ and G is strictly decreasing on $[0, +\infty)$. Then

$$\max_{\rho \in [0, +\infty)} G(\rho) = G(0)$$

So, one can conclude that

(i) If $G(0) \geq \lambda^{\frac{1}{p}}$, equation $G(\rho) = \lambda^{\frac{1}{p}}$ admits a unique solution.

(ii) If $G(0) < \lambda^{\frac{1}{p}}$, equation $G(\rho) = \lambda^{\frac{1}{p}}$ admits no solution.

Then, if one put $\hat{\lambda} = [G(0)]^p$, theorem (53) follows.

6.4.2 Example 2

Consider the following problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = -\lambda u^\alpha \text{ in } (0,1) \\ \lim\limits_{x\to 0^+} u(x) = \lim\limits_{x\to 1^-} u(x) = +\infty \end{cases} \tag{6.45}$$

where $\varphi_p(y) = |y|^{p-2}y$, $p > 1$, $\lambda > 0$ and $\alpha > p-1$ are real parameters.

Theorem 54 *If $\alpha > p-1$ then, for each fixed $\lambda > 0$, problem (6.45) admits a unique nonnegative solution and this solution is in A^+.*

Proof. In this example, one has for $u \geq 0$:

$$f(u) = -u^\alpha \text{ and } F(u) = \int_0^u -s^\alpha ds = -\frac{u^{\alpha+1}}{\alpha+1}$$

We have

$$I := \{s \geq 0 : f(s) < 0 \text{ and } F(s) > F(u), \forall u > s\} = (0, +\infty)$$

The function G is defined on $I = (0, +\infty)$ by

$$\begin{aligned} G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_\rho^{+\infty} \frac{du}{[F(\rho) - F(u)]^{\frac{1}{p}}} \\ &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} (\alpha+1)^{\frac{1}{p}} \rho^{-\alpha} \int_1^{+\infty} \frac{du}{[u^{\alpha+1} - 1]^{\frac{1}{p}}} \end{aligned}$$

Using the change of variables $v = u^{\alpha+1}$ one gets

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} (\alpha+1)^{\frac{1-p}{p}} \rho^{-\alpha} \int_1^{+\infty} v^{-\frac{\alpha}{\alpha+1}} (v-1)^{-\frac{1}{p}} dv$$

and using the change of variables $v = \dfrac{1}{\cos^2\theta}$, one gets

$$\begin{aligned} G(\rho) &= 4(\alpha+1)^{\frac{1-p}{p}}\rho^{-\alpha}\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_0^{\frac{\pi}{2}}\cos^{2\left(\frac{\alpha+1-p}{\alpha+1}\right)-1}\theta\sin^{2\left(\frac{p-1}{p}\right)-1}\theta d\theta \\ &= 2(\alpha+1)^{\frac{1-p}{p}}\left(\frac{p-1}{p}\right)^{\frac{1}{p}}B\left(\frac{\alpha+1-p}{p+1},\frac{p-1}{p}\right)\rho^{-\alpha} \end{aligned}$$

We observe that

$\lim\limits_{\rho\to 0^+} G(\rho) = +\infty$, $\lim\limits_{\rho\to+\infty} G(\rho) = 0^+$ and G is strictly decreasing on $[0,+\infty)$. Then the equation $G(\rho) = \lambda^{\frac{1}{p}}$ admits a unique solution for any $\lambda > 0$.

6.4.3 Example 3

Consider the following boundary value problem:

$$\begin{aligned} -(\varphi_p(u'))' &= -\lambda(u^\alpha + u^\beta) \text{ in } (0,1) \\ \lim_{x\to 0^+} u(x) &= \lim_{x\to 1^-} u(x) = +\infty \end{aligned} \tag{6.46}$$

where $\varphi_p(y) = |y|^{p-2}y$, $p > 1$, $\lambda > 0$ and $0 < \beta < p-1 < \alpha$ are real parameters.

The main result of this example is:

Theorem 55 *Assume that $p \in (1,2]$. Then for each fixed $\lambda > 0$, problem (6.46) admits a unique nonnegative solution and this solution is in A^+.*

Proof. In this example, one has

$$f(u) = -\left(u^\alpha + u^\beta\right) \text{ and } F(u) = \int_0^u f(s)\,ds = -\left(\frac{u^{\alpha+1}}{\alpha+1} + \frac{u^{\beta+1}}{\beta+1}\right)$$

We have

$$I := \{s \geq 0 : f(s) < 0 \text{ and } F(s) > F(u),\ \forall u > s\} = (0,+\infty)$$

The function G is defined on $I = (0,+\infty)$ by

$$
\begin{aligned}
G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{+\infty} \frac{du}{\left[F(\rho)-F(u)\right]^{\frac{1}{p}}} \\
&= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_{\rho}^{+\infty} \frac{du}{\left[\left(\frac{u^{\alpha+1}}{\alpha+1}+\frac{u^{\beta+1}}{\beta+1}\right)-\left(\frac{\rho^{\alpha+1}}{\alpha+1}+\frac{\rho^{\beta+1}}{\beta+1}\right)\right]^{\frac{1}{p}}}
\end{aligned}
$$

Using the preceding lemmas, we have

(i) $\lim_{\rho\to 0^+} G(\rho) = +\infty$ (lemma49).

(ii) $\lim_{\rho\to +\infty} G(\rho) = 0^+$ (lemma43).

(iii) G is strictly decreasing on $(0,+\infty)$ (lemma45).

Then, the equation $G(\rho) = \lambda^{\frac{1}{p}}$ admits a unique solution for any $\lambda > 0$.

6.4.4 Example 4

Consider the following boundary value problem:

$$
\begin{aligned}
&-\left(\varphi_p(u')\right)' = -\lambda\left|u-1\right|^p \text{ in } (0,1) \\
&\lim_{x\to 0^+} u(x) = \lim_{x\to 1^-} u(x) = +\infty
\end{aligned} \tag{6.47}
$$

where $\varphi_p(y) = |y|^{p-2}y$, $p > 1$ and $\lambda > 0$ are real parameters.

The main result of this example is:

Theorem 56 *There exists a real number $\tilde{\lambda} > 0$ such that:*

(i) *If $\lambda \geq \tilde{\lambda}$, problem (6.47) admits exactly two nonnegative solutions and these solutions are in A^+.*

(ii) *If $\lambda < \tilde{\lambda}$, problem (6.47) admits a unique nonnegative solution and this solution is in A^+.*

Proof. In this example, one has

$$f(u) = -|u-1|^p \text{ and } F(u) = \int_0^u f(s)\,ds = -\frac{|u-1|^p(u-1)}{p+1}$$

We have

$$I := \{s \geq 0 : f(s) < 0 \text{ and } F(s) > F(u),\ \forall u > s\} = [0,1) \cup (1,+\infty)$$

The function G is defined on I by

$$\begin{aligned} G(\rho) &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_\rho^{+\infty} \frac{du}{[F(\rho)-F(u)]^{\frac{1}{p}}} \\ &= 2\left(\tfrac{p-1}{p}\right)^{\frac{1}{p}} \int_\rho^{+\infty} \frac{du}{\left[\frac{|u-1|^p(u-1)}{p+1} - \frac{|\rho-1|^p(\rho-1)}{p+1}\right]^{\frac{1}{p}}} \end{aligned}$$

Using the change of variables $(u-1) = (\rho-1)\,v$, one gets

$$G(\rho) = \begin{cases} 2\left(\dfrac{p-1}{p}\right)^{\frac{1}{p}} \dfrac{(p+1)^{\frac{1}{p}}}{(1-\rho)^{\frac{1}{p}}} \displaystyle\int_{-\infty}^{1} \dfrac{dv}{[1-|v|^p\,v]^{\frac{1}{p}}} & \text{if } \rho < 1 \\ 2\left(\dfrac{p-1}{p}\right)^{\frac{1}{p}} \dfrac{(p+1)^{\frac{1}{p}}}{(\rho-1)^{\frac{1}{p}}} \displaystyle\int_{1}^{+\infty} \dfrac{dv}{[v^{p+1}-1]^{\frac{1}{p}}} & \text{if } \rho > 1 \end{cases} \tag{6.48}$$

Some easy computations shows that:

$$\int_{-\infty}^{1} \frac{dv}{[1-|v|^p\,v]^{\frac{1}{p}}} = \frac{B\left(\frac{1}{p(p+1)}, \frac{1}{p+1}\right) + B\left(\frac{1}{p}, \frac{p-1}{p}\right)}{p+1} \tag{6.49}$$

$$\int_{1}^{+\infty} \frac{dv}{[v^{p+1}-1]^{\frac{1}{p}}} = \frac{B\left(\frac{1}{p(p+1)}, \frac{p-1}{p}\right)}{p+1} \tag{6.50}$$

Then, from (6.48), (6.49) and (6.50) it follows that

$$G(\rho)=\begin{cases} 2\left(\dfrac{p-1}{p}\right)^{\frac{1}{p}}\dfrac{B\left(\dfrac{1}{p(p+1)},\dfrac{1}{p+1}\right)+B\left(\dfrac{1}{p},\dfrac{p-1}{p}\right)}{(p+1)^{\frac{p-1}{p}}(1-\rho)^{\frac{1}{p}}} & \text{if } \rho<1 \\ 2\left(\dfrac{p-1}{p}\right)^{\frac{1}{p}}\dfrac{B\left(\dfrac{1}{p(p+1)},\dfrac{p-1}{p}\right)}{(p+1)^{\frac{p-1}{p}}(\rho-1)^{\frac{1}{p}}} & \text{if } \rho>1 \end{cases}$$

We observe that

$\lim_{\rho\to 1^-} G(\rho)=\lim_{\rho\to 1^+} G(\rho)=+\infty$, $\lim_{\rho\to +\infty} G(\rho)=0^+$, G is strictly increasing on $[0,1)$ and strictly decreasing on $(1,+\infty)$.

So, one can conclude that

(i) If $G(0)\leq\lambda^{\frac{1}{p}}$, equation $G(\rho)=\lambda^{\frac{1}{p}}$ admits exactly two solutions.

(ii) If $G(0)>\lambda^{\frac{1}{p}}$, equation $G(\rho)=\lambda^{\frac{1}{p}}$ admits a unique solution.

Then, if one put $\tilde{\lambda}=[G(0)]^p$, theorem56 follows.

6.5 Appendix

In this section, we prove proposition39 which is a consequence of the following two lemmas.

Lemma 57 *Assume that (6.6) holds and u is a nonnegative solution of (6.3)-(6.4). Then*

(i) *The derivative u' vanishes exactly once in $(0,1)$.*

(ii) *The solution u is symmetric with respect to $\frac{1}{2}$.*

Lemma 58 *Assume that (6.7) holds and u is a nonnegative solution of problem (6.3)-(6.4). Then*

(i) *The solution u is symmetric with respect to $\frac{1}{2}$.*

(ii) *The derivative u vanishes exactly once in $(0,1)$.*

Proof. of lemma57

Assume that u is a nonnegative solution of problem (6.3)-(6.4).

Proof of Assertion **(i)**: By Rolle's theorem u' vanishes at least once. On other hand since $f(u) < 0$ for all $u \geq 0$ and $\lambda > 0$. By the equation (6.3) it follows that $\varphi_p(u'(.))$ is strictly increasing. By $u' = |\varphi_p(u')|^{\frac{2-p}{p-1}} \varphi_p(u')$ it follows that u' is strictly increasing. Then the derivative u' vanishes exactly once in $(0,1)$. Therefore Assertion **(i)** is proved.

Proof of Assertion **(ii)**: By Assertion **(i)** of the present lemma, let x_0 be the unique critical point of u. Thus, $u(x_0) = \min_{0<x<1} u(x)$ and

$$u' < 0 \text{ in } (0, x_0) \text{ and } u' > 0 \text{ in } (x_0, 1)$$

Multiplying the equation (6.3) by u' and integrating the resulting equation over (x, x_0) (respectively (x_0, x)), we obtain

$$u'(x) = -\left[\frac{p}{p-1}\lambda\left(F(u(x_0)) - F(u(x))\right)\right]^{\frac{1}{p}} \text{ for all } x \in (0, x_0]$$

and

$$u'(x) = \left[\frac{p}{p-1}\lambda\left(F(u(x_0)) - F(u(x))\right)\right]^{\frac{1}{p}} \text{ for all } x \in [x_0, 1)$$

Then,

$$x = \int_{u(x)}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\lambda\left(F(u(x_0)) - F(\xi)\right)\right]^{\frac{1}{p}}} \text{ for all } x \in (0, x_0] \tag{6.51}$$

and

$$1 - x = \int_{u(x)}^{+\infty} \frac{d\xi}{\left[\frac{p}{p-1}\lambda\left(F(u(x_0)) - F(\xi)\right)\right]^{\frac{1}{p}}} \text{ for all } x \in [x_0, 1) \tag{6.52}$$

Taking $x = x_0$ in (6.51) and (6.52) we obtain that $x_0 = \dfrac{1}{2}$.

Let x_1 and x_2 in $(0,1)$ such that $x_1+x_2=1$ and $x_1 \in \left(0,\frac{1}{2}\right]$. Take $x=x_1$ in (6.51) and $x=x_2$ in (6.52) and substracting to obtain

$$\int_{u(x_1)}^{u(x_2)} \frac{d\xi}{\left[\frac{p}{p-1}\lambda\left(F\left(u\left(x_0\right)\right)-F\left(\xi\right)\right)\right]^{\frac{1}{p}}}=0$$

then, $u\left(x_1\right)=u\left(x_2\right)$ which means that u is symmetric with respect to $\frac{1}{2}$. Therefore, Assertion **(ii)** is proved and lemma57 is proved.

To prove lemma58, we need the following lemma

Lemma 59 *Let $p>1$ and assume that u is a nonnegative solution of problem (6.3)-(6.4). Then*

$$\left(\left|u'\right|^p(x)+\frac{p}{p-1}\lambda F\left(u\left(x\right)\right)\right)'=0, \textit{ for all } x\in(0,1)$$

Proof. : The proof is similar to that of lemma 7 in [3].

Proof. of lemma58

Assume that u is a nonnegative solution of problem (6.3)-(6.4).

Proof of Assertion **(i):** Let x_0 be the point at which u attains its minimum. Denote $u\left(x_0\right)=\rho\geq 0$. Thus $u'\left(x_0\right)=0$, u is a solution of the following problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)'=\lambda f\left(u\right) \\ u\left(x_0\right)=\rho,\ u'\left(x_0\right)=0 \end{cases} \tag{6.53}$$

Let $v\left(x\right)=u\left(2x_0-x\right)$. This is also a solution of (6.53). Since f is locally Lipschitzian and $1<p\leq 2$ the problem (6.53) has a unique solution. Then $u\left(x\right)=v\left(x\right)$ for all x. Arguing by contradiction we obtain $x_0=\frac{1}{2}$. Assume that $x_0<\frac{1}{2}$, then

$$+\infty=\lim_{x\to 0^+} u\left(x\right)=\lim_{x\to 0^+} v\left(x\right)=u\left(2x_0\right)$$

which is impossible. The assumption $x_0>\frac{1}{2}$, leads to contradiction in a similar way. Then we must have $x_0=\frac{1}{2}$ and thus u is symmetric with respect to $\frac{1}{2}$. Therefore Assertion **(i)** is proved. .

Proof of Assertion **(ii)**: For any $\rho \geq 0$ and any $\lambda > 0$ we define $u(x, \lambda, \rho)$ to be the solution to the initial value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda f(u) \\ u\left(\frac{1}{2}\right) = \rho,\ u'\left(\frac{1}{2}\right) = 0 \end{cases} \tag{6.54}$$

Since f is locally Lipschitzian and $1 < p \leq 2$, the problem (6.54) has a unique maximal global nonnegative solution in the interval $\left[\frac{1}{2}, 1\right)$. Let us denote this solution by $u(x, \lambda, \rho)$. On other hand by Assertion **(i)** of the present lemma we see that the set of nonnegative solutions of (6.3)-(6.4) is precisely the set of solutions of (6.54) for which

$$u(x, \lambda, \rho) \geq 0 \text{ for all } x \in \left[\frac{1}{2}, 1\right) \text{ and } \lim_{x \to 1^-} u(x, \lambda, \rho) = +\infty \tag{6.55}$$

Let us show that the solution $u(x, \lambda, \rho)$ of problem (6.54) which satisfying (6.55) is strictly decreasing with respect to x. Arguing by contradiction, let $u(x) = u(x, \lambda, \rho)$ and let c be the minimal value for which there exists $x_1 < x_2$ such that $u(x_1) = u(x_2) = c$. Then one of the derivative $u'(x_1)$ and $u'(x_2)$ is 0, hence according to lemma59 the other one is 0, too. Using this we define the following solution of (6.54)

$$v(x) = \begin{cases} u(x) \text{ if } x < x_1 \\ u(x + x_2 - x_1) \text{ if } x \geq x_2 \end{cases}$$

Then $\lim_{x \to 1 - x_2 + x_1} v(x) = +\infty$. This contradicts the maximality of u. Then u is strictly increasing in $\left(\frac{1}{2}, 1\right)$. Therefore, Assertion **(ii)** is proved and lemma58 is proved.

Chapter 7

A quasilinear elliptic problem with nonlocal boundary conditions

Abstract.- In this work, we investigate the existence of solutions of the multipoint boundary value problem

$$\begin{cases} -\Delta_p u &= f(x,u) \text{ in } \Omega \\ u(x) &= \int\limits_{\Omega} \Phi(x,y)\, u(y)\, dy \text{ on } \partial\Omega \end{cases}$$

where $\Delta_p = div\left(|\nabla u|^{p-2} \nabla u\right)$, $p \in]1,+\infty[$; Ω is a bounded domain in $\mathbf{R}^N$ of a class $C^{1,\alpha}$, $0 < \alpha < 1$, with smooth boundary $\partial\Omega$, $f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and $\Phi : \partial\Omega \times \bar{\Omega} \to \mathbf{R}_+$ is a smooth function. We rely on the upper and lower solutions method to provide a constructive method for obtaining at least one solution.

7.1 Introduction

The purpose of this chapter is to study the existence of solutions for a class of second order quasilinear partial differential equations subject to nonlocal boundary conditions. More specifically, we consider the following nonlinear multipoint boundary value problem

$$\begin{cases} -\Delta_p u = f(x,u) \text{ in } \Omega \\ u(x) = \int_{\Omega} \Phi(x,y)\, u(y)\, dy \text{ on } \partial\Omega \end{cases} \tag{7.1}$$

where $\Delta_p = div\left(|\nabla u|^{p-2} \nabla u\right)$, $p \in]1,+\infty[$; Ω is a bounded domain of a class $C^{1,\alpha}$,$0 < \alpha < 1$, with smooth boundary $\partial\Omega$ $f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and $\Phi : \partial\Omega \times \bar{\Omega} \to \mathbf{R}_+$ is a smooth function.

Mathematical models leading to the above so called "nonlocal" boundary value problem, were first investigated by Il'in.V and Moiseev E (see [116]). Recently, several papers have been devoted to the study of problem (7.1) in the one dimensional case for $p = 2$. More precisely, the problem considered is

$$\begin{cases} -u'' = f_1(t,u) \quad 0 < t < 1 \\ u(0) = \sum_{i=1}^{m_1} b_i u(\eta_i) \\ u(1) = \sum_{j=1}^{m_2} c_j u(\delta_j) \end{cases} \tag{7.2}$$

where $f_1 : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function, $\eta_i \in (0,1)$, $b_i \in \mathbf{R}_+$, $i = 1,2,...,m_1$, $\delta_j \in (0,1)$ and $c_j \in \mathbf{R}_+$, $j = 1,2,...,m_2$ (see for instance [90], [104], [105] and [106]). In general, the analysis is done by reducing the problem (7.2) to a three point problem with boundary conditions $u(0) = 0$, $u(1) = bu(\sigma)$ where $b \in \mathbf{R}$ and $\sigma \in (0,1)$. The case $b = 1$ has been investigated in ([46], [106] and [148]).

In almost all the above papers, the main assumption is that f_1 is allowed to grow linearly (see [104], [105] and [106]). In [90], the authors assume $f_1 = g_1 + h$, where g_1 satisfies a sign condition and h is allowed a nonlinear growth. In [46], the authors used an integral monotonicity condition which generalizes the usual sign condition. All these condition have proven sufficient condition which generalizes the usual sign condition.

in order to obtain a priori bounds on solutions. Then degree theoretical methods are used to prove existence.

In the present work, we consider the case of the general boundary conditions in the higher dimension ($N > 1$) and assumptions on f which are more general than those used in the above papers. We shall rely on upper and lower solutions method. At our Knowledge, this is the first time that the method of upper and lower solutions is used in the context of quasilinear elliptic problem with nonlocal boundary conditions. As by product, we will provide a constructive method to get at least one solution of problem (7.1). The chapter is organized as follows. In section 2, we present some notations and definitions that will be used through the paper. In section 3, we state and prove a preliminary result. Section 4 is devoted to our main result.

7.2 Definitions and notations

Let Ω be a bounded domain in $\mathbf{R}^N$ of a class $C^{1,\alpha}$, $0 < \alpha < 1$ and $|\Omega|$ its Lebesgue measure. For $k = 0, 1, 2, ...$, let $C^k(\bar{\Omega})$ denote the space of real valued functions which are k-times continuously differentiable on $\bar{\Omega}$. For $u \in C^k(\bar{\Omega})$, we define its norm by

$$\|u\|_k = \max\left(\left\|D^j u\right\|_0\right)_{0 \leq j \leq k}$$

where $\|v\|_0 = \sup\left\{v(t)\,;\ t \in \bar{\Omega}\right\}$ and $D^j u = \frac{\partial^{|j|} u}{\partial x_1^{j_1} ... \partial x_N^{j_N}}$ with $j = (j_1, ..., j_N)$, j_i =integer≥ 0, and $|j| = j_1 + ... + j_N$. The space of real valued functions which are k-times continuously differentiable whose k-th order derivatives are Hölder continuous with exponent $\gamma \in (0,1)$ is denoted by $C^{k,\gamma}(\bar{\Omega})$. For $u \in C^{k,\gamma}(\bar{\Omega})$, we define its norm by

$$\|u\|_{k+\gamma} = \|u\|_k + \sup_{x \neq y \in \Omega} \frac{\left|D^k u(x) - D^k u(y)\right|}{|x-y|^\gamma}$$

The space of mesurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable over Ω is denoted by $L^p(\Omega)$. We denote the norm in $L^p(\Omega)$ by $\|u\|_p$. Also, we shall refer to the sobolev space $W^{1,p}(\Omega)$, which may be defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega)\,;\ \nabla u \in L^p(\Omega)\}$$

with the norm $\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p$ and $W_0^{1,p}(\Omega)$ the completition of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W^{1,p}}$, where $C_0^\infty(\Omega)$ is the space of smooth real-valued functions with compact support in Ω.

Definition 1: By a solution of problem (7.1), we mean a function $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{cases} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_\Omega f(x,u) \varphi dx \ \forall \varphi \in W_0^{1,p}(\Omega) \\ u(x) = \int_\Omega \Phi(x,y) u(y) dy \text{ on } \partial\Omega \end{cases}$$

Definition 2: A function $\bar{U} \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ is called an upper solution for problem (7.1) if

$$\begin{cases} \int_\Omega |\nabla \bar{U}|^{p-2} \nabla \bar{U} \nabla \varphi dx \geq \int_\Omega f(x,\bar{U}) \varphi dx \ \forall \varphi \in W_0^{1,p}(\Omega), \ \varphi \geq 0 \\ \bar{U}(x) \geq \int_\Omega \Phi(x,y) \bar{U}(y) dy \text{ on } \partial\Omega \end{cases}$$

Definition 3: A function $\underline{U} \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ is called a lower solution for problem (7.1) if

$$\begin{cases} \int_\Omega \left|\nabla \underline{U}\right|^{p-2} \nabla \underline{U} \nabla \varphi dx \leq \int_\Omega f\left(x,\underline{U}\right) \varphi dx \ \forall \varphi \in W_0^{1,p}(\Omega), \ \varphi \geq 0 \\ \underline{U}(x) \leq \int_\Omega \Phi(x,y) \underline{U}(y) dy \text{ on } \partial\Omega \end{cases}$$

7.3 Preliminary result

In this section, we state and prove a preliminary result. On the nonlinearity f, we shall impose the following condition:

(H1) $f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$, is a continuous function and such that there exists a continuous function $\Theta : \bar{\Omega} \to \mathbf{R}$ with the property that

$$f(x,u) - f(x,v) \geq -\Theta(x) \left(|u|^{p-2} u - |v|^{p-2} v\right) \text{ for all } x \in \bar{\Omega} \text{ and all } u, \ v \in \mathbf{R}.$$

Theorem 60 *Assume* ***(H1)*** *is satisfied. Suppose that problem (7.1) has a lower solution* $\underline{U}$ *and an upper solution* $\bar{U}$ *such that* $\underline{U} \leq \bar{U}$. *Then problem (7.1) has a solution* $\underline{U} \leq u \leq \bar{U}$.

Proof. It follows from **(H1)** that there exist a constant $M > 0$ such that the function

$$s \mapsto f(x, s) + M |s|^{p-2} s$$

is increasing for all $x \in \bar{\Omega}$.

Define a sequence of functions (u_k) in the following way

$$\begin{cases} u_0 = \bar{U} \\ -\Delta_p u_{k+1}(x) + M |u_{k+1}(x)|^{p-2} u_{k+1}(x) = L(x, u_k(x)), \; x \in \Omega \\ u_{k+1}(x) = \int\limits_{\Omega} \Phi(x, y) u_k(y) \, dy, \; x \in \partial\Omega \end{cases} \tag{7.3}$$

where $L(x, u_k(x)) := f(x, u_k(x)) + M |u_k(x)|^{p-2} u_k(x)$

Note that the sequence is well defined (see theorem71 in the Appendix).

Lemma 61 *We have* $\underline{U} \leq u_k \leq \bar{U}$ *for all* $k \in \mathbf{N}$.

Proof. Suppose, by induction, that $\underline{U} \leq u_j \leq \bar{U}$ for all $j = 0, 1, ..., k$.

Let $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$

It follows from the definition of the upper solution $\bar{U}$ and (7.3) that

$$\begin{aligned} & \int_{\Omega} \left(|\nabla u_{k+1}|^{p-2} \nabla u_{k+1} - \left|\nabla \bar{U}\right|^{p-2} \nabla \bar{U} \right) \nabla \varphi dx \\ & + M \int_{\Omega} \left(|u_{k+1}|^{p-2} u_{k+1} - \left|\bar{U}\right|^{p-2} \bar{U} \right) \varphi dx \\ \leq & \int_{\Omega} \left(L(x, u_k(x)) - L\left(x, \bar{U}(x)\right) \right) \varphi dx \\ \leq & \; 0 \end{aligned}$$

Then we obtain

$$\int_{\Omega} |\nabla u_{k+1}|^{p-2} \nabla u_{k+1} \nabla \varphi dx + M \int_{\Omega} |u_{k+1}|^{p-2} u_{k+1} \varphi dx$$
$$\leq \int_{\Omega} |\nabla \bar{U}|^{p-2} \nabla \bar{U} \nabla \varphi dx + M \int_{\Omega} |\bar{U}|^{p-2} \bar{U} \varphi dx$$

By the weak comparaison principle (see theorem72 in the Appendix), we have $u_{k+1} \leq \bar{U}$ Hence, we have that

$$u_k \leq \bar{U} \text{ for all } k \in \mathbf{N}$$

Similarly, we can prove that

$$\underline{U} \leq u_k \text{ for all } k \in \mathbf{N}$$

Lemma 62 *If* $\left(\underline{U}, \bar{U}\right) \in \left(C^{1,\alpha}\left(\bar{\Omega}\right)\right)^2$, *then there exists a positive number* $C = C\left(N, |\Omega|, K, \alpha, p\right) > 0$, *with* $K = \max\left(\left\|\underline{U}\right\|_0, \left\|\bar{U}\right\|_0\right)$ *such that* $u_k \in C^1\left(\bar{\Omega}\right)$ *and* $\|u_k\|_1 \leq C$ *for all* $k \in \mathbf{N}$.

Proof. See lemma2 p.54 in [94].

As $(u_k)_{k \in \mathbb{N}}$ is bounded in $C^1\left(\overline{\Omega}\right)$, this implies, by Ascoli-Arzelà compactness theorem, that, after passing to a subsequence,$u_k \to u$,the limit function u belongs to $C^1\left(\overline{\Omega}\right)$ and satisfies (7.1) in the sense of definition 1.

7.4 Main result

In this section, we state and prove our main result to prove existence of solutions of problem (7.1).

Assume that f satisfies the following conditions

(H2) $f(x, -u) = -f(x, u)$ for all $(x, u) \in \Omega \times \mathbf{R}$.

(H3) There exists a continuous function $g : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$f(x,u) \leq g(u) \text{ for all } (x,u) \in \Omega \times \mathbf{R}_+$$

In this case, we shall refer to problem (7.1) as problem (P_0).

Assume further that the function g in (**H3**) satisfies (**H4**):

(i) There exists a positive constant ε such that

$$g(u) \geq \varepsilon > 0 \text{ for all } u \in \mathbf{R}_+$$

(ii) $\lim\limits_{|u|\to+\infty} \dfrac{pG(u)}{u^p} < l$

where $G(u) := \int_0^u g(s)\,ds$ and $l = l(\Omega, p)$ is given by

$$l = \left[\frac{(p-1)^{\frac{1}{p}}\pi}{R(\Omega)\, p \sin\frac{\pi}{p}} \right]^p$$

where $R(\Omega)$ is the radius of the smallest ball containing Ω.

Theorem 63 *If the assumptions* (**H1**), (**H2**), (**H3**), *and* (**H4**), *are satisfied, then* (P_0) *has at least one solution.*

Proof. Let M_1 be a strictly positive number and consider the boundary value problem

$$\begin{cases} -\Delta_p v = g(v) \text{ in } B \\ v \geq M_1 \text{ in } B \end{cases} \tag{7.4}$$

where B is the smallest ball containing Ω.

We look for a radially symmetric solution of (7.4).

A radially symmetric solution of (7.4) satisfies

$$\begin{cases} -\left(|v'|^{p-2}v'\right)' - \dfrac{N-1}{r}|v'|^{p-2}v' = g(v) \\ v'(0) = 0 \text{ and } v \geq M_1 \text{ on } [0, R(\Omega)] \end{cases} \tag{7.5}$$

Consider the initial value problem

$$\begin{cases} -\left(|v'|^{p-2}v'\right)' - \dfrac{N-1}{r}|v'|^{p-2}v' = g(v) \\ v(0) = d,\ v'(0) = 0 \end{cases} \tag{7.6}$$

Put $\omega = |v'|^{p-2}v'$, the equation in (7.6) becomes

$$\begin{cases} v' = |\omega|^{\frac{2-p}{p-1}}\omega \\ \omega' = \dfrac{1-N}{r}\omega\text{-}g(v) \end{cases} \tag{7.7}$$

Lemma 64 *For any $d \in \mathbf{R}$ there exists a solution v of (7.6) defined on a maximal interval $[0, R_d[$, with $v \in C^1([0, R_d[)$ and $|v'|^{p\text{-}2}v' \in C^1([0, R_d[)$.*

Proof. Take $\eta > 0$ and consider the mapping defined by

$$(Lv)(t) = d - \int_0^t |(Hv)(s)|^{\frac{2-p}{p-1}} (Hv)(s)\, ds$$

in the Banach space $C([0, \eta])$, where

$$(Hv)(s) = s^{1\text{-}N}\int_0^s r^{N\text{-}1}g(v(r))\, dr$$

It is easily to show that the operator L is compact.

We will show that for $\eta > 0$ sufficiently small, L map some ball of $C([0, \eta])$ into itself. For that purpose take $\delta > 0$ such that $|g(s+d) - g(s)| \leq 1$ for $|s| \leq \delta$.

It follows that if $|v(t)\text{-}d| \leq \delta$, then

$$\begin{aligned} |(Lv)(t)\text{-}d| &= \left|\int_0^t \left[\left|s^{1\text{-}N}\int_0^s r^{N\text{-}1}g(v(r))\, dr\right|^{\frac{2-p}{p-1}} s^{1\text{-}N}\int_0^s r^{N\text{-}1}g(v(r))\, dr\right] ds\right| \\ &\leq \tfrac{p-1}{p}\left[\frac{g(d)+1}{N}\right]^{\frac{1}{p-1}} \eta^{\frac{p}{p-1}} \\ &\leq \delta \end{aligned}$$

for $\eta > 0$ sufficiently small.

Consequently, for such an η, L admits a fixed point v in $C([0,\eta])$ .It is easily seen that this function v is a solution of (7.6), with $v \in C^1([0,\eta])$ and $|v'|^{p-2} v' \in C^1([0,\eta])$ Applying the general theory of first order system with continuous right hand side, we can extend v over a maximal interval $[0, R_d[$.

Lemma 65 *The solution v of problem (7.6) is defined on* $[0, +\infty[$.

Proof. Suppose there exist a sequence $(t_n)_{n\in\mathbf{N}}$ which tends to $t_* \in [0, +\infty[$ such that

$$\lim_{n\to+\infty} \left(|v(t_n, d)| + |v'(t_n, d)|\right) = +\infty$$

The mean value theorem shows that

$$\lim_{n\to+\infty} v'(t_n, d) = v'(t_*, d) = +\infty$$

Let

$$E(t, d) := \frac{p-1}{p} |v'(t, d)|^p + G(v(t, d))$$

We have

$$\frac{dE(t, d)}{dt} = \frac{1-N}{r} |v'(t, d)|^p \leq 0$$

Then

$$\begin{aligned} E(t_*, d) &\leq E(0, d) \\ &= G(d) \end{aligned}$$

This is a contradiction. ■

Now consider the time-map

$$T(d) := \int_0^d \left[\frac{p}{p-1}(G(d) - G(u))\right]^{-\frac{1}{p}} du$$

Lemma 66 *We have* $\lim\limits_{d\to+\infty} T(d) > R(\Omega)$.

Proof. First, we have $T(d) < +\infty$ for all $d > 0$. In fact, the mean value theorem implies that there exists $\sigma \in (u, d)$ such that

$$G(d) - G(u) = (d-u)\, g(\sigma)$$

This gives that

$$T(d) = \int_0^d \left[\frac{p}{p-1} g(\sigma)(d-u)\right]^{-\frac{1}{p}} du$$

It follows from (H4.**i**) that

$$\begin{aligned} T(d) &\leq \left[\frac{p-1}{p\varepsilon}\right]^{\frac{1}{p}} \int_0^d [d-u]^{-\frac{1}{p}} du \\ &= \left[\frac{p-1}{p\varepsilon}\right]^{\frac{1}{p}} d^{\frac{p-1}{p}} \int_0^1 [1-v]^{-\frac{1}{p}} dv \end{aligned}$$

which shows that $T(d) < +\infty$. Moreover, the improper integral defining $T(d)$ is uniformly convergent in each subinterval of $\mathbf{R}_+^*$ which implies that T is continuous on $\mathbf{R}_+^*$.

Now we have

$$\begin{aligned} \lim_{d\to+\infty} T(d) &= \lim_{d\to+\infty} \int_0^d \left[\frac{p}{p-1}\left(G(d) - G(u)\right)\right]^{-\frac{1}{p}} du \\ &= \int_0^1 \lim_{d\to+\infty} \left[\frac{p}{p-1} \frac{\left(G(d) - G(dv)\right)}{d^p}\right]^{-\frac{1}{p}} dv \\ &= \frac{(p-1)^{\frac{1}{p}} \pi}{l_1^{\frac{1}{p}} p \sin\frac{\pi}{p}} > R(\Omega) \end{aligned}$$

with $l_1 = \lim\limits_{u\to+\infty} \dfrac{pG(u)}{u^p}$ ■

Proposition 67 *For each $M_1 > 0$, there exist a solution v of (7.5) defined on $[0, R(\Omega)]$ such* *that* $v \in C^2([0, R(\Omega)])$, $v'(0) = 0$ *and* $v \geq M_1$ *in* $[0, R(\Omega)]$.

Proof. Let v be a solution of (7.5) defined on $[0, +\infty[$. If v does not remain above M_1, then there exist a first instant $t_d < +\infty$ such that $v(t_d) = M_1$. Multiplying the first equation in (7.7) by $g(v)$, the second by $|\omega|^{\frac{2-p}{p-1}}\omega$ and adding, we obtain

$$g(v)\,v' + \omega' |\omega|^{\frac{2-p}{p-1}}\omega = \frac{1-N}{r}|\omega|^{\frac{p}{p-1}} \leq 0$$

That is

$$\frac{d}{dt}\left(G(v(t)) + \frac{p-1}{p}|v'(t)|^p\right) \leq 0$$

Then

$$G(v(t)) + \frac{p-1}{p}|v'(t)|^p \leq G(d)$$

Since $v' < 0$, we obtain

$$\frac{-v'(t)}{\left[\frac{p}{p-1}(G(d) - G(v(t)))\right]^{\frac{1}{p}}} \leq 1$$

on $]0, +\infty]$. Integrating from 0 to t_d, we obtain

$$\int_{M_1}^{d} \left[\frac{p}{p-1}(G(d) - G(v(t)))\right]^{-\frac{1}{p}} ds \leq t_d$$

Since

$$\int_{0}^{M_1} \left[\frac{p}{p-1}(G(d) - G(v(t)))\right]^{-\frac{1}{p}} ds \to 0 \text{ as } d \to +\infty$$

we obtain

$$\lim_{d\to+\infty} t_d \geq \lim_{d\to+\infty} T(d) > R(\Omega)$$

Lemma 68 *Let v be a solution of problem (7.4). Then $V = v\,|_{\Omega}$ the restriction of v to Ω is an upper solution for problem (P_0).*

Proof. The definition of V and condition (**H3**) imply that

$$-\Delta_p V = g(V) \geq f(x,V) \text{ for all } x \in \Omega$$

on the other hand if we choose $\int_{\Omega} \Phi(x,y)\,dy \leq \rho$ with ρ sufficiently small, we obtain

$$\int_{\Omega} \Phi(x,y)\,V(y)\,dy \leq M_1$$

Lemma 69 *-V is a lower solution for problem (P_0).*

Proof. We have

$$-\Delta_p V = g(V) \text{ on } \Omega$$

Hence, by (**H3**)

$$\begin{aligned} \Delta_p V &= -g(V) \\ &\leq -f(x,V) \text{ on } \Omega \end{aligned}$$

Now, (**H2**) yields

$$-\Delta_p(-V) \leq f(x,-V) \text{ on } \Omega$$

Also

$$
\begin{aligned}
-V(x) &\leq -M_1 \\
&\leq -\int_{\Omega} \Phi(x,y) V(y)\, dy
\end{aligned}
$$

This complete the proof.

Now, we see that all assumptions of theorem (60) are satisfied. Hence problem (P_0) has a solution.

7.5 Appendix

The following result will be needed in the sequel

Lemma 70 *Let x, $y \in \mathbf{R}^N$ and $\langle .,. \rangle$ the standard scalar product in $\mathbf{R}^N$. Then*

$$
\left\langle |x|^{p-2}x, |y|^{p-2}y \right\rangle \geq \begin{cases} c_p |x-y|^p, & \text{if } p \geq 2 \\ c_p \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2 \end{cases}
$$

Proof. See Lemma A.0.5 pp78 in [170].

Consider the following Dirichlet problem

$$
\begin{cases} -\Delta_p u + a|u|^{p-2}u = \tilde{g} \text{ on } \Omega \\ u = \tilde{h} \text{ on } \partial\Omega \end{cases} \tag{7.8}
$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$,
$\tilde{g} \in W^{-1,p'}(\Omega)$, $p' = \frac{p}{p-1}$, $\tilde{h} \in W^{\frac{1}{p'},1}(\partial\Omega)$ and $a > 0$.
We have the following result

Theorem 71 *The problem (7.8) has a unique solution $u \in W^{1,p}(\Omega)$ in the weak sense, namely*

$$
\begin{cases} \displaystyle\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi dx + a \int_{\Omega} |u|^{p-2} u\varphi dx = \int_{\Omega} \tilde{g}\varphi dx \;\; \forall \varphi \in W_0^{1,p}(\Omega) \\ u(x) = \tilde{h}(x) \;\; on \;\; \partial\Omega \end{cases}
$$

Proof. Let us introduce the subset of $W^{1,p}(\Omega)$

$$K = \left\{ v \in W^{1,p}(\Omega) : v = \tilde{h} \text{ on } \partial\Omega \right\}$$

It is easily to show that K is nonempty, closed and convex.

Note that u is a solution of (7.8) if and only if

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u \nabla (v-u)\, dx + a \int_\Omega |u|^{p-2} u (u-v)\, dx \\ &\geq \int_\Omega \tilde{g}(v-u)\, dx \quad \forall v \in K \end{aligned} \tag{7.9}$$

Applying theorem 8.2 pp247 in [144] the inequality (7.9) has a solution.

Now suppose that (7.8) has two solutions u_1 and u_2. Then we have

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla (v-u_1)\, dx + a \int_\Omega |u_1|^{p-2} u_1 (u_1-v)\, dx \\ &\geq \int_\Omega \tilde{g}(v-u_1)\, dx \quad \forall v \in K \end{aligned} \tag{7.10}$$

$$\begin{aligned} &\int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla (v-u_2)\, dx + a \int_\Omega |u_2|^{p-2} u_2 (u_2-v)\, dx \\ &\geq \int_\Omega \tilde{g}(v-u_2)\, dx \quad \forall v \in K \end{aligned} \tag{7.11}$$

If we put $v = u_2$ (resp $v = u_1$) in (7.10) (resp in (7.11)) and additing, we obtain

$$\begin{aligned} &\int_\Omega \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) \nabla (u_1-u_2)\, dx \\ &+a \int_\Omega \left(|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 \right) (u_1-u_2)\, dx \leq 0 \end{aligned}$$

Put

$$\begin{aligned} I \ : \ = &\int_\Omega \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) \nabla (u_1-u_2)\, dx \\ &+a \int_\Omega \left(|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 \right) (u_1-u_2)\, dx \end{aligned}$$

It follows from lemma70 that

$$I \geq \begin{cases} C_1 \int_\Omega |\nabla (u_1 - u_2)|^p \, dx + C_2 \int_\Omega |u_1 - u_2|^p \, dx & \text{if } p \geq 2 \\ C_3 \int\limits_\Omega \frac{|\nabla (u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx + C_4 \int\limits_\Omega \frac{|(u_1 - u_2)|^2}{(|u_1| + |u_2|)^{2-p}} dx & \text{if } 1 < p < 2 \end{cases}$$

where C_i are positive constants for all $i \in \{1, 2, 3, 4\}$

Then if $p \geq 2$, we have

$$C_1 \int\limits_\Omega |\nabla (u_1 - u_2)|^p \, dx + C_2 \int\limits_\Omega |u_1 - u_2|^p \, dx \leq 0$$

We obtain that

$$C_1 \|\nabla (u_1 - u_2)\|_p + C_2 \|(u_1 - u_2)\|_p \leq 0$$

This implies that

$$u_1 = u_2$$

If $1 < p < 2$, we have

$$C_3 \int\limits_\Omega \frac{|\nabla (u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx + C_4 \int\limits_\Omega \frac{|(u_1 - u_2)|^2}{(|u_1| + |u_2|)^{2-p}} dx \leq 0$$

Then by Hölder inequality

$$\begin{aligned} & C_3 \int_\Omega |\nabla (u_1 - u_2)|^p \, dx + C_4 \int_\Omega |u_1 - u_2|^p \, dx \\ & \leq C_3 \left(\int\limits_\Omega \frac{|\nabla (u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int\limits_\Omega (|\nabla u_1| + |\nabla u_2|)^p \right)^{\frac{2-p}{p}} \\ & + C_4 \left(\int\limits_\Omega \frac{|u_1 - u_2|^2}{(|u_1| + |u_2|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int\limits_\Omega (|u_1| + |u_2|)^p \right)^{\frac{2-p}{p}} \\ & \leq 0 \end{aligned}$$

This implies that

$$C_3 \left\|\nabla \left(u_1 - u_2\right)\right\|_p + C_4 \left\|\left(u_1 - u_2\right)\right\|_p \leq 0$$

Then consequently it follows that

$$u_1 = u_2$$

Theorem 72 (*Weak comparaison principle*) *Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. Let $u_1, u_2 \in W^{1,p}(\Omega)$ and $a > 0$ satisfy*

$$\int_\Omega \left(\left|\nabla u_1\right|^{p-2} \nabla u_1 \nabla \psi + a \left|u_1\right|^{p-2} u_1 \psi\right) dx \leq \int_\Omega \left(\begin{array}{c} \left|\nabla u_2\right|^{p-2} \nabla u_2 \nabla \psi \\ +a \left|u_2\right|^{p-2} u_2 \psi \end{array}\right) dx \tag{7.12}$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$, that is

$$-\Delta_p u_1 + a \left|u_1\right|^{p-2} u_1 \leq -\Delta_p u_2 + a \left|u_2\right|^{p-2} u_2$$

in the weak sense.

Then the inequality

$$u_1 \leq u_2 \text{ on } \partial\Omega$$

implies that

$$u_1 \leq u_2 \text{ in } \Omega$$

Proof. Let $\psi = \max\{u_1 - u_2, 0\}$. Since $u_1 \leq u_2$ on $\partial\Omega$, so ψ belongs to $W_0^{1,p}(\Omega)$. Inserting this function ψ into (7.12), we have

$$\int_{\{u_1 > u_2\}} \left(\left|\nabla u_1\right|^{p-2} \nabla u_1 - \left|\nabla u_2\right|^{p-2} \nabla u_2\right) \left(\nabla u_1 - \nabla u_2\right) dx$$

$$+a \int_{\{u_1>u_2\}} \left(|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 \right) (u_1 - u_2) \, dx \leq 0$$

Therefore using lemma 70, we obtain

$$u_1 \leq u_2 \text{ in } \Omega$$

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