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A mes parents

A ma sœur

A mon frère

HOMMAGE

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Acknowledgement

Chapter 1

Introduction

The work presented here belongs to a project titled "Modelling of the larval stage of the anchovy of the bay of Biscay. Estimation of the rate of recruitment in the juvenile stage". This project has mainly for goal modelling the growth and survival of larvae in an environment made up of both physical and biological traits; the model considered is a system of three partial of differential equations; one for the larvae, one for the weight of larvae and one for the phytoplankton assumed to be the main food for early larvae (before maturity i.e. until they acquire vertical movement activity by their own) The equation for the larvae describes the variation of concentrations due to physical process only. The equation for the phytoplankton focuses on the variation entailed by physical and biological processes. Coupling of larvae and phytoplankton is accounted for, in the third equation. This model was theoretically developed and studied in the simplest case where the coefficients of diffusion were neglected and so first order hyperbolic system is obtained. However in practice; the lack of data for the phytoplankton, notably on the production of food concentrations made the model consisting of system of three PDE's impracticable. So the first model was revised and the system has been reduced to a single equation encompassing both the physical influence on the variation of larval concentrations and temperature dependent growth laws. Because of the lack of data about phytoplankton the growth is described as function of the temperature, we deal in this thesis with the latter model. In the next paragraph we present the work done for collecting data. Biological data were extracted from egg surveys, the sampling protocol consists in taking samples of the tiny particles in suspension in the water, one sample for each cell 315 squared nautical miles of a grid covering the bay of Biscay. This was done in 2D, considering the whole mixed layer as a vertically homogeneous medium; calculating the average velocity in the mixed layer which required as preliminary, estimating the thermocline, averaging throughout the thermocline has the effect of cancelling the vertical term. Because concentration of eggs and biotic material are roughly constant throughout a station, horizontal diffusion was discarded. For egg data 78 stations were selected, they were divided into five groups; each group corresponding to all the stations sampled within a time interval less than 24 hours. Two options were considered; the first one assumes that what has been collected a given day at a given cell reflects the situation for this day only and can not be taken as a value of the egg production at this cell for the other days; the second one on the contrary assumes that what has been found one day at one place is what would be found any day at this place. Assuming that the maximum life span of eggs is three days initial cohort are built as follows : In the case of option 1 : On day 0 take the eggs aged 0 days from sample 1, on day 1 add the eggs aged 1 days from sample 2, on day 2 add to the above the eggs aged 2 days from sample 3. In the case of option 2 : Use the values for eggs aged 0 days found in each station as an estimate of the daily production of eggs in this section, this way is at day 0 it is made up of the distribution of eggs aged 0 days throughout the 78 stations. On of the most practical conclusions deduced from this is that the currents can act in such a way as to mix eggs of different stations within a small time.

P. Lazure and A-M Jegou see [32] are somewhat limited westward : some of the material is lost for westward migration outside the domain of the model; 2) the growth laws available are mostly restricted to the earliest larval stages. So, while the program set up could in principle simulate the dynamics for as long as the whole passive larval stage, its actual range is limited a little beneath its potential one by some limitations in the data. We also want to mention in this introduction some other works, old and new, related to ours. The account is not intended to be exhaustive : our intention when quoting such works is mainly to convince the reader, after we have convinced ourselves that, in spite of its absolute scarcity in the literature, the sort of domain we have undertaken has been considered of interest by several researchers and has even been attempted by some. Amongst the first model , one can quote one by W. J. Vlymen [51] who modelled the growth of the larvae at the beginning of exogenous nutrition from around 5mm of length to the one set of schooling for various levels of contagion of food organisms. Vlymen shows in particular that, with directed swimming, larvae could greatly enhance their growth rates by feeding in micropatchs of prey. This model is indeed complementary to ours, both by the period of the larva's life modelled, and also by the issue addressed , since the author focused on the relationship between a larva and its food, and did not consider the movement of the larvae within the sea. Closer to the approach followed in this thesis is a model of the drift of northern anchovy larvae in the California current by J. Power [39]. The paper [39] explores the role of the horizontal advection and diffusion on the movement of anchovy larvae. Two differences with our model are : 1) that the main emphasis is placed in horizontal effects of the physical environment, while our study stresses the role of vertical displacement; 2) no demographic processes are accounted for. The work which parallels our own's the most and was in fact a source of inspiration for our model is the work by J. S. Wroblewski, and subsequent work by Wroblewski and coworkers [52]. In his (1984) paper, Wroblewski investigates, by means of a simple model, the role of oceanographic conditions on the growth and mortality of anchovy larvae. The model considers on one hand, the prey, the phytoplankton , which being passive, is the most sensitive to the action of currents and turbulence both assumed to be essentially vertical, and on the other hand, the anchovy larvae biomass whose growth is supposed to vary as a function of the abundance of the prey. Assuming that the larvae are evenly distributed throughout the mixed layer, the growth and survival of anchovy will be affected by the distribution of the phytoplankton (in the mixed layer), and thus the environment impact indirectly on the anchovy recruitment. Later work by Wroblewski and his coworkers pursue the same line of thought with some merely technical improvements with respect to the early paper. Also relevant to this presentation is the work by P. Franks, alone or with coworkers: the main argument of the research undertaken by these authors is the strong link between the physical process and the growth and survival of planktonic species, counting possibly fish larvae in this category [22]. These authors combine the dynamics of water circulation and those of species growth in numerical simulations which for most of them, are organized as follows: the program solve in sequences the equations for the water circulation, assuming the demographic processes be suspended, then in the next time step, demographic processes (food uptake, birth, death, and possibly other such events) take a place in the absence of any movement. For the anchovy of the bay of Biscay, very few models seem to have been done along the lines we just explained. In fact, the only other model we know of is one by M. Gonzalez et al [27] where the main emphasis has been put in setting up a model of circulation in the bay of Biscay and using it to simulate the transport of a patch of eggs from the spawning areas located offshore of the Gironde estuary. While a comparison between the two oceanic models, the one by Gonzalez et al, and the one by Lazure and Jegou [32], is yet to be done, there are clear differences, as far as the population dynamics is concerned, since the model considered here accounts for heterogeneity in larval maturation due to the temperature-dependent maturation rates.

After introducing the work of modelization done upstream, the principal part of this work deals with solving and making qualitative study of problems born from modelization (by qualitative study we mean existence, positivity and uniqueness of solution); to this end we will use classical and less classical methods. The problem treated is

$$\begin{cases} \dfrac{\partial l}{\partial t} + \dfrac{\partial (fl)}{\partial s} + div\,(Vl) - \dfrac{\partial}{\partial z}(h\dfrac{\partial l}{\partial z}) + \mu l = 0, \\ l(t,1,x,y,z) = B(t,P), \end{cases}$$

with Newmann or Dirichlet conditions.
The main characteristic of this equation is that it has mixed parabolic-hyperbolic type, due to directional separation of the diffusion and convection effects: while a matter is convected along the y axis, it is simultaneously diffused along all orthogonal directions. Some authors call this equation ultraparabolic equation that is parabolic in many directions. Our goal is to prove existence, uniqueness and positivity to such problem. Firstly we will treat the case where the horizontal velocities V_1, V_2, and the growth function f do not depend in the vertical direction z. Under this assumption, it was possible to uncouple the vertical and the horizontal components in the following sense: the study was restricted to each of the horizontal streamlines and real line: the restriction to such a line reduces the functions of time horizontal components to functions of time so that the full model reduces on such a line to a diffusion equation in the vertical variable coupled with a first order growth equation, i.e. (one dimensional non autonomous parabolic equation coupled with a first hyperbolic equation). Time dependence is dealt with using results on time-dependent evolution equations by Acquistapace [1, 2, 3] and several other authors (Lunardi [34], Tanabe [45]). The main result of this case, ensures that, under some conditions on the coefficients of the equation, the Cauchy problem associated with the equation has a unique classical solution, which moreover is nonnegative if the initial value is non negative.

Secondly we will treat the general case where the horizontal current depends of all variables. In this situation we can not uncouple the vertical and the horizontal components. The principal difficulty is the lack of coercivity to our elliptic operator, i.e. equation with degenerated elliptic operator. The idea we exploit here is to perturb our equation by adding a vanishing artificial viscosity, in other terms a diffusion, in the missing directions (along the (x,y)-axis). The monotone operator theory can be applied see [30](p 316) which gives us existence, and uniqueness of the solution to the perturbed problem, after that we will establish the positivity of our solution. Passing to the limit in a suitable way, we get existence and positivity of a solution of the main model. Since the main operator is not coercive we obtain some extract regularity of the solution in the direction of x_3. After that we will treat the same model in the general case by another technique called multilayer methods, The idea is to approximate the model by one in which the above mentioned restriction is assumed to hold piecewise : this has been done by dividing the water column into thin layers in each of which it is reasonable to assume that the coefficients are constant throughout the vertical direction. The mathematical analysis of the problem leads to two main issues : 1) each approximating equation sets up a system of equations of parabolic type with time dependent coefficients and rather unusual boundary conditions. 2) The approximating solution converge in suitable way to a solution of the main equation, we will show existence and positivity of solution with such method. The advantage of this method is the fact that it give the practical approximate of the coefficients and this make numerical analysis of such equation easier, indeed we treat the parabolic and hyperbolic equation separately and then the convergence theorems remain the correct.
Finally we treat the following non linear model

$$\begin{cases} \dfrac{\partial l}{\partial t} + \operatorname{div}(Vl) - \sum_{i=1}^{i=3} \dfrac{\partial}{\partial x_i}\left(h_i \dfrac{\partial l}{\partial x_i}\right) + \mu(l)l = 0, \\ l(0,P) = l_0(P). \end{cases}$$

with homogeneous Dirichlet condition.

If we want to treat the above non linear model with neglecting the horizontal diffusion which is our future problem, among of the method is the same what treated in the section 3, that is perturbed the original problem to obtained a non linear parabolic equation. So the goal of this section is to show that the perturbed associated problem such above equation has a positive solution. In our knowledge the problem with neglecting the horizontal

diffusion remain an open problem. In chapter 4 we make a numerical analysis to such equation at least in the simple case where the horizontal diffusions do not depends of the vertical variable. As already mentioned, in this case we can uncouple our equation to first hyperbolic equation and one dimensional parabolic equation. The method treated here is to solve the system of ordinary equation by Runge-Kutta method and we injected the found solution in the parabolic equation. The main method treated here for parabolic equation is finite elements and finite volumes. We study the convergence for this last two methods. after we treat some example of ultraparabolic equation.
Finally we will give in appendix a large description of a work due to Escobedo, Vazquez and Zuazua [21], because it is -in our sense- the most important work done till now, for solving ultraparabolic problems with constant coefficients, we also believe that it can be generalized to some problems with non constant coefficients and this is the scope and the starting point of our future researches.

Chapter 2

Description of the model

2.1 Larval growth

The growth of larvae was described as a result of larvae eating phytoplankton. As already mentioned, the scarcity of data on the phytoplankton has rendered necessary to proceed differently. Inspired by the work of S.Regner [40] on the anchovy of the Adriatic Sea as well as the one by N. Lo [33] for Engraulix mordax, we considered data determined in the laboratory. Specific values for Engraulis encrasicolus of the bay of Biscay were obtained by L. Motos [35] in his thesis and are used here. We now explain the principle of determination of these data. In typical experiments, samples of anchovy in a primitive stage are thrown in basins raised at a given temperature and fed ad libitum. One then determines the number of days or hours that are necessary for the larvae to progress between to well identified stage. This yields duration of a certain stage as a function of temperature, for example :

$$D = A(1 + exp(B - CT))$$

with $A = 1.012896$, $B = 4.914322$, $C = 0.257451$ and T is a temperature. The egg stage, from fertilization to hatching, is divided into eleven stages whose total duration is any time between 35 and 130 hours dependent upon the temperature. Empirically, the mean age average in the i^{th} stage is given by a function of the type

$$y_{i,T} = ai^{\alpha} exp(bT + ci).$$

A specific formula is given by S.Regner [40]

$$y_{i,T} = 16.07i^{1.74} exp(-0.1145T + 0.0098i).$$

In his doctoral thesis, L. Motos [35] has determined a set of parameters for the bay of Biscay anchovy which differ slightly from those found by S. Regner, so $a = 15.45$, $b = -0.115$, $c = -0.147$ and $\alpha = 2.071$. It is convenient to extend to notion of stage so as to make the stage a continuous variable which will be further on denoted s. The above formulae give the age at stage: one can invert this formulae to determine the stage at age. Since this information has to be incorporated in a continuous time and space equation, it will be convenient to express it in terms of the instantaneous variation of stage as a function of age and temperature. Taking for the time being, the general formula, we obtain

$$dy = (cy + \alpha\frac{y}{s})ds$$

which yields

$$\frac{dy}{ds} = \frac{s}{csy + \alpha y}.$$

This equation has a singularity at the origin ($s = y = 0$), which explains that all the solutions go through this point. Each particular solution is associated to one the value of the temperature. Since we are going to follow trajectories of larvae in the ocean through possibly various values of the temperature, it is convenient to write the equation in a way which shows the role of temperature

$$\frac{dy}{ds} = \frac{exp(-bT)}{aexp(cs)(cs^{\alpha} + \alpha s^{\alpha-1})}$$

where $T = T(y)$ is allowed. Hatch occur at the end of the eleventh stage, which dependent upon the temperatures crossed by the egg throughout its development, will occur more or less rapidly. The period which goes from hatching to the resorption of the yolk-sac is counted as a twelfth stage. The duration of this stage is given by a function of the type

$$D_{\mu} = ET^{-F},$$

where, as for the previous stages, the parameters E, F and D_{μ} are determined in laboratory, at constant temperatures. Once again, this information is to be converted into instantaneous variation of stage. let us more generally show how to do this. Suppose we know the duration of a certain stage, say the i^{th} stage, as a function of temperature

$$D_i = D_i(T).$$

Let us introduce the variable m_i, the maturity in the i^{th} stage, with $m_i = 0$ at the beginning of the i^{th} stage and $m_i = 1$ at the end. Assuming that there is a function $f_i = f_i(T)$ of the progression of the maturity and this function is constant as long as the temperature is constant, it is natural to suggest

$$f_i(T) = \frac{1}{D_i} = \frac{1}{D_i(T)}.$$

If now T depends on t, we obtain

$$\frac{dm_i}{dt} = f_i(T(t)) = \frac{1}{D_i(T(t))},$$

from which we deduce the expression of maturity in stage i at time t

$$m_i(t) = \int_{t_i}^{t} \frac{1}{D_i(T(s))} ds,$$

the completion of the stage corresponding to the time t for which $m_i(t) = 1$. It is this formula that is used to describe the progression in the yolk-sac stage, with the function D_μ taken from the PhD work by L. Motos [35]. The formula obtained by L. Motos however is not specific of the yolk-sac stage: it covers the whole passive stage, from egg fertilization to the end of the yolk-sac stage, or in other words, the endogenous feeding period. In the absence of a specific model for the yolk-sac stage this was considered a possible choice. This choice is of course disputable : the parameters computed in this manner account for the whole growth process from fertilization to the end of the yolk-sac stage, wherein the specificity of the yolk-sac stage is likely to be dampened out. The values of the parameters determined by L.Motos [35] are stated as follows $E = exp(10.376)$ and $F = 2.1749$. During the part of the larval development which goes from the yolk-sac resorption to the beginning of the swim-bladder use, growth is modelled by means of the length of the larva. S.Regner [40] mentions two laws , corresponding to two temperatures. Generally, it is assumed that the length is given in terms of a Gompertz model

$$\lambda(t) = Gexp(-Hexp(-It)),$$

in which the parameters G, H and I are functions of the temperature, $\lambda(t)$ is the length at time t. As long as the temperature is constant, one gets, by taking the time derivative on both sides of the above identity

$$\frac{d\lambda}{dt} = I\lambda(t)(ln(G) - ln(\lambda(t))).$$

We assume the same formula holds in the case when the temperature changes with time. The only problem is that, as already mentioned, we only have value for two temperatures. on the other hand, field data in the bay of Biscay in 1997 and 1998 have provided values of length as a function of age. Age has been evaluated by analysis of the otolith. There are some distinctive features between the two years: generally, the data for 1997 do not exceed 17 days of age while, in 1998, data on both place go over 20 days, with larvae beyond 25 days of age found in the place called 'Fer a Cheval'. Although a linear formula of the weight as a function of age, for the post larval stage, has been proposed from these data, namely,

$$y = 4.9 + 0.35x, R_2 = 0.7.$$

The data demonstrate the heterogeneity of growth rate, with some samples growing faster and surviving more than others. So, clearly, environmental conditions should be incorporated in the formula of size at age. In the absence of further knowledge, we will use a model for linear growth of the length,

$$\frac{d\lambda(t)}{dt} = E - K\lambda(t),$$

for the part of change in length between the end of yolk-sac and the beginning of the swim bladder period, assuming once again that the parameters E and K are constants as long as the temperature is constant and are, otherwise, functions of the temperature. the above equation is derived from von Bertalanffy's principle (see the following further remarks). In contrast to the equation used in the pre-larval stage which depend on three parameters, the above equation depends only on two parameters: in principle, we should be able to compute them as soon as we know the length at the beginning of the post-yolk-sac period and the length at the onset of swim bladder.

2.1.1 Synopsis of the growth and maturity processes from birth to the beginning of the swim bladder period

The period going from birth to the beginning of active vertical movement has been subdivided into three main phases: the first one comprises the eleven stages going from the moment of egg release(or, rather, fertilization) onwards to the moment when the larva hatches out; the second one covers the time when larva feeds on the yolk-sac (endogenous feeding), and the third one goes from the beginning of exogenous feeding until the time when the larva uses actively its swim-bladder. Corresponding to each phase, there is a temperature-dependent differential equation, governing the instant progression within the phase, measured differently according to the phase: for the first phase, it is the actual stage the egg is in (anywhere between 1 and 11), for the second one, it is the maturity, converted for convenience to a value between 11 and 12; finally, the progression through the third phase is described in terms of the length. In the absence of the data for this third phase, the simulations have been limited to the first two ones.

2.1.2 Further remarks about the modelling of the growth rate

As explained above, the original model equation, in which growth rate was modelled as a result of larva eating phytoplankton, has been abandoned. The first reason for this is the lack of data for the phytoplankton of the bay of Biscay, and the lack of a model of growth and proliferation for the phytoplankton. Another reason which should be mentioned is the fact that no satisfactory model of the interactions of the larvae with the phytoplankton had been described. The model proposed was just using a contact rate principle, based on the simultaneous presence of phytoplankton and larvae. In fact, a model of the process of attack of the prey by the larva should be added since, in the very first days of larva's life, it is probably an important factor of the success or failure of feeding. The modelling chosen here is even more questionable since it indirectly assumes that temperature is a faithful indicator of larva's growth schedule, thus, indirectly, an indicator of a presence of the prey. One should however point out that the segment of larva's life that has been considered here corresponds to the period of endogenous feeding. To put it in a necessarily less systematic wording, this is a period when the larva does not depends for its survival on food availability, although it has been frequently noted in the literature that larva start to eat phytoplankton very soon, even before hatching. So, the environment reflected by temperature does not impact, during that period of larva's life, so much to better or on the contrary worsen food accessibility, as it does directly by establishing a thermal environment more or less suited to the larva metabolism. Assuming that the temperature is indeed the dominant growth parameter, the model itself is also questionable. Let us briefly repeat some of the points of a detailed discussion made nearly 50 years ago by Beverton and Holt who, in their famous treatise 'On the dynamics of exploited fish populations' [10], devoted a whole chapter (section 9 of their book) to growth and feeding. Their preference goes to the von Bertalanffy approach(section 3-4 in [10]) of growth on the ground that it is based on physiological principles, as opposed to other approaches which are based on empirical arguments. The model used in our work for the first part of the development, from fertilization to the end yolk-sac stage, is subject to the criticism made by Beverton and Holt [10], that is it is essentially built up as a best fit to data collected in laboratory, which no energy budget consideration.

2.2 Dynamics of the larvae

The survival of young fishes and larvas is completely related and dependent of the physical surroundings and current movement, so this relationship between renew of sea species and their surrounding is an evident fact but also a challenge to mathematical and numerical analysis of equations. A mathematical model of population dynamics considering the effects of current transport and vertical restlessness was developed, and studying anchovy larvae in bay of Biscay. The biological part of this model was in great part concentrated on weight and height growth of fishes, assuming that the growth is a function depending on the environment wealth which was represented by the temperature. The state variable for the dynamics of the larvae is the density of larvae. For the part of larval cycle which goes from fertilization to the end of stage 12, the density $l = l(t, s, P)$, where s denotes the location within the stages, and $P = (x, y, z)$ represents a generic point in the physical space. The region of observation is assimilated to the product of a horizontal plane and the vertical line. The origin is a point of the surface in the sea, the x axis is oriented westward, the y axis is oriented northward and the z axis is oriented downward. Of course, t is the chronological time. l is a density with respect to the stage and the position, numbers of individual are deduced from integration of l over products of the type $\Sigma \times \Omega$, where Σ is a subset of positive measure of the real line (corresponds to a subset of values of s) and Ω is a subset of positive measure of the physical space(corresponds to a subset of the ocean):

$$\int_{\Sigma\times\Omega} l(t,s,P)dsdP$$

is the number of larvae which, at time t, have age in the stage in the set Σ and are at a point P of the set Ω. In order to describe the equation verified by l, we first introduce some functions and parameters: Physical parameters include the current velocity $V(t,P)$. We will occasionally denote V_1, V_2, V_3 the components of the currents on the x axis, respectively the y and the z axis. The other important physical parameter is the mixing coefficient, supposed to be essentially vertical, $h = h(t,P)$. Incompressibility condition is assumed to hold, that is,

$$div(V) = 0.$$

The main biological parameter is the function $f = f(T,s)$ which gives the instantaneous rate of progression within the stages from egg fertilization to the end of the yolk-sac period.
Now suppose ω is any sub-volume of Ω. Suppose $d\sigma(P)$ is a small surface element in Ω with unit normal n. The number of individuals in the part $[s_1, s_2] \times \omega$ between time t and $t + \Delta t$

$$\int_{s_1}^{s_2}\int_{\omega}(l(t+\Delta t,s,P) - l(t,s,P))dPds.$$

In $[s_1, s_2] \times \omega$ there are exiting and entering new individuals in such ways:
1/ There will be new individuals after a time t included in stage $[s1, s2]$ and there will be others excluded because they will be to old for s_2, this phenomenon is modelled by

$$\int_{\omega} l(t,s_1,P)dPds_1 - \int_{\omega} l(t,s_2,P)dPds_2,$$

or

$$\int_{\omega} l(t,s_1,P)f(s_1)dPdt - \int_{\omega} l(t,s_2,P)f(s_2)dPdt,$$

which equal to

$$-\int_{s_1}^{s_2}\int_{\omega}\frac{\partial(fl)}{\partial s}dPdsdt.$$

2/ If we are in stage $[s_1, s_2]$ but near the boundary of ω "$\partial\omega$", the exist (excluding) individuals is given by:

$$-\int_{s_1}^{s_2}\int_{\partial\omega} l(t,s,P)V.nd\sigma(P)dsdt$$

which equal by applying the divergence theorem

$$-\int_{s_1}^{s_2}\int_{\omega} div(Vl)dPdsdt.$$

The flow of density is measured by the "larval flux field" q. $q.nd\sigma(P)$ is a quantity of larvae crossing $d\sigma(P)$ per unit time, per unit stage at time t, stage s. Then

$$\int_{\partial\omega} q.ndS$$

is the quantity of larvae leaving ω per unit time,per unit stage at time t, stage s. Applying the divergence theorem to the above integral, we obtain,

$$\int_{\partial\omega} q.ndS = \int_{\omega} div(q)dP.$$

Now, the larval flux field should be related to the larval density by

$$q = -h\frac{\partial l}{\partial z},$$

where h is the mixing coefficient. For the larvae mortality

$$-\int_{s_1}^{s_2}\int_{\omega} \mu(s)ldPdsdt.$$

Assembling all these equations we obtain the growth of larvae equation

$$\frac{\partial l}{\partial t} + \frac{\partial (fl)}{\partial s} + div\,(Vl) - \frac{\partial}{\partial z}\left(h\frac{\partial l}{\partial z}\right) + \mu l = 0. \tag{2.1}$$

This model takes into account both the physical and biological effects. For the physical part, the model considered here stresses two main factors: 1) Transport entailed by the currents: the currents are computed using Navier-Stokes equations and are introduced in the equations of the larvae as functions of space and time with sufficient regularity to allow existence and uniqueness of stream lines. 2) Vertical diffusion induced by vertical mixing in the upper part of the water column. For the biological part the main parameters are a function which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period. The model is expressed in a generality which encompasses a large variety of situations. The motivation at the origin of this work is the study of the dynamics of the bay of Biscay anchovy [6], that is to say, a region of the Atlantic ocean close to the French coast, bordered eastward by the continental shelf. The bay of Biscay goes from the Northern Spanish coast up to about 46^0 in "latitude". In this region at the end of May , a thermocline establishes itself: the top of the thermocline is roughly at the same distance z_{therm} from the surface. The thermocline divides the water column into three regions: the upper part, from the surface to z_{therm} deep, the so called mixed layer. This is where the larvae grow. Below is the thermocline, a rather thin layer where the temperature loses rapidly a few degrees and the vertical mixing coefficient is negligibly small. Below the thermocline is another well mixed layer where the temperature is only slowly changing with depth. This region is of no concern to us for the rest of the study. We will be confined to the mathematical issues related to the above model, and we study only the upper layer, the mixed layer of the water column; see Figure 2.1. The domain under consideration is $\Omega = D \times (0, z^*)$, where D is an open subset of the surface, that is D is a portion of the plane z^* is the distance from the surface to a region in the seabed. We denote by Q the product space $\Omega \times (0, T)$ and $\Sigma := \Gamma \times (0, T)$ the boundary of Q. The state variable for the dynamics of the larvae is the density of larvae. For the part of the larval cycle which goes from fertilization to the end of stage, the density $l = l(t, s, P)$, where s denotes the position within the stages, which we take specifically of the bay of Biscay anchovy in $[1, 12)$ [6] and $P = (x, y, z)$ represents a generic point in the physical space. The region of observation is assimilated to the product of the horizontal plane and a vertical line. The origin is a point of the surface in the sea, the x axis is oriented westward, the y axis is oriented northward, and the z axis is oriented downward. Of course t is the chronological time. l is a density with respect to the stage and the position. The larvae are characterized by their density, that is to say, at each time $t \in [0, T]$, where T is the maximal time of observation, $l(t, s, P)$ can be thought of as the larvae biomass per unit of volume evaluated at the point P, at that time.

water surface

$z_{\text{therm}}\left\{\begin{array}{l}\text{mixed layer,}\\ \text{where larvae grow}\end{array}\right.$

thermocline

another mixed layer

Figure 2.1: Water column divided into three regions

The full model is as follows

$$\left\{\begin{array}{l}\dfrac{\partial l}{\partial t}+\dfrac{\partial(fl)}{\partial s}+div\,(Vl)-\dfrac{\partial}{\partial z}(h\dfrac{\partial l}{\partial z})+\mu l=0,\\ l(t,1,P)=B(t,P),\\ h\dfrac{\partial l}{\partial z}=0,\quad z=0,\\ h\dfrac{\partial l}{\partial z}=0\quad z=z^*,\end{array}\right. \tag{2.2}$$

We can replace the Newmann conditions by the following homogeneous Dirichlet conditions $l=0$, that is there is no larvae in the boundary. We now discuss in detail the parameters and functions of the model.

The velocity. The velocity vector $V(t,P)=(V_1(t,P),V_2(t,P),V_3(t,P))$ describes the sea current which is supposed to be known. We assume that the sea water is incompressible, which yields:

$$\mathrm{div}(V)=0, \tag{2.3}$$

with

$$V_3(t,x,y,0)=V_3(t,x,y,z^*)=0. \tag{2.4}$$

The mixing coefficient. The mixing coefficient $h=h(t,P)$ gives the diffusion rate, supposed to be essentially vertical.

The growth function. The main biological parameters are functions $f(t,s)$, which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period. For the principle of determination see [35, 6].

The mortality of larvae. The mortality is modelled by the expression $\mu=\mu(t,s,P)$.

Demographic boundary conditions. Demographic boundary conditions are given at $s = 1$, at any time during the spawning period, the variable s takes its values in the interval $[1, 12)$, where $s = 1$ corresponds to the newly fertilized eggs, and $s = 12$, to the end of the yolk sac period.
Horizontal boundary conditions. Model (2.2) does not show any lateral boundary conditions. Choosing the right boundary in the x and y directions is a difficult issue that we mainly avoid here by assuming that the initial value has a compact support contained in the interior of the domain and we consider the solution within a time interval $[0, T]$ during which the horizontal projection of the support is contained in the interior of the domain D.
Vertical boundary conditions. Vertical boundary conditions are imposed at the surface and at z^*, here we are assuming a no flux conditions.
Initial conditions Initial conditions are given at $t = 0$ (beginning of the year). The standing assumption is that there is no larva alive at this period of the year, so that $l(0, s, x, y, z) = 0$.

Remark 2.2.1 What we call an initial value in the present context is not the value of the solution at a given time or rather, the only relevant information would be that at $t = 0$ (that is 1^{st} January) there is no larva in the sea. What we consider as an initial value is the distribution of newly fertilized eggs, that is the larvae at stage $s = 1$ all over the reproduction season.

Chapter 3

Mathematical issues

This chapter is devoted to a theoretical study of the main model either with Newmann or Dirichlet condition. We show that under some hypotheses we can obtain the existence, uniqueness and positivity of the solution.
The main characteristic of this equation is that it has mixed parabolic-hyperbolic type, due to the directional separation of the diffusion and convection effects. Such problem is called also non autonomous ultraparabolic equations, that is parabolic in many directions. So principal difficulty is the lack of coercivity to our elliptic operator, or equation with degenerated elliptic operator.

3.1 The uncoupled case

The purpose of this work is to perform a mathematical analysis of the model, notably, show existence, uniqueness and positivity of solutions. In a previous work coauthored by Boushaba, Boussouar; and Arino [13], a simplified version of the model of the phytoplankton had been investigated. It was assumed that the diffusion rate and the vertical current does not depend on time and the horizontal current is uniform throughout the water column that is $V_1(t,x,y,z) = V_1(t,x,y)$, $V_2(t,x,y,z) = V_2(t,x,y)$. Under this assumption, it was possible to uncouple the vertical and the horizontal components in the following sense: the study was restricted to each of the horizontal streamlines: the restriction to such a line reduces the functions of time horizontal components to functions of time so that the full model reduces on such a line to a diffusion equation in the vertical variable coupled with a first order growth equation. Our purpose in this work is to extend this method to the more realistic situation where the diffusion rate and the vertical current depends also on time and the horizontal current is uniform throughout the water column. The idea we exploit here is the same in the first time, that is to uncouple the vertical and the horizontal components but the restriction on such a line gives equations of parabolic type with time dependent coefficients. The study of such equations takes up the main part in this work. Time dependence is dealt with using results on time-dependent evolution equations by Acquistapace [1, 2, 3] and several other authors (Lunardi [34], Tanabe [45]). A valuable source of information of this work was a monograph by Tanabe [45]. The main result of this part, stated in Theorem 3.2.6, ensures that, under some conditions on the coefficients of the equation, the Cauchy problem associated with the equation has a unique classical solution, which moreover is nonnegative if the initial value is non negative.

3.2 Notation and preliminary results

Let Y be a Banach space and $[a,b]$ a finite interval of the real line, then we define the space

$$C([a,b];Y) = \{f : [a,b] \to Y : f \text{ is continuous}\}.$$

Note that $C([a,b];Y)$ is a Banach space with the norm

$$\|f\|_{C([a,b];Y)} = \sup_{s\in[a,b]} \|f(s)\|_Y.$$

We also consider the space $C^1([a,b];Y)$ consisting of functions $f \in C([a,b];Y)$ such that f is strongly differentiable in $[a,b]$ and $f' \in C([a,b];Y)$, with the norm

$$\|f\|_{C^1([a,b];Y)} = \|f\|_{C([a,b];Y)} + \|f'\|_{C([a,b];Y)}.$$

Let $\theta \in]0,1[$ then we define the following Holder type spaces

$$C^\theta([a,b];Y) = \{f \in C([a,b];Y) : [f]_{C^\theta([a,b];Y)} = \sup\{\frac{\|f(s)-f(t)\|_Y}{|t-s|^\theta} \quad t,s \in [a,b], t \neq s\} < \infty\},$$

which is equipped with the norm

$$\|f\|_{C^\theta([a,b];Y)} = \|f\|_{C([a,b];Y)} + [f]_{C^\theta([a,b];Y)}.$$

Also let

$$C^{1,\theta}([a,b];Y) = \{f \in C^1([a,b];Y) : f' \in C^\theta([a,b];Y)\},$$

with norm

$$\|f\|_{C^{1,\theta}([a,b];Y)} = \|f\|_{C([a,b];Y)} + \|f'\|_{C^\theta([a,b];Y)}.$$

We consider the problem

$$\begin{cases} u_t(t,x) - a(t,x)u_{xx}(t,x) - b(t,x)u_x(t,x) - c(t,x)u(t,x) = 0, \\ (t,x) \in [0,T] \times [0,1] \\ \alpha_0(t)u(t,0) - \beta_0(t)u_x(t,0) = \alpha_1(t)u(t,1) + \beta_1(t)u_x(t,1) = 0, \quad t \in [0,T], \\ u(0,x) = \Phi(x), \quad x \in [0,1], \end{cases} \tag{3.1}$$

under the following assumptions:

$$\begin{gathered} a,b,c \in C([0,T] \times [0,1]), \\ a(.,x), b(.,x), c(.,x) \in C^{1,\delta}([0,T]; I\!R) \\ \text{with norms indpendent of } x \in [0,1], \text{ for some } \delta \in]0,1[, \\ a > 0, c \leq 0 \quad \text{in } [0,T] \times [0,1], \end{gathered} \tag{3.2}$$

To recall some propositions, we set $E = C([0,1])$, $\|u\|_E = \sup_{x\in[0,1]} |u(x)|$, and define for each $t \in (0,T)$,

$$\begin{gathered} D(A(t)) = \{u \in C^2([0,1]) : \alpha_0(t)u(0) - \beta_0(t)u'(0) = \alpha_1(t)u(1) + \beta_1(t)u'(1) = 0,\} \\ A(t)u = a(t,.)u'' + b(t,.)u' + c(t,.)u\,. \end{gathered} \tag{3.3}$$

Proposition 3.2.1 ([2]) *Let a, b, c be as in (3.1), (3.2), and suppose that $u \in C^2([0,1])$ is a solution of*

$$\begin{cases} \lambda u - a(t,.)u'' - b(t,.)u' - c(t,.)u = f \in C([0,1]), \\ \alpha_0(t)u(0) - \beta_0(t)u'(0) = z_0 \in \mathcal{C}, \\ \alpha_1(t)u(1) + \beta_1(t)u'(1) = z_1 \in \mathcal{C}, \end{cases} \tag{3.4}$$

where $t \in [0,T]$ is fixed and λ is a complex number lying in the sector

$$\Sigma_K := \{z \in \mathcal{C} : \operatorname{Re} z \geq 0\} \cup \{z \in \mathcal{C} : |\operatorname{Im} z| > K|\operatorname{Re} z|\} \quad (K > 0.)$$

Then there exists $M > 0$, depending on K, a, b, c, but independent of t such that

$$(1+|\lambda|)\|u\|_E + (1+|\lambda|^{1/2})\|u'\|_E + \|u''\|_E \leq M(\|f\|_E + (1+|\lambda|^{1/2})(|z_0| + |z_1|)). \tag{3.5}$$

As a consequence of the above proposition we have the following result.

Proposition 3.2.2 ([2]) *Let a, b, c be as in (3.1), (3.2); let $\{A(t)\}_{t\in[0,T]}$ be defined by (3.3). Then we have:*

(i) $[0,\infty[\subseteq \rho(A(t))$ *for all* $t \in [0,T]$; *where* $\rho(A(t))$ *is the resolvent set of* $A(t)$ *and* $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$.

(ii) $\Sigma_K \subseteq \rho(A(t))$ *and for each* $K > 0$ *there exists* $M(K) > 0$ *(depending also on* a, b, c*) such that*

$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leq \frac{M(K)}{1+|\lambda|} \quad \forall \lambda \in \Sigma_K, \forall t \in [0,T],$$

where Σ_K *is defined above.*

Definition 3.2.3 ([45]) A classical solution of (3.1) is a function

$$u \in C([0,T],E) \cap C((0,T],D(A(t))) \cap C^1((0,T],E),$$

such that $u(0) = x$, $u'(t) - A(t)u(t) = 0$ in $(0,T]$.

Let us assume the following hypotheses:

(AT1) For each $t \in [0,T]$, $A(t) : D(A(t)) \subseteq E \to E$ is a closed linear operator and there exists $M > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$\rho(A(t)) \supseteq S_\theta := \{\lambda \in C : \lambda \neq 0, |\arg \lambda| < \theta\} \cup \{0\},$$
$$\|R(\lambda, A(t))\| \leq \frac{M}{1+|\lambda|} \quad \forall \lambda \in S_\theta \cup \{0\},\ \forall t \in [0,T],$$

(AT2) There exist $B > 0$ and $\delta_1, \ldots, \delta_k, \nu_1, \ldots, \nu_k$ with $0 \leq \nu_i < \delta_i \leq 2$ such that

$$\|A(t)R(\lambda, A(t))((A(s))^{-1} - (A(t))^{-1})\| \leq B \sum_{i=1}^{k} |t-s|^{\delta_i} |\lambda|^{\nu_i - 1},$$

for all $\lambda \in S_\theta - \{0\}$, $0 \leq s < t \leq T$.

It is obvious that Σ_K and S_θ are the same sets.

Theorem 3.2.4 ([45]) *Assume that (AT1) and (AT2) hold. Then, if* $x \in \bar{D}(A(0))$, *problem* (3.1) *has a unique classical solution.*

Remark 3.2.5 In general the function c in the problem (3.1)is not negative. Moreover by setting $u = ve^{\omega t}$ with $\omega \in I\!R$, the function v is solution of

$$\begin{cases} v'(t) - (A(t) - \omega I)v(t) = 0, & t \in (0, T] \\ v(0) = x\,. \end{cases} \tag{3.6}$$

Hence existence, uniqueness and positivity of solutions of problem (3.6) is equivalent to the same properties of problem (3.1).

3.2.1 Existence, uniqueness and positivity of the solution of the Cauchy problem

The aim of this section is to show that model (2.2) possesses a positive, unique solution. For this we use an approach by the method of characteristics to build a one dimensional time dependent parabolic equation whose solution will yield the solution of equation (2.2). We assume that

(H1) V_1, V_2 are functions in $C^1((0,T) \times D)$ and $f \in C^1((0,T) \times (1, s^*))$.

We introduce the flow generated by the horizontal current and the size growth, that is

$$\phi := \phi(\tau, t_0, 1, x_0, y_0),$$

and for each initial value $\tilde{\zeta} \equiv (t_0, 1, x_0, y_0)$, $\phi(\tau, \tilde{\zeta})$ is the solution of the equation

$$\left(\frac{dt}{d\tau}, \frac{ds}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) = (1,\ f(t,s), V_1(t,x,y), V_2(t,x,y)) \tag{3.7}$$

satisfying $t(0) = t_0$, $s(0) = 1$, $x(0) = x_0$, $y(0) = y_0$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$. We denote $\bar{l}(\tau, z) \equiv \bar{l}(\tau, \tilde{\zeta}, z) = l(\phi(\tau, \tilde{\zeta}), z)$ the restriction of l along the characteristic line. The equation verified by $\bar{l}$ reads

$$\frac{\partial \bar{l}(\tau, z)}{\partial \tau} + \bar{V}_3 \frac{\partial \bar{l}(\tau, z)}{\partial z} - \frac{\partial}{\partial z}\left(\bar{h}\frac{\partial \bar{l}(\tau, z)}{\partial z}\right) + \bar{\gamma}\bar{l}(\tau, z) = 0,$$

where $\bar{V}_3 := \bar{V}_3(\tau, \tilde{\zeta}, z)$, $\bar{h} := \bar{h}(\tau, \tilde{\zeta}, z)$, $\bar{\gamma} := \bar{\gamma}(\tau, \tilde{\zeta}, z)$ are the restrictions of V_3, h, B, γ respectively along the characteristic line and γ is equation of order 0. So to each $\tilde{\zeta}$, we have associated the following problem

$$\begin{cases} \dfrac{\partial \bar{l}}{\partial \tau} + \bar{V}_3 \dfrac{\partial \bar{l}}{\partial z} - \dfrac{\partial}{\partial z}(\bar{h}\dfrac{\partial \bar{l}}{\partial z}) + \bar{\gamma}\bar{l} = 0, \\ \bar{l}(0, z) = \bar{B}(z), \\ \bar{h}(\tau, 0)\dfrac{\partial \bar{l}}{\partial z}(\tau, 0) = 0, \\ \bar{h}(\tau, z^*)\dfrac{\partial \bar{l}}{\partial z}(\tau, z^*) = 0, \end{cases} \tag{3.8}$$

where $\bar{B}(z)$ is the restriction of B along the characteristic line. We consider the operator $A(\tau) : D(A(\tau)) \subseteq C([0, z^*]) \to C([0, z^*])$ defined by

$$A(\tau)u = \bar{V}_3(\tau, .)u' - (\bar{h}(\tau, .)u')' + \bar{\gamma}(\tau, .)u,$$
$$D(A(\tau)) = \{u \in C^2([0, z^*]),\ \ \bar{h}(\tau, 0)u'(0) = \bar{h}(\tau, z^*)u'(z^*) = 0\}.$$

We now state the assumptions of this section.

(H2) $h \in C^1([0, T] \times \bar{\Omega})$, $V_3 \in C([0, T] \times \bar{\Omega})$, $\gamma \in C([0, T] \times [1, s^*] \times \bar{\Omega})$.

(H3) h, $\frac{\partial h}{\partial z}$, $V_3 \in C^{1,\delta}([0, T]; C(\bar{\Omega}))$, and $\gamma \in C^{1,\delta}([0, T]; C([1, s^*] \times \bar{\Omega}))$.

(H4) $h \geq c_0$ in $[0, T] \times \bar{\Omega}$ where $c_0 > 0$.

Theorem 3.2.6 *Assume (H2)–(H4) hold. If the positive function $\bar{B}$ is in $C([0, z^*])$, then problem (3.8) has a unique non-negative classical solution.*

Proof. Without loss of generality we can assume that $\gamma \geq 0$, otherwise we can replace γ by $\gamma + \omega \geq 0$ see Remark 3.2.5. The main idea is to use theorem 3.2.4. The first assertion (AT1) follows from the proposition 3.2.2. Concerning the second assertion (AT2), for $f \in C([0, z^*])$, $t, s \in G_1$, where G_1 is some neighborhood of $\tau = 0$, $\lambda \in S_\theta - \{0\}$, we set $v = (A(s))^{-1}f$ and $u = R(\lambda, A(t))(\lambda - A(s))v$, then we have to estimate the $C([0, z^*])$-norm of

$$u - v = (A(t))R(\lambda, A(t))(A(t))^{-1} - (A(s))^{-1})f.$$

Now $u - v \in C^2([0, z^*])$ and u and v solve

$$\begin{cases} \lambda u - A(t, .)u = \lambda v - f, \\ \bar{h}(t, 0)u'(0) = \bar{h}(t, z^*)u'(z^*) = 0, \end{cases} \tag{3.9}$$

and

$$\begin{cases} A(s,.)v = f, \\ \bar{h}(s,0)v'(0) = \bar{h}(s,z^*)v'(z^*) = 0, \end{cases} \tag{3.10}$$

respectively. This shows that

$$\begin{cases} \lambda(u-v) - A(t,.)(u-v) = (A(t,.) - A(s,.))v, \\ \bar{h}(t,0)(u'-v')(0) = (\bar{h}(s,0) - \bar{h}(t,0))v'(0), \\ \bar{h}(t,z^*)(u'-v')(z^*) = (\bar{h}(s,z^*) - \bar{h}(t,z^*))v'(z^*). \end{cases} \tag{3.11}$$

Applying Proposition 3.2.1 to (3.10) with $\lambda = 0$, we have

$$\|v\|_E \leq c\|f\|_E. \tag{3.12}$$

Using again the proposition 3.2.1 to (3.11), we get

$$\begin{aligned} |\lambda|\|u-v\|_E \leq M &(\|(A(s,.) - A(t,.))v\|_E + (1+|\lambda|^{1/2}) \\ &\times (|(\bar{h}(s,0) - \bar{h}(t,0))v'(0)| + |(\bar{h}(s,z^*) - \bar{h}(t,z^*))v'(z^*)|)) . \end{aligned}$$

Using hypothesis (H3) and by virtue of (3.12),

$$\|u-v\|_E \leq c(|t-s||\lambda|^{-1}|t-s|^{\delta}|\lambda|^{-1} + |t-s||\lambda|^{-1/2})\|f\|_E.$$

Hence the hypothesis (AT2) holds. Since $D(A(0))$ is dense in $C([0,z^*])$ see [2], then according to Theorem 3.2.4, for $\bar{B} \in C([0,z^*])$ we have existence and uniqueness of a classical solution of problem (3.8). It remain to see that the solution is positive which can be proved by the standard argument. If u is solution of problem (3.8), we set $u = u^+ - u^-$ where u^+ and u^- are respectively the positive and negative part of u, so multiplying the equation (3.8) by u^-, and integrating over $(0,z^*)$ we have

$$\int_0^{z^*} (\frac{\partial u}{\partial \tau}u^- + \bar{h}(\tau,z)\frac{\partial u}{\partial z}\frac{\partial u^-}{\partial z} + \bar{V}_3(\tau,z)\frac{\partial u}{\partial z}u^- + \bar{\gamma}(\tau,z)uu^-)dz = 0,$$

hence

$$-\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} \geq c_0\int_0^{z^*} |\frac{\partial u^-}{\partial z}|^2 dz - m_3\int_0^{z^*} \frac{\partial u^-}{\partial z}u^- dz + m_4\int_0^{z^*} |u^-|^2 dz,$$

with

$$m_3 = \sup_{\tau,z} |\bar{V}_3(\tau,z)|, \quad m_4 = \inf_{\tau,z} |\bar{\gamma}(\tau,z)|.$$

Since

$$\int_0^{z^*} \frac{\partial u^-}{\partial z} u^- dz \leq \int_0^{z^*} (\rho |\frac{\partial u^-}{\partial z}|^2 + \frac{1}{\rho}|u^-|^2) dz, \quad \forall \rho > 0,$$

it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} + (c_0 - m_3\rho)\|\frac{\partial u^-(\tau)}{\partial z}\|^2_{L^2(0,z^*)} \\ &+ (m_4 - \frac{m_3}{\rho} + \omega)\|u^-(\tau)\|^2_{L^2(0,z^*)} \\ &\leq \omega \|u^-(\tau)\|^2_{L^2(0,z^*)}, \end{aligned}$$

choosing ρ and ω such that

$$c_0 - m_3\rho > 0 \quad \text{and} \quad m_4 - \frac{m_3}{\rho} + \omega > 0,$$

so

$$\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} \leq \omega \|u^-(\tau)\|^2_{L^2(0,z^*)},$$

then

$$\|u^-(\tau)\|^2_{L^2(0,z^*)} \leq \|u^-(0)\|^2_{L^2(0,z^*)} e^{2\omega\tau},$$

which gives $u^-(\tau) = 0$ provided $B \geq 0$, then the solution is positive. ■

Recall that for $z \in [0, z^*]$ the system

$$\begin{aligned} t &= T(\tau, t_0, x_0, y_0), \quad & s &= S(\tau, t_0, x_0, y_0), \\ x &= X(\tau, t_0, x_0, y_0), \quad & y &= Y(\tau, t_0, x_0, y_0), \end{aligned}$$

is a solution of the characteristic system (3.7) emanating from the point $\tilde{\zeta}$. We have also

$$\begin{aligned} t_0 &= T(0, t_0, x_0, y_0), \quad & 1 &= S(0, t_0, x_0, y_0), \\ x_0 &= X(0, t_0, x_0, y_0), \quad & y_0 &= Y(0, t_0, x_0, y_0). \end{aligned}$$

If

$$\operatorname{Jac}(T, S, X, Y) := \begin{vmatrix} \frac{\partial T}{\partial \tau} & \frac{\partial T}{\partial t_0} & \frac{\partial T}{\partial x_0} & \frac{\partial T}{\partial y_0} \\ \frac{\partial S}{\partial \tau} & \frac{\partial S}{\partial t_0} & \frac{\partial S}{\partial x_0} & \frac{\partial S}{\partial y_0} \\ \frac{\partial X}{\partial \tau} & \frac{\partial X}{\partial t_0} & \frac{\partial X}{\partial x_0} & \frac{\partial X}{\partial y_0} \\ \frac{\partial Y}{\partial \tau} & \frac{\partial Y}{\partial t_0} & \frac{\partial Y}{\partial x_0} & \frac{\partial Y}{\partial y_0} \end{vmatrix}_{\tau=0} \neq 0,$$

then the Jacobian does not vanish in a neighborhood of the initial curve. Therefore, the local inversion theorem guarantees that we can solve for

(τ, t_0, x_0, y_0) as function of (t, s, x, y) near the initial curve; that is, there exists a neighborhood G_1 of $(0, t_0, x_0, y_0)$ and a neighborhood G of (t, s, x, y) such that

$$(T, S, X, Y) : G_1 \to G$$

is a diffeomorphism. Then

$$\begin{aligned} \tau &= \psi_1(t, s, x, y), \quad & t_0 &= \psi_2(t, s, x, y), \\ x_0 &= \psi_3(t, s, x, y), \quad & y_0 &= \psi_4(t, s, x, y), \end{aligned}$$

and for initial data

$$\begin{aligned} 0 &= \psi_1(t_0, 1, x_0, y_0), \quad & t_0 &= \psi_2(t_0, 1, x_0, y_0), \\ x_0 &= \psi_3(t_0, 1, x_0, y_0), \quad & y_0 &= \psi_4(t_0, 1, x_0, y_0). \end{aligned}$$

Once problem (3.8) is solved, we have

$$l(t, s, x, y, z) = \bar{l}(\psi_1, \psi_2, \psi_3, \psi_4, z),$$

in a neighborhood of G. Indeed, by differentiation we obtain that

$$\begin{aligned} \frac{\partial l}{\partial t} &= \frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial t} + \frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial t} + \frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial t} + \frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial t}, \\ f\frac{\partial l}{\partial s} &= f(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial s} + \frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial s} + \frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial s} + \frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial s}), \\ V_1\frac{\partial l}{\partial x} &= V_1(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial x} + \frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial x} + \frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial x} + \frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial x}), \\ V_2\frac{\partial l}{\partial y} &= V_2(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial y} + \frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial y} + \frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial y} + \frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial y}). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial l}{\partial t} + f\frac{\partial l}{\partial s} + V_1\frac{\partial l}{\partial x} + V_2\frac{\partial l}{\partial y} = & \frac{\partial \bar{l}}{\partial \tau}(\frac{\partial \psi_1}{\partial t} + f\frac{\partial \psi_1}{\partial s} + V_1\frac{\partial \psi_1}{\partial x} + V_2\frac{\partial \psi_1}{\partial y}) \\ & + \frac{\partial \bar{l}}{\partial t_0}(\frac{\partial \psi_2}{\partial t} + f\frac{\partial \psi_2}{\partial s} + V_1\frac{\partial \psi_2}{\partial x} + V_2\frac{\partial \psi_2}{\partial y}) \\ & + \frac{\partial \bar{l}}{\partial x_0}(\frac{\partial \psi_3}{\partial t} + f\frac{\partial \psi_3}{\partial s} + V_1\frac{\partial \psi_3}{\partial x} + V_2\frac{\partial \psi_3}{\partial y}) \\ & + \frac{\partial \bar{l}}{\partial y_0}(\frac{\partial \psi_4}{\partial t} + f\frac{\partial \psi_4}{\partial s} + V_1\frac{\partial \psi_4}{\partial x} + V_2\frac{\partial \psi_4}{\partial y}). \end{aligned}$$

Then

$$\begin{aligned}\frac{\partial l}{\partial t}+f\frac{\partial l}{\partial s}+V_1\frac{\partial l}{\partial x}+V_2\frac{\partial l}{\partial y} &= \frac{\partial \bar{l}}{\partial \tau}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_1}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_1}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_1}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_1}{\partial y})\\ &+\frac{\partial \bar{l}}{\partial t_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_2}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_2}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_2}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_2}{\partial y})\\ &+\frac{\partial \bar{l}}{\partial x_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_3}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_3}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_3}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_3}{\partial y})\\ &+\frac{\partial \bar{l}}{\partial y_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_4}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_4}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_4}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_4}{\partial y}).\end{aligned}$$

For $Z=(T,S,X,Y)^T$ and $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$, we have $(Z\circ\psi)(t,s,x,y)=(t,s,x,y)$ which implies

$$\mathrm{Jac}(Z).\,\mathrm{Jac}(\psi)=\mathrm{Id}_4\,. \tag{3.13}$$

By identification in (3.13), we find

$$\begin{aligned}\frac{\partial T}{\partial \tau}\frac{\partial \psi_1}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_1}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_1}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_1}{\partial y}&=1,\\ \frac{\partial T}{\partial \tau}\frac{\partial \psi_2}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_2}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_2}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_2}{\partial y}&=0,\\ \frac{\partial \psi_3}{\partial t}+f\frac{\partial \psi_3}{\partial s}+V_1\frac{\partial \psi_3}{\partial x}+V_2\frac{\partial \psi_3}{\partial y}&=0,\\ \frac{\partial T}{\partial \tau}\frac{\partial \psi_4}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_4}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_4}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_4}{\partial y}&=0\,.\end{aligned}$$

Therefore,

$$\frac{\partial l}{\partial t}+f\frac{\partial l}{\partial s}+V_1\frac{\partial l}{\partial x}+V_2\frac{\partial l}{\partial y}=\frac{\partial \bar{l}}{\partial \tau}\,.$$

In addition,

$$\frac{\partial l}{\partial z}=\frac{\partial \bar{l}}{\partial z},$$

for the initial data

$$\begin{aligned}l(t,1,x,y,z)&=\bar{l}(\psi_1(t,1,x,y),\psi_2(t,1,x,y),\psi_3(t,1,x,y),\psi_4(t,1,x,y),z),\\ &=\bar{l}(0,t,x,y,z),\\ &=\bar{B}(t,x,y,z),\\ &=B(T(0,t,x,y),X(0,t,x,y),Y(0,t,x,y),z),\\ &=B(t,x,y,z)\,.\end{aligned}$$

Then l is a unique solution of (2.2). So the solution l of (2.2) can be determined in terms of the solution $\bar{l}$ of (3.8).

3.3 Perturbation method

In the present work, we investigate the same model which we had treated above in a general case namely, all a coefficients depends of all a variables. For convenience treatment we replace (x, y, z) by (x_1, x_2, x_3) in our problem, so

$$\frac{\partial l}{\partial t} + \frac{\partial (fl)}{\partial s} + div(Vl) - \frac{\partial}{\partial x_3}\left(h\frac{\partial l}{\partial x_3}\right) + \mu l = 0, \qquad (3.14)$$

where h, V and μ depends of all variables. As mentioned above the principal difficulty is the lack of coercivity to our elliptic operator.
The lack of the coercivity can be handled by using a convenient perturbation argument or by other terms adding a vanishing artificial viscosity. The monotone operator theory can be applied see [30](p 316) which gives us existence, and uniqueness of the solution to the perturbed problem, after that we will establish the positivity of our solution. Passing to the limit in a suitable way, we get existence and positivity of a solution of the main model. Since the main operator is not coercive we obtain some extract regularity of the solution in the direction of x_3. Uniqueness of solution seems to be a more difficult problem, see remark in the end of the chapter, and appendix.
For the boundary conditions we assume that there is no larvae in our boundary that is $w = 0$ in Σ.

3.3.1 Notation and preliminary results

We recall here some definitions and results that we will use in this paper. Let X be a separable and reflexive Sobolev space with norm $||.||$ and its dual X' with norm $||.||_*$. We denote by $<,>$ the duality bracket of $X' \times X$. We define the norm of $L^2(0,T;X)$ by

$$\left(\int_0^T ||v||^2 dt\right)^{1/2}.$$

for each $v \in L^2(0,T;X)$.
We denote by $\mathcal{D}(0,T;X)$ the space of infinitely differentiable functions with

compact support in $(0,T)$ with values in X, and $\mathcal{D}'(0,T;X)$ the space of distributions on $(0,T)$ with values in X. Consider $W(0,T;X,X') := \{v, v \in L^2(0,T;X), \frac{\partial v}{\partial t} \in L^2(0,T;X')\}$.

Definition 3.3.1 *We say that an operator A from X to X' is monotone, if*

$$< A(u) - A(v), u - v > \geq 0 \quad \forall u, v \in X. \tag{3.15}$$

The operator A is strictly monotone if we have a strict positivity in (3.15) for all $u, v \in X$ and $u \neq v$.

Remark 3.3.2 *If A is a linear operator, then the monotonicity is equivalent to*

$$< Au, u > \geq 0 \quad \forall \ u \in D(A).$$

Definition 3.3.3 *Let A be a monotone operator from X to X'. We say that A is a maximal monotone operator if its graph is a maximal subset of $X \times X'$ with respect to set inclusion.*

Lemma 3.3.4 *[30]*
Let L be a unbounded linear operator from X into X' with a dense domain $D(L)$ in X. Then L is maximal monotone if and only if L is a closed operator and such that

$$< Lv, v > \geq 0 \quad \forall \ v \in D(L)$$

and

$$< L^*v, v > \geq 0 \quad \forall \ v \in D(L^*).$$

where L^* is the adjoint operator of L.

Theorem 3.3.5 *[30]*
Assume that X is a reflexive Banach space. Let L be a linear operator of dense domain $D(L) \subset X$ and take its values in X'. Assume that L is maximal monotone and suppose that A is a monotone, coercive operator from X to X', i.e.

$$\frac{< A(v), v >}{||v||} \to \infty \text{ if } ||v|| \to \infty.$$

Then, for all $f \in X'$, there exists $u \in D(L)$ such that

$$Lu + A(u) = f.$$

Remark 3.3.6 *If we assume in addition that the operator A is strictly monotone then there exists a unique solution $u \in D(L)$ such that*

$$Lu + A(u) = f.$$

Remark 3.3.7 *One can easily see that in the case of a linear operator A, the coercivity implies strictly monotonicity.*

Remark 3.3.8 *Let u be a solution to the following problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f, \\ u(0) = u_0, \end{cases} \tag{3.16}$$

where A is a linear operator. We set $u = ve^{kt}$, $k \in I\!R$, then v is a solution of problem

$$\begin{cases} v'(t) + (A + kI)\, v(t) = f_1, \\ v(0) = u_0. \end{cases} \tag{3.17}$$

Hence, proving existence, uniqueness and positivity of solutions of problem (3.17) is equivalent to prove the same properties to problem (3.16). Throughout this paper we will deal with problem (3.17), where k is a real constant that we will choose later.

Remark 3.3.9 *We consider two Hilbert spaces V, H with $V \hookrightarrow H$, the continuous injection $\hookrightarrow$ having dense image in H. Then we can identify H with its dual H', and therefore*

$$V \hookrightarrow H \hookrightarrow V'.$$

From Remark 3.3.8 and Remark 3.3.9 we obtain the following Lemma.

Lemma 3.3.10 *[18]*
For $u_0 \in H$ there exists v in $W(0,T;V,V')$, such that $v = u_0$ in H. Thus $w = u - v$, solves the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + (A + kI)w = f_2, \\ w(0) = 0, \end{cases} \tag{3.18}$$

where u is solution of problem (3.17). So, we will consider the case where $u_0 \equiv 0$.

3.3.2 Existence, uniqueness and positivity of solution of the perturbed problem

The objective of this section is to study existence, uniqueness and positivity of solution of the associated perturbed problem (3.21). For this, we start by using the method of characteristic to reduce the number of variable. We assume that

(H1) $f \in C^1((0,T) \times (1, s^*))$.

We introduce the flow generated by the size growth, that is

$$\phi := \phi(\tau, t_0, 1),$$

and for each initial value $\tilde{\zeta} \equiv (t_0, 1)$, $\phi(\tau, \tilde{\zeta})$ is the solution of the equation

$$(\frac{dt}{d\tau}, \frac{ds}{d\tau}) = (1,\ f(t,s)), \tag{3.19}$$

satisfying $t(0) = t_0$, $s(0) = 1$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$. Let

$$t = T(\tau, t_0), \quad s = S(\tau, t_0),$$

be a solution of the characteristic system (3.19) emanating from the point $\tilde{\zeta}$. We assume that

$$\frac{\partial S}{\partial t_0} - f \neq 0$$

at $\tau = 0$. Without loss of generality we can assume that $t_0 = 0$, otherwise we replace V, h and l by those restriction along the characteristic line.
Then for each $\tilde{\zeta}$, we have associated the following problem see for instance [25]

$$\begin{cases} \dfrac{\partial l}{\partial t} + div(Vl) - \dfrac{\partial}{\partial x_3}\left(h\dfrac{\partial l}{\partial x_3}\right) + (\mu + k)l = 0, \\ l = 0, \text{ in } \Sigma, \\ l(0,P) = l_0(P), \end{cases} \tag{3.20}$$

where μ may be other function of order zero, and k is a real constant that we will chose later. We will use a perturbation method to get a time dependent parabolic equation whose resolution will yield to the solution of equation (2.2). Namely we consider the following perturbed problem

$$\begin{cases} \dfrac{\partial l}{\partial t} + div(Vl) - \sum_{i=1}^{3} \dfrac{\partial}{\partial x_i}\left(a_i^\varepsilon \dfrac{\partial l}{\partial x_i}\right) + (\mu + k)l = 0, \\ l = 0, \text{ in } \Sigma, \\ l(0,P) = l_0(P), \end{cases} \tag{3.21}$$

where

$$a_i^\varepsilon(t,P) = \begin{cases} \varepsilon \text{ if } i = 1,2 \\ h(t,P) + \varepsilon \text{ if } i = 3. \end{cases}$$

Let

$$Lu_\varepsilon = \frac{\partial u_\varepsilon}{\partial t} + div(Vu_\varepsilon) + (k+\mu)u_\varepsilon,$$

with

$$D(L) = \{v \in L^2(0,T; W_0^{1,2}(\Omega)); \frac{\partial v}{\partial t} \in L^2(0,T; W^{-1,2}(\Omega)), v(0) = 0\},$$

and

$$Au_\varepsilon = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_i^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\right),$$

defined by

$$< Au_\varepsilon, v >= \sum_{i=1}^{3} \int_Q a_i^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_i} dPdt,$$

for each $v \in L^2(0,T; W_0^{1,2}(\Omega))$. We state now the main assumptions of this section.

(H2) $h \in C^1(\bar{Q})$, $V_i \in C([0,T] \times \bar{\Omega})$, $i = 1,2,3$ and $\mu \in C([0,T] \times [1,12] \times \bar{\Omega})$.

(H3) $h \geq c_0 > 0$ in $[0,T] \times \bar{\Omega}$.

The main result in this section is the following theorem that gives conditions under which problem (3.21) has a unique positive solution.

Theorem 3.3.11 *Assume (H2)-(H3) hold. Let $l_0 \in L^2(\Omega)$, be such that $l_0 \geq 0$. Then problem (3.21) has a unique non negative solution $u_\varepsilon \in D(L)$.*

Proof. The main idea is to use Theorem 3.3.5. In the first step we will see that L is a closed operator with a dense domain ; indeed, let u_n in $D(L)$ be such that $u_n \to u$ in $L^2(0,T;W_0^{1,2}(\Omega))$ and $Lu_n \to y$ in $L^2(0,T;W^{-1,2}(\Omega))$, hence

$$u_n \to u \text{ in } \mathcal{D}'(0,T;W^{-1,2}(\Omega))$$

and

$$Lu_n \to y \text{ in } \mathcal{D}'(0,T;W^{-1,2}(\Omega)).$$

It follows that

$$Lu_n \to Lu \text{ in } \mathcal{D}'(0,T;W^{-1,2}(\Omega)).$$

Therefore we get $y = Lu$ and $u \in D(L)$. Hence L is a closed operator. It is not difficult to see that $\mathcal{D}(0,T;W_0^{1,2}(\Omega))$ is included in $D(L)$, then we deduce that $D(L)$ is dense in $L^2(0,T;W_0^{1,2}(\Omega))$. Concerning the monotonicity of L, we have for $u \in D(L)$,

$$\begin{aligned}
< Lu, u > &= \int_0^T < \frac{\partial u}{\partial t}, u > dt + \int_Q (div(Vu)u + (k+\mu)u^2)dPdt \\
&= \frac{1}{2}\int_0^T \frac{d}{dt}\|u(t)\|_*^2 dt + \int_\Sigma (V,\eta)u^2 d\sigma - \int_Q (V,\nabla u)udPdt \\
&+ \int_Q (k+\mu)u^2 dPdt \\
&= \frac{1}{2}\|u(T)\|_*^2 - \frac{1}{2}\int_Q (V,\nabla u^2)dPdt + \int_Q (k+\mu)u^2 dPdt,
\end{aligned}$$

with η is exterior normal, and $(,)$ is the scalar product. Hence integrating by parts we obtain that

$$< Lu, u > = \frac{1}{2}\|u(T)\|_*^2 + \int_Q (k + \frac{1}{2}div(V) + \mu)u^2 dPdt,$$

choosing k large enough such that

$$k + \frac{1}{2}div(V) + \mu \geq 0,$$

it follows that L is monotone for all $u \in D(L)$. In addition for $u \in D(L)$,

$$< Lu, v >= \int_0^T < \frac{\partial u}{\partial t}, v > dt + \int_Q (div(Vu)v + (k+\mu)uv)dPdt$$

$$= \int_0^T < u, -\frac{\partial v}{\partial t} > dt + < u(T), v(T) > + \int_Q (-(V,\nabla v) + (k+\mu)v)udPdt,$$

thus, the associated adjoint operator is given by

$$L^*v = -\frac{\partial v}{\partial t} - (V, \nabla v) + (k+\mu)v,$$

with

$$D(L^*) = \{v \in L^2(0,T;W_0^{1,2}(\Omega)); \frac{\partial v}{\partial t} \in L^2(0,T;W^{-1,2}(\Omega)), v(T) = 0\}.$$

The proof of monotonicity of L^* is similar to the one of L. Then L is a maximal monotone operator. It remain to see that A is coercive, indeed for $u \in L^2(0,T;W_0^{1,2}(\Omega))$ and applying the hypothesis on h, it holds

$$< Au, u >= \sum_{i=1}^{3} \int_Q a_i^\varepsilon |\frac{\partial u}{\partial x_i}|^2 dPdt \geq M_\varepsilon \|u\|^2_{L^2(0,T;W_0^{1,2}(\Omega))}.$$

According to Theorem 3.3.5 and Remark 3.3.6, we get the existence of a unique solution $u_\varepsilon \in D(L)$ to problem (3.21). Hence for all $v \in L^2(0,T;W_0^{1,2}(\Omega))$, we have

$$\begin{aligned} &\int_0^T < \frac{\partial u_\varepsilon}{\partial t}, v > dt + \int_Q (div(Vu_\varepsilon) + (\mu+k)u_\varepsilon)vdPdt \\ &+\sum_{i=1}^{3} \int_Q a_i^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_i} dPdt = 0. \end{aligned} \tag{3.22}$$

We prove now the positivity of the solution. We set $u = u^+ - u^-$, where u^+ and u^- are respectively the positive and negative part of u. Using u_ε^- as a test function in (3.22) and integrating on $(0,t)$, we get

$$\begin{aligned} -\int_0^t < \frac{\partial u_\varepsilon^-}{\partial t}, u_\varepsilon^- > dt - \int_0^t \int_\Omega (div(Vu_\varepsilon^-) + (\mu+k)u_\varepsilon^-)u_\varepsilon^- dPdt \\ -\sum_{i=1}^{3} \int_0^t \int_\Omega a_i^\varepsilon |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2 dPdt = 0. \end{aligned}$$

Then we conclude that

$$\begin{aligned} &-\frac{1}{2}\|u_\varepsilon^-(t)\|_*^2 \\ &= \int_0^t \int_\Omega (\frac{1}{2}div(V) + \mu + k)(u_\varepsilon^-)^2 dPdt + \sum_{i=1}^{3} \int_0^t \int_\Omega a_i^\varepsilon |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2 dPdt, \end{aligned}$$

hence,

$$-\frac{1}{2}\|u_\varepsilon^-(t)\|_*^2 \geq 0.$$

Then $u_\varepsilon^-(t) = 0$ for all $t \in (0,T)$. Hence we conclude ■

3.3.3 The exact solution

In this section we show that the perturbed solution defined in (3.22) tends to the desired solution of problem (3.20) in $L^2(Q)$ as ε tends to 0. Our main result is the following Theorem.

Theorem 3.3.12 *Let $l_0 \in L^2(\Omega)$ and consider u_ε the solution to problem (3.21), then u_ε converges weakly to u in $L^2(Q)$ where u is a distributional solution to problem* (3.20). *In addition we have $\frac{\partial u}{\partial x_3} \in L^2(Q)$ and u satisfies*

$$-\int_\Omega \int_0^T u\frac{\partial \phi}{\partial t} dxdt + \int_Q (-(V, \nabla\phi) + (\mu + k)\phi)udPdt + \int_Q h\frac{\partial u}{\partial x_3}\frac{\partial \phi}{\partial x_3}dPdt$$

$$= \int_\Omega l_0(P)\phi(0,P)dP, \tag{3.23}$$

for all $\phi \in K$ where

$$K \equiv \{\phi \in L^2(0,T; W_0^{1,2}(\Omega)); \frac{\partial \phi}{\partial t} \in L^2(0,T; W^{-1,2}(\Omega)) \cap L^2(Q) \text{ with } \phi(T) = 0\}. \tag{3.24}$$

Proof. Using u_ε as a test function in (3.22), we obtain that

$$\int_0^T < \frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon > dt + \int_Q (div(Vu_\varepsilon) + (\mu + k)u_\varepsilon)u_\varepsilon dPdt$$
$$+ \sum_{i=1}^3 \int_Q a_i^\varepsilon |\frac{\partial u_\varepsilon}{\partial x_i}|^2 dPdt = 0, \tag{3.25}$$

integrating by parts and using the definition of u_ε, we deduce

$$\frac{1}{2}\|u_\varepsilon(T)\|_*^2 + \int_Q (k + \frac{1}{2}div(V) + \mu)u_\varepsilon^2 dPdt + \sum_{i=1}^3 \int_Q a_i^\varepsilon \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2 dPdt$$
$$= \frac{1}{2}\|l_0\|_*^2. \tag{3.26}$$

Since $div(V)$ and μ are bounded functions we conclude that $\|u_\varepsilon\|^2_{L^2(Q)} \leq C$ and then there exists a subsequence called also u_ε such that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(Q)$. Notice that, in the same way, we obtain that

$$\sum_{i=1}^{3} \int_Q a_i^\varepsilon |\frac{\partial u_\varepsilon}{\partial x_i}|^2 dPdt \leq C_1.$$

By letting $\varepsilon \to 0$, we get

$$\limsup_{\varepsilon \to 0} \int_Q h|\frac{\partial u_\varepsilon}{\partial x_3}|^2 dPdt \leq C_1.$$

We claim that u is a solution of (3.20) in the sense of distribution. To proof the claim we consider $\phi \in C_0^\infty(\Omega \times (0,T))$, then using ϕ as a test function in (3.21) we obtain that

$$-\int_\Omega \int_0^T u_\varepsilon \phi_t dPdt + \int_Q (-(V, \nabla\phi) + (\mu + k)\phi) u_\varepsilon dPdt$$
$$+ \sum_{i=1}^{3} \int_Q u_\varepsilon \frac{\partial}{\partial x_i} \left(a_i^\varepsilon \frac{\partial \phi}{\partial x_i} \right) dPdt = \int_\Omega l_0(P)\phi(0,P)dP.$$

Since $\nabla h \in (L^2(Q))^3$ and $u_\varepsilon \rightharpoonup u$ weakly in $L^2(Q)$, then passing to the limit in the above equality we obtain that

$$-\int_\Omega \int_0^T u\phi_t dPdt + \int_Q (-(V, \nabla\phi) + (\mu + k)\phi) udPdt$$
$$+ \int_Q u \frac{\partial}{\partial x_3} \left(h \frac{\partial \phi}{\partial x_3} \right) dPdt = \int_\Omega l_0(P)\phi(0,P)dP.$$

Hence u is a distributional solution to problem (3.20) and the claim follows. To get a more regularity on u we set

$$\Psi_\varepsilon(t, x_3) = \int_D hu_\varepsilon dxdy,$$

where u_ε is the solution of (3.21). Using the hypothesis on h and V and by the classical result on the theory of regularity we obtain that $u_\varepsilon \in C^1([0,T] \times \bar{\Omega})$. Thus

$$\frac{\partial \Psi_\varepsilon}{\partial x_3} = \int_D (h\frac{\partial u_\varepsilon}{\partial x_3} + u_\varepsilon \frac{\partial h}{\partial x_3}) dxdy,$$

by integrating on $(0,T)\times(0,z^*)$ we get

$$\int_0^T\int_0^{z^*}|\frac{\partial\Psi_\varepsilon}{\partial x_3}|^2dx_3dt\le\int_Q h^2|\frac{\partial u_\varepsilon}{\partial x_3}|^2dPdt+C\int_Q|u_\varepsilon|^2dPdt\le C_2.$$

Since Ψ_ε is bounded in $L^2((0,T)\times(0,z^*))$, which can be proved easily, we conclude that Ψ_ε is bounded in $L^2(0,T;W_0^{1,2}(0,z^*))$, hence up to a subsequence, called also Ψ_ε, we obtain that Ψ_ε converges weakly in $L^2(0,T;W_0^{1,2}(0,z^*))$ to Ψ where

$$\Psi=\int_D h\,u\,dxdy.$$

Notice that the last identification follows by the fact that $u_\varepsilon\rightharpoonup u$ in weak topology of $L^2(Q)$ and by the uniqueness of the weak limit. We claim that $\frac{\partial u}{\partial x_3}\in L^2(Q)$. To get the claim we will prove that $\frac{\partial u}{\partial x_3}\in(L^2(Q))'\equiv L^2(Q)$. Notice that $\frac{\partial u}{\partial x_3}$ is well defined as a distribution. Let $\phi\in\mathcal{C}_0^\infty(Q)$, then we have

$$\begin{aligned}\int_Q\frac{\partial u}{\partial x_3}\phi dPdt &= -\int_Q\frac{\partial\phi}{\partial x_3}udPdt\\ &= -\lim_{\varepsilon\to0}\int_Q\frac{\partial\phi}{\partial x_3}u_\varepsilon dPdt=\lim_{\varepsilon\to0}\int_Q\frac{\partial u_\varepsilon}{\partial x_3}\phi dPdt\\ &\le \lim_{\varepsilon\to0}\Big(\int_Q|\frac{\partial u_\varepsilon}{\partial x_3}|^2dPdt\Big)^{\frac12}\Big(\int_Q|\phi|^2dPdt\Big)^{\frac12}.\end{aligned}$$

Then

$$|\int_Q\frac{\partial u}{\partial x_3}\phi dPdt|\le C\Big(\int_Q|\phi|^2dPdt\Big)^{\frac12},$$

for all $\phi\in\mathcal{C}_0^\infty(Q)$. Hence by density we conclude that $\frac{\partial u}{\partial x_3}\in(L^2(Q))'\equiv L^2(Q)$ and then the claim follows. Therefore we conclude that

$$\int_Q h\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial v}{\partial x_3}dPdt\to\int_Q h\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial x_3}dPdt,$$

for all $v\in L^2(0,T;W_0^{1,2}(0,z^*))$. Let $\phi\in\mathcal{C}_0^\infty(Q)$, using a density result, see for example [43], [11], there result that $\{\eta(t,x_1,x_2)\times\psi(t,x_3)\}$ is a total family in $C_0^\infty(Q)$. Hence using the above computation we get

$$\int_Q h\eta \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \psi}{\partial x_3} dPdt \to \int_Q h\eta \frac{\partial u}{\partial x_3}\frac{\partial \phi}{\partial x_3} dPdt.$$

Therefore by the density result obtained in [43] and in [11] we get the same conclusion for all $\phi \in C_0^\infty(Q)$, hence

$$\int_Q h \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \phi}{\partial x_3} dPdt \to \int_Q h \frac{\partial u}{\partial x_3}\frac{\partial \phi}{\partial x_3} dPdt.$$

Since $\phi \in C_0^\infty(Q)$ is dense in $L^2(0,T;W_0^{1,2}(D\times(0,z^*)))$ and by the fact that $\frac{\partial u}{\partial x_3} \in L^2(Q)$ we get that (3.23) holds for all $\phi \in L^2(0,T;W_0^{1,2}(D\times(0,z^*)))$. Moreover by letting $\varepsilon \to 0$ in (3.22), we obtain

$$\int_Q u\frac{\partial \phi}{\partial t} dxdt + \int_Q (-(V,\nabla\phi) + (\mu+k)\phi) u dPdt + \int_Q h\frac{\partial u}{\partial x_3}\frac{\partial \phi}{\partial x_3} dPdt = \int_\Omega l_0(P)\phi(0,P)dP,$$

for all $\phi \in K$, where K is defined in (3.24). Hence we conclude. ■

Remark 3.3.13 *We give a remark on the uniqueness of solution. In the paper by M. Escobedo, J. L. Vazquez and E. Zuazua, [21], the authors consider the following Cauchy problem*

$$u_t - \Delta_x u = \partial_y(f(u)), \quad x \in \mathbb{R}^{N-1},\ y \in \mathbb{R}, \quad t > 0, \tag{3.27}$$

then using the vanishing viscosity argument and the notion of Entropy solution, they obtain existence and uniqueness of solution to problem (3.27). *The argument used depends strongly on the presence of the linear operator* Δ_x, *and on the estimates obtained in [26], see appendix for the proof of the existence and uniqueness of the problem* (3.27). *The extension of the above uniqueness result to a non autonomous problem seems to be a more difficult technical problem.*

3.4 Multilayer method

In this section we treat the main model in the general situation, that is in the case where the horizontal current is not uniform throughout the water column, and then we can depends of all variables. In this situation we show existence, and positivity of solutions. The idea we exploit here is to approximate the model by one in which the above mentioned restriction is assumed to hold piecewise : this has been done by dividing the water column into thin layers in each of which it is reasonable to assume that the coefficients are constant throughout the vertical direction. The mathematical analysis of the problem leads to two main issues : 1) each approximating equation sets up a system of equations of parabolic type with time dependent coefficients and rather unusual boundary conditions. The study of such systems takes up the main part in this work.

Time dependence is dealt with using results on time-dependent evolution equations by P. Acquistapace [1],[3],[2], [34] and several other authors (A. Lunardi, H. Tanabe). A valuable source of information of this work was a monograph by H. Tanabe [45].

The main result of this part, stated in Theorem 3.4.1, ensures that, under some conditions on the coefficients of the equation, the Cauchy problem associated with the equation has a unique classical solution, which moreover is nonnegative if the initial value is non negative.

Approximate solutions converge in some sense to a solution of the full equation.

3.4.1 The multilayer model

The multilayer approach consists in dividing the water column into thin horizontal layers, assuming that the horizontal current is independent on z in each sub layer and the temperature and the mixing coefficient are constants in z on each sub layer. In order to satisfy the incompressibility condition, it is assumed that the vertical current depends linearly on z in each sub layer.

We divide the water column into n layers

$$\Omega = \bigcup_{i=1}^{n} D \times (z_{i-1}, z_i),$$

for $1 \leq i \leq n$ we denote respectively by :

$$V_k^i \equiv V_k^i(t,x,y) = \frac{1}{z_i - z_{i-1}} \int_{z_{i-1}}^{z_i} V_k(t,P)dz, \quad k = 1,2$$

$$V_3^i \equiv V_3^i(t,x,y,z) = \frac{V_3(t,x,y,z_i) - V_3(t,x,y,z_{i-1})}{z_i - z_{i-1}}(z - z_{i-1}) + V_3(t,x,y,z_{i-1}),$$

$$h^i \equiv h^i(t,x,y) = \frac{1}{z_i - z_{i-1}} \int_{z_{i-1}}^{z_i} h(t,P)dz,$$

$$f^i \equiv f^i(t,s,x,y) = \frac{1}{z_i - z_{i-1}} \int_{z_{i-1}}^{z_i} f(t,s,P)dz,$$

the averaged horizontal velocity of the sea currents, the affine approximation of the vertical velocity in the sea currents, the averaged mixing function and growth rate respectively in the layer $D \times (z_{i-1}, z_i)$. Note that V^i gives an approximation of the velocity of the sea currents and satisfies the incompressibility property in the layer $D \times (z_{i-1}, z_i)$, we have

$$\frac{\partial V_1^i}{\partial x} + \frac{\partial V_2^i}{\partial y} + \frac{\partial V_3^i}{\partial z} = \frac{1}{z_i - z_{i-1}} \int_{z_{i-1}}^{z_i} \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} \right) dz + \frac{1}{z_i - z_{i-1}} \int_{z_{i-1}}^{z_i} \frac{\partial V_3(t,P)}{\partial z} dz,$$

from the incompressibility property (2.3), we conclude that :

$$\operatorname{div}(V^i) = 0.$$

We denote by $l^i = l|_{[z_{i-1}, z_i)}$, the restriction of l to $[z_{i-1}, z_i)$. In each sub layer, for $P \in D \times [z_{i-1}, z_i)$, $1 \leq i \leq n$, equation (2.2) can be written as follows :

$$\frac{\partial l^i}{\partial t} + \frac{\partial (f^i l^i)}{\partial s} + div(V^i l^i) - h^i \left(\frac{\partial^2 l^i}{\partial z^2} \right) + \mu l^i = 0,$$

or

$$\frac{\partial l^i}{\partial t} + f^i \frac{\partial l^i}{\partial s} + \operatorname{div}(V^i l^i) - h^i \left(\frac{\partial^2 l^i}{\partial z^2} \right) + \gamma^i l^i = 0,$$

with

$$\gamma^i(t,s,x,y) = \frac{\partial f^i(t,s,x,y)}{\partial s} + \mu(s).$$

The s -boundary condition

$$l^i(t,1,P) = B^i(t,P).$$

The spatial boundary conditions

$$\begin{aligned} h^1 \frac{\partial l^1}{\partial z}(t,s,x,y,0) &= 0, \\ h^n \frac{\partial l^n}{\partial z}(t,s,x,y,z^*) &= 0, \end{aligned}$$

and the continuity of the solution and the flux at the interface of any two contiguous gives rise to the following supplementary boundary conditions

$$\begin{aligned} l^i(t,s,x,y,z_i) &= l^{i+1}(t,s,x,y,z_i), \\ h^i \frac{\partial l^i}{\partial z}(t,s,x,y,z_i) &= h^{i+1} \frac{\partial l^{i+1}}{\partial z}(t,s,x,y,z_i), \end{aligned}$$

for $1 \leq i \leq n$. Using the incompressibility property, the full multilayer model can be written in the form

$$\begin{cases} \dfrac{\partial l^i}{\partial t} + f^i \dfrac{\partial l^i}{\partial s} + V_1^i \dfrac{\partial l^i}{\partial x} + V_2^i \dfrac{\partial l^i}{\partial y} + V_3^i \dfrac{\partial l^i}{\partial z} - h^i \left(\dfrac{\partial^2 l^i}{\partial z^2} \right) + \gamma^i l^i = 0, \quad 1 \leq i \leq n, \\ l^i(t,1,P) = B^i(t,P), \\ l^i(t,s,x,y,z_i) = l^{i+1}(t,s,x,y,z_i), \\ h^i \dfrac{\partial l^i}{\partial z}(t,s,x,y,z_i) = h^{i+1} \dfrac{\partial l^{i+1}}{\partial z}(t,s,x,y,z_i), \qquad 1 \leq i \leq n-1, \\ h^1 \dfrac{\partial l^1}{\partial z}(t,s,x,y,0) = 0, \\ h^n \dfrac{\partial l^n}{\partial z}(t,s,x,y,z^*) = 0. \end{cases} \tag{3.28}$$

3.4.2 Existence, uniqueness and positivity of the solution of the Cauchy problem for the multilayer model.

The aim of this section is to show that the multilayer model (3.28) possesses a positive, unique solution. For this we use an approach by the method of characteristics to build a one dimensional time dependent parabolic equation whose resolution will yield the solution of equation (3.28).

We now state the assumptions of this section.

(H1) f, V_1, V_2 are functions in $C^1(\bar{\Gamma})$, where $\Gamma = (0,T) \times (1, s^*) \times D$.

(H2) $h \in C^1([0,T] \times \bar{\Omega})$, $V_3 \in C([0,T] \times \bar{\Omega})$, $\gamma \in C([0,T] \times [1, s^*] \times \bar{\Omega})$.

(H3) $h(.,P), V_3(.,P)$ and $\gamma(.,s,P) \in C^{1,\delta}([0,T]; I\!R)$.

(H4) $h \geq c_0$ in $[0,T] \times \bar{\Omega}$ where $c_0 > 0$.

(H5) $\sum_{i=1}^{3} V_i cos(\eta, x_i) \geq 0$ in Σ, where η is the exterior normal.

We introduce the flow generated by the horizontal current and the size growth the i^{th} layer, that is

$$\phi^i := \phi^i(\tau, t_0, 1, x_0, y_0),$$

and for each initial value $\tilde{\zeta} \equiv (t_0, 1, x_0, y_0)$, $\phi^i(\tau, \tilde{\zeta})$ is the solution of the equation

$$\left(\frac{dt^i}{d\tau}, \frac{ds^i}{d\tau}, \frac{dx^i}{d\tau}, \frac{dy^i}{d\tau}\right) = \left(1,\ f^i(t^i, s^i, x^i, y^i), V_1^i(t^i, x^i, y^i), V_2^i(t^i, x^i, y^i)\right), \tag{3.29}$$

satisfying $t(0) = t_0$, $s(0) = 1$, $x(0) = x_0$, $y(0) = y_0$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$.
We denote $\bar{l}^i(\tau, z) \equiv \bar{l}^i(\tau, \tilde{\zeta}, z) = l^i(\phi^i(\tau, \tilde{\zeta}), z)$ the restriction of l^i along the characteristic line. The equation verified by $\bar{l}^i$ reads

$$\frac{\partial \bar{l}^i(\tau, z)}{\partial \tau} + \bar{V}_3^i \frac{\partial \bar{l}^i(\tau, z)}{\partial z} - \bar{h}^i \left(\frac{\partial^2 \bar{l}^i(\tau, z)}{\partial z^2}\right) + \bar{\gamma}^i \bar{l}^i(\tau, z) = 0,$$

where $\bar{V}_3^i := \bar{V}_3^i(\tau, \tilde{\zeta}, z)$, $\bar{h}^i := \bar{h}^i(\tau, \tilde{\zeta})$, $\bar{\gamma}^i := \bar{\gamma}^i(\tau, \tilde{\zeta})$ are the restrictions of V_3^i, h^i, B^i, γ^i respectively along the characteristic line.
A natural condition to be satisfied at the common boundary $z = z_i$ of the i^{th} and the $(i+1)^{th}$ layers is

$$\bar{l}^{i+1}(\tau, \tilde{\zeta}, z_i) = \bar{l}^i(\tau, \phi^i(-\tau, \phi^{i+1}(\tau, \tilde{\zeta})), z_i),$$

together with

$$\bar{h}^{i+1}(\tau) \frac{\partial \bar{l}^{i+1}}{\partial z}(\tau, \tilde{\zeta}, z_i) = \bar{h}^i(\tau) \frac{\partial \bar{l}^i}{\partial z}(\tau, \phi^i(-\tau, \phi^{i+1}(\tau, \tilde{\zeta})), z_i).$$

These compatibility conditions are completed by a boundary condition at $z = 0$, and $z = z^*$,

$$\begin{aligned} \bar{h}^1 \frac{\partial \bar{l}^1}{\partial z}(\tau, \tilde{\zeta}, 0) &= 0, \\ \bar{h}^n \frac{\partial \bar{l}^n}{\partial z}(\tau, \tilde{\zeta}, z^*) &= 0. \end{aligned}$$

The compatibility conditions determined above would render very difficult the treatment of the approximate equation. One can observe that these conditions are non local, that is, relate the value of the solution at one point of the surface $z = z_i$ to its value at a different point. Noticing that

$$\phi^i(-\tau, \phi^{i+1}(\tau, \tilde{\zeta})) \longrightarrow \tilde{\zeta}, \tag{3.30}$$

as the number of layers becomes large and their maximum thickness becomes small, we will consider the following compatibility conditions

$$\bar{l}^{i+1}(\tau, \tilde{\zeta}, z_i) = \bar{l}^i(\tau, \tilde{\zeta}, z_i),$$

together with

$$\bar{h}^{i+1}(\tau) \frac{\partial \bar{l}^{i+1}}{\partial z}(\tau, \tilde{\zeta}, z_i) = \bar{h}^i(\tau) \frac{\partial \bar{l}^i}{\partial z}(\tau, \tilde{\zeta}, z_i).$$

Indeed; let the following problem

$$\left(\frac{dt^j}{d\tau}, \frac{ds^j}{d\tau}, \frac{dx^j}{d\tau}, \frac{dy^j}{d\tau} \right) = \left(1,\ f^j(t^j, s^j, x^j, y^j), V_1^j(t^j, x^j, y^j), V_2^j(t^j, x^j, y^j)\right), \tag{3.31}$$

satisfying $t^j(0) = t_0,\ s^j(0) = 1,\ x^j(0) = x_0,\ y^j(0) = y_0$, for $j = 1, .., n$. subtracting the vector (3.31) for $j = i$ from the one for $j = i + 1$, thus

$$|\frac{dt^{i+1}}{d\tau} - \frac{dt^i}{d\tau}| = 0,$$

$$|\frac{ds^{i+1}}{d\tau} - \frac{ds^i}{d\tau}| = |f^{i+1}(t^{i+1}, s^{i+1}, x^{i+1}, y^{i+1}) - f^i(t^i, s^i, x^i, y^i)|, \tag{3.32}$$

$$|\frac{dx^{i+1}}{d\tau} - \frac{dx^i}{d\tau}| = |V_1^{i+1}(t^{i+1}, s^{i+1}, x^{i+1}, y^{i+1}) - V_1^i(t^i, s^i, x^i, y^i)|,$$

$$|\frac{dy^{i+1}}{d\tau} - \frac{dy^i}{d\tau}| = |V_2^{i+1}(t^{i+1}, s^{i+1}, x^{i+1}, y^{i+1}) - V_2^i(t^i, s^i, x^i, y^i)|.$$

First of all, we show that

$$\lim_{n\to\infty} f^i(t, s, x, y, z) = f(t, s, x, y, z).$$

In fact, for $(t, s, x, y, z) \in \Gamma \times [0, z^*]$ there exists $i \equiv i(n) \in \{0, ..., n-1\}$ such that $z \in [z_i, z_{i+1})$ and

$$|z_{i+1} - z_i| \to 0 \quad \text{as } n \to \infty, \tag{3.33}$$

$$\lim_{n\to\infty} f^i(t, s, x, y) = \lim_{n\to\infty} \frac{1}{z_{i+1} - z_i} \int_{z_i}^{z_{i+1}} f(t, s, x, y, z')dz'. \tag{3.34}$$

Using the hypothesis over the sequences $\{z_{i+1}\}$ and $\{z_i\}$ we obtain that $\{z_{i+1}\}, \{z_i\}$ converge to z as $n \to \infty$. Hence we conclude that

$$\lim_{n\to\infty} f^i(t, s, x, y, z) = f(t, s, x, y, z). \tag{3.35}$$

Set now

$$R = |s^{i+1} - s^i| + |x^{i+1} - x^i| + |y^{i+1} - y^i|.$$

Remarking that f is globally lipschitzienne, then by the equation (3.32) we obtain

$$|\frac{ds^{i+1}}{d\tau} - \frac{ds^i}{d\tau}| \leq R + |(f^{i+1} - f^i)(t^i, s^i, x^i, y^i)|.$$

As shown above, the sequence $\{f^n\}$ is convergent, then

$$|\frac{ds^{i+1}}{d\tau} - \frac{ds^i}{d\tau}| \leq MR + \varepsilon.$$

In the similar way, we obtain

$$|\frac{dV_1^{i+1}}{d\tau} - \frac{dV_1^i}{d\tau}| \leq MR + \varepsilon,$$

$$|\frac{dV_2^{i+1}}{d\tau} - \frac{dV_2^i}{d\tau}| \leq MR + \varepsilon.$$

Summing this last three equation, we have

$$\frac{dR}{d\tau} \leq M_1 R + \varepsilon,$$

therefore

$$R(\tau) \leq \frac{\varepsilon}{M_1}(e^{M_1\tau} - 1).$$

Hence we conclude.
So to each $\tilde{\zeta}$, we have associated the following system of equations

$$\begin{cases} \dfrac{\partial \bar{l}^i}{\partial \tau} + \bar{V}_3^i(\tau, z)\dfrac{\partial \bar{l}^i}{\partial z} - \bar{h}^i(\tau)\left(\dfrac{\partial^2 \bar{l}^i}{\partial z^2}\right) + \bar{\gamma}^i(\tau)\bar{l}^i = 0, \qquad 1 \leq i \leq n, \\ \bar{l}^i(0, z) = \bar{B}^i(z), \\ \bar{l}^{i+1}(\tau, z_i) = \bar{l}^i(\tau, z_i), \\ \bar{h}^{i+1}(\tau)\dfrac{\partial \bar{l}^{i+1}(\tau, z_i)}{\partial z} = \bar{h}^i(\tau)\dfrac{\partial \bar{l}^i(\tau, z_i)}{\partial z}, \qquad 1 \leq i \leq n-1, \\ \bar{h}^1\dfrac{\partial \bar{l}^1(\tau, 0)}{\partial z} = 0, \\ \bar{h}^n\dfrac{\partial \bar{l}^n(\tau, z^*)}{\partial z} = 0, \end{cases} \tag{3.36}$$

where $\bar{B}^i(z) := \bar{B}^i(\tilde{\zeta}, z)$ is the restriction of B^i along the characteristic line. We will first use a change of variables to transform problem (3.36) into a system of partial differential equations over a single interval.
We set for $z \in [z_{i-1}, z_i]$

$$\begin{aligned} \bar{v}^i(\tau, \zeta) &= \bar{l}^i(\tau, z_{i-1} + \zeta(z_i - z_{i-1})), \quad 0 \leq \zeta \leq 1, \\ \alpha_3^i(\tau, \zeta) &= \bar{V}_3^i(\tau, z_{i-1} + \zeta(z_i - z_{i-1})), \quad 1 \leq i \leq n, \end{aligned}$$

the conditions yield :

$$\bar{v}^i(\tau, 1) = \bar{l}^i(\tau, z_i),$$

for $z \in [z_i, z_{i+1}]$

$$\bar{v}^{i+1}(\tau, 0) = \bar{l}^{i+1}(\tau, z_i),$$

so

$$\bar{l}^{i+1}(\tau, z_i) = \bar{l}^i(\tau, z_i) \Leftrightarrow \bar{v}^{i+1}(\tau, 0) = \bar{v}^i(\tau, 1),$$

in addition

$$\frac{1}{z_i - z_{i-1}}\frac{\partial \bar{v}^i(\tau, \zeta)}{\partial \zeta} = \frac{\partial \bar{l}^i(\tau, z)}{\partial z},$$

then

$$\bar{h}^{i+1}(\tau)\frac{\partial \bar{l}^{i+1}(\tau, z_i)}{\partial z} = \bar{h}^i(\tau)\frac{\partial \bar{l}^i(\tau, z_i)}{\partial z} \Leftrightarrow \frac{\bar{h}^{i+1}(\tau)}{z_{i+1} - z_i}\frac{\partial \bar{v}^{i+1}(\tau, 0)}{\partial \zeta} = \frac{\bar{h}^i(\tau)}{z_i - z_{i-1}}\frac{\partial \bar{v}^i(\tau, 1)}{\partial \zeta},$$

hence, the model becomes

$$\begin{cases} \dfrac{\partial \bar{v}^i}{\partial \tau} - \dfrac{\bar{h}^i(\tau)}{(z_i - z_{i-1})^2} \dfrac{\partial^2 \bar{v}^i}{\partial \zeta^2} + \dfrac{\alpha_3^i(\tau,\zeta)}{z_i - z_{i-1}} \dfrac{\partial \bar{v}^i}{\partial \zeta} + \bar{\gamma}^i(\tau)\bar{v}^i = 0, & 1 \le i \le n, \\ & 0 \le \zeta \le 1, \\ \bar{v}^i(0,\zeta) = \bar{B}^i(\zeta), \\ \bar{v}^{i+1}(\tau,0) = \bar{v}^i(\tau,1), \\ \dfrac{\bar{h}^{i+1}(\tau)}{z_{i+1} - z_i} \dfrac{\partial \bar{v}^{i+1}(\tau,0)}{\partial \zeta} = \dfrac{\bar{h}^i(\tau)}{z_i - z_{i-1}} \dfrac{\partial \bar{v}^i(\tau,1)}{\partial \zeta}, & 1 \le i \le n-1, \\ \bar{h}^1 \dfrac{\partial \bar{v}^1}{\partial \zeta}(\tau,0) = 0, \\ \bar{h}^n \dfrac{\partial \bar{v}^n}{\partial \zeta}(\tau,1) = 0. \end{cases} \tag{3.37}$$

We consider the operator $A(t) : D(A(t)) \subseteq C([0,z^*]) \to C([0,z^*])$ defined by

$$\begin{aligned} A(t)u &= ((A^i(t))u^i)_{i=1}^n, \quad where \\ A^i(t)u^i &= -\left(\frac{\bar{h}^i(t)}{(z_i - z_{i-1})^2}\right) \frac{d^2u^i}{d\zeta^2} + \left(\frac{\alpha_3^i(t,\zeta)}{z_i - z_{i-1}}\right) \frac{du^i}{d\zeta} + \left(\bar{\gamma}^i(t)\right) u^i, \end{aligned}$$

$$\begin{aligned} D(A(t)) &= \{u \in C^2([0,1]; I\!R^n), \frac{du^1(0)}{d\zeta} = \frac{du^n(1)}{d\zeta} = 0,\ u^{i+1}(0) = u^i(1), \\ & \frac{\bar{h}^{i+1}(t)}{z_{i+1} - z_i} \frac{du^{i+1}(0)}{d\zeta} = \frac{\bar{h}^i(t)}{z_i - z_{i-1}} \frac{du^i(1)}{d\zeta}, \quad 1 \le i \le n-1\}, \end{aligned} \tag{3.38}$$

where $u = (u^1, ..., u^n)^T$.
Therefore we obtain

Theorem 3.4.1 *Assume (H2)-(H4) hold. If the positive function $\bar{B}$ is in $C([0,z^*])$, then problem (3.36) has a unique non-negative classical solution.*

Proof. Without loss of generality we can assume that $\gamma \ge 0$, otherwise we can replace γ by $\gamma + \omega \ge 0$ see Remark 3.2.5. The main idea is to use theorem 3.2.4. The first assertion (AT1) follows from the proposition 3.2.2. Concerning the second assertion (AT2), for $f \in C([0,z^*])$, $t, t_1 \ge 0$, $\lambda \in S_\theta - \{0\}$, we set $v = (A(t_1))^{-1}f$ and $u = R(\lambda, A(t))(\lambda - A(t_1))v$, then we have to estimate the $C([0,z^*])$-norm of

$$u - v = (A(t))R(\lambda, A(t))(A(t))^{-1} - (A(t_1))^{-1})f.$$

Now $u - v \in C^2([0, z^*]; I\!R^n)$ and u and v solve respectively

$$\begin{cases} \lambda u^i - A^i(t,.)u^i = \lambda v^i - f, & 1 \le i \le n, \\ \dfrac{du^1(0)}{d\zeta} = \dfrac{du^n(1)}{d\zeta} = 0, & \\ u^{i+1}(0) = u^i(1), & \\ \dfrac{\bar{h}^{i+1}(t)}{z_{i+1} - z_i}\dfrac{du^{i+1}(0)}{d\zeta} = \dfrac{\bar{h}^i(t)}{z_i - z_{i-1}}\dfrac{du^i(1)}{d\zeta}, & 1 \le i \le n-1, \end{cases} \tag{3.39}$$

$$\begin{cases} A^i(t_1,.)v^i = f, & 1 \le i \le n, \\ \dfrac{dv^1(0)}{d\zeta} = \dfrac{dv^n(1)}{d\zeta} = 0, & \\ v^{i+1}(0) = v^i(1), & \\ \dfrac{\bar{h}^{i+1}(t_1)}{z_{i+1} - z_i}\dfrac{dv^{i+1}(0)}{d\zeta} = \dfrac{\bar{h}^i(t_1)}{z_i - z_{i-1}}\dfrac{dv^i(1)}{d\zeta}, & 1 \le i \le n-1, \end{cases} \tag{3.40}$$

this shows that

$$\begin{cases} \lambda(u^i - v^i) - A^i(t,.)(u^i - v^i) = (A^i(t,.) - A^i(t_1,.))v^i, & 1 \le i \le n, \\ \dfrac{d(u^1 - v^1)(0)}{d\zeta} = \dfrac{d(u^n - v^n)(1)}{d\zeta} = 0, & \\ (u^{i+1} - v^{i+1})(0) = (u^i - v^i)(1), & \\ \dfrac{\bar{h}^{i+1}(t)}{z_{i+1} - z_i}\dfrac{d(u^{i+1} - v^{i+1})(0)}{d\zeta} - \dfrac{\bar{h}^i(t)}{z_i - z_{i-1}}\dfrac{d(u^i - v^i)(1)}{d\zeta} = & \\ \dfrac{(\bar{h}^{i+1}(t_1) - \bar{h}^{i+1}(t))}{z_{i+1} - z_i}\dfrac{dv^{i+1}(0)}{d\zeta} - \dfrac{(\bar{h}^i(t_1) - \bar{h}^i(t))}{z_i - z_{i-1}}\dfrac{dv^i(1)}{d\zeta}, & 1 \le i \le n-1, \end{cases} \tag{3.41}$$

We can write the previous problem as

$$\begin{cases} \lambda(u - v) - A(t,.)(u - v) = (A(t,.) - A(t_1,.))v, \\ A_0(u - v)(0) - B_0(u - v)(1) = 0, \\ A_1(t)\dfrac{d(u - v)(0)}{d\zeta} - B_1(t)\dfrac{d(u - v)(1)}{d\zeta} = C_1(t), \end{cases} \tag{3.42}$$

where $A_0, B_0, A_1(t), B_1(t), C_1(t)$ are the appropriate matrices. Applying Proposition 3.2.1 to (3.40) with $\lambda = 0$, we have

$$||v||_E \le c||f||_E. \tag{3.43}$$

Using again the Proposition 3.2.1 to (3.41), we get

$$|\lambda|||u - v||_E \le M(||(A(t,.) - A(t_1,.))v||_E + (1 + |\lambda|^{1/2}) \times |C_1(t)|) .$$

Using hypothesis (H3) and by virtue of (3.43),

$$\|u-v\|_E \le c(|t-t_1||\lambda|^{-1}|t-t_1|^{\delta}|\lambda|^{-1}+|t-t_1||\lambda|^{-1/2})\|f\|_E.$$

Hence the hypothesis (AT2) holds. Since $D(A(0))$ is dense in $C([0,z^*])$ see [2], then according to Theorem 3.2.4, for $\bar{B}\in C([0,z^*])$ we have existence and uniqueness of a classical solution of problem (3.36). It remain to see that the solution is positive which can be proved by the standard argument. We set $u=u^+-u^-$ where u^+ and u^- are respectively the positive and negative part of u, so multiplying the equation (3.36) by u^-, and integrating over $(0,z^*)$ we have

$$\int_0^{z^*}\left(\frac{\partial u^i}{\partial\tau}u^{i^-}+\bar{h}^i(\tau)\frac{\partial u^i}{\partial z}\frac{\partial u^{i^-}}{\partial z}+\bar{V}_3^i(\tau,z)\frac{\partial u^i}{\partial z}u^{i^-}+\bar{\gamma}^i(\tau)u^iu^{i^-}\right)dz=0,$$

for $1\le i\le n$, hence

$$-\frac{1}{2}\frac{d}{d\tau}\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)}\ge c_0\int_0^{z^*}|\frac{\partial u^{i^-}}{\partial z}|^2dz-m_3\int_0^{z^*}\frac{\partial u^{i^-}}{\partial z}u^{i^-}dz+m_4\int_0^{z^*}|u^{i^-}|^2dz,$$

with

$$m_3=\sup_{\tau,z}|\bar{V}_3^i(\tau,z)|,\quad m_4=\inf_{\tau}|\bar{\gamma}^i(\tau)|.$$

Since

$$\int_0^{z^*}\frac{\partial u^{i^-}}{\partial z}u^{i^-}dz\le\int_0^{z^*}\left(\rho|\frac{\partial u^{i^-}}{\partial z}|^2+\frac{1}{\rho}|u^{i^-}|^2\right)dz,\quad\forall\rho>0,$$

it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)}+(c_0-m_3\rho)\|\frac{\partial u^{i^-}(\tau)}{\partial z}\|^2_{L^2(0,z^*)}\\&+(m_4-\frac{m_3}{\rho}+\omega)\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)}\\&\le\omega\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)},\end{aligned}$$

choosing ρ and ω such that

$$c_0-m_3\rho>0\quad\text{and}\quad m_4-\frac{m_3}{\rho}+\omega>0,$$

so

$$\frac{1}{2}\frac{d}{d\tau}\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)}\le\omega\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)},$$

then

$$\|u^{i^-}(\tau)\|^2_{L^2(0,z^*)} \leq \|u^{i^-}(0)\|^2_{L^2(0,z^*)} e^{2\omega\tau},$$

which gives $u^{i^-}(\tau) = 0$ provided $B \geq 0$, then the solution is positive. ■
For each $z \in [z_{i-1}, z_i]$

$$\begin{cases} t = T^i(\tau, t_0, x_0, y_0), \\ s = S^i(\tau, t_0, x_0, y_0), \\ x = X^i(\tau, t_0, x_0, y_0), \\ y = Y^i(\tau, t_0, x_0, y_0), \end{cases} \tag{3.44}$$

is solution of characteristic system (3.29) emanating from the point $\tilde{\zeta}$. We have

$$\begin{cases} t_0 = T^i(0, t_0, x_0, y_0), \\ 1 = S^i(0, t_0, x_0, y_0), \\ x_0 = X^i(0, t_0, x_0, y_0), \\ y_0 = Y^i(0, t_0, x_0, y_0), \end{cases} \tag{3.45}$$

if

$$Jac(T^i, S^i, X^i, Y^i) := \begin{vmatrix} \frac{\partial T^i}{\partial \tau} & \frac{\partial T^i}{\partial t_0} & \frac{\partial T^i}{\partial x_0} & \frac{\partial T^i}{\partial y_0} \\ \frac{\partial S^i}{\partial \tau} & \frac{\partial S^i}{\partial t_0} & \frac{\partial S^i}{\partial x_0} & \frac{\partial S^i}{\partial y_0} \\ \frac{\partial X^i}{\partial \tau} & \frac{\partial X^i}{\partial t_0} & \frac{\partial X^i}{\partial x_0} & \frac{\partial X^i}{\partial y_0} \\ \frac{\partial Y^i}{\partial \tau} & \frac{\partial Y^i}{\partial t_0} & \frac{\partial Y^i}{\partial x_0} & \frac{\partial Y^i}{\partial y_0} \end{vmatrix} \neq 0, \tag{3.46}$$

then we can solve for (τ, t_0, x_0, y_0) as function of (t, s, x, y), so

$$\begin{cases} \tau = \psi^i_1(t, s, x, y), \\ t_0 = \psi^i_2(t, s, x, y), \\ x_0 = \psi^i_3(t, s, x, y), \\ y_0 = \psi^i_4(t, s, x, y), \end{cases} \tag{3.47}$$

and for initial data

$$\begin{cases} 0 = \psi^i_1(t_0, 1, x_0, y_0), \\ t_0 = \psi^i_2(t_0, 1, x_0, y_0), \\ x_0 = \psi^i_3(t_0, 1, x_0, y_0), \\ y_0 = \psi^i_4(t_0, 1, x_0, y_0). \end{cases} \tag{3.48}$$

Once problem (3.36) is solved, we have

$$l^i(t, s, x, y, z) = \bar{l}^i(\psi^i_1, \psi^i_2, \psi^i_3, \psi^i_4, z), \quad 1 \leq i \leq n. \tag{3.49}$$

Indeed, by differentiation we obtain that

$$\begin{aligned}
\frac{\partial l^i}{\partial t} &= \frac{\partial \bar{l}^i}{\partial \tau}\frac{\partial \psi_1^i}{\partial t} + \frac{\partial \bar{l}^i}{\partial t_0}\frac{\partial \psi_2^i}{\partial t} + \frac{\partial \bar{l}^i}{\partial x_0}\frac{\partial \psi_3^i}{\partial t} + \frac{\partial \bar{l}^i}{\partial y_0}\frac{\partial \psi_4^i}{\partial t},\\
f^i\frac{\partial l^i}{\partial s} &= f^i\left(\frac{\partial \bar{l}^i}{\partial \tau}\frac{\partial \psi_1^i}{\partial s} + \frac{\partial \bar{l}^i}{\partial t_0}\frac{\partial \psi_2^i}{\partial s} + \frac{\partial \bar{l}^i}{\partial x_0}\frac{\partial \psi_3^i}{\partial s} + \frac{\partial \bar{l}^i}{\partial y_0}\frac{\partial \psi_4^i}{\partial s}\right),\\
V_1^i\frac{\partial l^i}{\partial x} &= V_1^i\left(\frac{\partial \bar{l}^i}{\partial \tau}\frac{\partial \psi_1^i}{\partial x} + \frac{\partial \bar{l}^i}{\partial t_0}\frac{\partial \psi_2^i}{\partial x} + \frac{\partial \bar{l}^i}{\partial x_0}\frac{\partial \psi_3^i}{\partial x} + \frac{\partial \bar{l}^i}{\partial y_0}\frac{\partial \psi_4^i}{\partial x}\right),\\
V_2^i\frac{\partial l^i}{\partial y} &= V_2^i\left(\frac{\partial \bar{l}^i}{\partial \tau}\frac{\partial \psi_1^i}{\partial y} + \frac{\partial \bar{l}^i}{\partial t_0}\frac{\partial \psi_2^i}{\partial y} + \frac{\partial \bar{l}^i}{\partial x_0}\frac{\partial \psi_3^i}{\partial y} + \frac{\partial \bar{l}^i}{\partial y_0}\frac{\partial \psi_4^i}{\partial y}\right),
\end{aligned}$$

thus

$$\begin{aligned}
\frac{\partial l^i}{\partial t} + f^i\frac{\partial l^i}{\partial s} + V_1^i\frac{\partial l^i}{\partial x} + V_2^i\frac{\partial l^i}{\partial y} &= \frac{\partial \bar{l}^i}{\partial \tau}(\frac{\partial \psi_1^i}{\partial t} + f^i\frac{\partial \psi_1^i}{\partial s} + V_1^i\frac{\partial \psi_1^i}{\partial x} + V_2^i\frac{\partial \psi_1^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial t_0}(\frac{\partial \psi_2^i}{\partial t} + f^i\frac{\partial \psi_2^i}{\partial s} + V_1^i\frac{\partial \psi_2^i}{\partial x} + V_2^i\frac{\partial \psi_2^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial x_0}(\frac{\partial \psi_3^i}{\partial t} + f^i\frac{\partial \psi_3^i}{\partial s} + V_1^i\frac{\partial \psi_3^i}{\partial x} + V_2^i\frac{\partial \psi_3^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial y_0}(\frac{\partial \psi_4^i}{\partial t} + f^i\frac{\partial \psi_4^i}{\partial s} + V_1^i\frac{\partial \psi_4^i}{\partial x} + V_2^i\frac{\partial \psi_4^i}{\partial y}).
\end{aligned}$$

then

$$\begin{aligned}
\frac{\partial l^i}{\partial t} + f^i\frac{\partial l^i}{\partial s} + V_1^i\frac{\partial l^i}{\partial x} + V_2^i\frac{\partial l^i}{\partial y} &= \frac{\partial \bar{l}^i}{\partial \tau}(\frac{\partial T^i}{\partial \tau}\frac{\partial \psi_1^i}{\partial t} + \frac{\partial S^i}{\partial t_0}\frac{\partial \psi_1^i}{\partial s} + \frac{\partial X^i}{\partial x_0}\frac{\partial \psi_1^i}{\partial x} + \frac{\partial Y^i}{\partial y_0}\frac{\partial \psi_1^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial t_0}(\frac{\partial T^i}{\partial \tau}\frac{\partial \psi_2^i}{\partial t} + \frac{\partial S^i}{\partial t_0}\frac{\partial \psi_2^i}{\partial s} + \frac{\partial X^i}{\partial x_0}\frac{\partial \psi_2^i}{\partial x} + \frac{\partial Y^i}{\partial y_0}\frac{\partial \psi_2^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial x_0}(\frac{\partial T^i}{\partial \tau}\frac{\partial \psi_3^i}{\partial t} + \frac{\partial S^i}{\partial t_0}\frac{\partial \psi_3^i}{\partial s} + \frac{\partial X^i}{\partial x_0}\frac{\partial \psi_3^i}{\partial x} + \frac{\partial Y^i}{\partial y_0}\frac{\partial \psi_3^i}{\partial y})\\
&+ \frac{\partial \bar{l}^i}{\partial y_0}(\frac{\partial T^i}{\partial \tau}\frac{\partial \psi_4^i}{\partial t} + \frac{\partial S^i}{\partial t_0}\frac{\partial \psi_4^i}{\partial s} + \frac{\partial X^i}{\partial x_0}\frac{\partial \psi_4^i}{\partial x} + \frac{\partial Y^i}{\partial y_0}\frac{\partial \psi_4^i}{\partial y}).
\end{aligned}$$

Moreover for

$$Z^i = (T^i, S^i, X^i, Y^i)^T, \text{ and } \psi^i = (\psi_1^i, \psi_2^i, \psi_3^i, \psi_4^i)^T,$$

we have

$$(Z^i \circ \psi^i)(t, s, x, y) = (t, s, x, y),$$

which implies

$$Jac(Z^i).Jac(\psi^i) = Id_4, \tag{3.50}$$

by identification in (3.50), we find

$$\begin{cases} \dfrac{\partial T^i}{\partial \tau}\dfrac{\partial \psi_1^i}{\partial t} + \dfrac{\partial S^i}{\partial t_0}\dfrac{\partial \psi_1^i}{\partial s} + \dfrac{\partial X^i}{\partial x_0}\dfrac{\partial \psi_1^i}{\partial x} + \dfrac{\partial Y^i}{\partial y_0}\dfrac{\partial \psi_1^i}{\partial y} = 1, \\ \dfrac{\partial T^i}{\partial \tau}\dfrac{\partial \psi_2^i}{\partial t} + \dfrac{\partial S^i}{\partial t_0}\dfrac{\partial \psi_2^i}{\partial s} + \dfrac{\partial X^i}{\partial x_0}\dfrac{\partial \psi_2^i}{\partial x} + \dfrac{\partial Y^i}{\partial y_0}\dfrac{\partial \psi_2^i}{\partial y} = 0, \\ \dfrac{\partial \psi_3^i}{\partial t} + f^i\dfrac{\partial \psi_3^i}{\partial s} + V_1^i\dfrac{\partial \psi_3^i}{\partial x} + V_2^i\dfrac{\partial \psi_3^i}{\partial y} = 0, \\ \dfrac{\partial T^i}{\partial \tau}\dfrac{\partial \psi_4^i}{\partial t} + \dfrac{\partial S^i}{\partial t_0}\dfrac{\partial \psi_4^i}{\partial s} + \dfrac{\partial X^i}{\partial x_0}\dfrac{\partial \psi_4^i}{\partial x} + \dfrac{\partial Y^i}{\partial y_0}\dfrac{\partial \psi_4^i}{\partial y} = 0. \end{cases}$$

Thus

$$\frac{\partial l^i}{\partial t} + f^i\frac{\partial l^i}{\partial s} + V_1^i\frac{\partial l^i}{\partial x} + V_2^i\frac{\partial l^i}{\partial y} = \frac{\partial \bar{l}^i}{\partial \tau}.$$

In addition

$$\frac{\partial l^i}{\partial z} = \frac{\partial \bar{l}^i}{\partial z},$$

and

$$h^{i+1}\frac{\partial l^{i+1}(t,s,x,y,0)}{\partial z} = h^i\frac{\partial l^i(t,s,x,y,0)}{\partial z}.$$

For the initial data

$$\begin{aligned} l^i(t,1,P) &= \bar{l}^i(\psi_1^i(t,1,x,y),\psi_2^i(t,1,x,y),\psi_3^i(t,1,x,y),\psi_4^i(t,1,x,y),z), \\ &= \bar{l}^i(0,t,P) \\ &= \bar{B}^i(t,P), \\ &= B^i(T^i(0,t,x,y),X^i(0,t,x,y),Y^i(0,t,x,y),z), \\ &= B^i(t,P), \end{aligned}$$

then l^i is solution of the problem (3.28). So the solution $l^i$of problem (3.28) can be determined in terms of the solution $\bar{l}^i$ of problem (3.36).
We define the following functions

$$L^n(t,s,P) = \sum_{i=1}^{n} l^i(t,s,P)\chi_{[z_{i-1},z_i)}, \tag{3.51}$$

where$(l^i)_{i=1}^n$ is solution of problem (3.28), and χ_I the characteristic function for set I. We define the following problem

$$\begin{cases} \dfrac{\partial l}{\partial t} + \dfrac{\partial (f^n l)}{\partial s} + V_1^n \dfrac{\partial l}{\partial x} + V_2^n \dfrac{\partial l}{\partial y} + V_3^n \dfrac{\partial l}{\partial z} - \dfrac{\partial}{\partial z}\left(H^n \dfrac{\partial l}{\partial z}\right) + \mu l = 0, \\ l(t,1,P) = B^n(t,P), \\ (H^n \dfrac{\partial l}{\partial z})(t,s,x,y,0) = 0, \\ (H^n \dfrac{\partial l}{\partial z})(t,s,x,y,z^*) = 0, \end{cases} \tag{3.52}$$

where

$$\begin{aligned} H^n(t,P)|_{[z_{i-1},z_i)} &= h^i(t,x,y), \ \ V_3^n(t,P)|_{[z_{i-1},z_i)} = V_3^i(t,P), \\ f^n(t,s,P)|_{[z_{i-1},z_i)} &= f^i(t,s,x,y), V_k^n(t,P)|_{[z_{i-1},z_i)} = V_k^i(t,x,y), \ k = 1,2, \\ B^n(t,P)|_{[z_{i-1},z_i)} &= B^i(t,P), \ 1 \le i \le n \ . \end{aligned}$$

Theorem 3.4.2 *Let L^n be defined by (3.51). Then L^n is a solution of problem (3.52) .*

Proof. For each $(t,s,x,y) \in \Gamma$, $L^n(t,s,x,y,.) \in L^2(0,z^*)$, indeed

$$\int_0^{z^*} |L^n(t,s,P)|^2 \, dz = \sum_{i=1}^n \int_{z_{i-1}}^{z_i} \left|l^i(t,s,P)\right|^2 dz < \infty, \tag{3.53}$$

since for each $(t,s,x,y) \in \Gamma$, $l^i(t,s,x,y,.) \in L^2(z_{i-1},z_i)$. In addition, for any $\varphi \in \mathcal{D}(0,z^*)$

$$\begin{aligned} \int_0^{z^*} L^n(t,s,P)\frac{d\varphi(z)}{dz}dz &= \sum_{i=1}^n \int_{z_{i-1}}^{z_i} l^i(t,s,P)\frac{d\varphi(z)}{dz}dz, \\ &= \sum_{i=1}^n \left[l^i(t,s,P)\varphi(z)\right]_{z_{i-1}}^{z_i} \\ &- \sum_{i=1}^n \int_{z_{i-1}}^{z_i} \frac{\partial l^i(t,s,P)}{\partial z}\varphi(z)dz, \\ &= -\sum_{i=1}^n \int_{z_{i-1}}^{z_i} \frac{\partial l^i(t,s,P)}{\partial z}\varphi(z)dz, \end{aligned}$$

so, the fact that $l^i(t,s,x,y,.) \in H^2(z_{i-1},z_i)$, and by the Cauchy inequality we have

$$\left| \int_0^{z^*} L^n(t,s,P)\frac{d\varphi(z)}{dz}dz \right| \le c_n \left|\varphi\right|_{L^2(0,z^*)}, \tag{3.54}$$

then $L^n(t,s,x,y,.) \in H^1(0,z^*)$. Moreover since $H^n(t,x,y,.) \in L^\infty(0,z^*)$, then $H^n\dfrac{\partial L^n}{\partial z} \in L^2(0,z^*)$. In addition

$$\begin{aligned}\int_0^{z^*} H^n \frac{\partial L^n}{\partial z}\frac{d\varphi}{dz}dz &= \sum_{i=1}^{n}\int_{z_{i-1}}^{z_i} h^i\frac{\partial l^i}{\partial z}\frac{d\varphi}{dz}dz,\\ &= \sum_{i=1}^{n}\left[h^i\frac{\partial l^i}{\partial z}\varphi\right]_{z_{i-1}}^{z_i} - \sum_{i=1}^{n}\int_{z_{i-1}}^{z_i} h^i\frac{\partial^2 l^i}{\partial z^2}\varphi dz,\end{aligned}$$

from the compatibility conditions,

$$\int_0^{z^*} H^n \frac{\partial L^n}{\partial z}\frac{d\varphi}{dz}dz = -\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i} h^i\frac{\partial^2 l^i}{\partial z^2}\varphi dz, \tag{3.55}$$

and we have

$$\left|\int_0^{z^*} H^n \frac{\partial L^n}{\partial z}\frac{d\varphi}{dz}dz\right| \le \sum_{i=1}^{n}\int_{z_{i-1}}^{z_i} h^i\left|\frac{\partial^2 l^i}{\partial z^2}\right|\left|\varphi\right| dz, \tag{3.56}$$

the fact that $l^i(t,s,x,y,.) \in H^2(z_{i-1},z_i)$ and the Cauchy inequality entail that

$$\left|\int_0^{z^*} H^n \frac{\partial L^n}{\partial z}\frac{d\varphi}{dz}dz\right| \le c_n' \left|\varphi\right|_{L^2(0,z^*)}, \tag{3.57}$$

then

$$H^n\frac{\partial L^n(t,s,x,y,.)}{\partial z} \in H^1(0,z^*). \tag{3.58}$$

We know that

$$\frac{\partial l^i}{\partial t} + f^i\frac{\partial l^i}{\partial s} + V_1^i\frac{\partial l^i}{\partial x} + V_2^i\frac{\partial l^i}{\partial y} + V_3^i\frac{\partial l^i}{\partial z} - h^i\left(\frac{\partial^2 l^i}{\partial z^2}\right) + \gamma^i l^i = 0, \qquad 1 \le i \le n, \tag{3.59}$$

so for any regular function φ

$$\int_{\Gamma}\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}\left(\frac{\partial l^i}{\partial t}+f^i\frac{\partial l^i}{\partial s}+V_1^i\frac{\partial l^i}{\partial x}+V_2^i\frac{\partial l^i}{\partial y}+V_3^i\frac{\partial l^i}{\partial z}-h^i\left(\frac{\partial^2 l^i}{\partial z^2}\right)+\ \gamma^i l^i\right)\varphi dzd\sigma=0,$$

where

$$d\sigma = dtdsdxdy,$$

this shows that

$$\int_{\Gamma}\int_{0}^{z^*}\left((\frac{\partial L^n}{\partial t}+\frac{\partial(f^nL^n)}{\partial s}+V_1^n\frac{\partial L^n}{\partial x}+V_2^n\frac{\partial L^n}{\partial y}+V_3^n\frac{\partial L^n}{\partial z}+\mu L^n)\varphi+H^n\frac{\partial L^n}{\partial z}\frac{\partial\varphi}{\partial z}\right)dzd\sigma=0, \tag{3.60}$$

hence L^n is solution of problem (3.52). The positivity of L^n follows readily from the positivity of $\{l^i\}_{i=1}^n$. ■

3.4.3 The exact solution

In this section we show that the approximate solution defined in each layer tends to the desired weak solution of problem (2.2) as the number of layers n tends to infinity.

Theorem 3.4.3 *Let $B \in L^2(\Gamma\times(0,z^*))$ then there exists $l \in L^2(\Gamma\times(0,\ z^*))$ such that L^n solution of (3.52)*

$$L^n \rightharpoonup l, \tag{3.61}$$

weakly in $L^2(\Gamma\times(0,\ z^*))$. Finally l is a distributional solution of (2.2).

Proof. Let B^n such that

$$B^n \to B \quad \text{as} \quad n\to\infty \tag{3.62}$$

in $L^2(\Gamma\times(0,z^*))$.
Choosing $\varphi = L^n$ in (3.60), it yields

$$\int_{\Gamma}\int_{0}^{z^*}(\frac{\partial L^n}{\partial t}+\frac{\partial(f^nL^n)}{\partial s}+V_1^n\frac{\partial L^n}{\partial x}+V_2^n\frac{\partial L^n}{\partial y}+V_3^n\frac{\partial L^n}{\partial z}+\mu L^n)L^n+H^n|\frac{\partial L^n}{\partial z}|^2dzd\sigma=0,$$

integrating by parts, we obtain

$$\frac{1}{2}\int_{\Omega}\int_{1}^{s^*}|L^n(T,s,P)|^2 dsdP+\frac{1}{2}\int_{Q}[f^n|L^n|^2]_1^{s^*}dPdt$$

$$+\ \frac{1}{2}\int_{\Sigma}(V.\eta)|L^n|^2+\int_{\Gamma}\int_{0}^{z^*}(H^n|\frac{\partial L^n}{\partial z}|^2+(\mu+\frac{1}{2}\frac{\partial f^n}{\partial s})|L^n|^2)dzd\sigma=0,$$

applying the hypothesis $(H5)$ and the fact that f^n is uniformly bounded, yields

$$||L^n||^2_{L^2(\Gamma\times(0,z^*))}\leq c, \tag{3.63}$$

and

$$\int_{\Gamma}\int_{0}^{z^*}H^n|\frac{\partial L^n}{\partial z}|^2 dzd\sigma\leq c,$$

Therefore, since $V_i\in C([0,T]\times\bar{\Omega}), i=1,3$ then V_i^n is uniformly bounded in n, using the Lebesgue dominated theorem, we may conclude that

$$V_i^n\to V_i,$$

strongly in $L^2(\Gamma\times(0,z^*))$. therefore

$$V_i^n L^n\rightharpoonup V_i l,$$

weakly in $L^2(\Gamma\times(0,z^*))$. Analogously we have

$$f^n L^n\to fl,$$

and

$$\mu L^n\to\mu l$$

weakly in $L^2(\Gamma\times(0,z^*))$.

Let n tend to ∞ in (3.60), we find a distributional solution of problem (2.2). ■

Conclusion 3.4.4 *our intent in this work was to prove the existence and positivity of a solution to the problem of larval distribution. We studied an approximate layered problem and proved there exist unique positive solutions* *which converge to the original problem's solution. The technique developed here should prove useful in the analysis of similar problems where turbulence is a factor, such as the distribution of semi-passive organisms in stream flow and the distribution of organisms in atmosphere flows. The technique may also prove useful in the study of sound in stratified media with background flow. The original equations model the dynamics of fish larvae. Our work shows that one technique for numerical solving the problem would be to solve the approximate layered problem thus reducing the dimension of the system from three to several coupled two dimensional problems. Due to the manner the data is collected with data points taken at fixed depths, more is known about the change in the vertical component of the current and thus approximating the horizontal component as constant is reasonable.*

3.5 Non linear model

Our purpose in this work is to prove existence and positivity of solution to the situation where the mortality functions depends on the population density but without neglect the horizontal diffusion. The last problem is more difficult and in our knowledge is open problem. Our idea if we want to resolve this problem i.e. (with neglecting the horizontal diffusion), then among possible methods is to approximate this problem in the one with complete diffusion i.e.(without neglecting the horizontal diffusion) and we try to passe at the limit in the suitable ways. So in this section we treat the approximate problem namely the model with complete diffusion only, that is a non linear non autonomous parabolic equation.
Choosing the space $L^2(0,T;W_0^{1,2}(\Omega))$ rather than $L^2(0,T;W_0^{1,p}(\Omega)$, provided $2 \leq p \leq 6$, by this hypothesis we have the continuous injection between $W_0^{1,2}(\Omega)$ and $L^p(\Omega)$. So one can treated in this case our problem by assuming a certain restricted hypothesis to the mortality function, namely the inequality in (H_3) is satisfied for $2 \leq p \leq 6$. It is possible to generalize this inequality, namely for all $p \geq 2$, if we choose the space $L^2(0,T;W_0^{1,p}(\Omega)$, rather than $L^2(0,T;W_0^{1,2}(\Omega))$. The main difficulty in our problem is the lack of the coercivity of the operator in the space $L^2(0,T;W_0^{1,p}(\Omega))$, which will be handled by using a convenient perturbation argument. An existence result of Lions see [30](p 316) give us existence to the perturbed problem. Passing to the limit in a suitable way, we obtain the existence and positivity of a solution to the main model. The model which we will treated in this section is given

by

$$\left\{\begin{array}{l}\dfrac{\partial l}{\partial t}+\operatorname{div}(Vl)-\sum_{i=1}^{i=3}\dfrac{\partial}{\partial x_i}\left(h_i\dfrac{\partial l}{\partial x_i}\right)+\mu(l)l=0,\\ l=0, \qquad \text{for } (t,P)\in\Sigma\\ l(0,P)=l_0(P).\end{array}\right. \tag{3.64}$$

3.5.1 Notation and preliminary results

We recall here some definitions and results that we will use in this section. Let X be a separable and reflexive Sobolev space with norm $||.||$ and it dual X' with norm $||.||_*$. We denote by $<,>$ the duality bracket of $X' \times X$. We define the norm of $L^p(0,T;X)$ by

$$\left(\int_0^T ||v||^p dt\right)^{1/p},$$

for each $v \in L^p(0,T;X)$, $p \in [1,+\infty.)$
We denote by $\mathcal{D}(0,T;X)$ the space of infinitely differentiable functions which have compact support in $(0,T)$ and with values in X, and $\mathcal{D}'(0,T;X)$ the space of distribution on $(0,T)$ with values in X. We set also $W(0,T;X,X') := \{v, v \in L^p(0,T;X), \frac{dv}{dt} \in L^{p'}(0,T;X')\}$. We consider a family $\{A(t,.), t \in [0,T]\}$ of operators from X to X', we define $\mathbf{A} : L^p(0,T;X) \to L^{p'}(0,T;X')$ by

$$\forall u \in L^p(0,T;X), \quad \mathbf{A}(u)(t) = A(t,u(t)) \text{ a.e on } [0,T].$$

p' stands for the conjugate exponent of p.

Definition 3.5.1 *The operator A from X to X′ is said to be hemicontinuous if its satisfied the following property :*

$$\left\{\begin{array}{l}\forall\ u,v,w \in X, \text{ the function}\\ \lambda \to < A(u+\lambda v), w >\\ \text{is continuous from } I\!R \to I\!R,\end{array}\right. \tag{3.65}$$

Definition 3.5.2 *The operator A from X to X′ is said to be a variational calculus if it is bounded and if it can be to show as $A(v) = A(v,v)$, where $u,v \to A(u,v)$ from $X \times X$ to X' has the following properties :*

$$\left\{\begin{array}{l}\forall\ u \in X, v \to A(u,v), \text{ hemicontinuous,}\\ \text{and } < A(u,u) - A(u,v), u-v > \ \geq 0.\end{array}\right. \tag{3.66}$$

$$\forall \ v \in X, u \to A(u,v) \text{ is bounded and hemicontinuous,} \tag{3.67}$$

$$\begin{cases} \text{if } u_n \to u \text{ in } X \text{ weakly and if } < A(u_n,u_n) - A(u_n,u), u_n - u > \to 0, \\ \text{then } A(u_n,v) \to A(u,v) \text{ in } X' \text{ weakly,} \end{cases} \tag{3.68}$$

$$\begin{cases} \text{if } u_n \to u \text{ in } X \text{ weakly and if } A(u_n,v) \to \psi \text{ in } X' \text{ weakly} \\ \text{then } < A(u_n,v), u_n > \to < \psi, u > . \end{cases} \tag{3.69}$$

Definition 3.5.3 *The operator A from X to X' is said to be pseudo-monotone if :*

(i) A is bounded

(ii) As $u_n \to u$ weakly in X and limsup $< A(u_n), u_n - u > \leq 0$ then

$$liminf < A(u_n), u_n - v > \ \geq \ < A(u), u - v > \quad \forall v \in X.$$

The next proposition gives a relationship between the pseudo-monotonicity and calculus variational,(see [30] for a complete proof).

Proposition 3.5.4 *We have the following implication :*

A is variational calculus $\Rightarrow$ A is pseudo-monotone.

Definition 3.5.5 *Let L be a monotone operator from X to X'. We say that L is a maximal monotone operator if its graph is a maximal subset of $X \times X'$ with respect to set inclusion.*

We will use the following results, where the proof can be found in [30].

Lemma 3.5.6 *Let L be a unbounded linear operator, with a dense domain $D(L)$ in X and take its values in X'. Then L is maximal monotone if and only if L is a closed operator and such that*

$$< Lv - Lu, v - u > \ \geq 0 \quad \forall \ v, u \in D(L),$$

and

$$< L^*v - L^*u, v - u > \ \geq 0 \quad \forall \ v, u \in D(L^*).$$

where L^* is the adjoint operator of L.

Theorem 3.5.7 *Let X be a reflexive Banach space. Let L be a linear operator of dense domain $D(L) \subset X$ and take its values in X'. Assume that L is maximal monotone and consider A, an operator from X to X', pseudo-monotone, coercive, namely*

$$\frac{< A(v), v >}{||v||} \to \infty \text{ if } ||v|| \to \infty.$$

Then, for all $f \in X'$, there exists $u \in D(L)$ such that

$$Lu + A(u) = f.$$

Remark 3.5.8 *Let H be a Hilbert space with $X \hookrightarrow H$, the continuous injection $\hookrightarrow$having its image dense in H. Then we can identify H with its dual H', and therefore*

$$X \hookrightarrow H \hookrightarrow X'.$$

The following Lemma give the compact injection in the general case, for the proof see for instance [30].

Lemma 3.5.9 *Let B_0, B, B_1 be the Banach spaces such that*

$$B_0 \hookrightarrow B \hookrightarrow B_1,$$

with B_0, B_1 reflexive and the injection $B_0 \to B_1$ is compact. We define

$$W := \{v, v \in L^{p_0}(0,T; B_0), \frac{dv}{dt} \in L^{p_1}(0,T; B_1)\}$$

where $1 < p_i < \infty, i = 0, 1$. With the following norm

$$||v||_{L^{p_0}(0,T;B_0)} + ||\frac{dv}{dt}||_{L^{p_1}(0,T;B_1)},$$

W is a Banach space and then the injection of W to $L^{p_0}(0,T; B_0)$ is compact.

3.5.2 Existence and positivity of solution of the perturbed problem

The objective of this section is to study existence and positivity of solution of the associate perturbed problem (3.71).

Using the imcopressibility hypothesis, the model (3.64) can be written in the form:

$$\begin{cases} \frac{\partial l}{\partial t}+\sum_{i=1}^{3} V_i\frac{\partial l}{\partial x_i}-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i\frac{\partial l}{\partial x_i}\right)+\mu(l)l=0, \\ l=0, \text{ in } \Sigma, \\ l(0,P)=l_0(P), \end{cases} \tag{3.70}$$

We will use a perturbation method to get a time dependent non linear parabolic equation whose resolution will yield to the solution of equation (3.70). We consider for $p \geq 2$ and $\varepsilon > 0$ the following perturbed problem

$$\begin{cases} \frac{\partial l}{\partial t}+\sum_{i=1}^{3} V_i\frac{\partial l}{\partial x_i}-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i\frac{\partial l}{\partial x_i}\right)+\mu(l)l-\varepsilon\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(|\frac{\partial l}{\partial x_i}|^{p-2}\frac{\partial l}{\partial x_i}\right)=0, \\ l=0, \text{ in } \Sigma, \\ l(0,P)=l_0(P). \end{cases} \tag{3.71}$$

Let

$$Lu=\frac{\partial u}{\partial t}+\sum_{i=1}^{3} V_i\frac{\partial u}{\partial x_i},$$

with

$$D(L)=\{v\in L^2(0,T;W_0^{1,p}(\Omega));\frac{dv}{dt}\in L^2(0,T;W^{-1,p}(\Omega)),v(0)=0\},$$

and

$$Au=-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i\frac{\partial u}{\partial x_i}\right)+\mu(u)u-\varepsilon\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

defined by

$$< Au,v >=\sum_{i=1}^{3}\int_Q h_i\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dPdt+\int_Q \mu(u)uvdPdt+\varepsilon\sum_{i=1}^{3}\int_Q|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dPdt,$$

for each $v\in L^2(0,T;W_0^{1,p}(\Omega))$. The main result of this section is the following Theorem that gives conditions under which problem (3.71) has a positive solution.

We now state the assumptions of this section.

(H_1) $\{h\}_{i=1,3} \in L^\infty((0,T)\times\Omega)$ and $\{V_i\}_{i=1,3} \in L^\infty((0,T)\times\Omega)$.

(H_2) $\sum_{i=1}^{3} h_i\zeta_i^2 \geq c_0(\zeta_1^2+\zeta_2^2+\zeta_3^2)$, $c_0>0$, $\forall\zeta\in I\!R$, *a.e* in Ω.

(H_3) the function μ is continuous from $I\!R$ to $I\!R^+$ and satisfied $\mu(s) \leq M|s|^{p-2}$, for all $s\in I\!R$, and $p\geq 2$.

Theorem 3.5.10 *Assume $(H_1)--(H_3)$ hold. If the positive function l_0 is in $L^2((0,T)\times\Omega)$, then problem (3.71) have a non negative solution $u\in D(L)$.*

Proof. We set $F(s)=s\mu(s)$, for all $s\in I\!R$. In the first step we will see that L is a closed operator with a dense domain ; indeed, let u_n in $D(L)$ be such that

$$u_n \to u,$$

in $L^2(0,T;W_0^{1,p}(\Omega))$, and

$$Lu_n \to y,$$

in $L^2(0,T;W^{-1,p}(\Omega))$, hence

$$u_n \to u,$$

in $\mathcal{D}'(0,T;W^{-1,p}(\Omega))$, and

$$Lu_n \to y,$$

in $\mathcal{D}'(0,T;W^{-1,p}(\Omega))$, it follows that

$$Lu_n \to Lu,$$

in $\mathcal{D}'(0,T;W^{-1,p}(\Omega))$, therefore $y=Lu$ and $u\in D(L)$, so L is closed. Moreover it is clear that $\mathcal{D}(0,T;W_0^{1,p}(\Omega))$ is included in $D(L)$, we deduce that $D(L)$ is dense in $L^2(0,T;W_0^{1,p}(\Omega))$. Concerning the monotonicity of L, we have for $u\in D(L)$,

$$\begin{aligned} < Lu,u > &= \int_0^T < \frac{\partial u}{\partial t}, u > dt + \sum_{i=1}^{i=3}\int_Q V_i\frac{\partial u}{\partial x_i} u dP dt, \\ &= \frac{1}{2}\int_0^T \frac{d}{dt}\|u(t)\|_*^2 dt + \frac{1}{2}\int_\Sigma (V,\eta)u^2 d\sigma - \frac{1}{2}\int_Q div(V)u^2, dPdt, \end{aligned}$$

with η is exterior normal, and $(,)$ is the scalar product. Using the incompressibility hypothesis we obtain that,

$$< Lu,u > = \frac{1}{2}\|u(T)\|_*^2,$$

it follows that L is monotone for all $u \in D(L)$. In addition for $u \in D(L)$,

$$\begin{aligned} < Lu, v > &= \int_0^T < \frac{\partial u}{\partial t}, v > dt + \sum_{i=1}^{i=3} \int_Q V_i \frac{\partial u}{\partial x_i} v dP dt, \\ &= \int_0^T < u, -\frac{\partial v}{\partial t} > dt + < u(T), v(T) > \\ &- \sum_{i=1}^{i=3} \int_Q u V_i \frac{\partial v}{\partial x_i} dP dt, \end{aligned}$$

so, the associated adjoint operator is given by

$$L^* v = -\frac{\partial v}{\partial t} - \sum_{i=1}^{i=3} V_i \frac{\partial v}{\partial x_i},$$

with

$$D(L^*) = \{v \in L^2(0,T; W_0^{1,2}(\Omega)); \frac{dv}{dt} \in L^2(0,T; W^{-1,2}(\Omega)), v(T) = 0\}.$$

The proof of monotonicity of L^* is similar to the one of L. Then L is a maximal monotone operator. For the coercivity of A : for $u \in L^2(0,T; W_0^{1,p}(\Omega))$ and applying the hypothesis on (H_2), it holds

$$\begin{aligned} < Au, u > &= \sum_{i=1}^{3} \int_Q h_i |\frac{\partial u}{\partial x_i}|^2 dP dt + \int_Q \mu(u) u^2 dP dt \\ &+ \varepsilon \sum_{i=1}^{3} \int_Q |\frac{\partial u}{\partial x_i}|^p dP dt \geq \varepsilon \|u\|^p_{L^2(0,T;W_0^{1,p}(\Omega))}, \end{aligned}$$

because μ is positive. It remain to see that A is pseudo-monotone, for this it suffice to prove that A is variational calculus and according to Proposition 3.5.4, and Theorem 3.5.7, we get the existence of a solution $u_\varepsilon \in D(L)$ of the problem (3.71). Indeed we set

$$a_1(u, v, w) = \sum_{i=1}^{3} \int_Q h_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} dP dt + \varepsilon \sum_{i=1}^{3} \int_Q |\frac{\partial v}{\partial x_i}|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} dP dt,$$

and

$$a_2(u, w) = \int_Q \mu(u) u w dP dt,$$

the form $w \to a_1(u,v,w) + a_2(u,w)$ is continuous in $W_0^{1,p}(\Omega)$, so

$$a_1(u,v,w) + a_2(u,w) = a(u,v,w) =< A(u,v), w >$$

with $A(u,v) \in W^{-1,p}(\Omega)$. As easily seen $A(u,u) = A(u)$. We will have the desired result if we verified (3.66),(3.67),(3.68),(3.69).
Verification of (3.66) :
the application $v \to A(u,v)$ is bounded and hemicontinuous, indeed

$$a(u, v_1 + \lambda v_2, w) \to a(u,v,w)$$

as $\lambda \to 0$ and $u, v_i, w \in W_0^{1,p}(\Omega)$. Moreover, taking into account that the operator $\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\right)$ is monotone, see for instance ([28], [30]), we have

$$\begin{aligned}
< A(u,u) - A(u,v), u - v > &= a_1(u,u,u-v) + a_2(u,u-v) \\
&- a_1(u,v,u-v) - a_2(u,u-v), \\
&= a_1(u,u,u-v) - a_1(u,v,u-v), \\
&= \sum_{i=1}^{i=3} \int_Q h_i |\frac{\partial(u-v)}{\partial x_i}|^2 dPdt \\
&+ \varepsilon \sum_{i=1}^{3} \int_Q (|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i} - |\frac{\partial v}{\partial x_i}|^{p-2}\frac{\partial v}{\partial x_i})\frac{\partial(u-v)}{\partial x_i} dPdt \\
&\geq 0.
\end{aligned}$$

Verification of (3.67) :

$$a(u_1 + \lambda u_2, v, w) = a_1(u,v,w) + a_2(u_1 + \lambda u_2, w),$$

we have

$$a_2(u_1 + \lambda u_2, w) = \int_Q \mu(u_1 + \lambda u_2)(u_1 + \lambda u_2)w,$$

according to (H_3), F is continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$ see for instance [28], then

$$a(u_1 + \lambda u_2, v, w) \to a(u_1, v, w)$$

as $\lambda \to 0$, and $u_i, v, w \in W_0^{1,p}(\Omega)$.
Verification of (3.68):
Let be $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and

$$\begin{aligned} < A(u_n,u_n) - A(u_n,u), u_n - u > &= \sum_{i=1}^{i=3} \int_Q h_i |\frac{\partial(u_n-u)}{\partial x_i}|^2 dPdt \\ &+ \varepsilon \sum_{i=1}^{3} \int_Q (|\frac{\partial u_n}{\partial x_i}|^{p-2} \frac{\partial u_n}{\partial x_i} \\ &- |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i}) \frac{\partial(u_n-u)}{\partial x_i} dPdt \to 0, \end{aligned}$$

so $u_n \to u$ strongly in $L^p(\Omega)$, since F is continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$ then $F(u_n) \to F(u)$ strongly in $L^{p'}(\Omega)$.
Verification of (3.69) :
Let be $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and $A(u_n, v) \to \psi$ weakly in $W^{-1,p'}(\Omega)$, so since

$$u_n \to u,$$

weakly in $W_0^{1,2}(\Omega)$, and

$$\sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega),$$

we obtain

$$a_1(u_n, u, u_n) \to a_1(u, u, u),$$

$$a_2(u_n, u_n - u) = \int_Q \mu(u_n) u_n (u_n - u) dPdt,$$

so

$$\begin{aligned} a_2(u_n, u_n - u) &\leq (\int_Q |\mu(u_n)u_n|^{p'})^{1/p'} (\int_Q |u_n - u|^p)^{1/p}, \\ &\leq M(\int_Q |u_n|^p)^{(p-1)/p} (\int_Q (u_n - u)^p)^{1/p}, \end{aligned}$$

so $a_2(u_n, u_n - u) \to 0$ as $n \to \infty$. In addition

$$a_2(u_n, u) = < A(u_n, v), u > - a_1(u, v, u),$$

thus

$$a_2(u_n, u) \to < \psi, u > -a_1(u, v, u),$$

we have also

$$a_2(u_n, u_n) = a_2(u_n, u_n - u) + a_2(u_n, u),$$

then

$$a_2(u_n, u_n) \to < \psi, u > -a_1(u, v, u),$$

hence

$$< A(u_n, v), u_n > \to < \psi, u > .$$

Then the operator A is variational calculus. Hence for all $v \in L^2(0, T; W_0^{1,p}(\Omega))$, we have

$$\begin{aligned}\int_0^T < \frac{\partial u_\varepsilon}{\partial t}, v > dt \; &+ \; \sum_{i=1}^3 \int_Q V_i \frac{\partial u_\varepsilon}{\partial x_i} v dP dt + \sum_{i=1}^3 \int_Q h_i \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dP dt \\ &+ \; \int_Q \mu(u_\varepsilon) u_\varepsilon v dP dt + \varepsilon \sum_{i=1}^3 \int_Q (|\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dP dt = 0,\end{aligned}$$

We prove now that the positivity of the solution. We set $u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$, where u_ε^+ and u_ε^- are respectively the positive and negative part of u. Multiplying the equation (3.70) by u_ε^- and integrating on $(0, t)$, we get

$$\begin{aligned}-\int_0^t < \frac{\partial u_\varepsilon^-}{\partial t}, u_\varepsilon^- > dt \; &- \; \sum_{i=1}^3 \int_0^t \int_\Omega V_i \frac{\partial u_\varepsilon^-}{\partial x_i} u_\varepsilon^- dP dt \\ &- \; \sum_{i=1}^3 \int_0^t \int_\Omega h_i |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2 dP dt \\ &- \; \int_0^t \int_\Omega \mu(u_\varepsilon) |u_\varepsilon^-|^2 dP dt \\ &- \; \varepsilon \sum_{i=1}^3 \int_0^t \int_\Omega \left| \frac{\partial u_\varepsilon^-}{\partial x_i} \right|^p dP dt = 0,\end{aligned}$$

integrating by parts and applying the incompressibility hypothesis, we obtain

$$\begin{aligned}-\frac{1}{2} \|u_\varepsilon^-(t)\|_*^2 \; &= \; \sum_{i=1}^3 \int_0^t \int_\Omega h_i |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2 dP dt \\ &+ \; \int_0^t \int_\Omega \mu(u_\varepsilon) |u_\varepsilon^-|^2 dP dt + \varepsilon \sum_{i=1}^3 \int_0^t \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p dP dt,\end{aligned}$$

so,

$$-\frac{1}{2}||u_\varepsilon^-(t)||_*^2 \geq 0,$$

then $u_\varepsilon^-(t) = 0$ for all $t \in (0,T)$. Therefore u_ε is a positive solution to problem (3.71), which complete the proof. ■

3.5.3 The exact solution

In this section we show that the perturbed solution defined in (3.71) tends to the desired solution of problem (3.70) in $L^2(0,T;W_0^{1,2}(\Omega))$ as ε tends to 0. Our main result is the following Theorem.

Theorem 3.5.11 *Let $l_0 \in L^\infty(\Omega)$ and consider u_ε the solution to problem (3.71), then u_ε converges weakly to u in $L^2(0,T;W_0^{1,2}(\Omega))$. Moreover u satisfies the equation*

$$\int_0^T < \frac{\partial u}{\partial t}, \phi > dt + \sum_{i=1}^3 \int_Q V_i \frac{\partial u}{\partial x_i} \phi dPdt + \sum_{i=1}^3 \int_Q h_i \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} dPdt + \int_Q \mu(u)u\phi = 0,$$

for all $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$.

Proof. Multiplying the equation (3.71) by u_ε and integrating by parts, by applying the incompressibility hypothesis, i.e. $div(V) = 0$, we get

$$\frac{1}{2}||u_\varepsilon(T)||_*^2 + \sum_{i=1}^3 \int_Q h_i \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2 dPdt + \varepsilon \sum_{i=1}^3 \int_Q \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^p dPdt = \frac{1}{2}||l_0||_*^2.$$

Since $\{h_i\}_{i=1,3}$ are bounded functions we conclude that

$$||u_\varepsilon||^2_{L^2(0,T;W_0^{1,2}(\Omega))} \leq C,$$

and as a consequence there exists a subsequence which we denote also by u_ε, weakly convergent to some function $u \in L^2(0,T;W_0^{1,2}(\Omega))$. We have also $||\frac{du_\varepsilon}{dt}||_{L^2(0,T;W^{-1,p}(\Omega))} \leq c$, thus by the Lemma 3.5.9 we have

$$u_\varepsilon \rightarrow u$$

strongly in $L^2(Q)$ and almost everywhere (exactly we extract a subsequence which it converge almost everywhere). As a consequence

$$\sum h_i \frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \sum h_i \frac{\partial u}{\partial x_i}$$

weakly in $L^2(Q)$.
Multiplying the other time the equation (3.71) by the test function $w := \dfrac{u_\varepsilon^\alpha}{1+\frac{1}{k}u_\varepsilon^\alpha}$ for a large constant α and integer k. We remark that w belong to $L^\infty(Q)$. We set Θ the primitive of w that is

$$\Theta(s) = \int_0^s \frac{s'^\alpha}{1+\frac{1}{k}s'^\alpha} ds'$$

then we get

$$\begin{aligned}\int_0^T < \frac{\partial u_\varepsilon}{\partial t}, \Theta'(u_\varepsilon) > dt \quad &+ \quad \sum_{i=1}^{3} \int_Q V_i \frac{\partial u_\varepsilon}{\partial x_i} \Theta'(u_\varepsilon) dPdt + \sum_{i=1}^{3} \int_Q h_i \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \Theta'(u_\varepsilon)}{\partial x_i} dPdt \\ &+ \quad \int_Q \mu(u_\varepsilon) u_\varepsilon \Theta'(u_\varepsilon) dPdt \\ &+ \quad \varepsilon \sum_{i=1}^{3} \int_Q (|\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \Theta'(u_\varepsilon)}{\partial x_i} dPdt = 0,\end{aligned}$$

integrating by parts the second element of the first member and applying the incompressibility hypothesis, we get

$$\begin{aligned}\|\Theta(u_\varepsilon(T))\|_* \quad &+ \quad \sum_{i=1}^{3} \int_Q h_i \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 \frac{\alpha u_\varepsilon^{\alpha-1}}{(1+\frac{1}{k}u_\varepsilon^\alpha)^2} dPdt \\ &+ \quad \varepsilon \sum_{i=1}^{3} \int_Q \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p \frac{\alpha u_\varepsilon^{\alpha-1}}{(1+\frac{1}{k}u_\varepsilon^\alpha)^2} dPdt + \\ &+ \quad \int_Q \mu(u_\varepsilon) u_\varepsilon \Theta'(u_\varepsilon) dPdt = \|\Theta(u_\varepsilon(0))\|_*.\end{aligned}$$

Since $\{h_i\}_{i=1,3}$ are bounded functions, the initial condition l_0 belong to $L^\infty(\Omega)$ and u_ε is positive we have

$$\sum_{i=1}^{3}\int_Q \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2 \frac{\alpha u_\varepsilon^{\alpha-1}}{(1+\frac{1}{k}u_\varepsilon^\alpha)^2} dPdt \leq C.$$

As k tends to ∞ and by Fatou lemma we obtain

$$\sum_{i=1}^{3}\int_Q \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2 \alpha u_\varepsilon^{\alpha-1} dPdt \leq C,$$

and then

$$\sum_{i=1}^{3}\int_Q \left|\frac{\partial u_\varepsilon^{\frac{\alpha+1}{2}}}{\partial x_i}\right|^2 dPdt \leq C,$$

by Poincarre inequality

$$\int_Q |u_\varepsilon|^{\alpha+1} dPdt \leq C,$$

then u_ε is bounded in $L^p(Q)$ for all $p \geq 2$. According to (H_3) we have $F(u_\varepsilon)$ is bounded in $L^p(Q)$ for all $p \geq 2$, then $F(u_\varepsilon)$ converge weakly in $L^p(Q)$ and as

$$u_\varepsilon \to u \quad a.e \ \ in \ \ Q$$

this conclude that

$$F(u_\varepsilon) \rightharpoonup F(u)$$

in $L^p(Q)$. So when ε tends to 0 in (3.71) we obtain that

$$\int_0^T < \frac{\partial u}{\partial t}, \phi > dt + \sum_{i=1}^{3}\int_Q V_i \frac{\partial u}{\partial x_i}\phi dPdt + \sum_{i=1}^{3}\int_Q h_i \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} dPdt + \int_Q \mu(u)u\phi = 0,$$

for all $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$. ■

Chapter 4

Numerical analysis

In this chapter, we present the construction of the numerical program which has been used in the simulations of the main model in the particular case where V_1, V_2 are independent of the vertical variable. Let the following problem

$$\begin{cases} \dfrac{\partial l}{\partial t} + \dfrac{\partial (f_1 l)}{\partial s} + div(Vl) - \dfrac{\partial}{\partial z}\left(h\dfrac{\partial l}{\partial z}\right) + (\mu + k)l = f, \\ l(t,1,x,y,z) = B(t,P), \end{cases} \tag{4.1}$$

with $f_1 = f_1(t,s), V_1 = V_1(t,x,y)$ and $V_2 = V_2(t,x,y)$. We solve the problem along the characteristic line, so that

$$(\frac{dt}{d\tau}, \frac{ds}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}) = (1,\ f_1(t,s), V_1(t,x,y), V_2(t,x,y)), \tag{4.2}$$

satisfying $t(0) = t_0,\ s(0) = 1,\ x(0) = x_0,\ y(0) = y_0$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$. Among the most popular explicit one step methods we will use the Runge-Kutta method of order four; its algorithm is defined as follows

$$U_{n+1} = U_n + \frac{\delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{4.3}$$

where

$$\begin{cases} K_1 = F(t_n, U_n) \\ K_2 = F(t_n + \dfrac{\delta t}{2}, U_n + \delta t\dfrac{K_1}{2}) \\ K_3 = F(t_n + \dfrac{\delta t}{2}, U_n + \delta t\dfrac{K_2}{2}) \\ K_4 = F(t_n + \delta t, U_n + \delta t K_3) \end{cases} \tag{4.4}$$

The restriction of the solution in the characteristic line solve the following problem

$$\begin{cases} \dfrac{\partial \bar{l}}{\partial \tau} + \bar{V_3}\dfrac{\partial \bar{l}}{\partial z} - \dfrac{\partial}{\partial z}(\bar{h}\dfrac{\partial \bar{l}}{\partial z}) + (\bar{\gamma} + k)\bar{l} = \bar{f}, \\ \bar{l}(0,z) = \bar{B}(z). \end{cases} \tag{4.5}$$

Throughout this chapter we replace the coefficients $\bar{h}$, $\bar{V_3}$, $\bar{\gamma}$, and $\bar{f}$ by h, V_3, μ, and f for simplification. So every problem similar to the one given by (4.1) can be treated in the following way : first we solve the system of equations by numerical methods for instance Runge-Kutta methods, second we inject the obtained result in the original model and then we obtain a purely parabolic equations which can be treated by one of the well known numerical methods, namely finite element and finite volume, it is the goal of the next sections.

4.1 Finite element

We present in this section, for second order parabolic differential equations, semi and fully-discrete finite element methods. The construction as well as the theoretical analysis of these scheme are discussed. We consider the parabolic differential equation (4.5) together with the Newmann conditions $h\dfrac{\partial l}{\partial z}(t,0) = g(t)$ and $h\dfrac{\partial l}{\partial z}(t,1) = g_1(t)$ where the coefficients are sufficiently smooth functions.

4.1.1 Construction of the Galerkin finite element scheme

Let us discretize the interval $\Omega = (0,1)$ into a set of points (or nodes). For the construction of approximation $u_{\delta z}$ of u, we will choose a subspace of $H^1(\Omega)$ consisting of affine functions continuous by intervals. For this let I be an integer constant and $\delta z = \dfrac{1}{I+1}$. We associated the following points $z_i = i\delta z$ for $0 \leq i \leq I+1$ which subdivide the interval $\bar{\Omega}$ in $I+1$ intervals $K_i = [z_i, z_{i+1}]$, $0 \leq i \leq I$ of length δz. One choose then for a subspace of finite dimension of $H^1(\Omega)$ the space

$$V_{\delta z} = \{v \in C(\bar{\Omega}), v|_{K_i} \in P_1, 0 \leq i \leq I\}, \tag{4.6}$$

where P_1 is the set of all the polynomials of degrees less than or equal to 1. So by (4.6) and Newmann condition it is clear that the dimension of $V_{\delta z}$ is

$(I+2)$. The function $v_i \in V_{\delta z}$, $0 \leq i \leq I+1$ defined by $v_i(z_j) = \delta_{i,j}$ where $\delta_{i,j}$ is Kronecker functions constitute the basis of $V_{\delta z}$. Such a functions are given explicitly by

$$v_i(z) = \begin{cases} 1 - \dfrac{|z - z_i|}{\delta z} & \text{if} \quad z \in [z_{i-1}, z_{i+1}] \\ 0 \quad \text{otherwise} \end{cases} \tag{4.7}$$

for $1 \leq i \leq I$ and

$$v_0(z) = \begin{cases} 1 - \dfrac{z}{\delta z} & \text{if} \quad z \in [0, \delta z] \\ 0 \quad \text{otherwise} \end{cases} \tag{4.8}$$

$$v_{I+1}(z) = \begin{cases} 1 - \dfrac{|1 - z_i|}{\delta z} & \text{if} \quad z \in [1 - \delta z, 1] \\ 0 \quad \text{otherwise.} \end{cases} \tag{4.9}$$

We will determine the function $u_{\delta z} \in V_{\delta z}$ by its coordinate in the base $(v_i)_{0 \leq i \leq I+1}$, that is the number

$$u_i = u_{\delta z}(z_i) \tag{4.10}$$

for $0 \leq i \leq I+1$. Then any $u_{\delta z} \in V_{\delta z}$ has the following expression

$$u_{\delta z} = \sum_{i=0}^{I+1} u_i(t) v_i(z). \tag{4.11}$$

4.1.2 Semi-discrete finite element scheme

We assume that

$$\begin{aligned} a(t, u, u) &= \int_\Omega (h|\frac{\partial u}{\partial z}|^2 + V_3 \frac{\partial u}{\partial z} u + (\mu + k) u^2) dz \\ &\geq \alpha \|u\|_1^2, \quad \forall u \in W^{1,2}(\Omega). \end{aligned} \tag{4.12}$$

The variational problem related to (4.5) is : Find $u = u(.,t) \in W^{1,2}(\Omega)$, $0 \leq t \leq T$ such that

$$\begin{cases} (u_t, v) + a(t, u, v) = L(t, v) \\ u(0, z) = u_0(z), \quad z \in \Omega, \end{cases} \tag{4.13}$$

where $(.,.)$ denotes the inner product of $L^2(\Omega)$, $u_t = \dfrac{\partial u}{\partial t}$ and

$$a(t,u,v) = \int_\Omega (h\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + V_3\frac{\partial u}{\partial z}v + (\mu+k)uv)dz. \qquad (4.14)$$

$$L(t,v) = (f(t),v) + g_1(t)v(1) - g(t)v(0).$$

We have

$$|a(t,u,v)| \leq M\|u\|_1\|v\|_1, \qquad (4.15)$$

and

$$|L(t,v)| \leq M_1\|v\|_1, \qquad (4.16)$$

for all u, v belonging to $W^{1,2}(\Omega)$. The solution of (4.13) is called the generalized solution of (4.5). Then the semi-discrete finite element scheme for (4.5) is : Find $u_{\delta z} = u_{\delta z}(t,.) \in V_{\delta z}(0 \leq t \leq T)$ such that

$$\begin{cases} (u_{\delta z,t}, v_{\delta z}) + a(t, u_{\delta z}, v_{\delta z}) = L(t, v_{\delta z}) \\ u_{\delta z}(0,z) = u_{0\delta z}(z), \quad z \in \Omega, \end{cases} \qquad (4.17)$$

where $u_{0\delta z}$ is an approximation of u_0 on $V_{\delta z}$. Another way is to replace the above equality for the initial condition by

$$(u_{\delta z}(0,.), v_{\delta z}) = (u_0, v_{\delta z}), \quad \forall v_{\delta z} \in V_{\delta z}.$$

(4.17) can be expressed as : Find a solution of the form

$$u_{\delta z} = \sum_{i=0}^{I+1} u_i(t)v_i(z)$$

such that its coefficients $u_1(t), u_2(t), ...u_{I+1}(t)$ satisfy

$$\begin{cases} \sum_{j=0}^{I+1}[\frac{du_j(t)}{dt}(v_j,v_i) + u_j(t)a(t,v_j,v_i)] = L(t,v_i), \quad t>0, \quad i=0,...,I+1, \\ u_j(0) = \alpha_j, \quad j=0,...,I+1, \end{cases} \qquad (4.18)$$

where α_j's are the coefficients in $u_{0\delta z} = \sum_{j=0}^{I+1}\alpha_j v_j$. Let us introduce the following matrix and vector notations :

$$M = [m_{ij}] = [(v_j,v_i)], \quad K(t) = [k_{ij}] = [a(t,v_j,v_i)]$$

$$u = [u_1(t),...,u_{I+1}(t)]^T, \quad F(t) = [L(t,v_1),...,L(t,v_{I+1})]^T,$$

$$\alpha = [\alpha_1, ..., \alpha_{I+1}]^T.$$

Then we can rewrite (4.18) as

$$\begin{cases} M\dfrac{du}{dt} + K(t)u = F(t) \\ u(0) = \alpha. \end{cases} \tag{4.19}$$

Let us introduce an elliptic projection operator

$$P_{\delta z} : W^{2,2}(\Omega) \to V_{\delta z},$$

defined by the following finite element scheme :

$$a(t, P_{\delta z}u, v_{\delta z}) = a(t, u, v_{\delta z}), \quad \forall v_{\delta z} \in V_{\delta z}, \quad t > 0. \tag{4.20}$$

By (4.12) and (4.15) we see that $P_{\delta z}$ is uniquely defined by (4.20) for any $u \in W^{1,2}(\Omega)$. We call $P_{\delta z}u$ the elliptic projection of u. We have the following estimate :

Lemma 4.1.1 *Let $P_{\delta z}$ be the elliptic projection of u defined by (4.20), then*

$$\|u - P_{\delta z}u\|_1 \leq C(\delta z)\|u\|_2. \tag{4.21}$$

The proof of this Lemma can be found in [42].

L^2-error estimate

Theorem 4.1.2 *Let u and $u_{\delta z}$ be the solutions to the problem (4.13) and (4.17), respectively. Then we have*

$$\|u - u_{\delta z}\|_0 \leq C\{\|u_0 - u_{0,\delta z}\|_0 + (\delta z)[\|u_0\|_2 + \textstyle\int_0^t(\|u\|_2 + \|u_\tau\|_2)d\tau]\}. \tag{4.22}$$

Proof. Write

$$\rho = u - P_{\delta z}u, \quad e = P_{\delta z}u - u_{\delta z}, \tag{4.23}$$

where $P_{\delta z}$ is the elliptic projection operator. Then we have

$$u - u_{\delta z} = \rho + e. \tag{4.24}$$

It follows from (4.21) that

$$\begin{aligned}\|\rho\|_0 &\leq C(\delta z)\|u\|_2 = C(\delta z)\|u_0 + \textstyle\int_0^t u_\tau d\tau\|_2 \\ &\leq C(\delta z)[\|u_0\|_2 + \textstyle\int_0^t \|u_\tau\|_2 d\tau].\end{aligned} \tag{4.25}$$

We turn to estimate e. Since u and $u_{\delta z}$ satisfy (4.13) and (4.17) respectively, we have

$$(u_t - u_{\delta z,t}, v_{\delta z}) + a(t, u - u_{\delta z}, v_{\delta z}) = 0, \quad \forall v_{\delta z} \in V_{\delta z}, \quad t > 0. \tag{4.26}$$

This together with (4.20) gives

$$(e_t, v_{\delta z}) + a(t, e, v_{\delta z}) = -(\rho_t, v_{\delta z}), \quad \forall v_{\delta z} \in V_{\delta z}. \tag{4.27}$$

Choosing $v_{\delta z} = e$ and using the coercivity of the operator, yield

$$\frac{1}{2}\frac{d}{dt}\|e\|_0^2 \leq \|\rho_t\|\|e\|_0,$$

so

$$\frac{d}{dt}\|e\|_0 \leq \|\rho_t\|.$$

Integrating it with respect to t, then we have

$$\|e\|_0 \leq \|e(0)\|_0 + \int_0^t \|\rho_\tau\|_0 d\tau. \tag{4.28}$$

By virtue of Lemma (4.1.1) we have

$$\begin{aligned}\|e(0)\|_0 &\leq \|P_{\delta z}u_0 - u_0\|_0 + \|u - u_{0,\delta z}\|_0 \\ &\leq C(\delta z)\|u_0\|_2 + \|u - u_{0,\delta z}\|_0,\end{aligned} \tag{4.29}$$

On the other hand remarking that

$$a(t, (P_{\delta z}u)_t - P_{\delta z}u_t, v_{\delta z}) = B(t, u - P_{\delta z}u, v_{\delta z}) \quad \forall v_{\delta z} \in V_{\delta z}, \tag{4.30}$$

where

$$B(t, u, v) = \int_\Omega (h_t \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + (V_3)_t \frac{\partial u}{\partial z} v + \mu_t uv) dz.$$

Since

$$|B(t,u,v)| \le M\|u\|_1\|v\|_1,$$

then set $v_{\delta z} = (P_{\delta z}u)_t - P_{\delta z}u_t$ in (4.30), applying the coercivity of the bilinear form a, and the continuity of B, we obtain

$$\begin{aligned} \|(P_{\delta z}u)_t - P_{\delta z}u_t\|_1 &\le M'\|u - P_{\delta z}\|_1 \\ &\le CM'(\delta z)\|u\|_2. \end{aligned} \tag{4.31}$$

So it follows that

$$\begin{aligned} \|\rho_\tau\|_0 &= \|u_\tau - (P_{\delta z}u)_\tau\|_0 \\ &\le \|u_\tau - P_{\delta z}u_\tau\|_0 + \|P_{\delta z}u_\tau - (P_{\delta z}u)_\tau\|_0 \\ &\le C(\delta z)\|u_\tau\|_2 + C_1(\delta z)\|u\|_2 \\ &\le C'(\delta z)(\|u_\tau\|_2 + \|u\|_2). \end{aligned} \tag{4.32}$$

A combination of (4.25) and (4.28)-(4.32) leads to (4.22). This complete the proof. ■

4.2 Fully-discrete finite element schemes

4.2.1 Fully-discrete schemes

In the last section the semi-discrete schemes are obtained by discretizing the variable space. In order to finally get numerical solutions we need to further discretize the time variable to obtain fully-discrete schemes. To this end we use the implicit Euler's scheme. Let δt denote the time step size, and $t_n = n\delta t$, $(n = 0, 1, ...)$, $u_{\delta z}^n = u_{\delta z}(t_n)$. At time $t = t_n$, if we use the backward difference quotient

$$\bar{\partial}_t u_{\delta z}^n = \frac{u_{\delta z}^n - u_{\delta z}^{n-1}}{dt}$$

to approximate the differential quotient $u_{\delta z,t}$, then we obtain the discrete scheme : Find $u_{\delta z}^n \in V_{\delta z}$, $(n = 1, 2, ...)$ such that

$$\begin{cases} (\bar{\partial}_t u_{\delta z}^n, v_{\delta z}) + a(t_n, u_{\delta z}^n, v_{\delta z}) = (f(t_n), v_{\delta z}), \quad \forall v_{\delta z} \in V_{\delta z} \\ u_{\delta z}^0 = u_{0\delta z}. \end{cases} \tag{4.33}$$

Or we can write it as

$$\begin{cases} (u_{\delta z}^n, v_{\delta z}) + \delta t a(t_n, u_{\delta z}^n, v_{\delta z}) = (u_{\delta z}^{n-1} + \delta t f(t_n), v_{\delta z}), & \forall v_{\delta z} \in V_{\delta z} \\ u_{\delta z}^0 = u_{0\delta z}. \end{cases} \tag{4.34}$$

This scheme is referred to as a backward Euler finite element scheme. Remarking that

$$a(t_n, u_{\delta z}^n, u_{\delta z}^n) + \frac{1}{\delta t}(u_{\delta z}^n, u_{\delta z}^n) \geq \alpha \|u_{\delta z}^n\|_1^2, \quad \forall u_{\delta z}^n \in V_{\delta z}.$$

This guarantees the existence and uniqueness of the solution $u_{\delta z}^n$ to (4.33) for a given $u_{\delta z}^{n-1}$.

4.2.2 Error estimate for backward Euler finite element schemes

Theorem 4.2.1 *Let u and $u_{\delta z}^n$ be the solutions to the parabolic equation (5-1-3) and the backward Euler finite element scheme(5-2-1), respectively. Then*

$$\|u(t_n) - u_{\delta z}^n\|_0$$

$$\leq C\{\|u_0 - u_{0,\delta z}\|_0 + (\delta z)[\|u_0\|_2 + \textstyle\int_0^{t_n}(\|u\|_2 + \|u_t\|_2)dt] \tag{4.35}$$

$$+\delta t \textstyle\int_0^{t_n} \|u_{tt}\|_0 dt\}, \quad n = 1, 2, \ldots.$$

Proof. Set

$$\rho^n = u(t_n) - P_{\delta z}u(t_n), \quad e^n = P_{\delta z}u(t_n) - u_{\delta z}^n,$$

then

$$u(t_n) - u_{\delta z}^n = \rho^n + e^n.$$

It follows from (4.21) that

$$\|\rho^n\|_0 \leq C(\delta z)\|u(t_n)\|_2 \leq C(\delta z)[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt]. \tag{4.36}$$

Set $t = t_n$ in (5-1-3), and subtract it from (4.33), then we have

$$(u_t(t_n) - \bar{\partial}_t u_{\delta z}^n, v_{\delta z}) + a(t_n, \rho^n + e^n, v_{\delta z}) = 0, \quad \forall v_{\delta z} \in V_{\delta z}. \tag{4.37}$$

By virtue of (4.20) we have

$$(\bar{\partial}_t e^n, v_{\delta z}) + a(t_n, e^n, v_{\delta z}) = (\bar{\partial}_t P_{\delta z} u(t_n) - u_t(t_n), v_{\delta z}), \quad \forall v_{\delta z} \in V_{\delta z}. \quad (4.38)$$

Write

$$r^n = \bar{\partial}_t P_{\delta z} u(t_n) - u_t(t_n),$$

set $v_{\delta z} = e^n$, then we have

$$(\bar{\partial}_t e^n, e^n) \le (r^n, e^n).$$

So

$$\|e^n\|_0^2 \le (e^{n-1}, e^n) + \delta t (r^n, e^n),$$
$$\|e^n\|_0^2 \le (\|e^{n-1}\|_0 + \delta t \|r^n\|_0)\|e^n\|_0.$$

Eliminating $\|e^n\|_0$ and using the above recursion relation, we have

$$\|e^n\|_0 \le \|e^0\|_0 + \delta t \sum_{j=1}^{n} \|r^j\|_0. \quad (4.39)$$

Write $r^j = r_1^j + r_2^j$, where

$$r_1^j = \bar{\partial}_t P_{\delta z} u(t_j) - \bar{\partial}_t u(t_j) = \frac{1}{\delta t} \int_{t_{j-1}}^{t_j} ((P_{\delta z} u)_t - u_t) dt,$$

$$r_2^j = \bar{\partial}_t u(t_j) - u_t(t_j) = \frac{1}{\delta t} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) u_{tt} dt.$$

Then by (4.21) and (4.31)

$$\begin{aligned}\textstyle\sum_{j=1}^n \|r_1^j\|_0 &\le \frac{1}{\delta t} \textstyle\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (\|(P_{\delta z} u)_t - P_{\delta z} u_t\|_0 + \|P_{\delta z} u_t - u_t\|_0) dt \\ &\le C(\delta z) \textstyle\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (\|u\|_2 + \|u_t\|_2) dt \\ &= \frac{C}{\delta t}(\delta z) \textstyle\int_0^{t_n} \|u_t\|_2 dt.\end{aligned} \quad (4.40)$$

Similarly

$$\sum_{j=1}^{n} \|r_2^j\|_0 \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|u_{tt}\|_0 dt = \int_0^{t_n} \|u_{tt}\|_0 dt. \tag{4.41}$$

Again by (4.21)

$$\|e^0\|_0 \leq \|P_{\delta z} u_0 - u_0\|_0 + \|u_0 - u_{0\delta z}\|_0. \tag{4.42}$$

Substituting (4.40)-(4.42) into (4.39) yields

$$\|e^n\|_0 \leq C\{\|u_0 - u_{0\delta z}\|_0 + (\delta z)[\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 dt] + \delta t \int_0^{t_n} \|u_{tt}\|_0 dt\}. \tag{4.43}$$

Finally (4.35) follows from (4.36) and (4.43). This complete the proof. ■

4.3 The algorithm

By using the implicit Euler scheme,

$$\frac{1}{\delta t}(u^{n+1}(z) - u^n(z)) + V_3(t_{n+1}, z)\frac{\partial u^{n+1}}{\partial z} - \frac{\partial}{\partial z} h(t_{n+1}, z)\frac{\partial u^{n+1}}{\partial z} + (\mu(t_{n+1}, z) + k)u^{n+1} = f(t_{n+1}, z) exp(-kt_{n+1}), \tag{4.44}$$

Multiplying the equation (4.44) by the test function v and integrating over $\Omega = (0,1)$

$$\int_\Omega \frac{u^{n+1} - u^n}{\delta t} v dz + \int_\Omega V_3(t_{n+1}, z)\frac{\partial u^{n+1}}{\partial z} + \int_\Omega h(t_{n+1}, z)\frac{\partial u^{n+1}}{\partial z}\frac{\partial v}{\partial z} dz + \int_\Omega (\mu(t_{n+1}, z) + k)u^{n+1} v dz = \int_\Omega f(t_{n+1}, z) exp(-kt_{n+1}) v dz + g_1(t_{n+1}) exp(-kt_{n+1}) v(1) - g(t_{n+1}) exp(-kt_{n+1}) v(0) \tag{4.45}$$

for all $v \in H^1(\Omega)$.

We obtain by replacing u^n by its formula (4.11) and the test function v by v_j

$$\sum_{i=0}^{I+1} \int_\Omega u_i^{n+1}(\frac{v^i v^j}{\delta t} + V_3(t_{n+1}, z)\frac{\partial v_i}{\partial z} v_j$$

$$+h(t_{n+1}, z)\frac{\partial v_i}{\partial z}\frac{\partial v_j}{\partial z} + (\mu(t_{n+1}, z) + k)v_i v_j)dz \tag{4.46}$$

$$= \int_\Omega f(t_{n+1}, z)exp(-kt_{n+1})v_j dz + \sum_{i=0}^{I+1} \int_\Omega u_i^n \frac{v^i v^j}{\delta t} dz$$

$$+g_1(t_{n+1})exp(-kt_{n+1})v_j - g(t_{n+1})exp(-kt_{n+1})v_j,$$

for all $0 \leq j \leq I+1$. We denote by

$$h^{n+1}(i) = \frac{1}{\delta z}\int_{z_i}^{z_{i+1}} h(t_{n+1}, z)dz,$$

$$\mu_0^{n+1}(i) = \frac{3}{(\delta z)^3}\int_{z_i}^{z_{i+1}} (z_{i+1} - z)^2 \mu(t_{n+1}, z)dz$$

$$\mu^{n+1}(i) = \frac{6}{(\delta z)^3}\int_{z_i}^{z_{i+1}} (z_{i+1} - z)(z - z_i)\mu(t_{n+1}, z)dz$$

$$\mu_1^{n+1}(i) = \frac{3}{(\delta z)^3}\int_{z_i}^{z_{i+1}} (z - z_i)^2 \mu(t_{n+1}, z)dz$$

$$v_0^{n+1}(i) = \frac{2}{(\delta z)^2}\int_{z_i}^{z_{i+1}} (z_{i+1} - z)v(t_{n+1}, z)dz$$

$$v_1^{n+1}(i) = \frac{2}{(\delta z)^2}\int_{z_i}^{z_{i+1}} (z - z_i)v(t_{n+1}, z)dz$$

$$f_0^{n+1}(i) = \frac{2}{(\delta z)^2} exp(-kt_{n+1}) \int_{z_i}^{z_{i+1}} (z_{i+1} - z) f(t_{n+1}, z) dz$$

$$f_1^{n+1}(i) = \frac{2}{(\delta z)^2} exp(-kt_{n+1}) \int_{z_i}^{z_{i+1}} (z - z_i) f(t_{n+1}, z) dz.$$

The numerical scheme for the approximation of problem (4.1) with Newmann conditions is therefore
for $i = 0$,

$$\begin{aligned}
&(\frac{\delta z}{3}(\frac{1}{\delta t} + k + \mu_0^{n+1}(0)) + \frac{1}{\delta z} h^{n+1}(0) - \frac{1}{2} V_0^{n+1}(0)) u_0^{n+1} \\
&+(\frac{\delta z}{6}(\frac{1}{\delta t} + k + \mu^{n+1}(0)) - \frac{1}{\delta z} h^{n+1}(0) + \frac{1}{2} V_0^{n+1}(0)) u_1^{n+1} \\
&= \frac{\delta z}{2} f_0^{n+1}(0) + \frac{\delta z}{3\delta t} u_0^n + \frac{\delta z}{6\delta t} u_1^n - g(t_{n+1}) exp(-kt_{n+1})
\end{aligned}$$

for $1 \leq i \leq I$

$$\begin{aligned}
&(\frac{\delta z}{6}(\frac{1}{\delta t} + k + \mu^{n+1}(i-1)) - \frac{1}{\delta z} h^{n+1}(i-1) - \frac{1}{2} V_1^{n+1}(i-1)) u_{i-1}^{n+1} \\
+ \quad &(\frac{\delta z}{3}(\frac{1}{\delta t} + k + \mu_1^{n+1}(i-1)) + \frac{1}{\delta z} h^{n+1}(i-1) \\
+ \quad &\frac{1}{2} V_1^{n+1}(i-1) + \frac{\delta z}{3}(\frac{1}{\delta t} + k + \mu_0^{n+1}(i)) + \frac{1}{\delta z} h^{n+1}(i) - \frac{1}{2} V_0^{n+1}(i)) u_i^{n+1} \\
+ \quad &(\frac{\delta z}{6}(\frac{1}{\delta t} + k + \mu^{n+1}(i)) - \frac{1}{\delta z} h^{n+1}(i) + \frac{1}{2} V_0^{n+1}(i)) u_{i+1}^{n+1} \\
= \quad &\frac{\delta z}{2}(f_1^{n+1}(i-1) + f_0^{n+1}(i)) + \frac{\delta z}{6\delta t} u_{i-1}^n + \frac{2\delta z}{3\delta t} u_i^n + \frac{\delta z}{3\delta t} u_{i+1}^n
\end{aligned}$$

for $i = I + 1$

$$\begin{aligned}
&(\frac{\delta z}{6}(\frac{1}{\delta t} + k + \mu^{n+1}(I)) - \frac{1}{\delta z} h^{n+1}(I) - \frac{1}{2} V_1^{n+1}(I)) u_I^{n+1} \\
&+(\frac{\delta z}{3}(\frac{1}{\delta t} + k + \mu_1^{n+1}(I)) + \frac{1}{\delta z} h^{n+1}(I) + \frac{1}{2} V_1^{n+1}(I)) u_{I+1}^{n+1} \\
&= \frac{\delta z}{2} f_1^{n+1}(I) + \frac{\delta z}{6\delta t} u_I^n + \frac{\delta z}{3\delta t} u_{I+1}^n + g_1(t_{n+1}) exp(-kt_{n+1}).
\end{aligned}$$

If we apply the numerical integration, for instance the trapezoid rule, it yields

$$h^{n+1}(i) = \frac{1}{2}(h(t_{n+1}, z_{i+1}) + h(t_{n+1}, z_i))$$

$$\mu_0^{n+1}(i) = \frac{3}{2}\mu(t_{n+1}, z_i)$$

$$\mu^{n+1}(i) = 0$$

$$\mu_1^{n+1}(i) = \frac{3}{2}\mu(t_{n+1}, z_{i+1})$$

$$f_0^{n+1}(i) = exp(-kt_{n+1})f(t_{n+1}, z_i)$$

$$f_1^{n+1}(i) = exp(-kt_{n+1})f(t_{n+1}, z_{i+1})$$

$$V_0^{n+1}(i) = V_3(t_{n+1}, z_i)$$

$$V_1^{n+1}(i) = V_3(t_{n+1}, z_{i+1})$$

so the algorithm is written as
$i = 0$

$$a_{00} = \frac{\delta z}{3}(\frac{1}{\delta t} + k + \frac{3}{2}\mu(t_{n+1}, z_0)) + \frac{1}{2\delta z}(h(t_{n+1}, z_1) + h(t_{n+1}, z_0)) - \frac{1}{2}V_3(t_{n+1}, z_0)$$

$$a_{01} = \frac{\delta z}{6}(\frac{1}{\delta t} + k) - \frac{1}{2\delta z}(h(t_{n+1}, z_1) + h(t_{n+1}, z_0)) + \frac{1}{2}V_3(t_{n+1}, z_0)$$

$$b_0 = \frac{\delta z}{2}exp(-kt_{n+1})f(t_{n+1}, z_0) + \frac{\delta z}{3\delta t}u_0^n + \frac{\delta z}{6\delta t}u_1^n - g(t_{n+1})exp(-kt_{n+1})$$

for $1 \leq i \leq I$

$$a_{i,i-1} = \frac{\delta z}{6}(\frac{1}{\delta t} + k) - \frac{1}{2\delta z}(h(t_{n+1}, z_i) + h(t_{n+1}, z_{i-1})) - \frac{1}{2}V_3(t_{n+1}, z_i)$$

$$\begin{aligned} a_{i,i} &= \frac{\delta z}{3}(\frac{1}{\delta t}+k+\frac{3}{2}\mu(t_{n+1},z_i))+\frac{1}{2\delta z}(h(t_{n+1},z_i)+h(t_{n+1},z_{i-1})) \\ &+ \frac{1}{2}V(t_{n+1},z_i)+\frac{\delta z}{3}(\frac{1}{\delta t}+k+\frac{3}{2}\mu(t_{n+1},z_i)) \\ &+ \frac{1}{2\delta z}(h(t_{n+1},z_{i+1})+h(t_{n+1},z_i))-\frac{1}{2}V_3(t_{n+1},z_i) \end{aligned}$$

$$a_{i,i+1}=\frac{\delta z}{6}(\frac{1}{\delta t}+k)-\frac{1}{2\delta z}(h(t_{n+1},z_{i+1})+h(t_{n+1},z_i))+\frac{1}{2}V_3(t_{n+1},z_i)$$

$$b_i=\delta z exp(-kt_{n+1})f(t_{n+1},z_i)+\frac{\delta z}{6\delta t}u_{i-1}^n+\frac{2\delta z}{3\delta t}u_i^n+\frac{\delta z}{6\delta t}u_{i+1}^n$$

for $i=I+1$

$$a_{I+1,I}=\frac{\delta z}{6}(\frac{1}{\delta t}+k)-\frac{1}{2\delta z}(h(t_{n+1},z_{I+1})+h(t_{n+1},z_I))-\frac{1}{2}V_3(t_{n+1},z_{I+1})$$

$$\begin{aligned} a_{I+1,I+1} &= \frac{\delta z}{3}(\frac{1}{\delta t}+k+\frac{3}{2}\mu(t_{n+1},z_{I+1}))+\frac{1}{2\delta z}(h(t_{n+1},z_{I+1})+h(t_{n+1},z_I)) \\ &+ \frac{1}{2}V_3(t_{n+1},z_{I+1}) \end{aligned}$$

$$b_{I+1}=\frac{\delta z}{2}f(t_{n+1},z_{I+1})exp(-kt_{n+1})+\frac{\delta z}{6\delta t}u_I^n+\frac{\delta z}{3\delta t}u_{I+1}^n+g_1(t_{n+1})exp(-kt_{n+1})$$

4.4 Finite volume

This section is devoted to study the parabolic differential equation (4.5) together with the Dirichlet conditions $l(t,0)=g(t)$ and $l(t,1)=g_1(t)$ by the finite volume method. This one is a discretization method for conservation laws. As suggested by its name, a conservation law expresses the conservation of some matter $q(u)$, where q is a given function of the unknown u and states that the variation in time of the quantity of $q(u)$ in any bounded domain D of $I\!R^N$ is equal to the overall outward flux of matter denoted by $\Phi(u)$ through the boundary of the domain which is considered. Hence a conservation law can be written as

$$(\int_D q(u)(t,x)dx)_t+\int_{\partial D}\Phi(u)(t,x)d\sigma(x)=0,$$

where $(.)_t$ denotes the time partial derivative of the entity within the parentheses. The local expression of a conservation law (which is obtained by considering an infinitesimal domain) is of the form :

$$(q(u))_t(t,x) + div(F(u))(t,x) = 0,$$

where F is a given vector functional of the function u, such that

$$\int_{\partial D} F(u).n(t,x)d\sigma = \int_{\partial D} \Phi(t,x)d\sigma,$$

$n(t,x)$ is the outward normal to the boundary ∂D.

Definition 4.4.1 *(Admissible one dimensional mesh) An admissible mesh of $(0,L)$, denoted by τ, is given by a family $(K_i)_{i=1,...N}$, $N \in I\!N^*$, such that $K_i = (z_{i-\frac{1}{2}}, z_{i+\frac{1}{2}})$ and a family $(z_i)_{i=0,...N+1}$ such that*
$z_0 = z_{\frac{1}{2}} = 0 < z_1 < z_{\frac{3}{2}} < ... < z_{i-\frac{1}{2}} < z_i < z_{i+\frac{1}{2}} < ... < z_N < z_{N+\frac{1}{2}} = z_{N+1} = L$

in order to simplify the notations, we shall choose a constant space, time step, so we denote $\delta z = m(K_i) = z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}$, $i = 1,...,N$ and $\delta t \in (0,T)$. Let $N_{\delta t} \in I\!N^*$ such that

$$N_{\delta t} = max\{n \in I\!N, n\delta t < T\}$$

and we shall denote $t_n = n\delta t$ for $n \in \{0,...,N_{\delta t}+1\}$. The discrete unknown are denoted by u_i^n for $1 \le i \le N$, and are expected to be some approximation of u in the cell K_i (the discrete unknown u_i^n can be viewed as an approximation of the mean value of u over K_i, or of the value of $u(t_n, x_i)$, or of other value of u in the control volume K_i...). In order to obtain the numerical scheme, let us integrate the equation (4.5) over each control volume K_i of τ, and time interval (t_n, t_{n+1}) for $n \in \{0,...,N_{\delta t}\}$. Then

$$\begin{aligned} &\int_{K_i}(u^{n+1} - u^n)dz - \int_{t_{n+1}}^{t_n}\int_{K_i}(\frac{\partial}{\partial z}(h\frac{\partial u}{\partial z}))dzdt \\ &+ \int_{t_{n+1}}^{t_n}\int_{K_i}\frac{(\partial(V_3 u))}{\partial z}dzdt + \int_{t_{n+1}}^{t_n}\int_{K_i}(\mu + \frac{\partial V_3}{\partial z} + k)udzdt \qquad (4.47) \\ &= \int_{t_{n+1}}^{t_n}\int_{K_i} f exp(-kt)dzdt, \end{aligned}$$

Recall that, as usual, the stability condition for an explicit discretization of a parabolic equation require the time step to be limited by a power two of the space step, which is generally too strong a condition in terms of computational cost. Hence the choice of an implicit formulation in (4.47) which yields

$$\frac{1}{\delta t}\int_{K_i}(u(t_{n+1},z)-u(t_n,z))dz-(h\frac{\partial u}{\partial z})(t_{n+1},z_{i+\frac{1}{2}})+(h\frac{\partial u}{\partial z})(t_{n+1},z_{i-\frac{1}{2}})$$

$$+(V_3u)(t_{n+1},z_{i+\frac{1}{2}})-(V_3u)(t_{n+1},z_{i-\frac{1}{2}})$$

$$+\int_{K_i}(\mu(t_{n+1},z)-\frac{\partial V_3(t_{n+1},z)}{\partial z}+k)u(t_{n+1},z)dz$$

$$=\int_{K_i}f(t_{n+1},z)exp(-kt_{n+1})dz. \tag{4.48}$$

There now remains to replace in equation (4.48) each term by its approximation with respect to the discrete unknowns. The convective term $(V_3u)(t_{n+1},z_{i+\frac{1}{2}})$ is approximated by $V_3(t_{n+1},z_{i+\frac{1}{2}})u_i^{n+1}$ ("upstream") if $V_3(t_{n+1},z_{i+\frac{1}{2}})\geq 0$ and is approximated by $V_3(t_{n+1},z_{i+\frac{1}{2}})u_{i+1}^{n+1}$ if $V_3(t_{n+1},z_{i+\frac{1}{2}})\leq 0$. A reasonable choice for the approximation of $\frac{\partial u}{\partial z}(t_{n+1}z_{i+\frac{1}{2}})$ seems to be the differential quotient $\frac{u_{i+1}^{n+1}-u_i^{n+1}}{\delta z}$. This approximation is consistent in the sense that, if $u(t,.)\in C^2([0,1])$, then

$$\frac{u(t_{n+1},z_{i+1})-u(t_{n+1},z_i)}{\delta z}=u_z(t_{n+1},z_{i+\frac{1}{2}})+O(\delta z)$$

where $|O(\delta z)|\leq C\delta z$, $C\in I\!R^+$ only depending on u.

Remark 4.4.2 *Assume that z_i is the center of K_i. Let $\widetilde{u}_i^{n+1}$ denotes the mean value over K_i of the exact solution u to problem (4.5). One may then remark that*

$$|\widetilde{u}_i^{n+1}-u(t_{n+1},x_i)|\leq C\delta z^2,$$

with some C only depending on u; it follows that

$$\frac{\widetilde{u}_{i+1}^{n+1}-\widetilde{u}_i^{n+1}}{\delta z}=u_z(t_{n+1},z_{i+\frac{1}{2}})+O(\delta z),$$

also holds, for $i = 1, ..., N-1$. Hence the approximation of the flux is also consistent if the discrete unknowns u_i, $i = 1, ..., N$are viewed as approximations of the mean value of u in the control volumes.

The term

$$\int_{K_i} (\mu(t_{n+1}, z) - \frac{\partial V_3}{\partial z}(t_{n+1}, z) + k)u(t_{n+1}, z)dz$$

is approximated by

$$\delta z(\mu_i^{n+1} - \frac{\partial V_{3_i}^{n+1}}{\partial z} + k)u_i^{n+1},$$

with

$$\mu_i^{n+1} = \frac{1}{\delta z}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \mu(t_{n+1}, z)dz,$$

and

$$\frac{\partial V_{3_i}^{n+1}}{\partial z} = \frac{1}{\delta z}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \frac{\partial V_3}{\partial z}(t_{n+1}, z)dz.$$

The numerical scheme for the approximation of problem (4.1) with Dirichlet conditions is therefore

$$\frac{\delta z}{\delta t}(u_i^{n+1} - u_i^n) + G_{i+\frac{1}{2}}^{n+1} - G_{i-\frac{1}{2}}^{n+1} + F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1} + H_i^{n+1} = \delta z f_i^{n+1}, \qquad (4.49)$$

for $i = 0, ...N$ with

$$G_{i+\frac{1}{2}}^{n+1} = \begin{cases} V_3(t_{n+1}, z_{i+\frac{1}{2}})u_i^{n+1} & \text{if } V_3(t_{n+1}, z_{i+\frac{1}{2}}) \geq 0 \\ V_3(t_{n+1}, z_{i+\frac{1}{2}})u_{i+1}^{n+1} & \text{if } V_3(t_{n+1}, z_{i+\frac{1}{2}}) < 0 \end{cases} \qquad (4.50)$$

and

$$F_{i+\frac{1}{2}}^{n+1} = -h(t_{n+1}, z_{i+\frac{1}{2}})\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\delta z} \qquad (4.51)$$

for $i = 1, ..., N$

$$H_i^{n+1} = \delta z(\mu_i^{n+1} - \frac{\partial V_{3_i}^{n+1}}{\partial z} + k)u_i^{n+1} \qquad (4.52)$$

with

$$u_0^{n+1} = g(t_{n+1}) \qquad (4.53)$$

$$u_{N+1}^{n+1} = g_1(t_{n+1}) \qquad (4.54)$$

and

$$f_i^{n+1} = \frac{1}{\delta z}\int_{K_i} f(t_{n+1}, z)dz, \tag{4.55}$$

for the initial condition

$$u_i^0 = u_0(z_i) \tag{4.56}$$

or

$$u_i^0 = \frac{1}{\delta z}\int_{K_i} u_0(z)dz \tag{4.57}$$

4.4.1 Error estimate

Theorem 4.4.3 *Let $T > 0$ and $u \in C^2(\mathbb{R}^+ \times [0,1])$. Let $u_0 \in C^2([0,1])$ be defined by $u_0 = u(.,0)$, let $f \in C(\mathbb{R}^+ \times [0,1])$ and $g, g_1 \in C(\mathbb{R}^+)$, let τ be an admissible mesh. Then there exists a unique vector $(u_i^{n+1})_{i=1,...N}$ satisfying (4.49)-(4.57). There exists c only depending on u_0, T, f such that*

$$sup\{|u_i^n|, i \in \{1, ...N\}, n \in \{0, ..., N_{\delta t}+1\}\} \leq c. \tag{4.58}$$

Furthermore, let $e_i^n = u(t_n, z_i) - u_i^n$, for $i \in \{1, ...N\}$ and $n \in \{0, ..., N_{\delta t}+1\}$. Then there exists $C \in \mathbb{R}^+$ such that

$$(\sum_{i=1}^{N} \delta z (e_i^n)^2)^{\frac{1}{2}} \leq C(\delta z + \delta t). \tag{4.59}$$

Proof. For simplicity let us assume that z_i is the center of K_i.
(i) Existence, uniqueness and L^∞ estimate :
Without loss of generality we assume that $V_3 \geq 0$. For a given $n \in \{0, ..., N_{\delta t}\}$, set $f_i^n = 0$ and $u_i^n = 0$ in (4.49) and $g(t_{n+1}) = g_1(t_{n+1}) = 0$. Multiplying

(4.49) by u_i^{n+1} and summing for $i = 1, ...N$ yields

$$\sum_{i=1}^{N}(G_{i+\frac{1}{2}}^{n+1} - G_{i-\frac{1}{2}}^{n+1})u_i^{n+1} = \sum_{i=1}^{N}(V_3(t_{n+1}, z_{i+\frac{1}{2}})u_i^{n+1} - V_3(t_{n+1}, z_{i-\frac{1}{2}})u_{i-1}^{n+1})u_i^{n+1}$$
$$= \sum_{i=1}^{N}(V_3(t_{n+1}, z_{i+\frac{1}{2}})(u_i^{n+1})^2$$
$$+ \sum_{i=1}^{N}(V_3(t_{n+1}, z_{i-\frac{1}{2}})(\tfrac{1}{2}(u_i^{n+1} - u_{i-1}^{n+1})^2$$
$$-\tfrac{1}{2}(u_i^{n+1})^2 - \tfrac{1}{2}(u_{i-1}^{n+1})^2)$$
$$= \tfrac{1}{2}\sum_{i=1}^{N} V_3(t_{n+1}, z_{i-\frac{1}{2}})(u_i^{n+1} - u_{i-1}^{n+1})^2$$
$$+\tfrac{1}{2}\sum_{i=1}^{N}(V_3(t_{n+1}, z_{i+\frac{1}{2}}) - V_3(t_{n+1}, z_{i-\frac{1}{2}})(u_i^{n+1})^2$$
$$+\tfrac{1}{2}V_3(t_{n+1}, z_{N+\frac{1}{2}})u_N^2 - \tfrac{1}{2}V_3(t_{n+1}, z_{\frac{1}{2}})u_0^2 \tag{4.60}$$

$$\sum_{i=1}^{N}(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1})u_i^{n+1} = -\sum_{i=1}^{N} h(t_{n+1}, z_{i+\frac{1}{2}})\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\delta z}u_i^{n+1}$$
$$+\sum_{i=1}^{N} h(t_{n+1}, z_{i-\frac{1}{2}})\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\delta z}u_i^{n+1} \tag{4.61}$$

after straightforward calculus we obtain

$$\sum_{i=1}^{N}(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1})u_i^{n+1} = \sum_{i=1}^{N} h(t_{n+1}, z_{i+\frac{1}{2}})\frac{(u_{i+1}^{n+1} - u_i^{n+1})^2}{\delta z}$$
$$-h(t_{n+1}, z_{N+\frac{1}{2}})\frac{u_{N+1}^{n+1} - u_N^{n+1}}{\delta z}u_N^{n+1} \tag{4.62}$$
$$+h(t_{n+1}, z_{\frac{1}{2}})\frac{u_1^{n+1} - u_0^{n+1}}{\delta z}u_1^{n+1}.$$

$$\sum_{i=1}^{N} H_i^{n+1}u_i^{n+1} = \sum_{i=1}^{N} \delta z b_i^{n+1}(u_i^{n+1})^2, \tag{4.63}$$

with

$$b_i^{n+1} = \mu_i^{n+1} - \frac{\partial V_{3_i}^{n+1}}{\partial z} + k$$

(4.60),(4.62),(4.63) yields for $u_0^{n+1} = u_{N+1}^{n+1} = 0$, so $u_i^{n+1} = 0$ for all $i \in \{1, ...N\}$. This gives existence and uniqueness of $U = (u_1^{n+1}, ..., u_N^{n+1})^t$ solution to (4.49)-(4.57) for given $u_i^n, f_i^{n+1}, i \in \{1, ...N\}$. Let us prove the estimate (4.58). Set $m_f = min\{f(t,z), t \in (0, 2T), z \in (0, 1)\}$, $m_g = min\{g(t), t \in (0, 2T)\}$, and $m_{g_1} = min\{g_1(t), t \in (0, 2T)\}$, let $n \in \{0, ...N_{\delta t}\}$ we claim that $min\{u_i^{n+1}, i \in \{1, ...N\}\} \geq min\{min\{u_i^n, i \in \{1, ...N\} + \delta m_f\}, 0, m_g, m_{g_1}\}$. Indeed, if $min\{u_i^{n+1}, i \in \{1, ...N\}\} < min\{0, m_g, m_{g_1}\}$, let $i_0 \in \{1, ...N\}$ such that

$$u_{i0}^{n+1} = min\{u_i^{n+1}, i = 1, ...N\},$$

writing (4.49) with $i = i_0$ leads to

$$\frac{\delta z}{\delta t}(u_{i0}^{n+1} - u_{i0}^n) + (\delta z(\mu_i^{n+1} - \frac{\partial V_{3_i}^{n+1}}{\partial z} + k) - V_3(t_{n+1}, z_{i-\frac{1}{2}}))u_{i0}^{n+1} \geq \delta z f_{i0}^{n+1}, \tag{4.64}$$

since $V_3(t_{n+1}, z_{i-\frac{1}{2}})u_{i0-1}^{n+1} \geq V_3(t_{n+1}, z_{i-\frac{1}{2}})u_{i0}^{n+1}$ thus

$$u_{i0}^{n+1} - u_{i0}^n \geq \delta t f_{i0}^{n+1},$$

then

$$u_{i0}^{n+1} \geq min\{u_i^n, i \in \{1, ...N\}\} + \delta t m_f,$$

which yields by induction that

$$min\{u_i^n, i \in \{1, ...N\}\} \geq min\{\{min\{u_i^0, i \in \{1, ...N\}\}, 0, m_g, m_{g_1}\} + n\delta t m_f,$$

similarly

$$max\{u_i^n, i \in \{1, ...N\}\} \leq max\{\{max\{u_i^0, i \in \{1, ...N\}\}, 0, M_g, M_{g_1}\} + n\delta t M_f,$$

with $M_f = max\{f(t,z), t \in (0, 2T), z \in (0, 1)\}$, $M_g = max\{= g(t), t \in (0, 2T)\}$, and $M_{g_1} = max\{g_1(t), t \in (0, 2T), \}$. Then

$$sup\{|u_i^n|, i \in 1, ...N, n \in \{0, ..., N_{\delta t}\}\} \leq c$$

with $c = \|u_0\|_\infty + \|g\|_\infty + \|g_1\|_\infty + 2T\|f\|_\infty$.

(ii) Error estimate:

One uses the regularity of the solution to write an equation for the error $e_i^n = u(t_n, z_i) - u_i^n$ defined for $i \in \{1, ..., N\}$ and $n \in \{1, ..., N_{\delta t} + 1\}$. Note that $e_i^0 = 0$ for $i \in \{1, ..., N\}$. Let $n \in \{1, ..., N_{\delta t}\}$. Integration (in space) of equation (4.49) over each control volume K_i of τ at time $t = t_{n+1}$ gives

$$\begin{aligned}\int_{K_i} u_t(t,z)dz \quad & -(h\frac{\partial u}{\partial z})(t_{n+1}, z_{i+\frac{1}{2}}) + (h\frac{\partial u}{\partial z})(t_{n+1}, z_{i-\frac{1}{2}}) \\ & +(V_3 u)(t_{n+1}, z_{i+\frac{1}{2}}) - (V_3 u)(t_{n+1}, z_{i-\frac{1}{2}}) \\ & + \int_{K_i}(\mu(t_{n+1}, z) - \frac{\partial V_3(t_{n+1}, z)}{\partial z} + k)u(t_{n+1}, z)dz \\ & = \int_{K_i} f(t_{n+1}, z)exp(-kt_{n+1})dz.\end{aligned} \tag{4.65}$$

So that, for all $z \in K_i$, $i \in \{1, ..., N\}$, a Taylor expansion yields, thanks to the regularity of u

$$u_t(t_{n+1}, z) = \frac{1}{\delta t}(u(t_{n+1}, z_i) - u(t_n, z_i)) + s_i^{n+1}(z) \tag{4.66}$$

with

$$|s_i^{n+1}(z)| \leq c_1(\delta z + \delta t).$$

Therefore defining

$$S_i^{n+1} = \int_{K_i} s_i^{n+1}(z)dz, \tag{4.67}$$

one has

$$|S_i^{n+1}| \leq c_1 \delta z(\delta z + \delta t)$$

subtracting (4.49) to (4.65) yields

$$\begin{aligned}\frac{\delta z}{\delta t}(e_i^{n+1} - e_i^n) \quad & +S_i^{n+1} + V_3(t_{n+1}, z_{i+\frac{1}{2}})(u(t_{n+1}, z_{i+\frac{1}{2}}) - u_i^{n+1}) \\ & -V_3(t_{n+1}, z_{i-\frac{1}{2}})(u(t_{n+1}, z_{i-\frac{1}{2}}) - u_{i-1}^{n+1}) \\ & -h(t_{n+1}, z_{i+\frac{1}{2}})(\frac{\partial u}{\partial z}(t_{n+1}, z_{i+\frac{1}{2}}) - \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\delta z}) \\ & +h(t_{n+1}, z_{i-\frac{1}{2}})(\frac{\partial u}{\partial z}(t_{n+1}, z_{i-\frac{1}{2}}) - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\delta z}) \\ & +\delta z b_i^{n+1}(e_i^{n+1} + T_i^{n+1}) = 0\end{aligned} \tag{4.68}$$

with

$$|T_i^{n+1}| \leq c\delta z.$$

In addition

$$u(t_{n+1}, z_{i+\frac{1}{2}}) = u(t_{n+1}, z_i) + T'^{n+1}_{i+\frac{1}{2}}$$

with

$$|T'^{n+1}_{i+\frac{1}{2}}| \leq c'\delta z,$$

and

$$\frac{\partial u}{\partial z}(t_{n+1}, z_{i+\frac{1}{2}}) = \frac{u(t_{n+1}, z_{i+1}) - u(t_{n+1}, z_i)}{\delta z} + T''^{n+1}_{i+\frac{1}{2}},$$

with

$$|T''^{n+1}_{i+\frac{1}{2}}| \leq c''\delta z.$$

This yields

$$\begin{aligned}
\frac{\delta z}{\delta t}(e_i^{n+1} - e_i^n) \quad & + S_i^{n+1} + V_3(t_{n+1}, z_{i+\frac{1}{2}})(e_i^{n+1} + T'^{n+1}_{i+\frac{1}{2}}) \\
& - V_3(t_{n+1}, z_{i-\frac{1}{2}})(e_{i-1}^{n+1} + T'^{n+1}_{i-\frac{1}{2}}) \\
& - h(t_{n+1}, z_{i+\frac{1}{2}})(\frac{e_{i+1}^{n+1} - e_i^{n+1}}{\delta z} + T''^{n+1}_{i+\frac{1}{2}}) \\
& + h(t_{n+1}, z_{i-\frac{1}{2}})(\frac{e_i^{n+1} - e_{i-1}^{n+1}}{\delta z} + T''^{n+1}_{i-\frac{1}{2}}) \\
& + \delta z b_i^{n+1}(e_i^{n+1} + T_i^{n+1}) = 0.
\end{aligned} \tag{4.69}$$

Multiplying (4.69) by e_i^{n+1} and summing over K_i, $i \in \{1, ..., N\}$ and reordering the terms, yields

$$
\begin{aligned}
\frac{\delta z}{\delta t}\sum_{i=1}^{N}(e_i^{n+1})^2 \quad & +\tfrac{1}{2}\sum_{i=1}^{N} V_3(t_{n+1}, z_{i-\frac{1}{2}})(e_i^{n+1} - e_{i-1}^{n+1})^2 \\
& +\sum_{i=1}^{N-1} h(t_{n+1}, z_{i+\frac{1}{2}})\frac{(e_{i+1}^{n+1} - e_i^{n+1})^2}{\delta z} \\
& +\delta z \sum_{i=1}^{N} b'^{n+1}_i (e_i^{n+1})^2 \\
& = \frac{\delta z}{\delta t}\sum_{i=1}^{N} e_i^{n+1} e_i^n - \sum_{i=1}^{N} V_3(t_{n+1}, z_{i+\frac{1}{2}}) T'^{n+1}_{i+\frac{1}{2}}) e_i^{n+1} \\
& +\sum_{i=1}^{N} V_3(t_{n+1}, z_{i-\frac{1}{2}}) T'^{n+1}_{i-\frac{1}{2}}) e_i^{n+1} \\
& +\sum_{i=1}^{N} h(t_{n+1}, z_{i+\frac{1}{2}}) T''^{n+1}_{i+\frac{1}{2}}) e_i^{n+1} \\
& -\sum_{i=1}^{N} h(t_{n+1}, z_{i-\frac{1}{2}}) T''^{n+1}_{i-\frac{1}{2}}) e_i^{n+1} \\
& -\delta z \sum_{i=1}^{N} b_i^{n+1} T_i^{n+1} e_i^{n+1} \\
& -\sum_{i=1}^{N} S_i^{n+1} e_i^{n+1}
\end{aligned}
\tag{4.70}
$$

with

$$
b'^{n+1}_i = \mu_i^{n+1} - \frac{1}{2}\frac{\partial V_{3_i}^{n+1}}{\partial z} + k.
$$

One sets

$$I_1 = -\sum_{i=1}^N V_3(t_{n+1}, z_{i+\frac{1}{2}}) T'^{n+1}_{i+\frac{1}{2}}) e_i^{n+1}$$

$$+ \sum_{i=1}^N V_3(t_{n+1}, z_{i-\frac{1}{2}}) T'^{n+1}_{i-\frac{1}{2}}) e_i^{n+1} \tag{4.71}$$

$$= \sum_{i=0}^N V_3(t_{n+1}, z_{i+\frac{1}{2}}) T'^{n+1}_{i+\frac{1}{2}}) (e_{i+1}^{n+1} - e_i^{n+1})$$

we apply the Cauchy Schwartz inequality

$$|I_1| \leq (\sum_{i=1}^N \frac{\delta z V_3^2}{h} |T'^{n+1}_{i+\frac{1}{2}}|^2)^{\frac{1}{2}} (\sum_{i=1}^N \frac{h}{\delta z} (e_{i+1}^{n+1} - e_i^{n+1})^2)^{\frac{1}{2}} \tag{4.72}$$

using the Young's inequality

$$|I_1| \quad \leq \frac{1}{2} \sum_{i=1}^N \frac{\delta z V_3^2}{h} |T'^{n+1}_{i+\frac{1}{2}}|^2 + \frac{1}{2} \sum_{i=1}^N \frac{h}{\delta z} (e_{i+1}^{n+1} - e_i^{n+1})^2 \tag{4.73}$$

$$\leq c_1' |\delta z|^2 + \frac{1}{2} \sum_{i=1}^N \frac{h}{\delta z} (e_{i+1}^{n+1} - e_i^{n+1})^2.$$

$$I_2 = \sum_{i=1}^N h(t_{n+1}, z_{i+\frac{1}{2}}) T''^{n+1}_{i+\frac{1}{2}} e_i^{n+1} - \sum_{i=1}^N h(t_{n+1}, z_{i-\frac{1}{2}}) T''^{n+1}_{i-\frac{1}{2}} e_i^{n+1} \tag{4.74}$$

reordering the terms it yields

$$I_2 = \sum_{i=1}^N h(t_{n+1}, z_{i+\frac{1}{2}}) T''^{n+1}_{i+\frac{1}{2}} (e_{i+1}^{n+1} - e_i^{n+1}). \tag{4.75}$$

We apply again the Young's inequality

$$|I_2| \leq c_1 |\delta z|^2 + \frac{1}{2} \sum_{i=1}^N h(t_{n+1}, z_{i+\frac{1}{2}}) \frac{(e_{i+1}^{n+1} - e_i^{n+1})^2}{\delta z} \tag{4.76}$$

$$|I_3| \quad = |\delta z \sum_{i=1}^N b_i^{n+1} T_i^{n+1} e_i^{n+1}| \tag{4.77}$$

$$\leq c |\delta z|^2 + \frac{1}{2} \sum_{i=1}^N \delta z b_i^{n+1} |e_i^{n+1}|^2$$

$$I_4 \quad = | - \sum_{i=1}^N S_i^{n+1} e_i^{n+1}| \tag{4.78}$$

$$\leq c_1 \delta z (\delta z + \delta t) \sum_{i=1}^N |e_i^{n+1}|.$$

Thus we obtain

$$\frac{\delta z}{\delta t}\sum_{i=1}^{N}|e_i^{n+1}|^2+\delta z\sum_{i=1}^{N}b_i'^{n+1}|e_i^{n+1}|^2-\frac{\delta z}{2}\sum_{i=1}^{N}b_i^{n+1}|e_i^{n+1}|^2$$
$$\leq C_2|\delta z|^2+C_3\delta z(\delta z+\delta t)\sum_{i=1}^{N}|e_i^{n+1}|+\frac{\delta z}{\delta t}\sum_{i=1}^{N}|e_i^{n+1}||e_i^n| \tag{4.79}$$

remarking that

$$b_i'^{n+1}-\frac{1}{2}b_i^{n+1}\geq 0 \tag{4.80}$$

then

$$\sum_{i=1}^{N}\delta z|e_i^{n+1}|^2\leq C_2\delta t|\delta z|^2+C_3\delta t(\delta z+\delta t)\sum_{i=1}^{N}\delta z|e_i^{n+1}|+\sum_{i=1}^{N}\delta z|e_i^{n+1}||e_i^n| \tag{4.81}$$

by Young's the inequality (4.81) yields

$$\sum_{i=1}^{N}\delta z|e_i^{n+1}|^2 \leq 2C_2\delta t|\delta z|^2+2C_3\delta t(\delta z+\delta t)\sum_{i=1}^{N}\delta z|e_i^{n+1}|+\sum_{i=1}^{N}\delta z|e_i^n|^2$$
$$\leq \sum_{i=1}^{N}\delta z|e_i^n|^2+C_4(\delta t|\delta z|^2+\delta t(\delta z+\delta t)\sum_{i=1}^{N}\delta z|e_i^{n+1}|). \tag{4.82}$$

Remarking that for $\epsilon>0$ the following inequality holds

$$C_4\delta t(\delta z+\delta t)\sum_{i=1}^{N}\delta z|e_i^{n+1}|\leq\epsilon\sum_{i=1}^{N}\delta z|e_i^{n+1}|^2+\frac{1}{\epsilon}\sum_{i=1}^{N}\delta zC_4^2|\delta t|^2|\delta z+\delta t|^2 \tag{4.83}$$

taking

$$\epsilon=\frac{\delta t}{\delta t+1}$$

(4.82) yields

$$\sum_{i=1}^{N}\delta z|e_i^{n+1}|^2\leq(1+\delta t)\sum_{i=1}^{N}\delta z|e_i^n|^2+C_4(1+\delta t)\delta t(\delta z)^2+C_4^2(1+\delta t)^2\delta t(\delta z+\delta t)^2. \tag{4.84}$$

Then if

$$\sum_{i=1}^{N}\delta z|e_i^n|^2\leq c_n(\delta t+\delta z)^2 \tag{4.85}$$

with $c_n \in I\!R_+$ one deduce from (4.84), using $\delta t \leq \delta t + \delta z$ and $\delta t < T$, that

$$\sum_{i=1}^N \delta z |e_i^{n+1}|^2 \leq c_{n+1}(\delta t + \delta z)^2 \tag{4.86}$$

with

$$c_{n+1} = (1 + \delta t)c_n + C_5 \delta t \tag{4.87}$$

where

$$C_5 = C_4(1+T) + C_4^2(1+T)^2.$$

Choosing $c_0 = 0$ since $e_0^n = 0$, $i \in \{i = 1, ..., N\}$. The relation between c_n and c_{n+1} yields (by induction)

$$c_n \leq C_5 e^{2\delta t n}. \tag{4.88}$$

Estimate (4.59) follows with $C^2 = C_5^2 e^{4T}$. ■

Remark 4.4.4 *The error estimate given in Theorem 4.4.3 may be generalized to the case of discontinuous coefficients. The admissibility of the mesh redefined so that the data and the solution are piecewise regular on the control volumes.*

Let the following example

Example 4.4.5 *If $h(t,.) \in L^\infty(0,1)$ such that $a \leq h \leq b$ a.e with $a, b \in I\!R_+^*$. Let $\mathcal{T} = (K_i)_{i=1,...,N}$ be an admissible mesh, in the sense of Definition 4.4.1, such that the discontinuities of h coincide with the interface of the mesh. Let us turn to the approximation $H_{i+\frac{1}{2}}^{n+1}$ of $(h\frac{\partial u}{\partial z})(t_{n+1}, z_{i+\frac{1}{2}})$, let*

$$h_i^{n+1} = \frac{1}{\delta z}\int_{K_i} h(t_{n+1}, z)dz,$$

since $h|_{K_i} \in C^1(\bar{K}_i)$ there exists $c_h \in I\!R^+$, only depending on h, such that

$$|h_i^{n+1} - h(t_{n+1}, z)| \leq c_h h$$

$\forall z \in K_i$. In order that the scheme be conservative the discretization of the flux at $z_{i+\frac{1}{2}}$ should have the same value on $K_i$and K_{i+1}. To this purpose, we introduce the auxiliary unknown $u_{i+\frac{1}{2}}$(approximation of u at $z_{i+\frac{1}{2}}$.) Since on

K_i and K_{i+1}, h is continuous, the approximation of $(h\frac{\partial u}{\partial z})(t_{n+1}, z_{i+\frac{1}{2}})$ may be performed on each side of $z_{i+\frac{1}{2}}$ by using the finite difference principle :

$$H^{n+1}_{i+\frac{1}{2}} = h^{n+1}_i \frac{u^{n+1}_{i+\frac{1}{2}} - u^{n+1}_i}{\frac{\delta z}{2}}$$

on K_i, $1 \leq i \leq N$

$$H^{n+1}_{i+\frac{1}{2}} = h^{n+1}_{i+1} \frac{u^{n+1}_{i+1} - u^{n+1}_{i+\frac{1}{2}}}{\frac{\delta z}{2}}$$

on K_{i+1}, $0 \leq i \leq N-1$. Requiring the two above approximation of $(h\frac{\partial u}{\partial z})(t_{n+1}, z_{i+\frac{1}{2}})$ to be equal (conservatively of the flux) yields the value of $u^{n+1}_{i+\frac{1}{2}}$ for $i = 1, ..., N-1$

$$u^{n+1}_{i+\frac{1}{2}} = \frac{h^{n+1}_{i+1} u^{n+1}_{i+1} + h^{n+1}_i u^{n+1}_i}{h^{n+1}_{i+1} + h^{n+1}_i},$$

which in turn, allows to give the expression of the approximation $H^{n+1}_{i+\frac{1}{2}}$ of $(h\frac{\partial u}{\partial z})(t_{n+1}, z_{i+\frac{1}{2}})$,

$$H^{n+1}_{i+\frac{1}{2}} = \frac{2h^{n+1}_{i+1} h^{n+1}_i}{h^{n+1}_{i+1} + h^{n+1}_i} \frac{u^{n+1}_{i+1} - u^{n+1}_i}{\delta z} \tag{4.89}$$

for $i = 1, ...N-1$.

4.5 Comparison with other discretization technique

The finite volume method is quite different from the finite difference method or the finite element method as we saw. On these methods see e.g. Dahlquist and Bjorck [17], Thome [47], Ciarlet [15], [16], Roberts and Thomas [41]. Roughly speaking, the principle of the finite difference method is, given a number of discretization points which may be defined by a mesh, to assign one discrete unknown per discretization point, and to write one equation per discretization point. At each discretization point, the derivatives of the unknown are replaced by finite differences through the use of Taylor expansions. The finite difference method becomes difficult to use when the coefficients involved in the equation are discontinuous. With the finite volume method, discontinuities of the coefficients will not be any problem if the mesh is chosen such that the discontinuities of the coefficients occur on the boundaries of the control volumes. Note that the finite volume is often called "finite difference scheme" or "cell centered difference scheme". Indeed, in the finite volume method, the finite difference approach can be used for the approximation of the fluxes on the boundary of the control volumes. Thus, the finite volume schemes differ from the finite difference scheme in that the finite difference approximation is used for the flux rather than for the operator itself. The finite element method is based in the variational formulation, which is written for both the continuous and the discrete problems. The variational formulation is obtained by multiplying the original equation by a test function. The continuous unknown is then approximated by a linear combination of shape functions; these shape functions are the test functions for the discrete variational formulation (this is the so called Galerkin expansion); the resulting equation is integrated over the domain. The finite volume method is sometimes called a discontinuous finite element method, since the original equation is multiplied by the characteristic function of each grid cell, and the discrete unknown may be considered as a linear combination of shape functions.However, the techniques used to prove the convergence of finite element methods do not generally apply for this choice of test functions. From the industrial point of view, the finite volume method is known as a robust and cheap method for the discretization of conservation laws (by robust, we mean a scheme which behaves well even for particulary difficult equations, such as non linear systems of hyperbolic equations and which can easily be extended to more realistic and physical contexts than the classical academic problems). The finite volume method is cheap thanks of short and reliable computational coding for complex problems. It may be more adequate than the finite difference method which in particular requires a simple geometry. However, in some cases, it is difficult to design schemes which give enough precision. Indeed, the finite element method can be much more precise than the finite volume method when using higher order polynomials, but it requires an adequate functional framework which is not always available in industrial problems.

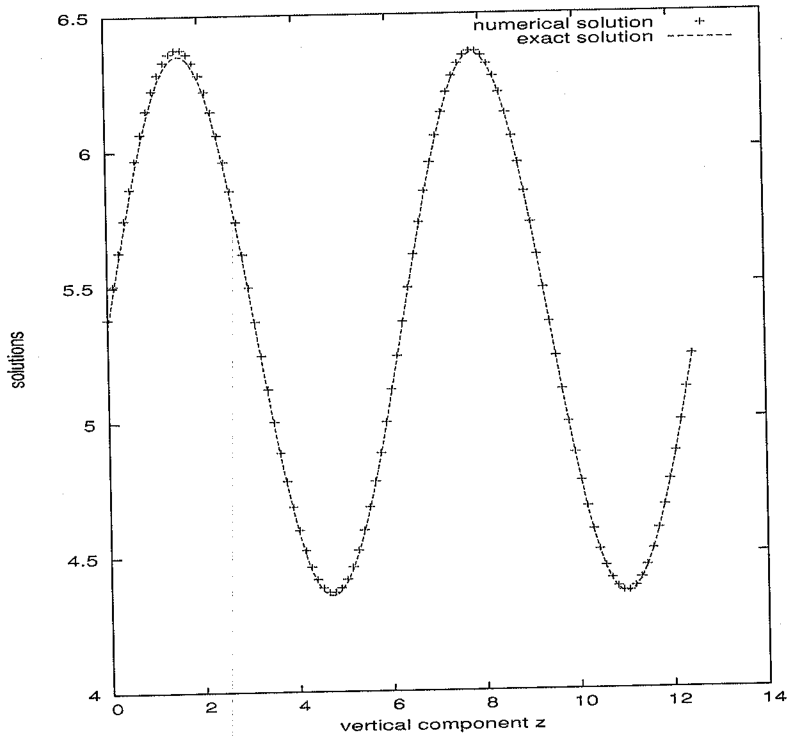

Figure 4.1: Comparison between the numerical solution R-K-F-E method and the exact solution.

Let us consider the same example by replacing the Newmann conditions by the following Dirichlet conditions

$$u(t, x, y, 0) = t + x + y,$$

$$u(t, x, y, 4\pi) = t + x + y.$$

The associated scheme

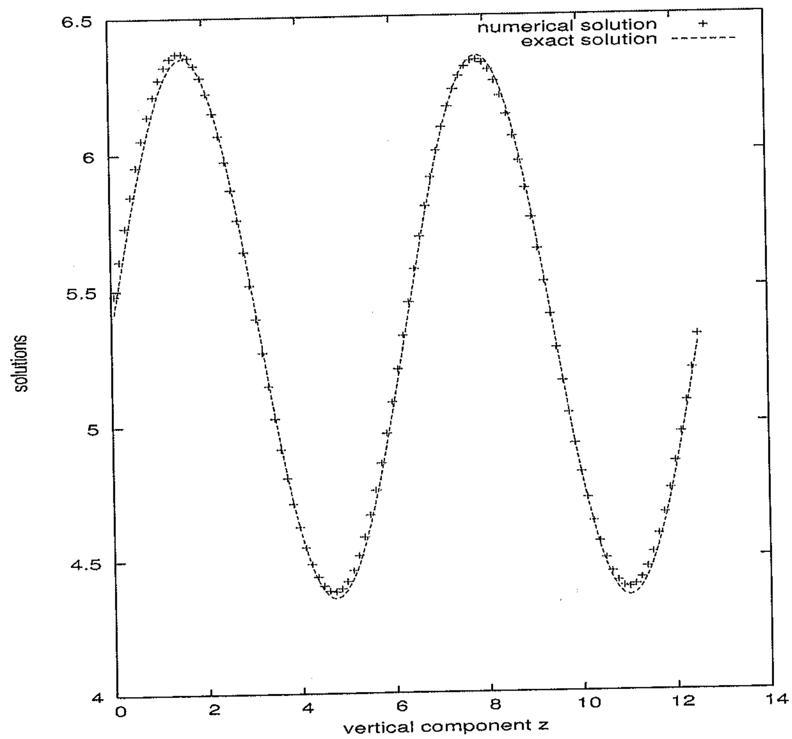

Figure 4.2: Comparison between the numerical solution R-K-F-V method and the exact solution.

In the following example we compare the R-K-E-F and R-K-F-V methods. We set

$$V_1 = x, V_2 = y, V_3 = t + x + y + z,$$
$$h = \frac{1}{1 + t^2 + x^2 + y^2 + z^2}, \mu = t$$

and

$$\begin{aligned} f &= 2t + 2x^2 + 2y^2 + 2z(t + x + y + z) \\ &- \frac{2(1 + t^2 + x^2 + y^2 - z^2)}{(1 + t^2 + x^2 + y^2 + z^2)^2} + t(t^2 + x^2 + y^2 + z^2), \end{aligned}$$

with the Newmann conditions

$$(h\frac{\partial u}{\partial z})(t, x, y, 0) = 0,$$

and

$$(h\frac{\partial u}{\partial z})(t,x,y,10) = \frac{20}{1+t^2+x^2+y^2+100}.$$

For the final time $t = 1$, the time step size $\delta t = 0.01$, and the space step size $\delta z = \frac{10}{100}$, we obtain the following figure,

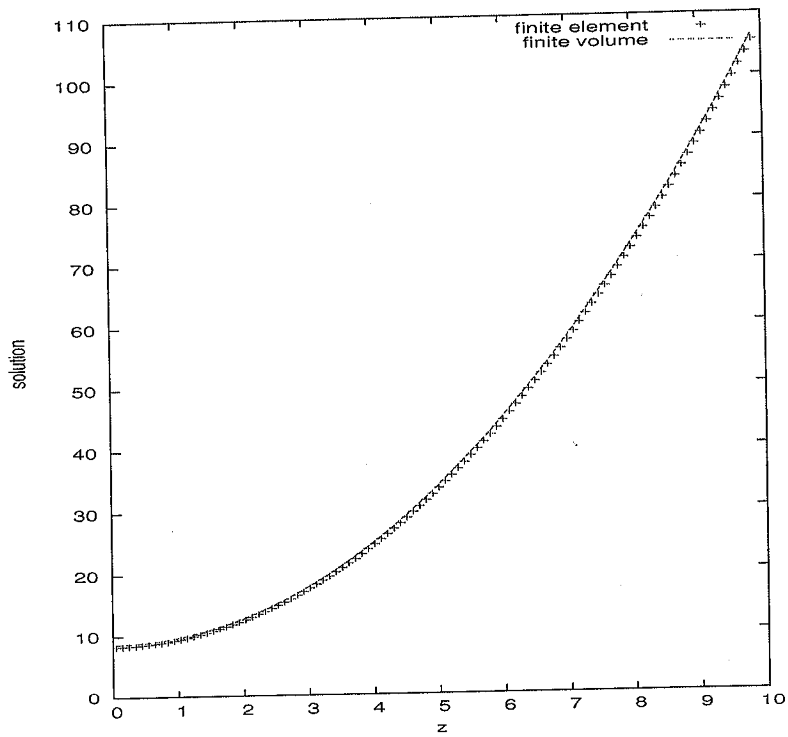

Figure 4.3: Comparison between the numerical solution R-K-F-E method and R-K-F-V method.

Remark During the simulation we remark that the quadratic error issue of the R-K-F-E method is less than the quadratic error issue of the R-K-F-V method. So as already mentioned in the last section, the finite element method can be much more precise than the finite volume, but the first method requires an adequate functional framework which is not always available in industrial problems and the second is cheap method.

Chapter 5

General conclusion and perspectives

Throughout this thesis we made a mathematical analysis of a model representing the growth of larvae; which is itself described as a consequence of larvae eating phytoplankton, but the scarcity of data leads us to proceed differently. The model focuses on the passive stage, a relatively short period after the egg has been released. During this period, the animal, first in its yolk-sac, then in the early larval stage is unable to move by itself and is subject to movements of advection and convection of the water. This period which, in the literature, is both recognized and at the same time not very well defined, is also considered to be crucial for the survival of the larvae. Since at this point the larvae are not able to move themselves, their feeding regime is ensured by the food present in their vicinity. The food in this first stage is mainly made up of phytoplankton, therefore it is also carried by the current. The model will be completing by a model describing the other stages of the life history : active larvae, juveniles and adults. A renewal equation will then be deduce, which should give a way of evaluating the recruitment rate.

Concerning the mathematical issue, the found model equation was a linear partial differential equation, but even of its linear aspect, it was not easy to handle. So we started by considering the uncoupled model where the horizontal current and the growth function are assumed to be independent of the vertical component. In this case we could prove existence, uniqueness and positivity of solution. The idea used here is to decompose the study into two steps. First, the problem was solved in (in the horizontal and growth component) along the characteristic lines. On these lines, it reduced to a one dimensional parabolic equation with respect to the vertical component, which was solved by means of evolution system theory elaborated by Acquistapace et all. After that we treated the model in the general case, where the coefficients depends of all of variables. In this situation we could not uncouple our equation, and this one remain ultraparabolic or equation with degenerated elliptic operator; for overcoming this degenerated type, we perturbed our main equation by adding the missing diffusions (x-y directions); so we obtained a non autonomous parabolic equations, which we treated by the monotonicity operator theory; then we passed to the limit in a suitable ways to obtain the existence of a weak solution of the main problem. The principal problem encountered by passing to the limit is the loss of the regularity, and this make the proof of uniqueness of solution more difficult. After we prove the existence and the positivity of the solution by another less classical method called multilayer methods. This approach consists in dividing the water column into horizontal layers, assuming that the horizontal current is independent on z in each sublayer and the temperature is vertically constant on each sublayer. In order to satisfy the incompressibility condition, it is assumed that the vertical current depends linearly on z in each sublayer, thus we can treat the approximated problem as we made in the first method that is uncouple this equation and solve separately the first order hyperbolic method and the one dimensional parabolic equation. As done before we passed then to the limit in a suitable way. The advantage of this last method is in the numerical treatment since by separating the main equation we can apply for this one, the same convergence theorem namely for parabolic and ordinary differential equations. We finished this chapter by treating the non autonomous non linear parabolic equation. As already mentioned our close perspective is to establish the existence and positivity of non autonomous non linear ultraparabolic equation. So the first step to prove it, is to approximate this one by a non linear parabolic equation, and one can prove by classical arguments the existence of the solution. The uniqueness is a difficult question and the works presented for this type of equation is scarce. The idea which we found is to impose extra conditions, so called entropy conditions, and so we set a new definition to the solution, which we called the entropy solution, given by Kruzkov [29] for hyperbolic equations and Escobedo et all. for parabolic equations. For all this it seems to us that it is preferable to give in appendix a large description of a work of this last authors. Remarking that if we can establish some regularity of our solution with respect to all

Appendix

We present a theory of existence and uniqueness for suitable entropy solutions of a non linear equation which describes the combined effects of diffusion and convection of matter. It has the form

$$u_t - \Delta_x u = \partial_y(f(u)), \;\; x \in I\!R^{N-1}, \; y \in I\!R, \; t > 0, \tag{1}$$

posed in the space $I\!R^n$, $n \geq 2$ denoted by the variable $z = (x, y)$, $x \in I\!R^{n-1}$, $y \in I\!R$. The main characteristic of this equation is that it has mixed parabolic hyperbolic type, due to the directional separation of the diffusion and convection effects : while the matter is convected along the y axis, it is simultaneously diffused along all orthogonal directions. The existence of solutions of equation (1) can be obtained by the method of adding a vanishing artificial viscosity, in other terms a diffusion, in the missing direction (along the y axis) or as we had seen in chapter 3 for the variable coefficients. The main interest in this appendix lies in the uniqueness part. It is well known that the solutions of conservation laws are not characterized in a unique way unless we impose extra conditions, so called entropy conditions. such problems arise for equation (1). In dealing with uniqueness the authors will be inspired in the entropy conditions in the form given by Kruzkov, [29]. However, in view of the presence of the diffusion term $\Delta_x u$ in equation (1) Kruzkov's entropy criterion has to be modified. This is done by introducing as entropy test functions all functions of the form $|u - \psi(x)|$ and ψ smooth, while Kruzkov's definition asks for ψ to be constant. This change also implies the modification of the criterion formula which now reads for a candidate solution $u(x, y, t)$:

$$\begin{aligned} &\frac{\partial}{\partial t}|u - \psi(x)| - \Delta_x |u - \psi(x)| \\ &\leq \frac{\partial}{\partial y}[|f(u) - f(\psi(x))| sign(u - \psi(x))] + sign(u - \psi(x))\Delta_x \psi(x), \end{aligned} \tag{2}$$

which is to be understood in the sense of distributions. The function *sign* is defined as $sign(s) = 1$ for $s > 0$, -1 for $s < 0$ and 0 for $s = 0$. In this setting uniqueness is established for entropy solutions of the Cauchy problem with initial data in $L^1 \cap L^\infty$. Most of the tools we use are in the theory of viscous or hyperbolic scalar conservation laws. However, equation (1) presents important new difficulties due to its mixed hyperbolic-parabolic character depending on directions. Before proceeding with the proofs, let us make some comments. Equation (1) admits as solutions functions of the form $u(y,t)$ as long as they are solutions of the non viscous conservation law $u_t = \partial_y f(u)$. This points out the need for some kind of entropy condition in equation (1), notwithstanding the fact that a viscosity term is present. On the other hand, solutions of the form $u = u(x,t)$ coincide with the solutions of the heat equation $u_t = \Delta_x u$, where no such additional condition is needed. In fact the restriction of condition (2) to such equations holds as a consequence of the regularity of the solutions. The condition we suggest modifies Kruzkov's condition to take into account those facts.

0.0.1 Statement of the main results

We will study the existence, uniqueness, and properties of entropy solutions for the Cauchy problem associated to the diffusion-convection equation of the reduced type

$$u_t - \Delta_x u = \partial_y(f(u)), \tag{3}$$

$$u(x,0) = u_0(x,y), \tag{4}$$

with f a locally Lipschitz continuous real function such that $f(0) = 0$ and the initial data $u_0 \in L^1(I\!R^n) \cap L^\infty(I\!R^n)$. Let us next consider the space BV of functions with bounded variation. given a function $g \in L^1_{loc}(\Omega)$, where Ω is an open subset. We define the total variation of g by

$$TV_\Omega(g) = \sup\{\int_\Omega g div\varphi dx, \varphi \in C^1_0(\Omega), \|\varphi\|_\infty \leq 1\}.$$

In general, we have $TV_\Omega(g) = +\infty$. Hence it make sense to introduce the following definition

Definition 1 *A function $g \in L^1_{loc}(\Omega)$ is said to have a bounded variation in Ω if $TV_\Omega(g) < +\infty$. We set*

$$TV_\Omega(g) = \{g \in L^1_{loc}(\Omega), TV_\Omega(g) < +\infty\}$$

Definition 2 *(i) By a solution of (3) we will understand a function*

$$u \in C((0,\infty) : L^1(I\!R^n) \cap L^\infty(Q), Q = I\!R^n \times (0,\infty),$$

which satisfies (3) in the sense of distributions.
(ii) A solution of problem (3)-(4) is a solution of (3) such that $u(.,t) \to u_0$ *in* $L^1(I\!R^n)$ *as* $t \to 0$. *Thus,* $u = u(x,y,t)$ *is continuous at* $t = 0$ *as a function* $:[0,\infty) \mapsto L^1(I\!R^n)$.
(iii) A solution $u(x,y,t)$ *is called entropy solution if the entropy criterion (2) is satisfied for all smooth function* $\psi = \psi(x)$. *As said above, it extends Kruzkov's definition by allowing* ψ *to be nonconstant. This forces to introduce the extra term*

$$sign(u - \psi(x))\Delta_x \psi(x)$$

on the right hand side, and the term $-\Delta_x|u - \psi(x)|$ *on the left hand side.*

The main results are

Theorem 3 *For every* $u_0 \in L^1(I\!R^n) \cap L^\infty(I\!R^n)$ *there exists an entropy solution of problems (3)-(4). This solution can be constructed by the vanishing-viscosity method.*

Theorem 4 *The entropy solution of problems (3)-(4) is unique. Moreover, comparison holds :*

$$If \quad u_0 \leq v_0 \quad a.e. \quad in \quad I\!R^n, \quad then \quad u \leq v \quad in \quad Q. \tag{5}$$

Finally the following L^1- *contraction property is true : if* u *and* v *are two entropy solutions with initial data* u_0, v_0 *resp., then for every* $t > 0$:

$$\|u(.,t) - v(.,t)\|_1 \leq \|u_0 - v_0\|_1. \tag{6}$$

Moreover,

$$\int_{I\!R^{n-1}} \int_{|y| \leq R} |u(x,y,t) - v(x,y,t)| dydx \leq \int_{I\!R^{n-1}} \int_{|y| \leq R+\alpha t} |u_0(x,y) - v_0(x,y)| dydx, \tag{7}$$

where

$$\alpha = max\{\|f'(u)\|_\infty, \|f'(v)\|_\infty\}. \tag{8}$$

The main properties of the solution are summarized as follows :

Theorem 5 *Let u be an entropy solution of (3) with initial data $u_0 \in L^1(I\!R^n)$. Then for every $t > 0$ we have conservation of mass*

$$\int u(x,y,t)dxdy = \int u_0(x,y)dxdy. \tag{9}$$

We have also

$$\|u(.,t)\|_p \leq \|u_0\|_p \tag{10}$$

for every $p \in [1,\infty]$.

0.0.2 Existence of solutions for smooth data

We establish here the existence part of Theorem 3 in the case where the initial data are smooth, precisely under the condition

$$u_0 \in L^\infty(I\!R^n) \cap BV(I\!R^n) \cap L^1(I\!R^n, 1+|x|+|y|), \quad \textit{and} \quad \nabla_x u_0 \in ((BV)(I\!R^n))^{n-1}. \tag{11}$$

The construction of the solution uses the vanishing-viscosity method. Define $u_{0,\epsilon} = u_0 * \zeta_\epsilon$, where $*$ denotes the convolution in $I\!R^n$, $\zeta_\epsilon = \zeta_\epsilon(x,y) = \epsilon^{-n}\zeta(x/\epsilon, y/\epsilon)$, ζ being a smooth cut-off function in $\mathcal{D}(I\!R^n)$ with the following properties :
(i) ζ is nonnegative and its support is contained in the unit ball of $I\!R^n$.
(ii) The integral of ζ over $I\!R^n$ is 1.
(iii) $\zeta(-x,-y) = \zeta(x,y)$.
We consider the regularized parabolic problem

$$\begin{cases} u_t = \Delta_x u + \epsilon \partial_y^2 u + \partial_y(f(u)), \\ u(x,y,0) = u_{0,\epsilon}(x,y). \end{cases} \tag{12}$$

For all $\epsilon \geq 0$ this problem has a unique solution (see for instance [20])

$$u_\epsilon \in C([0,\infty) : L^1(I\!R^n)) \cap L^\infty(Q).$$

In addition $u_\epsilon \in C([0,\infty) : W^{2,p}(I\!R^n)) \cap C^1([0,\infty); L^p(I\!R^n))$ for all $1 < p < \infty$. On the other hand, u_ϵ satisfies the properties (5) to (10). Proceeding as in the proof of Lemma 3-2, p. 68, of Godlewski and Raviart [26], we deduce that for every t and $\epsilon > 0$:

$$\int u_\varepsilon(x,y,t)dxdy = \int u_{0,\varepsilon}(x,y)dxdy, \tag{13}$$

$$\|\nabla u_\varepsilon(t)\|_1 \le \|\nabla u_{\varepsilon,0}(t)\|_1 \le TV(u_0), \tag{14}$$

$$\|\partial_t u_\varepsilon(t)\|_1 \le C[TV(u_0) + TV(\nabla_x u_0)]. \tag{15}$$

On the other hand, multiplying (12) by $u|u|^{p-2}$ and integrating gives

$$\|u_\varepsilon\|_p \le \|u_0\|_p \quad forall \quad 1 \le p \le \infty. \tag{16}$$

This estimates allow us to pass to the limit and to get a solution

$$u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty(Q)$$

satisfying (10). In order to prove equality (9), i.e. the conservation of mass for u, we need to prove a uniform estimate for the tails of u_ε. Taking into account that $u_0 \in L^1(\mathbb{R}^n; 1 + |x| + |y|)$ we can prove that $u_{0,\varepsilon}$ is uniformly bounded in $L^1(\mathbb{R}^n; 1+|x|+|y|)$. Then by doing L^1-estimates on the equation that $x_i u_\varepsilon$ and $y u_\varepsilon$ satisfy we get that

$$\int |u_\varepsilon(x,y,t)|(|x| + |y|)dxdy \le C(t) \tag{17}$$

for every t and $\varepsilon > 0$. This allows to estimate the tails of u_ε and to obtain the conservation of mass for u. Finally, since u_ε satisfies the entropy condition (2) for all $\varepsilon > 0$, passing to the limit as $\varepsilon \to 0$ we conclude that u also satisfies the entropy condition.

0.0.3 Uniqueness of entropy solutions with smooth data

We establish the following results

Theorem 6 *We assume that*

$$u_0 \in L^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, 1+|x|+|y|), \quad and \quad \nabla_x u_0 \in ((BV)(\mathbb{R}^n))^{n-1}. \tag{18}$$

Then there is a unique bounded entropy solution of problem (3),(4). Moreover, (5) and (6) hold.

Proof. Let denote by u the solution obtained in the subsection above by the vanishing viscosity method, and let $v = v(x,y,t)$ be another entropy solution with the same initial data. We will prove that $u = v$.
Remark For notational simplicity the expression

$$\partial_y[(f(u) - f(\psi))sign(u - \psi)],$$

which appears frequently in the calculations is replaced by $\partial_y|f(u)-f(\psi)|$. This is only true if f is nondecreasing.

(I) We shall take advantage of the fact that u has been constructed by the vanishing-viscosity method. Indeed, since u_ε is solution of the regularized parabolic problem we have

$$|u_\varepsilon-\psi|_t-(\Delta_x u_\varepsilon)sign(u_\varepsilon-\psi)-\varepsilon\partial_y^2|u_\varepsilon-\psi|\leq\partial_y|f(u_\varepsilon)-f(\psi)| \tag{19}$$

in $\mathcal{D}'(Q)$, $Q=I\!R_x^{n-1}\times I\!R_y\times I\!R_t^+$, for all $\varepsilon>0$ and all $\psi=\psi(x)\in L^\infty(I\!R^n)$. For fixed $z\in I\!R$ and $\tau\in I\!R^+$ we now take $\psi(x)=v(x,z,\tau)$ in (19) to get

$$\begin{aligned}&|u_\varepsilon(x,y,t)-v(x,z,\tau)|_t-(\Delta_x u_\varepsilon(x,y,t))sign(u_\varepsilon(x,y,t)-v(x,z,\tau))\\&-\varepsilon\partial_y^2|u_\varepsilon(x,y,t)-v(x,z,\tau)|\leq\partial_y|f(u_\varepsilon(x,y,t))-f(v(x,z,\tau))|\end{aligned} \tag{20}$$

in $\mathcal{D}'(Q)$, with respect to the same variable. On the other hand, since u_ε is smooth, we may take $\psi(x)=u_\varepsilon(x,y,t)$ in the entropy condition satisfied by $v=v(x,z,\tau)$. We thus get

$$\begin{aligned}&|v(x,z,\tau)-u_\varepsilon(x,y,t)|_\tau-\Delta_x|v(x,z,\tau)-u_\varepsilon(x,y,t)|\\&\leq\partial_z|f(v(x,z,\tau))-f(u_\varepsilon(x,y,t))|\\&+\Delta_x u_\varepsilon(x,y,t)sign(v(x,z,\tau)-u_\varepsilon(x,y,t))\end{aligned} \tag{21}$$

in $\mathcal{D}'(Q)$, where now $Q=I\!R_x^{n-1}\times I\!R_z\times I\!R_\tau^+$. This two inequalities are the cornerstone of the proof.

(II) We now establish the main distribution inequality for differences of solutions.

Lemma 7 *Let u be the entropy solution of (3) constructed by the vanishing-viscosity method, taking initial data $u_0=u_0(x,y)$ with the assumptions (18). Let $v=v(x,y,t)$ be any other uniformly bounded entropy solution of (3). Then*

$$\partial_t|u-v|-\Delta_x|u-v|-\partial_y|f(u)-f(v)|\leq 0 \;\; in \;\; \mathcal{D}'(Q). \tag{22}$$

Proof. We follow familiar ground, the textbook [26]. Firstly we take a cut-off function $r=r(y,t)\in\mathcal{D}'(I\!R\times I\!R^+)$ and set

$$r_\delta(y,t)=\delta^{-2}r(y/\delta,t/\delta).$$

We also take a nonnegative test function, $\Phi(x,\zeta,s)\in\mathcal{D}'(I\!R^n\times(0,\infty))$, and define two functions :

$$\phi_{z,\tau}(x,y,t)=\eta_{y,t}(x,z,\tau)=\Phi(x,\frac{y+z}{2},\frac{t+\tau}{2})r_\delta(\frac{y-z}{2},\frac{t-\tau}{2}).$$

(In the sequel the sub indices (z,τ) in ϕ and (y,t) in η will be frequently omitted.) Using ϕ as test function in (20) and η in (21) and integrating the inequalities we get with respect to (y,t) and (z,τ) respectively, we obtain, with $u_\varepsilon = u_\varepsilon(x,y,t)$ and $v = v(x,z,\tau)$

$$\begin{aligned} 0 \le \int_0^\infty \int_0^\infty \int_{I\!R} \int_{I\!R} \int_{I\!R^{n-1}} \quad & \{|u_\varepsilon - v|(\frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial \tau}) \\ & -|f(u_\varepsilon) - f(v)|(\frac{\partial \phi}{\partial y} + \frac{\partial \eta}{\partial z}) \\ & +\varepsilon|u_\varepsilon - v|\partial_y^2\phi + |u_\varepsilon - v|\Delta_x \eta\} dxdydzdtd\tau. \end{aligned}$$

Passing to the limit as $\varepsilon rightarrow 0$ we obtain that

$$\begin{aligned} 0 \le \int_0^\infty \int_0^\infty \int_{I\!R} \int_{I\!R} \int_{I\!R^{n-1}} \quad & \{|u - v|(\frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial \tau}) \\ & -|f(u) - f(v)|(\frac{\partial \phi}{\partial y} + \frac{\partial \eta}{\partial z}) \\ & +|u - v|\Delta_x \eta\} dxdydzdtd\tau, \end{aligned} \tag{23}$$

where $u = u(x,y,t)$. We note take into account that

$$\frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial \tau} = \frac{\partial \Phi}{\partial s}(x, \frac{y+z}{2}, \frac{t+\tau}{2}) r_\delta(\frac{y-z}{2}, \frac{t-\tau}{2})$$

and

$$\frac{\partial \phi}{\partial y} + \frac{\partial \eta}{\partial z} = \frac{\partial \Phi}{\partial \zeta}(x, \frac{y+z}{2}, \frac{t+\tau}{2}) r_\delta(\frac{y-z}{2}, \frac{t-\tau}{2}).$$

Let us introduce the change of variables

$$Y = \frac{y+z}{2}, \quad Z = \frac{y-z}{2}, \quad T = \frac{t+\tau}{2}, \quad S = \frac{t-\tau}{2},$$

That maps $I\!R^2 \times (I\!R^+)^2$ into $\Omega = \{(Y,Z,T,S) \in I\!R^2 \times I\!R^2 : T+S \ge 0, T-S \ge 0\}$. Setting

$$\begin{aligned} G \quad &= G(x,Y,Z,T,S) \\ &= |u(x,Y+Z,T+S) - v(x,Y-Z,T-S)|\frac{\partial \Phi}{\partial s}(x,Y,T) \\ &-|f(u(x,Y+Z,T+S)) - f(v(x,Y-Z,T-S))|\frac{\partial \Phi}{\partial \zeta}(x,Y,T) \\ &+|u(x,Y+Z,T+S) - v(x,Y-Z,T-S)|\Delta_x \Phi(x,Y,T), \end{aligned}$$

we may rewrite (23) as follows :

$$J_\delta \equiv \int_{I\!R^{n-1}} \int_\Omega G(x,Y,Z,T,S) r_\delta(Z,S)\, dx dY dT dZ dS \geq 0. \tag{24}$$

We now claim that

$$J_\delta \to \int_0^\infty \int_{I\!R^n} G(x,Y,0,T,0)\, dx dY dT = J_0 \quad as \quad \delta \to 0. \tag{25}$$

But then,

$$J_0 = \int_0^\infty \int_{I\!R^n} \{|u-v|\frac{\partial \Phi}{\partial s}(x,Y,T) - |f(u)-f(v)|\frac{\partial \Phi}{\partial \zeta} + |u-v|\Delta_x \Phi\} dx dY dT \geq 0$$

i.e. (22). In order to prove this claim we introduce the function

$$\chi(T,S) = \begin{cases} 1 & \text{if } T+S \geq 0, \quad T-S \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

With this notation we have

$$J_\delta = \int_{I\!R^{n+3}} G(x,Y,Z,T,S) r_\delta(Z,S) \chi(T,S)\, dx dY dT dZ dS.$$

On the other hand, since $\int_{I\!R^2} r_\delta(Z,S) dZ dS = 1$, we have also

$$J_0 = \int_{I\!R^{n+3}} G(x,Y,0,T,0) r_\delta(Z,S) \chi(T,0)\, dx dY dT dZ dS.$$

Let us denote by K the support of Φ and by C_δ the support of r_δ. We may assume that

$$C_\delta \subset \{(Z,S); |Z| \leq \delta, |S| \leq \delta\}.$$

We have

$$|J_\delta - J_0| \leq \int_K dx dY dT \int_{C_\delta} \quad \begin{aligned} &|G(x,Y,Z,T,S)\chi(T,S) \\ &-G(x,Y,0,T,0)\chi(T,0)| r_\delta(Z,S) dZ dS, \end{aligned}$$

and therefore, $|J_\delta - J_0| \leq A_\delta + B_\delta$, with

$$A_\delta = \int_K dx dY dT \int_{C_\delta} |G(x,Y,Z,T,S) - G(x,Y,0,T,0)| r_\delta(Z,S) dZ dS,$$

and

$$B_\delta = \int_K |G(x,Y,0,T,0)| [\int_{C_\delta} |\chi(T,S) - \chi(T,0)| r_\delta(Z,S) dZdS] dYdTdx.$$

Proceeding as in p.75 of [26] we deduce that

$$\begin{aligned} |G(x,Y,Z,T,S) - G(x,Y,0,T,0)| \quad & \leq C_1\{|u(x,Y+Z,T+S) - u(x,Y,T)| \\ & +|v(x,Y-Z,T-S) - v(x,Y,T)|\}, \end{aligned}$$

with $C_1 = C(\|u\|_\infty, \|v\|_\infty)$. Since $|r_\delta| \leq C_2\delta^{-2}$ we find

$$\begin{aligned} A_\delta \leq C_3\delta^{-2} \textstyle\int_K dxdYdT \int_{C_\delta} \quad & \{|u(x,Y+Z,T+S) - u(x,Y,T)| \\ & +|v(x,Y-Z,T-S) - v(x,Y,T)|\}dZdS. \end{aligned}$$

From Lebesgue's differentiation theorem we know that, for a.e. x,

$$\lim_{\delta\to 0} \frac{1}{meas(C_\delta)} \textstyle\int_{C_\delta} |u(x,Y+Z,T+S) - u(x,Y,T)| dZdS = 0,$$

for almost all $(Y,T) \in \mathbb{R} \times \mathbb{R}_+$. Of course, we have an analogous result for v. Then, applying Lebesgue dominated convergence theorem we deduce that :

$$\begin{aligned} \lim_{\delta\to 0} \delta^{-2} \textstyle\int_K dxdYdT \int_{C_\delta} \quad & \{|u(x,Y+Z,T+S) - u(x,Y,T)| \\ & +|v(x,Y-Z,T-S) - v(x,Y,T)|\}dZdS = 0, \end{aligned}$$

and this implies that $A_\delta \to 0$ as $\delta \to 0$. On the other hand,

$$\int_{C_\delta} |\chi(T,S) - \chi(T,0)| r_\delta(Z,S) dZdS \leq C_4\delta^{-1} \int_{-\delta}^{\delta} |\chi(T,S) - \chi(T,0)| dS,$$

and therefore

$$B_\delta \leq \int_R dT \int_{-\delta}^{\delta} |\chi(T,S) - \chi(T,0)| dS.$$

It is then easy to check that $B_\delta \leq C_5\delta$. This concludes the proof of claim (25) and the Lemma. ∎

Lemma 8 *Let $u_0 = u_0(x,y)$ be as above and u the solution of (3) obtained by the vanishing-viscosity method. Let v be any other uniformly bounded entropy solution with initial data $v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R})^n$. Then for any $R,T > 0$ we have*

$$\begin{aligned} \textstyle\int_{\mathbb{R}^{n-1}} \quad & \textstyle\int_{|y|\leq R} |u(x,y,T) - v(x,y,T)| dydx \\ & \leq \textstyle\int_{\mathbb{R}^{n-1}} \int_{|y|\leq R+\alpha T} |u_0(x,y) - v_0(x,y)| dydx, \end{aligned} \tag{26}$$

with $\alpha = max\{\|f'(u)\|_\infty, \|f'(v)\|_\infty\}$.

Remark 9 *Of course, this Lemma implies the uniqueness of the entropy solution of (3) for smooth initial data satisfying (18).*

Proof. We begin by approximating the characteristic function of the set

$$K_{R,T} = \{(y,t) \in I\!R \times I\!R_+; |y| \leq R + \alpha(T-t), t \in [0,T]\}.$$

We introduce the following approximation of Heaviside's function :

$$Y_\varepsilon(t) = \int_\infty^t \zeta_\varepsilon(s)ds,$$

with $\zeta_\varepsilon \in \mathcal{D}(I\!R)$ a cut-off function with support on $[-\varepsilon, \varepsilon]$. Then, for δ, ε, $\theta > 0$ with $\delta < T$ we set

$$\varphi(y,t) = (Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))(1 - Y_\theta(|y| - R - \alpha(T-t))).$$

Clearly $\varphi \in \mathcal{D}(I\!R \times I\!R^+)$ is nonnegative and satisfies

$$\varphi(y,t) = \begin{cases} 1 & \text{if} \quad |y| \leq R + \alpha(T-t) - \theta \quad \text{and} \quad \varepsilon + \delta \leq t \leq T - \varepsilon \\ 0 & \text{if} \quad |y| \geq R + \alpha(T-t) + \theta \quad \text{or} \quad t \leq \varepsilon + \delta. \end{cases}$$

Let us now suppose that $\varepsilon < \delta$. Given any nonnegative $\psi = \psi(x) \in \mathcal{D}(I\!R^{n-1})$, we use $\psi(x)\varphi(y,t)$ as test function in (22) to get

$$\int_0^\infty \int_{I\!R^n} |u-v|\psi(x)(\zeta_\varepsilon(t-\delta) - \zeta_\varepsilon(t-T))(1 - Y_\theta(|y| - R - \alpha(T-t)))dydxdt$$

$$+\int_0^\infty \int_{I\!R^n} |u-v|\Delta_x\psi(x)(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))$$

$$.(1 - Y_\theta(|y| - R - \alpha(T-t)))dydxdt$$

$$+\int_0^\infty \int_{I\!R^n} [|f(u) - f(v)| - \alpha|u-v|]\psi(x)(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))$$

$$.\zeta_\theta(|y| - R - \alpha(T-t))(\text{sign}y)dydxdt \geq 0.$$

Since $|f(u) - f(v)| \leq \alpha|u-v|$ and $Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T) \geq 0$, we deduce that the last integral is negative. Therefore we get

$$\begin{aligned} 0 \leq \int_0^\infty & \int_{I\!R^n} |u-v|\psi(x)(\zeta_\varepsilon(t-\delta) - \zeta_\varepsilon(t-T))(1 - Y_\theta(|y| - R - \alpha(T-t)))dydxdt \\ & +\int_0^\infty \int_{I\!R^n} |u-v|\Delta_x\psi(x)(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T)) \\ & .(1 - Y_\theta(|y| - R - \alpha(T-t)))dydxdt. \end{aligned}$$

Now we let θ tend to 0. Applying Lebesgue's dominated convergence theorem we get

$$0 \leq \int_{\mathbb{R}^{n-1}} \int_{K^{\varepsilon}_{R,T}} |u-v|\psi(x)(\zeta_{\varepsilon}(t-\delta)-\zeta_{\varepsilon}(t-T))dydxdt + \int_{\mathbb{R}^{n-1}} \int_{K^{\varepsilon}_{R,T}} |u-v|(\Delta_x\psi(x))(Y_{\varepsilon}(t-\delta)-Y_{\varepsilon}(t-T))dxdydt$$

with

$$K^{\varepsilon}_{R,T} = \{(y,t) \in \mathbb{R} \times \mathbb{R}^+; |y| \leq R + \alpha(T-t), \delta - \varepsilon \leq t \leq T + \varepsilon\}.$$

This can be equivalently rewritten as follows :

$$0 \leq \int_0^{\infty}(\zeta_{\varepsilon}(t-\delta)-\zeta_{\varepsilon}(t-T)) \int_{\mathbb{R}^{n-1}} \int_{S_t} \psi(x)|u-v|dydxdt + \int_0^{\infty}(Y_{\varepsilon}(t-\delta)-Y_{\varepsilon}(t-T)) \int_{\mathbb{R}^{n-1}} \int_{S_t} \Delta_x\psi(x)|u-v|dydxdt, \tag{27}$$

where S_t is as follows : $S_t = \{y \in \mathbb{R}; |y| \leq R + \alpha(t-T)\}$. Now,

$$\lim_{\varepsilon\to 0} \int_0^{\infty}(Y_{\varepsilon}(t-\delta)-Y_{\varepsilon}(t-T)) \int_{\mathbb{R}^{n-1}} \int_{S_t} \Delta_x\psi(x)|u-v|dydxdt = \int_{\delta}^{T} \int_{\mathbb{R}^{n-1}} \int_{S_t} \Delta_x\psi(x)|u-v|dydxdt,$$

and then clearly

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_0^{\infty}(Y_{\varepsilon}(t-\delta)-Y_{\varepsilon}(t-T)) \int_{\mathbb{R}^{n-1}} \int_{S_t} \Delta_x\psi(x)|u-v|dydxdt = \int_0^{T} \int_{\mathbb{R}^{n-1}} \int_{S_t} \Delta_x\psi(x)|u-v|dydxdt.$$

On the other hand, if we introduce the function$w : \mathbb{R}^{n-1} \times \mathbb{R}^+ \mapsto \mathbb{R}$, defined by

$$w(x,t) = \int_{S_t} |u(x,y,t)-v(x,y,t)|dy,$$

and set $w_{\varepsilon} = w * \zeta_{\varepsilon}$ (convolution in the time variable), defined for $t \geq \varepsilon$, we have

$$\int_0^{\infty} (\zeta_{\varepsilon}(t-\delta)-\zeta_{\varepsilon}(t-T)) \int_{\mathbb{R}^{n-1}} \psi(x) \int_{S_t} |u-v|dydxdt = \int_{\mathbb{R}^{n-1}}(w_{\varepsilon}(x,\delta)-w_{\varepsilon}(x,T))\psi(x)dx.$$

Letting first, $\varepsilon \to 0$ and then $\delta \to 0$ we obtain that

$$\int_{\mathbb{R}^{n-1}} \psi(x) \int_{|y|\leq R+\alpha t} |u_0(x,y) - v_0(x,y)|dydx$$

$$- \int_{\mathbb{R}^{n-1}} \psi(x) \int_{|y|\leq R} |u(x,y,T) - v(x,y,T)|dydx$$

$$= \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_0^\infty (\zeta_\varepsilon(t-\delta) - \zeta_\varepsilon(t-T)) \int_{\mathbb{R}^{n-1}} \int_{S_t} \psi(x)|u-v|dydxdt. \tag{28}$$

From (27)-(28) we deduce that

$$\int_{\mathbb{R}^{n-1}} \psi(x) \int_{|y|\leq R+\alpha t} |u_0(x,y) - v_0(x,y)|dydx$$

$$- \int_{\mathbb{R}^{n-1}} \psi(x) \int_{|y|\leq R} |u(x,y,T) - v(x,y,T)|dydx \tag{29}$$

$$+ \int_0^T \int_{\mathbb{R}^{n-1}} \Delta_x \psi(x) \int_{S_t} |u-v|dydxdt \geq 0.$$

Let us choose now $\varphi \in \mathcal{D}(\mathbb{R})$ such that

$$\varphi = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}$$

and $0 \leq \varphi \leq 1$ when $1 \leq |x| \leq 2$. We take in (29), $\psi(x) = \varphi(x/\beta)$ and let $\beta \to \infty$. We obtain in this way

$$\int_{\mathbb{R}^{n-1}} \int_{|y|\leq R} |u(x,y,T) - v(x,y,T)|dydx$$

$$\leq \int_{\mathbb{R}^{n-1}} \int_{|y|\leq R+\alpha t} |u_0(x,y) - v_0(x,y)|dydx,$$

with completes the proof of Lemma (8) . ■

We complete this section the proof of the main results by dealing with bounded and integrable initial data, possibly non smooth. Given $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we construct a sequence of smooth data $u_{0,n} \subset \mathcal{D}(\mathbb{R}^n)$, such that $u_{0,n}$ is uniformly bounded in $\mathbb{R}^n$ and

$$u_{0,n} \to u_0 \quad \text{in} \quad L^1(\mathbb{R}^n) \quad \text{as} \quad n \to \infty. \tag{30}$$

Let us denote by u_n the entropy solution of (3) with data $u_{0,n}$. In view of (30) and the concentration (6) we conclude that the sequence $\{u_n\}$ converges in $C([0,\infty);L^1(I\!R^n)$ to a function $u = u(x,y,t)$ as $n \to \infty$. In view of (10) the u_n are also uniformly bounded on $L^\infty((I\!R^n) \times (0,\infty))$. By interpolation we deduce that

$$u_n \to u \quad \text{in} \quad C([0,\infty);L^p(I\!R^n))$$

for every $p \in [1,\infty)$. This allows us to pass to the limit in equation (3) and entropy condition (2), which are of course satisfied by u_n. Thus the limit is an entropy solution of (3). It is also verifies the initial data u_0. For the uniqueness, since the solutions are supposed to be bounded and belong to $C([0,\infty);L^1(I\!R^n)$, proceeding as above, we may prove that for every pair of entropy functions u and v we have

$$\int_{I\!R^n} |u(x,y,T) - v(x,y,T)|dydx \leq \int_{I\!R^{n-1}} |u(x,y,t) - v(x,y,t)|dydx,$$

whenever $0 < t < T$. Taking a limits as $t \to 0$ on the right hand side, we deduce the L^1–contraction property (6), hence uniqueness. ■

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