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Je dédie ce modeste travail à
mon défunt père
ma chère mère
mes enfants Yacine et Malak.

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Abstract

The objective of this thesis is to present both some results on the existence, stability and controllability of the solutions of some classes of fractional diferential equations with delay and impulses in finite and infinite dimensional Banch spaces.

We shall make us the notion of the measure of noncompactness, the semigroup theory and the fixed point approach ;in particular we use the banach contraction priciple, Schauder fixed point theorem, Darbo fixed point theorem, Burton Kirk fixed point theorem.

Key words: Banach space, delay, fixed point, fractional differential equations,impulses, measure of noncompactness, semigroup, Ulam stability.

AMS Subject Classification : 26A33, 34A08, 34A37, 34G20, 34G25, 34K20, 34K30.

Résumé

Cette thèse vise à présenter des résultats sur l'existence, la stabilité et la contrôlabilité des solutions de certaines classes d'équations différentielles fractionnaires avec retard et impulsions dans des espaces de Banach de dimensions finies et infinies.

Nous utiliserons la théorie des semi-groupes, la mesure de la noncompacité et l'approche du point fixe, en particulier le principe de contraction de Banach, le théorème du point fixe de Schauder, le théorème du point fixe de Darbo et le théorème du point fixe de Burton-Kirk.

Mots clefs: Espace de Banach, Equations différentielles fractionnaires, Impulsion, mesure de noncompacité, , Point fixe, Retard, Semi-groupe, Solution, stabilité au sens de Ulam.
Classification AMS: 26A33, 34A08, 34A37, 34G20, 34G25, 34K20, 34K30.

Contents

Preface **1**

Introduction **1**

1 Preliminaries **8**
- 1.1 Introduction . . . 8
- 1.2 Definitions and notations . . . 8
- 1.3 Fixed Point Theorems . . . 10

2 Neutral Implicit Fractional q-Difference Equations with Delay **12**
- 2.1 Introduction . . . 12
- 2.2 Preliminaries . . . 13
- 2.3 Existence and Stability Results problem with Finite Delay . . 18
- 2.4 Existence and Stability Results problem with case of Infinite Delay . . . 23
- 2.5 Existence and Stability Results problem with State Dependent Delay . . . 26
  - 2.5.1 The Finite Delay Case . . . 26
  - 2.5.2 The Infinite Delay Case . . . 27
- 2.6 Some Examples . . . 28

3 Implicit Deformable Fractional Differential Boundary Value Problems **31**
- 3.1 Introduction . . . 31
- 3.2 Preliminaries . . . 32
- 3.3 Existence and Uniqueness of Solutions . . . 34
- 3.4 Successive Approximations and Uniqueness Results . . . 38
- 3.5 Some Examples . . . 43

4 Implicit Improved Conformable Fractional Differential Equations 46
4.1 Introduction . . . 46
4.2 Preliminaries . . . 47
4.3 Main Results . . . 49
4.3.1 Existence and uniqueness of solutions . . . 49
4.3.2 Ulam-Hyers-Rassias stability . . . 54
4.3.3 Successive approximations and uniqueness results . 56
4.4 Examples . . . 61

5 Abstract Fractional Differential Equations with Delay and non Instantaneous Impulses 64
5.1 Introduction . . . 64
5.2 Preliminaries . . . 66
5.3 Uniqueness and Ulam stabilities results with finite delay . . 71
5.4 The phase space $\mathbb{k}$ . . . 76
5.5 Uniqueness and Ulam stabilities results with infinite delay . 78
5.6 Uniqueness and Ulam stabilities results with state-dependent delay . . . 81
5.7 Examples . . . 82

6 Controllability Results for Second-Order Integro-differential Equations with State-Dependent Delay 86
6.1 Introduction . . . 86
6.2 Preliminaries . . . 87
6.3 Existence of mild solutions . . . 90
6.4 Controllability results . . . 96
6.4.1 Complete controllability . . . 96
6.4.2 Approximate Controllability . . . 99
6.5 An Example . . . 102

7 Conclusion and Perspective 106

Bibliography 107

Introduction

Fractional calculus and fractional differential equations have been found in several areas of engineering, mathematics, physics, and other applied sciences [4], [5], [6], [25], [26], [122], [133]. Recently, in [1], [7] [93]; the authors studied the existence of solutions of Caputo's fractional differential equations and inclusions, a considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivative; [5], [7], [40] [75].

A Fractional calculus has been a captivating field of study within functional space theory for a significant period, attracting scholars owing to its diverse range of applications across various disciplines. This domain of research focuses on employing non-integer derivatives of fractional order to model and comprehend complex natural phenomena. Some of the noteworthy areas where fractional calculus has found use include electrochemistry and viscoelasticity.

The application of fractional derivatives has demonstrated efficacy in extending the fundamental laws of nature, facilitating a more comprehensive and nuanced comprehension of these processes. Moreover, fractional calculus has been vital in capturing the memory and hereditary effects that emerge in several systems, which traditional integer-order derivatives fail to explain.

The use of fractional derivatives has proven to be an effective way of generalizing the fundamental laws of nature, providing a more comprehensive and nuanced understanding of these processes. The fractional calculus has been instrumental in capturing the memory and hereditary effects that arise in many systems, which traditional integer-order derivatives cannot account for. For those looking to develop into the subject, we recommend reading [2], [4], [44] [52], [66], [73], [75], [91] and its referenced works. Recently in [92], Khalil et al. gave a novel definition of fractional derivative which is a natural extension to the standard first derivative.

We note allso that the Fractional calculus is a highly effective tool in applied mathematics, offering a means to investigate a wide range of problems in various scientific and engineering fields. Remarkable breakthroughs have been made in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology, and bioengineering. In recent years, there has been significant progress in both ordinary and partial fractional differential equations. For further exploration, one can refer to the monographs by Abbas et al. [3], [1], Benchohra et al. [42], Kilbas et al. [22], [133], the papers of [4, 5, 34], and the references therein.

Concerning the stability problem Ulam initially introduced the topic of stability in functional equations during a talk at Wisconsin University in 1940. The problem he presented was as follows: Under what conditions does the existence of an additive mapping near an approximately additive mapping hold? Hyers provided the first solution to Ulam's question in 1941, specifically for the case of Banach spaces [18].

Considerable attention has been devoted to investigating Ulam-Hyers and Ulam-Hyers-Rassias stability in various forms of functional equations, as discussed in the monographs by [19, 20]. Ulam-Hyers stability in operatorial equations and inclusions has been examined by Bota-Boriceanu and Petrusel [13], Petru et al. [28], and Rus [31, 33]. Castro and Ramos [14] explored Hyers-Ulam-Rassias stability for a specific class of Volterra integral equations.

Wang et al. [39, 40] proposed Ulam stability for fractional differential equations involving the Caputo derivative. For further historical insights and recent developments in these stabilities, consult the monographs by [19–21] and the papers by [21, 25, 31, 39, 40].

The study of differential equations with impulses was initially explored by Milman and Myshkis [26]. In several fields such as physics, chemical technology, population dynamics, and natural sciences, numerous phenomena and evolutionary processes can undergo sudden changes or short-term disturbances [24] and references therein. These brief disturbances can be interpreted as impulses. Impulsive problems also arise in various practical applications including communications, chemical technology, mechanics (involving jump discontinuities in velocity), electrical engineering, medicine, and biology. These perturbations can be perceived as impulses. For instance, in the periodic treatment of certain diseases, impulses correspond to the administration of drug treatment. In environmental sciences, impulses represent seasonal changes in water levels in artificial reservoirs. Mathematical models involving impulsive differential equations and inclusions are used to describe these situations. Several mathematical results, such as the existence of solutions and their asymptotic behavior, have been obtained thus far [10, 23, 24, 36] and references therein. In [16,29,38] the authors studied some new classes of differential equations with not instantaneous impulses. For more recent results we refer, for instance to the book [9] and the papers [6–8,12].

Controllability theory is critical for understanding the behavior and dynamics of abstract control systems. The basic goal of controllability is to find a suitable control function that will allow us to direct the system's state towards a desired final state. The capacity to steer the system to an exact end state is known as exact controllability, whereas approximation controllability allows us to direct the system to an arbitrarily small neighborhood of the final state. As a result, approximation controllability becomes more desired and applicable to real-world systems, which frequently display some amount of uncertainty or imprecision. Many researchers have studied the approximate or complete controllability of control systems throughout the years, and various papers have been published in this field (see references [7, 14, 39–41, 43, 45, 47, 47, 105–108] and the references therein).

Let us now briefly describe the organization of this thesis.

We first give some general preliminaries and fixed point theorems.
In chapter 2 we first give preliminaries of the notion of $q-$calculus (quantum calculus), the deformable fractional derivatives then we prove some existence and Ulam stability results for the Cauchy problem of implicit neutral fractional $q-$difference equation with finite delay of the form.

$$\begin{cases} u(t) = \varphi(t);\ t \in [-r, 0], \\ {}^cD_q^\alpha(u(t) - h(t, u_t)) = f(t, u(t), {}^cD_q^\alpha(u(t) - h(t, u_t)));\ t \in I := [0, T], \end{cases}$$

where $q \in (0, 1),\ \alpha \in (0, 1],\ T, r > 0,\ \varphi \in \mathcal{C},\ h : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α, and $\mathcal{C} := C([-r, 0], \mathbb{R})$ is the space of continuous functions on $[-r,0]$.

For any $t\in I$, we define u_t by

$$u_t(s)=u(t+s),\ \text{for}\ s\in[-r,0].$$

In Section 2.3, we consider the Cauchy problem of implicit neutral fractional q-difference equation with infinite delay of the form.

$$\begin{cases} u(t)=\varphi(t);\ t\in(-\infty,0],\\ {}^cD_q^\alpha(u(t)-h(t,u_t))=f(t,u(t),{}^cD_q^\alpha(u(t)-h(t,u_t)));\ t\in I,\end{cases}$$

where $\varphi:(-\infty,0]\to\mathbb{R},\ h:I\times\mathcal{B}\to\mathbb{R},\ f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, and $\mathcal{B}$ is a phase space.

For any $t\in I$, we define $u_t\in\mathcal{B}$ by

$$u_t(s)=u(t+s);\ \text{for}\ s\in(-\infty,0].$$

In Section 2.4, we study the Cauchy problem of implicit neutral fractional q-difference equation with state-dependent delay of the form.

$$\begin{cases} u(t)=\varphi(t);\ t\in[-r,0],\\ {}^cD_q^\alpha(u(t)-h(t,u_{\rho(t,u_t)}))=f(t,u(t),{}^cD_q^\alpha(u(t)-h(t,u_{\rho(t,u_t)})));\ t\in I,\end{cases}$$

where $\varphi\in\mathcal{C},\ \rho:I\times\mathcal{C}\to\mathbb{R},\ h:I\times\mathcal{C}\to\mathbb{R},\ f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

In Section 2.5, we treat the last Cauchy problem of implicit neutral fractional q-difference equation with state dependent delay of the form.

$$\begin{cases} u(t)=\varphi(t);\ t\in(-\infty,0],\\ {}^cD_q^\alpha(u(t)-h(t,u_{\rho(t,u_t)}))=f(t,u(t),{}^cD_q^\alpha(u(t)-h(t,u_{\rho(t,u_t)})));\ t\in I,\end{cases}$$

where $\varphi:(-\infty,0]\to\mathbb{R},\ \rho:I\times\mathcal{B}\to\mathbb{R},h:I\times\mathcal{B}\to\mathbb{R},\ f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

In Chapter 3 we present an existence results of the problem.

$$\begin{cases} \left(\mathfrak{D}_0^\gamma\xi\right)(\zeta)=\aleph\left(\zeta,\xi(\zeta),\mathfrak{D}_0^\gamma\xi(\zeta)\right),\ \ \zeta\in\nabla:=[0,\varpi],\\ \imath\xi(0)+\jmath\xi(\varpi)=\varrho,\end{cases}$$

where $\mathfrak{D}_0^{\gamma}\xi(\zeta)$ is the deformable fractional derivative starting from the initial time 0 of the function $\aleph$ of order $\gamma \in (0,1)$, $\aleph : \nabla \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function $0 < \varpi < +\infty$ and $\imath, \jmath, \varrho$ are real constants where $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$.

In Chapter 4, we present two results on existence and uniqueness of the problem.

$$\begin{cases} {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = f\left(t, y(t), {}_0^C\tilde{\mathcal{T}}_\vartheta y(t)\right), \ t \in [0,T], \\ y(0) = 0, \end{cases}$$

where $0 < \vartheta < 1$, ${}_0^C\tilde{\mathcal{T}}_\vartheta$ is the improved Caputo-type conformable fractional derivative of order ϑ defined in [69], $I := [0,T]$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function such that $f(t,0,0) \neq 0$ for all $t \in I$.

In chapter 5, we investigate the uniqueness and Ulam-Hyers-Rassias stability of the following abstract impulsive fractional differential equations with finite delay of the form.

$$\begin{cases} {}^cD^\zeta_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_\vartheta); \text{ if } \vartheta \in \Im_\jmath, \ \jmath = 0, \dots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta)); \text{ if } \vartheta \in \widehat{\Im}_\jmath, \ \jmath = 1, \dots, \omega, \\ \chi(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

where $\Im_0 := [0, \vartheta_1], \ \widehat{\Im}_\jmath := (\vartheta_\jmath, \delta_\jmath], \ \Im_\jmath := (\delta_\jmath, \vartheta_{\jmath+1}]; \ \jmath = 1, \dots, \omega, \ {}^cD^\zeta_{\delta_\jmath}$ is the fractional Caputo derivative of order $\zeta \in (0,1], \ 0 = \delta_0 < \vartheta_1 \leq \delta_1 \leq \vartheta_2 < \cdots < \delta_{\omega-1} \leq \vartheta_\omega \leq \delta_\omega \leq \vartheta_{\omega+1} = \kappa_1, \ \kappa_2, \kappa_1 > 0, \ \aleph : \Im_\jmath \times \mathcal{C} \to \Xi; \ \jmath = 0, \dots, \omega, \ \widehat{\aleph}_\jmath : \widehat{\Im}_\jmath \times \Xi \to \Xi; \ \jmath = 1, \dots, \omega, \ \wp : [-\kappa_2, 0] \to \Xi$ are continuous functions, Ξ is a Banach space, Θ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{\mathfrak{H}(\vartheta); \ \vartheta > 0\}$ in Ξ and $\mathcal{C}$ is the Banach space defined by

$$\begin{aligned} \mathcal{C} = C_{\kappa_2} = \{\chi : [-\kappa_2, 0] \to \Xi : \ &\text{continuous and there exist } \ \varepsilon_\jmath \in (-\kappa_2, 0); \\ \jmath = 1, \dots, \omega, \ \text{such that } \ &\chi(\varepsilon_\jmath^-) \ \text{ and } \chi(\varepsilon_\jmath^+) \text{ exist with } \chi(\varepsilon_\jmath^-) = \chi(\varepsilon_\jmath)\big\}, \end{aligned}$$

with the norm

$$\|\chi\|_{\mathcal{C}} = \sup_{\vartheta \in [-\kappa_2, 0]} \|\chi(\vartheta)\|_\Xi.$$

We denote by χ_ϑ the element of $\mathcal{C}$ defined by

$$\chi_\vartheta(\varepsilon) = \chi(\vartheta + \varepsilon);\ \varepsilon \in [-\kappa_2, 0],$$

here $\chi_\vartheta(\cdot)$ represents the history of the state from time $\vartheta - \kappa_2$ up to the present time ϑ.

In section 5.5, we consider the abstract impulsive fractional differential equations with infinite delay of the form.

$$\begin{cases} {}^cD^{\zeta}_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_\vartheta);\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases}$$

where Θ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.1), $\aleph : \Im_\jmath \times \Bbbk \to \Xi;\ \jmath = 0, \ldots, \omega,\ \wp : \mathbb{R}_- \to \Xi$ are given continuous functions, and $\Bbbk$ is called a phase space that will be specified in Section 5.4.

The third problem is the abstract impulsive fractional differential equations with state-dependent delay and it is in section 5.6.

$$\begin{cases} {}^cD^{\zeta}_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_{\rho(\vartheta,\chi_\vartheta)});\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

where $\Theta,\ \aleph,\ \wp$ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.1) and $\rho : \Im_\jmath \times \mathcal{C} \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is a given continuous function.

The fourth problem is in section 5.6, where we consider the abstract impulsive fractional differential equations with state-dependent delay of the form.

$$\begin{cases} {}^cD^{\zeta}_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_{\rho(\vartheta,\chi_\vartheta)});\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in \mathbb{R}_-, \end{cases}$$

where $\Theta, \aleph, \wp$ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.2) and $\rho : \Im_\jmath \times \Bbbk \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is a given continuous function.

Finally in chapter 6, we discuss the approximate controllability and complete controllability for second-order Integro-differential equations with state-dependent delay described in the form.

$$\begin{cases} \vartheta''(\varsigma) = A(\varsigma)\vartheta(\varsigma) + \mathcal{K}\left(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (\Psi\vartheta)(\varsigma)\right) + \int_0^\varsigma \Upsilon(\varsigma,s)\vartheta(s)ds + \mathcal{P}u(\varsigma), \text{ if } \varsigma \in J, \\ \vartheta'(0) = \zeta_0 \in E, \;\; \vartheta(\varsigma) = \Phi(\varsigma), \text{ if } \varsigma \in \mathbb{R}_-, \end{cases}$$

where $J = [0,T]$, $A(\varsigma) : D(A(\varsigma)) \subset E \to E$, $\Upsilon(\varsigma, s)$ are closed linear operators on E, with dense domain $D(A(\varsigma))$, which is independent of t, and $D(A(s)) \subset D(\Upsilon(\varsigma, s))$, the operator Ψ is defined by

$$(\Psi\vartheta)(\varsigma) = \int_0^T \Xi(\varsigma, s, \vartheta(s))ds, \;\; a > 0,$$

the nonlinear terms $\Xi : J \times J \times E \to E$, $\mathcal{K} : J \times \mathcal{B} \times E \to E$, $\Phi : \mathbb{R}_- \to E$, $\rho : J \times \mathcal{B} \to (-\infty, \infty)$, are a given functions, the control function u is give function in $L^2(J, U)$ Banach space of admissible control with U as a Banach space. $\mathcal{P}$ is a bounded linear operator from U into E, and $(E, \|\cdot\|)$ is a Banach space.

We illustrate our main results with examples.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we give some general definitions that are useful in our thesis, we give also some fixed point theorems.

1.2 Definitions and notations

Let $(C(I), \|\cdot\|_\infty)$ be the Banach space of continuous functions $v : I \to \mathbb{R}$ with norm

$$\|v\|_\infty := \sup_{t\in I} |v(t)|,$$

and let $L^1(I)$ be the space of measurable functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_I |v(t)|dt.$$

Definition 1.2.1. *[109] A function $f : \mathbb{R} \to E$ is called strongly measurable if there exists a sequence of simple functions $(f_n)_n$ such that*

$$\lim_{n\to\infty} |f_n(t) - f(t)| = 0.$$

Definition 1.2.2. *[109] A function $f : \mathbb{R} \to E$ is said Bochner integrable on J if it is strongly measurable and such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(t) - f(t)|dt = 0$$

for any sequence of simple functions $(f_n)_n$.

Theorem 1.2.1. *[109] A strongly measurable function* $f : \mathbb{R} \to E$ *is Bochner integrable if and only if* $|f|$ *is measurable.*

The reader can find the Bochner integral in many books, e.g. [109,132].

Definition 1.2.3. *[30] A map* $f : I \times E \to E$ *is Carathéodory if*

(i) $t \longmapsto f(t, y)$ *is measurable for all* $y \in E$*, and*

(ii) $y \longmapsto f(t, y)$ *is continuous for almost each* $t \in I$.

If, in addition,

(iii) *for each* $r > 0$*, there exists* $g_r \in L^1(I, \mathbb{R}_+)$ *such that*

$$|f(t, y) \leq g_r(t) \ \text{ for all } \ |y| \leq r \ \text{ and almost each } t \in I,$$

then we say that the map is L^1*-Carathéodory.*

Definition 1.2.4. *[30] Let* X *be a Banach space and* Ω_X *the bounded subsets of* X. *The Kuratowski measure of noncompactness is the map* $\mu : \Omega_E \to [0, \infty]$ *defined by*

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \ \text{where } B \subset \Omega_X,$$

and

$$diam(B_i) = sup\{\|u - v\|_E : u, v \in B_i\},$$

where μ *satisfies the following properties.*

- $\mu(B) = 0$ *if and only if* $\overline{B}$ *is compact (regularity).*
- $\mu(B) = \mu(\overline{B})$*, invariance under closure.*
- $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$ *(semi-additivity).*
- $A \cup B \implies \mu(A) \leq \mu(B)$.
- $\mu(A + B) \leq \mu(A) + \mu(B)$.
- $\mu(cB) = |c|\mu(B), \ c \in \mathbb{R}$.

- $\mu(conB) = \mu(B)$.

  $\overline{B}$ denotes the closure and con denotes the convex hull of the bounded set B.

Lemma 1.2.1. *[62] If Y is a bounded subset of a Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu(Y) \leq 2\mu\left(\{y_k\}_{k=1}^{\infty}\right) + \epsilon.$$

Lemma 1.2.2. *[110] If $\{y_k\}_{k=0}^{\infty} \subset L^1$ is uniformly integrable, then the function $\varsigma \to \alpha(\{y_k(\varsigma)\}_{k=0}^{\infty})$ is measurable and*

$$\mu\left(\left\{\int_0^{\varsigma} y_k(s)ds\right\}_{k=0}^{\infty}\right) \leq 2\int_0^{\varsigma} \mu\left(\{y_k(s)\}_{k=0}^{\infty}\right)ds.$$

1.3 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

Theorem 1.3.1 (Schauder's fixed point theorem, [63])**.** *Let C be a nonempty closed convex bounded subset of a Banach space E. Then every continuous compact mapping $T : C \to C$ has a fixed point.*

Theorem 1.3.2 (Burton-Kirk's fixed point theorem [20])**.** *Let X Banach space, and $A, B : X \to X$ two operators. Suppose that B is a contraction and A a compact operator. Then either*

(i) *$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$ has a solution for $\lambda = 1$, or*

(ii) *the set $\{x \in X : x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax,\ \lambda \in (0,1)\}$ is unbounded.*

Theorem 1.3.3. *(Krasnoselskii fixed point theorem) [12, 15] Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*

(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 1.3.4 (Darbo's fixed point theorem, [55])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$, such that, for all subset M of Ω.*

$$\mu(TM) \leq k\mu(M),$$

where μ is the measure of non-compactenss of Kuratowski . Then, T has a fixed point in set Ω.

Let B be any bounded subset of a Banach space E, the Kuratowski measure of non-compactness of B, $\mu(B)$ is defined as the infimum of those $\varepsilon > 0$ such that B can be covered with a finite number of subsets of B having diameter less or equal to ε

Chapter 2

Neutral Implicit Fractional q-Difference Equations with Delay[(1)]

2.1 Introduction

In this chapter, we will treat the existence results and stability for four classes of implicit neutral fractional $q-$difference equations with delay.

$$\begin{cases} u(t) = \varphi(t);\ t \in [-r, 0], \\ {}^cD_q^\alpha(u(t) - h(t, u_t)) = f(t, u(t), {}^cD_q^\alpha(u(t) - h(t, u_t)));\ t \in I := [0, T], \end{cases} \tag{2.1}$$

where $q \in (0,1),\ \alpha \in (0,1],\ T, r > 0,\ \varphi \in \mathcal{C},\ h : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α, and $\mathcal{C} := C([-r, 0], \mathbb{R})$ is the space of continuous functions on $[-r, 0]$.

For any $t \in I$, we define u_t by

$$u_t(s) = u(t + s),\ \text{for } s \in [-r, 0].$$

[(1)] [35] A. Benchaib, A. Salim, S. Abbas and M. Benchohra, Qualitative Analysis of Neutral Implicit Fractional $q-$Difference Equations with Delay, *Differential Equation and Application*, **2024**, **16**, 19-38.

$$\begin{cases} u(t) = \varphi(t);\ t \in (-\infty, 0], \\ {}^{c}D_q^{\alpha}(u(t) - h(t, u_t)) = f(t, u(t), {}^{c}D_q^{\alpha}(u(t) - h(t, u_t)));\ t \in I, \end{cases} \tag{2.2}$$

where $\varphi : (-\infty, 0] \to \mathbb{R},\ h : I \times \mathcal{B} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\mathcal{B}$ is a phase space.

For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s) = u(t + s);\ \text{for } s \in (-\infty, 0].$$

$$\begin{cases} u(t) = \varphi(t);\ t \in [-r, 0], \\ {}^{c}D_q^{\alpha}(u(t) - h(t, u_{\rho(t,u_t)})) = f(t, u(t), {}^{c}D_q^{\alpha}(u(t) - h(t, u_{\rho(t,u_t)})));\ t \in I, \end{cases} \tag{2.3}$$

where $\varphi \in \mathcal{C},\ \rho : I \times \mathcal{C} \to \mathbb{R},\ h : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

$$\begin{cases} u(t) = \varphi(t);\ t \in (-\infty, 0], \\ {}^{c}D_q^{\alpha}(u(t) - h(t, u_{\rho(t,u_t)})) = f(t, u(t), {}^{c}D_q^{\alpha}(u(t) - h(t, u_{\rho(t,u_t)})));\ t \in I, \end{cases} \tag{2.4}$$

where $\varphi : (-\infty, 0] \to \mathbb{R},\ \rho : I \times \mathcal{B} \to \mathbb{R}, h : I \times \mathcal{B} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Some techniques are made of a fixed point theorem do to Krasnoselskii in Banach spaces, and the notion of the stability of Ulam kind.

2.2 Preliminaries

Let us recall some definitions and properties of fractional $q-$calculus. For $a \in \mathbb{R},\ 0 < q < 1$ we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

Definition 2.2.1. *[96] The q analogue of the power $(a - b)^n$ is defined by*

$$(a - b)^{(0)} = 1,\ (a - b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k);\ a, b \in \mathbb{R},\ n \in \mathbb{N}.$$

In general, we define

$$(a - b)^{(\alpha)} = a^{\alpha}\Pi_{k=0}^{\infty}\left(\frac{a - bq^k}{a - bq^{k+\alpha}}\right);\ a, b, \alpha \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^{\alpha}$.

Definition 2.2.2. *[96] The q-gamma function of* $\xi \in \mathbb{R} - \{0, -1, -2, \ldots\}$; *is defined by*

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}}.$$

Notice that $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 2.2.3. *[96] The* q*-derivative of order* $n \in \mathbb{N}$ *of a function* $u : I \to \mathbb{R}$ *is defined by* $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1-q)t};\ t \neq 0,\ \ (D_q u)(0) = \lim_{t \to 0} (D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t);\ t \in I,\ n \in \{1, 2, \ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.2.4. *[96] The* q*-integral of a function* $u : I_t \to \mathbb{R}$ *is defined by*

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n u(tq^n),$$

provided that the series converges.

Notice that $(D_q I_q u)(t) = u(t)$, and if u is continuous at 0, then

$$u(t) = u(0) + (I_q D_q u)(t).$$

Definition 2.2.5. *[9] The Riemann-Liouville fractional* q*-integral of order* $\alpha \in \mathbb{R}_+ := [0, \infty)$ *of a function* $u : I \to \mathbb{R}$ *is defined by* $(I_q^0 u)(t) = u(t)$, *and*

$$(I_q^{\alpha} u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

Lemma 2.2.1. *[118] For* $\alpha \in \mathbb{R}_+$ *and* $\lambda \in (-1, \infty)$, *we have*

$$(I_q^{\alpha}(t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)};\ 0 < a < t < T.$$

In particular, we have

$$(I_q^\alpha 1)(t) = \frac{t^{(\alpha)}}{\Gamma_q(1+\alpha)} = \frac{t^\alpha}{\Gamma_q(1+\alpha)}.$$

Definition 2.2.6. *[119] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$, and*

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]+1} I_q^{[\alpha]+1-\alpha} u)(t);\ t \in I,$$

where $[\alpha]$ denotes the integer part of α.

Definition 2.2.7. *[119] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and*

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t);\ t \in I.$$

Lemma 2.2.2. *[119] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above Lemma, and in order to define the solution for the problem 2.1. We conclude the following Lemma

Lemma 2.2.3. *Let $h : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $h(\cdot, w) \in C(I)$ and $f(\cdot, u, v) \in C(I)$, for each $w \in \mathcal{C}$, and $u, v \in \mathbb{R}$. Then the problem (2.1) is equivalent to the problem of obtaining the solutions of the integral equation*

$$\begin{cases} u(t) = \varphi(t);\ t \in [-r, 0], \\ g(t) = f(t, h(t, u_t) + \varphi(0) - h(0, u_0) + (I_q^\alpha g)(t), g(t));\ t \in I, \end{cases}$$

and if $g(\cdot) \in C(I)$, is the solution of this equation, then

$$\begin{cases} u(t) = \varphi(t);\ t \in [-r, 0], \\ u(t) = h(t, u_t) + \varphi(0) - h(0, u_0) + (I_q^\alpha g)(t);\ t \in I. \end{cases}$$

From lemma 2.2.3, we conclude the following corollary:

Corollary 2.2.1. *The solutions of the problem (2.1) are the fixed points of the operator $N : C([-r,T]) \to C([-r,T])$ defined by*

$$\begin{cases} (Nu)(t) = \varphi(t);\ t \in [-r,0], \\ (Nu)(t) = h(t,u_t) + \varphi(0) - h(0,u_0) + (I_q^\alpha g)(t);\ t \in I, \end{cases} \tag{2.5}$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)),$$

or

$$g(t) = f(t, h(t,u_t) + \varphi(0) - h(0,u_0) + (I_q^\alpha g)(t), g(t)).$$

Let $\epsilon > 0$ and $\Phi : I \to \mathbb{R}$ be a continuous and positive function. We put the following inequalities

$$|(Nu)(t) - u(t)| \le \epsilon;\ t \in I. \tag{2.6}$$

$$|(Nu)(t) - u(t)| \le \Phi(t);\ t \in I. \tag{2.7}$$

$$|(Nu)(t) - u(t)| \le \epsilon\Phi(t);\ t \in I. \tag{2.8}$$

Definition 2.2.8. [6, 121] *The problem (2.1) is Ulam-Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (2.6) there exists a solution $v \in C(I)$ of the problem (2.1) with*

$$|u(t) - v(t)| \le \epsilon c_N;\ t \in I.$$

Definition 2.2.9. [6,121] *The problem (2.1) is generalized Ulam-Hyers stable if there exists $c_N : C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (2.6) there exists a solution $v \in C(I)$ of (2.1) with*

$$|u(t) - v(t)| \le c_N(\epsilon);\ t \in I.$$

Definition 2.2.10. [6,121] *The problem (2.1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (2.8) there exists a solution $v \in C(I)$ of (2.1) with*

$$|u(t) - v(t)| \le \epsilon c_{N,\Phi}\Phi(t);\ t \in I.$$

Definition 2.2.11. [6, 121] *The problem (2.1) is generalized Ulam-Hyers-Rassias stable with respect to* Φ *if there exists a real number* $c_{N,\Phi} > 0$ *such that for each solution* $u \in C(I)$ *of the inequality (2.7) there exists a solution* $v \in C(I)$ *of (2.1) with*

$$|u(t) - v(t)| \leq c_{N,\Phi}\Phi(t); \ t \in I.$$

Remark 2.2.1. *(i) Definition 2.2.8* $\Rightarrow$ *Definition 2.2.9,*

(ii) Definition 2.2.10 $\Rightarrow$ *Definition 2.2.11,*

(iii) Definition 2.2.10 for $\Phi(\cdot) = 1 \ \Rightarrow$ *Definition 2.2.8.*

One can have similar remarks for the inequalities (2.6) and (2.8).

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a phase space. It is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms introduced by Hale and Kato [73]:

(A_1) If $z : (-\infty, T] \to \mathbb{R}$ continuous on I and $z_t \in \mathcal{B}$, for all $t \in (-\infty, 0]$, then there are constants $H, K, M > 0$ such that for any $t \in I$, the following conditions hold:

(i) z_t is in $\mathcal{B}$;

(ii) $\|z(t)\| \leq H\|z_t\|_{\mathcal{B}}$,

(iii) $\|z_t\|_{\mathcal{B}} \leq K \sup_{s\in[0,t]} \|z(s)\| + M \sup_{s\in(-\infty,0]} \|z_s\|_{\mathcal{B}}$,

(A_2) For the function $z(\cdot)$ in (A_1), z_t is a $\mathcal{B}$-valued continuous function on I.

(A_3) The space $\mathcal{B}$ is complete.

Example 2.2.1. *Let* $\mathcal{B}$ *be the set of all functions* $\phi : (-\infty, 0] \to \mathbb{R}$ *which are continuous on* $[-r, 0]$, $r \geq 0$, *with the semi-norm*

$$\|\phi\|_{\mathcal{B}} = \sup_{t\in[-r,0]} \|\phi(t)\|.$$

Then we have $H = K = M = 1$. *The quotient space* $\widehat{\mathcal{B}} = \mathcal{B}/\|\cdot\|_{\mathcal{B}}$ *is isometric to the space* $C([-r, 0], \mathbb{R})$ *of all continuous functions from* $[-r, 0]$ *into* $\mathbb{R}$ *with the supremum norm, this means that functional differential equations with finite delay are included in our axiomatic model.*

2.3 Existence and Stability Results with Finite Delay

We prove in this section some existence and Ulam stability results for the Cauchy problem of implicit fractional $q-$difference equation with finite delay

Let $C := C([-r,T],\mathbb{R})$ denotes the Banach space of continuous functions from $[-r,T]$ into $\mathbb{R}$ with the norm

$$\|u\|_C := \sup_{t\in[-r,T]} |u(t)|.$$

We start by defining solution of the problem (2.1).

Definition 2.3.1. *A solution of the problem (2.1) is a function $u \in C$ that satisfies the initial condition $u(t) = \varphi(t)$ on $[-r,0]$, and the equation ${}^cD_q^\alpha(u(t)-h(t,u(t)) = f(t,u_t,({}^cD_q^\alpha u)(t))$ on I.*

We will need to introduce the following hypotheses which are assumed there after:

(H_1) The function h satisfies the Lipschitz condition:

$$|h(t,u) - h(t,v)| \le \phi\|u-v\|_{\mathcal{C}},$$

for $t \in I$ and $u,v \in \mathcal{C}$, where $0 < \phi < 1$.

(H_2) There exist continuous functions $p,d,r : I \to \mathbb{R}_+$ with $r(t) < 1$ such that

$$|f(t,u,v)| \le p(t) + d(t)|u| + r(t)|v|, \text{ for each } t \in I \text{ and } u,v \in \mathbb{R}.$$

Theorem 2.3.1. *Suppose that the hypotheses (H_1), (H_2), and the condition*

$$2\phi + \frac{T^\alpha d^*}{(1-r^*)\Gamma_q(1+\alpha)} < 1,$$

hold. Then the problem (2.1) has at least one solution defined on $[-r,T]$.

Proof. Consider the operators $A,B : C([-r,T]) \to C([-r,T])$ defined by

$$\begin{cases} (Au)(t) = 0;\ t \in [-r,0], \\ (Au)(t) = \varphi(0) - h(0,u_0) + (I_q^\alpha g)(t);\ t \in I, \end{cases} \tag{2.9}$$

where $g \in C(I)$ with $g(t) = f(t, u(t), g(t))$, and

$$\begin{cases} (Bu)(t) = \varphi(t); \ t \in [-r, 0], \\ (Bu)(t) = h(t, u_t); \ t \in I. \end{cases} \tag{2.10}$$

Set

$$R \geq \max \left\{ \varphi^*, \frac{2h^* + \varphi^* + \dfrac{T^\alpha (p^* + d^* R)}{(1 - r^*)\Gamma_q(1 + \alpha)}}{1 - 2\phi - \dfrac{T^\alpha d^*}{(1 - r^*)\Gamma_q(1 + \alpha)}} \right\},$$

and let $B_R = \{u \in C([-r, T]) : \|u\|_C \leq R\}$ be the closed and convex ball in C.
We shall prove in three steps that A and B satisfy the conditions of the Theorem 1.3.3.

Step 1. $Au + Bv \in B_R$ whenever $u, v \in B_R$.
Let $u, v \in B_R$. Then, for each $t \in [-r, 0]$ we have

$$|Au(t) + Bv(t)| = \varphi(t) \leq \varphi^* \leq R,$$

and for each $t \in I$, we have

$$|(Au)(t) + (Bv)(t)| \leq |h(t, v_t)| + |\varphi(0)| + |h(0, u_0)| + \int_0^t \frac{(t - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)} |g(s)| d_q s,$$

where $g \in C(I)$ with

$$g(t) = f(t, u(t), g(t)).$$

By using (H_2), for each $t \in I$ we have

$$\begin{aligned} |g(t)| &\leq p(t) + d(t)|u(t)| + r(t)|g(t)| \\ &\leq p^* + d^* R + r^* |g(t)|. \end{aligned}$$

This gives

$$|g(t)| \leq \frac{p^* + d^* R}{1 - r^*}.$$

Thus

$$\begin{aligned}
\|A(u)+B(v)\|_\infty &\leq |\varphi(0)|+|h(0,0)|+|h(0,u_0)-h(0,0)|+\frac{T^\alpha(p^*+d^*R)}{(1-r^*)\Gamma_q(1+\alpha)}\\
&+ |h(t,v_t)-h(t,0)|+|h(t,0)|\\
&\leq \varphi^*+h^*+\phi\|u_0\|_{\mathcal{C}}+\frac{T^\alpha(p^*+d^*R)}{(1-r^*)\Gamma_q(1+\alpha)}+\phi\|v_t\|_{\mathcal{C}}+h^*\\
&\leq \varphi^*+h^*+\phi R+\frac{T^\alpha(p^*+d^*R)}{(1-r^*)\Gamma_q(1+\alpha)}+\phi R+h^*\\
&= 2h^*+\varphi^*+\frac{T^\alpha(p^*+d^*R)}{(1-r^*)\Gamma_q(1+\alpha)}+R\left(2\phi+\frac{T^\alpha d^*}{(1-r^*)\Gamma_q(1+\alpha)}\right)\\
&\leq R.
\end{aligned}$$

Hence, we get

$$\|A(u)+B(v)\|_C \leq R.$$

This proves that $Au+Bv\in B_R$ whenever $u,v\in B_R$.

Step 2. $A: B_R \to B_R$ is compact and continuous.

Claim 1. A is continuous.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\to u$ in B_R. Then we have

$$|(Au_n)(t)-(Au)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|(g_n(s)-g(s))|d_qs;\ t\in I,$$

where $g_n, g\in C(I)$ such that

$$g_n(t)=f(t,u_n(t),g_n(t)),$$

and

$$g(t)=f(t,u(t),g(t)).$$

Since $u_n\to u$ as $n\to\infty$ and f is continuous, we get

$$g_n(t)\to g(t) \text{ as } n\to\infty,\ \text{ for each } t\in I.$$

Hence

$$\|A(u_n)-A(u)\|_\infty \leq \frac{p^*+d^*R}{1-r^*}\|g_n-g\|_\infty \to 0 \ \ as\ \ n\to\infty.$$

Claim 2. $A(B_R)$ is bounded and equicontinuous.
We have $A(B_R)\subset B_R$ and B_R is bounded, thus $A(B_R)$ is bounded. Next,let

$t_1, t_2 \in I$, such that $t_1 < t_2$ and let $u \in B_R$. Then, there exists $g \in C(I)$ with $g(t) = f(t, u(t), g(t))$, such that

$$|(Au)(t_1) - (Au)(t_2)| \leq \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |g(s)| d_q s + \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |g(s)| d_q s.$$

Hence

$$|(Au)(t_1) - (Au)(t_2)| \leq \frac{p^* + d^* R}{1 - r^*} \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s + \frac{p^* + d^* R}{1 - r^*} \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \to 0 \text{ as } t_1 \to t_2.$$

As a consequence of the above claims, the Arzelá-Ascoli theorem implies that $A : B_R \to B_R$ is continuous and compact.

Step 3. B is a contraction mapping.
Let $u, v \in B_R$. From (H_1), for each $t \in I$, we have

$$\begin{aligned} |(Bu)(t) - (Bv)(t)| &\leq |h(t, u_t) - h(t, v_t)| \\ &\leq \phi \|u_t - v_t\|_{\mathcal{C}}. \end{aligned}$$

Thus

$$\|B(u) - B(v)\|_\infty \leq \phi \|u - v\|_\infty.$$

Hence

$$\|B(u) - B(v)\|_C \leq \phi \|u - v\|_C,$$

which implies that the operator B is a contraction.

As a consequence of the three above steps, from Theorem 1.3.3, the operator equation $(A + B)(u) = u$ has at least a solution.

Now, we prove a result about the generalized Ulam-Hyers-Rassias stability of the problem (2.1)

The following hypotheses will be used in the sequel.

(H_3) There exist functions $p_1, p_2, p_3, p_4 \in C(I, [0, \infty))$ with $p_3(t) < 1$ such that

$$(1 + |u|)|f(t, u, v)| \leq p_1(t)\Phi(t) + p_2(t)\Phi(t)|u| + p_3(t)|v|,$$

for each $t \in I$ and $u, v \in \mathbb{R}$, and

$$(1 + \|w - z\|_{\mathcal{C}})|h(t, w) - h(t, z)| \le p_4(t)\Phi(t)\|w - z\|_{\mathcal{C}},$$

for each $t \in I$ and $w, z \in \mathcal{C}$,

(H_4) There exists $\lambda_\Phi > 0$ such that for each $t \in I$, we have

$$(I_q^\alpha \Phi)(t) \le \lambda_\Phi \Phi(t).$$

Set $\Phi^* = \sup_{t \in I} \Phi(t)$ and

$$p_i^* = \sup_{t \in I} p_i(t), \; i \in \{1, 2, 3, 4\}.$$

□

Theorem 2.3.2. *Suppose that the hypotheses* (H_3), (H_4) *and the conditions* $p_4^*\Phi^* < 1$, $p_3^* + 2p_4^*\Phi^* + \dfrac{T^\alpha p_2^* \Phi^*}{\Gamma_q(1+\alpha)} - 2p_3^* p_4^* \Phi^* < 1$, *hold. Then the problem (2.1) is generalized Ulam-Hyers-Rassias stable.*

Proof. Let N be the operator defined in (2.5). It's clear that (H_3) implies (H_1) with $\phi = p_4^*\Phi^*$, and; (H_3) implies (H_2) with $p \equiv p_1\Phi$, $d \equiv p_2\Phi$ and $r \equiv p_3$.
Let u be a solution of the inequality (2.7), and let us assume that v is a solution of problem (2.1). Thus, we have $v(t) = \varphi(t);\ t \in [-r, 0]$, and

$$v(t) = h(t, v_t) + \varphi(0) - h(0, v_0) + (I_q^\alpha z)(t); \; t \in I,$$

where $z \in C(I)$ such that $z(t) = f(t, v(t), z(t))$.
From the inequality (2.7) for each $t \in I$, we have

$$|u(t) - h(t, u_t) - \varphi(0) + h(0, u_0) - (I_q^\alpha g)(t)| \le (I_q^\alpha \Phi)(t),$$

where $g \in C(I)$ such that $g(t) = f(t, u(t), g(t))$.

From the hypotheses (H_3) and (H_4), for each $t \in I$, we get

$$\begin{aligned}
|u(t)-v(t)| &\leq |u(t)-h(t,u_t)-\varphi(0)+h(0,u_0)-(I_q^{\alpha}g)(t)| \\
&+ |h(t,u_t)-h(t,v_t)+|h(t,u_0)-h(t,v_0)|+(I_q^{\alpha}(g-z))(t)| \\
&\leq (I_q^{\alpha}\Phi)(t)+2p_4^*\Phi(t)+\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|(g(s)|+|z(s))|)d_qs \\
&\leq (I_q^{\alpha}\Phi)(t)+2p_4^*\Phi(t)+\frac{p_1^*+p_2^*}{1-p_3^*}(I_q^{\alpha}\Phi)(t) \\
&\leq \lambda_\phi\Phi(t)+2p_4^*\Phi(t)+\lambda_\phi\frac{p_1^*+\frac{p_2^*|u(t)|}{1+|u(t)|}}{1-p_3^*}\Phi(t) \\
&\leq \left[2p_4^*+\lambda_\phi\left(1+\frac{p_1^*+p_2^*}{1-p_3^*}\right)\right]\Phi(t) \\
&:= c_{f,h,\Phi}\Phi(t).
\end{aligned}$$

Hence, we conclude the generalized Ulam-Hyers-Rassias stability of problem (2.1).

□

2.4 Existence and Stability Results problem with case of Infinite Delay

Consider the space

$$\Omega := \{u : (-\infty, T] \to \mathbb{R} : u_t \in \mathcal{B} \text{ for } t \in I \text{ and } u|_I \in C(I)\}.$$

In the present section, we are concerned with the problem.

$$\begin{cases} u(t) = \varphi(t);\ t \in (-\infty, 0], \\ {}^cD_q^{\alpha}(u(t)-h(t,u_t)) = f(t,u(t),{}^cD_q^{\alpha}(u(t)-h(t,u_t)));\ t \in I, \end{cases} \tag{2.11}$$

where $\varphi : (-\infty, 0] \to \mathbb{R},\ h : I \times \mathcal{B} \to \mathbb{R},\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\mathcal{B}$ is a phase space.

For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s) = u(t+s);\ \text{for } s \in (-\infty, 0].$$

Let us introduce the following hypotheses:

(H_{01}) The function h satisfies the Lipschitz condition:

$$|h(t,u) - h(t,v)| \leq \phi \|u - v\|_{\mathcal{B}},$$

for $t \in I$ and $u, v \in \mathcal{B}$, where $0 < \phi < 1$.

(H_{02}) There exist functions $p_1, p_2, p_3, p_4 \in C(I, \mathbb{R}_+)$ with $p_3(t) < 1$ such that

$$(1 + |u|)|f(t,u,v)| \leq p_1(t)\Phi(t) + p_2(t)\Phi(t)|u| + p_3(t)|v|,$$

for each $t \in I$ and $u, v \in \mathbb{R}$, and

$$(1 + \|w - z\|_{\mathcal{B}})|h(t,w) - h(t,z)| \leq p_4(t)\Phi(t)\|w - z\|_{\mathcal{B}},$$

for each $t \in I$ and $w, z \in \mathcal{B}$,

Theorem 2.4.1. *Assume that hypotheses (H_{01}), (H_2) hold and the condition*

$$\frac{T^{\alpha} d^*}{\Gamma_q(1+\alpha)} + r^* + \phi - \phi r^* < 1,$$

then the problem (2.11) has at least one solution defined on $(-\infty, T]$.

Proof. Define the operators $A, B : \Omega \to \Omega$ by

$$\begin{cases} (Au)(t) = 0; \ t \in (-\infty, 0], \\ (Au)(t) = u_0 - h(0, u_0) + (I_q^{\alpha} g)(t); \ t \in I, \end{cases} \tag{2.12}$$

where $g \in C(I)$ with $g(t) = f(t, u(t), g(t))$, and

$$\begin{cases} (Bu)(t) = \varphi(t); \ t \in (-\infty, 0], \\ (Bu)(t) = h(t, u_t); \ t \in I. \end{cases} \tag{2.13}$$

Let $v(\cdot) : (-\infty, T] \to \mathbb{R}$ be a function defined by,

$$v(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ 0; & t \in I. \end{cases}$$

Then $v_t = \varphi(t)$ for all $t \in (-\infty, 0]$. For each $w \in C(I)$ with $w(t) = 0$ for each $t \in (-\infty, 0]$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ w(t) & t \in I. \end{cases}$$

If $u(\cdot)$ satisfies,

$$u(t) = h(t, u_t),$$

then, $u(t) = \overline{w}(t) + v(t);\ t \in I$, and then $u_t = \overline{w}_t + v_t$, for every $t \in I$. Thus, the function $w(\cdot)$ satisfies

$$w(t) = h(t, u_t).$$

Let

$$C_0 = \{w \in \Omega :\ w(t) = 0 \text{ for } t \in (-\infty, 0]\},$$

be the Banach space with norm $\| \cdot \|_T$, with

$$\|w\|_T = \sup_{t\in(-\infty,0]} \|w_t\|_{\mathcal{B}} + \sup_{t\in I} \|w(t)\| = \sup_{t\in I} \|w(t)\|,\ w \in C_0.$$

Consider the operator $P : C_0 \to C_0$ be defined by

$$(Pw)(t) = h(t, u_t). \tag{2.14}$$

Then the operators $A + B$ and $A + P$ have the same fixed points. Set

$$R \geq \frac{(1 - r^*)[2h^* + |u_0|(1 + \phi)] + \frac{T^\alpha p^*}{\Gamma_q(1+\alpha)}}{1 - r^* - \phi + \phi r^* - \frac{T^\alpha d^*}{\Gamma_q(1+\alpha)}},$$

and define the ball $B_R = \{u \in \Omega : \|u\|_T \leq R\}$ in Ω. We can prove as in Theorem 2.4.1 that the operators P and B satisfy the conditions of the Theorem 1.3.3. This implies that the operator $A + B$ has at least a fixed point which is a solution of problem (2.11).

From Theorem 2.4.1, we can conclude the following result about the generalized Ulam-Hyers-Rassias stability of problem (2.11).

Theorem 2.4.2. *Assume that the hypotheses* (H_{02}) *and* (H_4) *hold. If* $p_4^*\Phi^* < 1$, *and*

$$p_3^* + 2p_4^*\Phi^* + \frac{T^\alpha p_2^*\Phi^*}{\Gamma_q(1+\alpha)} - 2p_3^*p_4^*\Phi^* < 1,$$

then the problem (2.11) has a solution and it is generalized Ulam-Hyers-Rassias stable.

2.5 Existence and Stability Results problem with State Dependent Delay

In this section we study the existence and stability; first for finite delay, then for infinite delay of the two following problems.

$$\begin{cases} u(t) = \varphi(t); \ t \in [-r, 0], \\ {}^cD_q^\alpha(u(t) - h(t, u_{\rho(t,u_t)})) = f(t, u(t), {}^cD_q^\alpha(u(t) - h(t, u_{\rho(t,u_t)}))); \ t \in I, \end{cases} \tag{2.15}$$

where $\varphi \in \mathcal{C}, \ \rho : I \times \mathcal{C} \to \mathbb{R}, \ h : I \times \mathcal{C} \to \mathbb{R}, \ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

$$\begin{cases} u(t) = \varphi(t); \ t \in (-\infty, 0], \\ {}^cD_q^\alpha(u(t) - h(t, u_{\rho(t,u_t)})) = f(t, u(t), {}^cD_q^\alpha(u(t) - h(t, u_{\rho(t,u_t)}))); \ t \in I, \end{cases} \tag{2.16}$$

where $\varphi : (-\infty, 0] \to \mathbb{R}, \ \rho : I \times \mathcal{B} \to \mathbb{R}, h : I \times \mathcal{B} \to \mathbb{R}, \ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

2.5.1 The Finite Delay Case

Set $\mathcal{R} := \mathcal{R}_{\rho^-} = \{\rho(t, u) : (t, u) \in I \times C(I), \ \rho(t, u) \leq 0\}$. We always assume that $\rho : I \times C(I) \to \mathbb{R}$ is continuous and the function $t \longmapsto u_t$ is continuous from $\mathcal{R}$ into $C(I)$.

As in Theorems 2.3.1 and 2.3.2, we conclude the following results:

Theorem 2.5.1. *Assume that the hypotheses (H_1) and (H_2) hold. If*

$$2\phi + \frac{T^\alpha d^*}{(1 - r^*)\Gamma_q(1 + \alpha)} < 1,$$

then the problem (2.15) has at least one solution defined on $[-r, T]$.

Theorem 2.5.2. *Assume that the hypotheses (H_3) and (H_4) hold. If $p_4^*\Phi^* < 1$, and*

$$p_3^* + 2p_4^*\Phi^* + \frac{T^\alpha p_2^*\Phi^*}{\Gamma_q(1 + \alpha)} - 2p_3^*p_4^*\Phi^* < 1,$$

then the problem (2.15) has at least a solution and it is generalized Ulam-Hyers-Rassias stable.

2.5.2 The Infinite Delay Case

Set $\mathcal{R}' := \mathcal{R}'_{\rho^-} = \{\rho(t,u) : (t,u) \in I \times \mathcal{B}\ \rho(t,u) \leq 0\}$. We always assume that the functions $\rho : I \times \mathcal{B} \to \mathbb{R}$ and $t \in \mathcal{R}' \longmapsto u_t \in \mathcal{B}$ are continuous.

In the sequel we will make use of the following hypothesis:

(C_φ) There exists a continuous bounded function $L : \mathcal{R}'_{\rho^-} \to (0,\infty)$ such that

$$\|\varphi_t\|_{\mathcal{B}} \leq L(t)\|\varphi\|_{\mathcal{B}}, \text{ for any } t \in \mathcal{R}'.$$

Also, we need the following generalization of a consequence of the phase space axioms ([[87], Lemma 2.1]).

Lemma 2.5.1. *If $u \in \Omega$, then*

$$\|u_t\|_{\mathcal{B}} = (M + L')\|\varphi\|_{\mathcal{B}} + K \sup_{\theta\in[0,\max\{0,t\}]} \|u(\theta)\|,$$

where

$$L' = \sup_{t\in\mathcal{R}'} L(t).$$

As in Theorems 2.4.1 and 2.4.2, we conclude the following result:

Theorem 2.5.3. *Assume that the hypotheses (C_φ), (H_{01}) and (H_2) hold. If*

$$\frac{T^\alpha d^*}{\Gamma_q(1+\alpha)} + r^* + \phi - \phi r^* < 1,$$

then the problem (2.16) has at least one solution defined on $(-\infty, T]$.

Theorem 2.5.4. *Assume that the hypotheses (C_φ), (H_{02}) and (H_4) hold. If $p_4^*\Phi^* < 1$, and*

$$p_3^* + 2p_4^*\Phi^* + \frac{T^\alpha p_2^*\Phi^*}{\Gamma_q(1+\alpha)} - 2p_3^*p_4^*\Phi^* < 1,$$

then the problem (2.16) has a solution and it is generalized Ulam-Hyers-Rassias stable.

2.6 Some Examples

Example 1. Consider the implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases} {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t)=f(t,u(t),{}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t));\ t\in[0,1], \\ u(t)=2+t^2;\ t\in[-2,0], \end{cases} \tag{2.17}$$

where

$$f(t,x,y)=\frac{t^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{t+5}}\right)(t^2+xt^2+y);\ t\in[0,1],x,y\in\mathbb{R},$$

and

$$h(t,z)=\frac{t^4}{1+|z-2|}\left(e^{-7}+\frac{1}{e^{t+5}}\right);\ t\in[0,1],\ z\in C([-2,0]),$$

The hypothesis (H_1) is satisfied with $\phi=2e^{-5}$. Also, the hypothesis (H_2) is satisfied with $\Phi(t)=t^2$ and $p(t)=d(t)=r(t)=\left(e^{-7}+\frac{1}{e^{t+5}}\right)t$. A simple computation show that all conditions of Theorems 2.3.1 and 2.3.2 are satisfied. Hence, our problem (2.17) has at least a solution defined on $[-2,1]$, and it is generalized Ulam-Hyers-Rassias stable.

Example 2. Consider now the following problem

$$\begin{cases} {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t)=f(t,u(t),{}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t));\ t\in[0,1], \\ u(t)=1+t^2;\ t\in(-\infty,0], \end{cases} \tag{2.18}$$

where

$$f(t,x,y)=\frac{t^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{t+5}}\right)(t^2+xt^2+y);\ t\in[0,1],\ x,y\in\mathbb{R},$$

and

$$h(t,z)=\frac{t^4}{1+z_t}\left(e^{-7}+\frac{1}{e^{t+5}}\right);\ t\in[0,1],\ z\in\mathcal{B},$$

where

$$\mathcal{B}_\gamma=\{u\in C((-\infty,0],\mathbb{R}):\lim_{\|\theta\|\to\infty}e^{\gamma\theta}u(\theta)\text{ exists in }\mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|u(\theta)|.$$

Let $u : (-\infty, 1] \to \mathbb{R}$ such that $u_t \in \mathcal{B}_\gamma$ for $t \in (-\infty, 0]$, then

$$\lim_{\|\theta\|\to\infty} e^{\gamma\theta}u_t(\theta) = \lim_{\|\theta\|\to\infty} e^{\gamma(\theta-t)}u(\theta)$$

$$= e^{-\gamma t}\lim_{\|\theta\|\to\infty} e^{\gamma\theta}u(\theta) < \infty.$$

Hence $u_t \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_t\|_\gamma = K\sup\{|u(s)| : s \in [0,t]\} + M\sup\{\|u_s\|_\gamma : s \in (-\infty, 0]\},$$

where $K = M = 1$ and $H = 1$.
If $t + \theta \le 0$ we get

$$\|u_t\|_\gamma = \sup\{|u(t)| : t \in (-\infty, 0]\},$$

and if $t + \theta \ge 0$, then we have

$$\|u_t\|_\gamma = \sup\{|u(s)| : s \in [0,t]\}.$$

Thus for all $t + \theta \in [0, 1]$, we get

$$\|u_t\|_\gamma = \sup\{|u(s)| : s \in (-\infty, 0]\} + \sup\{|u(s)| : s \in [0,t]\}.$$

Then

$$\|u_t\|_\gamma = \sup\{\|u_s\|_\gamma : s \in (-\infty, 0]\} + \sup\{|u(s)| : s \in [0,t]\}.$$

$(\mathcal{B}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space. Simple computations show that all conditions of Theorems 2.4.1 and 2.4.2 are satisfied.

Example 3. In this example, we consider the following problem

$$\begin{cases} {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t) - h(t, u_{\rho(t,u_t)})) = f(t, u(t), {}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t) - h(t, u_{\rho(t,u_t)})); \ t \in [0,1], \\ u(t) = 2 + t^2; \ t \in [-2, 0], \end{cases} \tag{2.19}$$

where

$$f(t,x,y)=\frac{t^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{t+5}}\right)(t^2+xt^2+y);\ t\in[0,1],x,y\in\mathbb{R},$$

and

$$h(t,z)=\frac{t^4}{1+\|z-\sigma(z(t))\|}\left(e^{-7}+\frac{1}{e^{t+5}}\right);\ t\in[0,1],\ z\in C([-2,0]),$$

where $\sigma\in C(\mathbb{R},[0,2])$,

$$\rho(t,\varphi)=t-\sigma(\varphi(0)),\ \ (t,\varphi)\in I\times C([-2,0],\mathbb{R}).$$

The hypothesis (H_1) is satisfied with $\phi=2e^{-5}$. Also, the hypothesis (H_2) is satisfied with

$$\Phi(t)=t^2\quad p(t)=d(t)=r(t)=\left(e^{-7}+\frac{1}{e^{t+5}}\right)t.$$

A simple computation show that all conditions of Theorems 2.5.1 and 2.5.2 are satisfied.

Example 4. Now, we treat the following implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases}{}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t)=f(t,u(t),{}^cD^{\frac{1}{2}}_{\frac{1}{4}}(u(t)-h(t,u_t));\ t\in[0,1],\\ u(t)=1+t^2;\ t\in(-\infty,0],\end{cases}\tag{2.20}$$

where

$$f(t,x,y)=\frac{t^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{t+5}}\right)(t^2+xt^2+y);\ t\in[0,1],\ x,y\in\mathbb{R},$$

and

$$h(t,z)=\frac{t^4}{1+|z(t-\sigma(u(t)))|}\left(e^{-7}+\frac{1}{e^{t+5}}\right);\ t\in[0,1],\ z\in\mathcal{B},$$

where $\sigma\in C(\mathbb{R},[0,\infty))$ and $\mathcal{B}_\gamma$ is the phase space defined in Example 2. Simple computations show that from the Theorem 2.4.1, the problem (2.20) has at least one solution on $[-\infty,1]$, and the Theorem 2.4.2 implies the generalized Ulam-Hyers-Rassias stability.

Chapter 3

Implicit Deformable Fractional Differential Boundary Value Problems(2)

3.1 Introduction

In this chapter, we will treat the existence results and the global convergence of successive approximations for Implicit deformable fractional diffferential boundary value problems .

$$\begin{cases} \left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \aleph\left(\zeta, \xi(\zeta), \mathfrak{D}_0^{\gamma}\xi(\zeta)\right), \quad \zeta \in \nabla := [0, \varpi], \\ \imath\xi(0) + \jmath\xi(\varpi) = \varrho, \end{cases} \tag{3.1}$$

where $\mathfrak{D}_0^{\gamma}\xi(\zeta)$ is the deformable fractional derivative starting from the initial time 0 of the function of order $\gamma \in (0,1)$, $\aleph : \nabla \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function $0 < \varpi < +\infty$ and $\imath, \jmath, \varrho$ are real constants where $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$.
Our main results are based on Schauder's fixed point theorem

(2) [36] A. Benchaib, S Krim, A. Salim and M. Benchohra, Existence and Successive Approximations for Implicit Deformable Fractional Differential Boundary Value Problems, (submitted).

3.2 Preliminaries

We denote by $C(\nabla, \mathbb{R})$ the Banach spaces of all continuous functions from ∇ into $\mathbb{R}$, with the following norms

$$\|\xi\|_{\infty} = \sup_{\zeta\in\nabla}\{|\xi(\zeta)|\}$$

Let $\digamma := \digamma(\nabla, \mathbb{R})$ be the Banach space defined by:

$$\digamma = \{\xi \in C(\nabla, \mathbb{R}) : \mathfrak{D}_0^{\gamma}\xi \text{ exists and continuous on } \nabla\},$$

with the norm

$$\|\xi\|_{\digamma} = \max\left\{\|\xi\|_{\infty}; \|\mathfrak{D}_0^{\gamma}\xi\|_{\infty}\right\}.$$

Consider the space $X_b^p(0, \varpi)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions $\aleph$ on $[0, \kappa]$ for which $\|\aleph\|_{X_b^p} < \infty$, where the norm is given by:

$$\|\aleph\|_{X_b^p} = \left(\int_0^{\varpi} |\zeta^b \aleph(\zeta)|^p \frac{d\zeta}{\zeta}\right)^{\frac{1}{p}}, \ \ (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 3.2.1 (The deformable fractional derivative [94, 134]). *Let* $\aleph : [0, +\infty) \longrightarrow \mathbb{R}$ *be a given function, the deformable fractional derivative of* $\aleph$ *of order* γ *is defined by*

$$(\mathfrak{D}_0^{\gamma}\aleph)(\zeta) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\chi)\aleph(\zeta+\varepsilon\gamma) - \aleph(\zeta)}{\varepsilon},$$

where $\gamma + \chi = 1$ *and* $\gamma \in (0, 1]$. *If the deformable fractional derivative of* $\aleph$ *of order* γ *exists, then we simply say that* $\aleph$ *is* γ*-differentiable.*

Definition 3.2.2 (The γ-fractional integral [94]). *For* $\gamma \in (0, 1]$ *and a continuous function* $\aleph$*, let*

$$\left(\mathcal{J}_{0^+}^{\gamma}\aleph\right)(\zeta) = \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\aleph(s)ds.$$

Lemma 3.2.1 ([94]). *If* $\gamma, \gamma_1 \in (0, 1]$ *such that* $\gamma + \chi = 1$, $\aleph$ *and* $\widehat{\aleph}$ *are two* γ*-differentiable functions at a point* ζ *and* m, n *are two given numbers, then the deformable fractional derivative satisfies the following properties:*

- $\mathfrak{D}_0^{\gamma}(\lambda) = \chi\lambda$, *for any constant* λ;
- $\mathfrak{D}_0^{\gamma}(m\aleph + n\widehat{\aleph}) = m\mathfrak{D}_0^{\gamma}(\aleph) + n\mathfrak{D}_0^{\gamma}(\widehat{\aleph})$;
- $\mathfrak{D}_0^{\gamma}(\aleph\widehat{\aleph}) = \widehat{\aleph}\mathfrak{D}_0^{\gamma}(\aleph) + \gamma\aleph\widehat{\aleph}',\ \widehat{\aleph}'$ *exists;*
- $\mathcal{J}_{0^+}^{\gamma}\ \mathcal{J}_{0^+}^{\gamma_1}\aleph = \mathcal{J}_{0^+}^{\gamma+\gamma_1}\aleph$.

Lemma 3.2.2 ([94]). *If* $\gamma \in (0,1]$, $\aleph$ *is continuous function, then we have:*

- $\left(\mathcal{J}_{0^+}^{\gamma}\ \mathfrak{D}_0^{\gamma}(\aleph)\right)(\zeta) = \aleph(\zeta) - e^{\frac{-\chi}{\gamma}\zeta}\aleph(0)$;
- $\mathfrak{D}_0^{\gamma}\left(\mathcal{J}_{0^+}^{\gamma}\aleph\right)(\zeta) = \aleph(\zeta)$.

Lemma 3.2.3. *Let* $\widehat{\aleph} \in L^1(\nabla)$, $0 < \gamma \leq 1$ *and* $\imath, \jmath, \varrho$ *are real constants where* $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$. *Then the problem*

$$\begin{cases} \left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \widehat{\aleph}(\zeta);\ \zeta \in \nabla := [0,\varpi], \\ \imath\xi(0) + \jmath\xi(\varpi) = \varrho, \end{cases} \tag{3.2}$$

has a unique solution defined by

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds. \tag{3.3}$$

Proof. Applying the γ-fractional integral of order γ to both sides the equation $\left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \widehat{\aleph}(\zeta)$, and by using Lemma 3.2.2 and if $\zeta \in \nabla$, we get

$$\xi(\zeta) - \xi(0)e^{\frac{-\chi}{\gamma}\zeta} = \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds. \tag{3.4}$$

Hence, we get

$$\xi(\zeta) = \xi(0)e^{\frac{-\chi}{\gamma}\zeta} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds. \tag{3.5}$$

Thus,

$$\xi(\varpi) = \xi(0)e^{\frac{-\chi}{\gamma}\varpi} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\varpi}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds.$$

From the mixed boundary conditions $\imath\xi(0) + \jmath\xi(\varpi) = \varrho$, we get

$$\imath\xi(0) + \jmath\left(\xi(0)e^{\frac{-\chi}{\gamma}\varpi} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\varpi}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds\right) = \varrho.$$

Thus,

$$\xi(0) = \frac{\varrho - \frac{\jmath}{\gamma}e^{\frac{-\chi}{\gamma}\varpi}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}.$$

Hence, we obtain

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds.$$

Conversely, we can easily show by Lemma 4.2.2 that if ξ verifies equation (3.3) then it satisfied the problem (3.2).

□

3.3 Existence and Uniqueness of Solutions

In this section, we are concerned with the existence results of the problem (3.1).

Definition 3.3.1. *A solution of problem* (3.1) *is a function* $\xi \in C(\nabla)$ *where*

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds,$$

such that $\widehat{\aleph} \in C(\nabla, \mathbb{R})$*, with* $\widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))$ *and* $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0.$

The hypotheses:

(H_1) There exist constants $\omega_1 > 0,\ 0 < \omega_2 < 1$ such that

$$|\aleph(\zeta, \xi_1, \Im_1) - \aleph(\zeta, \xi_2, \Im_2)| \leq \omega_1|\xi_1 - \xi_2| + \omega_2|\Im_1 - \Im_2|,$$

for any $\xi_1, \xi_2, \Im_1, \Im_2 \in \mathbb{R}$, and each $\zeta \in \nabla$.

Remark 3.3.1. *We note that for any* $\xi, \Im \in \mathbb{R}$, *and each* $\zeta \in \nabla$, *hypothesis* (H_1) *implies that*

$$|\aleph(\zeta, \xi, \Im)| \leq \omega_1|\xi| + \omega_2|\Im| + \aleph^*,$$

where $\aleph^* = \sup_{\zeta \in [0,\varpi]} \aleph(\zeta, 0, 0)$.

Now, we will give our existence result that is based on Schauder's fixed point theorem [71].

Theorem 3.3.1. *If* (H_1) *holds, and*

$$\frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right)|\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)|\omega_1}{\chi|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|(1 - \omega_2)} < 1, \tag{3.6}$$

then problem (3.1) *has at least one solution on* $[0, \varpi]$.

Proof. Consider the operator $\mathcal{H} : C(\nabla, \mathbb{R}) \to C(\nabla, \mathbb{R})$, such that

$$(\mathcal{H}\xi)(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds, \tag{3.7}$$

where $\widehat{\aleph} \in C(\nabla, \mathbb{R})$, with $\widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))$.

Let $\delta > 0$ such that

$$\delta \geq \frac{\frac{|\varrho|}{|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right)|\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)|\aleph^*}{\chi|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|(1 - \omega_2)}}{1 - \frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right)|\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)|\omega_1}{\chi|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|(1 - \omega_2)}}. \tag{3.8}$$

Consider the ball

$$\Xi_\delta = \{\xi \in C([0, \varpi], \mathbb{R}),\ \|\xi\|_\infty \leq \delta\}.$$

Claim 1. $\mathcal{H}$ is continuous.

Let $\{\xi_n\}_n$ be a sequence such that $\xi_n \to \xi$ on Ξ_δ. For each $\zeta \in \nabla$, we have

$$|(\mathcal{H}\xi_n)(\zeta) - (\mathcal{H}\xi)(\zeta)| \leq \frac{|\jmath| e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{|\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}|} \int_0^\varpi e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}_n(s) - \widehat{\aleph}(s)| ds + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^\zeta e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}_n(s) - \widehat{\aleph}(s)| ds,$$

where $\widehat{\aleph}_n,\ \widehat{\aleph} \in C(\nabla, \mathbb{R})$ such that

$$\widehat{\aleph}_n(\zeta) = \aleph(\zeta, \xi_n(\zeta), \widehat{\aleph}_n(\zeta)) \quad and \quad \widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta)).$$

Since

$$\|\xi_n - \xi\|_\infty \to 0 \text{ as } n \to \infty$$

and $\aleph, \widehat{\aleph}$ and $\widehat{\aleph}_n$ are continuous, we deduce that

$$\|\mathcal{H}(\xi_n) - \mathcal{H}(\xi)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Hence, $\mathcal{H}$ is continuous.

Claim 2. $\mathcal{H}(\Xi_\delta) \subset \Xi_\delta$.
Let $\xi \in \Xi_\delta$. From Remark 3.3.1, for each $\zeta \in \nabla$, we have

$$\begin{aligned} |\widehat{\aleph}(\zeta)| &\leq |\aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))| \\ &\leq \omega_1 \|\xi\|_\infty + \omega_2 \|\widehat{\aleph}\|_\infty + \aleph^* \\ &\leq \omega_1 \delta + \omega_2 \|\widehat{\aleph}\|_\infty + \aleph^*. \end{aligned}$$

Then

$$\|\widehat{\aleph}\|_\infty \leq \frac{\delta\omega_1 + \aleph^*}{1 - \omega_2}.$$

Thus,

$$\begin{aligned} |(\mathcal{H}\xi)(\zeta)| &\leq \left| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^\varpi e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s) ds + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^\zeta e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s) ds \right| \\ &\leq \frac{|\varrho|}{|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{|\jmath|}{|\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}|} \int_0^\varpi e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}(s)| ds + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^\zeta e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}(s)| ds \\ &\leq \frac{|\varrho|}{|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right) |\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)| (\delta\omega_1 + \aleph^*)}{\chi |\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}| (1 - \omega_2)} \\ &\leq \delta. \end{aligned}$$

Hence,

$$\|\mathcal{H}(\xi)\|_\infty \leq \delta.$$

Consequently, $\mathcal{H}(\Xi_\delta) \subset \Xi_\delta$.

Claim 3. $\mathcal{H}(\Xi_\delta)$ is equicontinuous. For $0 \leq \zeta_1 \leq \zeta_2 \leq \varpi$, and $\xi \in \Xi_\delta$, we get

$$\begin{aligned}
&|\mathcal{H}(\xi)(\zeta_2) - \mathcal{H}(\xi)(\zeta_1)| \\
&\leq \left| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_2}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta_2+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^\varpi e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta_2} \int_0^{\zeta_2} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds \right. \\
&\quad \left. - \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_1}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} + \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta_1+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^\varpi e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds - \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta_1} \int_0^{\zeta_1} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}\varpi}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^\varpi e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds \right] \right| \\
&\quad + \frac{1}{\gamma} \left| e^{\frac{-\chi}{\gamma}\zeta_2} \int_{\zeta_1}^{\zeta_2} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds - \int_0^{\zeta_1} \left[e^{\frac{-\chi}{\gamma}\zeta_1} - e^{\frac{-\chi}{\gamma}\zeta_2} \right] e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)ds \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath\left(1 - e^{\frac{-\chi}{\gamma}\varpi}\right)(\delta\omega_1 + \aleph^*)}{\chi\left(\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}\right)(1-\omega_2)} \right] \right| \\
&\quad + \frac{\delta\omega_1 + \aleph^*}{\chi(1-\omega_2)} \left| e^{\frac{-\chi}{\gamma}\zeta_2} \left(e^{\frac{\chi}{\gamma}\zeta_2} - e^{\frac{\chi}{\gamma}\zeta_1}\right) - \left(e^{\frac{-\chi}{\gamma}\zeta_1} - e^{\frac{-\chi}{\gamma}\zeta_2}\right)\left(e^{\frac{\chi}{\gamma}\zeta_1} - 1\right) \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath\left(1 - e^{\frac{-\chi}{\gamma}\varpi}\right)(\delta\omega_1 + \aleph^*)}{\chi\left(\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}\right)(1-\omega_2)} \right] \right| \\
&\quad + \frac{\delta\omega_1 + \aleph^*}{\chi(1-\omega_2)} \left| e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1} \right|.
\end{aligned}$$

As $\zeta_2 \to \zeta_1$ then $|\mathcal{H}(\xi)(\zeta_1) - \mathcal{H}(\xi)(\zeta_2)| \to 0$. We deduce that $\mathcal{H}(\Xi_\delta)$ is equicontinuous. Consequently, Arzelá-Ascoli theorem implies that $\mathcal{H}$ is continuous and compact. Thus, by Schauder's fixed point theorem [71], we deduce that $\mathcal{H}$ has at least a fixed point which is a solution of (3.1).

□

3.4 Successive Approximations and Uniqueness Results

This section is devoted to giving the main result of the global convergence of successive approximations of our problem (3.1). We will study the solution in $\mathcal{F}$ of our problem.

Set $\nabla_\lambda := [0, \lambda\varpi]$ for any $\lambda \in [0,1]$. In what follows, we need the following hypotheses:

(H_2) There exist a constant $\varkappa > 0$ and a continuous function $h : \nabla \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$, such that $h(\zeta, \cdot, \cdot)$ is nondecreasing for all $\zeta \in \nabla$ and the inequality

$$|\aleph(\zeta, \xi_1, \bar{\xi_1}) - \aleph(\zeta, \xi_2, \bar{\xi_2})| \leq h\left(\zeta, |\xi_1 - \xi_2|, |\bar{\xi_1} - \bar{\xi_2}|\right) \tag{3.9}$$

holds for $\zeta \in \nabla$ and $\xi_1, \xi_2, \bar{\xi_1}, \bar{\xi_2} \in \mathbb{R}$, with $|\xi_1 - \xi_2| \leq \varkappa$ and $|\bar{\xi_1} - \bar{\xi_2}| \leq \varkappa$.

(H_3) $R \equiv 0$ is the only function in $\mathcal{F}(\nabla_\theta, [0, \varkappa])$ which satisfies the integral inequality

$$R(\zeta) \leq \int_0^\varpi |G(\zeta, s)| h\left(s, R(s), \left(\mathfrak{D}_0^\gamma R\right)(s)\right) ds,$$

with $\lambda \leq \theta \leq 1$,

$$G(\zeta, s) = \frac{1}{\gamma} \begin{cases} \dfrac{\jmath e^{\frac{\chi}{\gamma}(s-\zeta-\varpi)}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - e^{\frac{\chi}{\gamma}(s-\zeta)}, & \text{if} \quad 0 \leq s \leq \zeta \leq \varpi, \\ \\ \dfrac{\jmath e^{\frac{\chi}{\gamma}(s-\zeta-\varpi)}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}, & \text{if} \quad 0 \leq \zeta \leq s \leq \varpi. \end{cases}$$

Here $G(\zeta, s)$ is called the Green function of the boundary value problem (3.1).

(H_4) For each $\zeta \in \nabla$, the set

$$\{\zeta \longmapsto \aleph(\zeta, \xi_1, \bar{\xi_1}) : \xi_1, \bar{\xi_1} \in \mathbb{R}\} \text{ is equicontinuous.}$$

For $\zeta \in \nabla$, we define the successive approximations of the problem (3.1) as follows:

$$\xi_0(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}},$$

$$\xi_{n+1}(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^{\varpi} G(t,s)\aleph(s, \xi_n(s), \left(\mathfrak{D}_0^{\gamma}\xi_n\right)(s))ds.$$

Theorem 3.4.1. *Assume that the hypotheses $(H_2) - (H_4)$ hold. Then, the successive approximations $\xi_n; n \in \mathbb{N}$ are well defined and converge to the unique solution of the problem uniformly in $\digamma$.*

Proof. Since the function $\aleph$ is continuous, then the successive approximations are well defined. Differentiating the two sides of the successive approximations ξ_n; $n \in \mathbb{N}$ by using the improved deformable fractional derivative of order γ, by Lemma 3.2.1 and Lemma 3.2.2, we have

$$\left(\mathfrak{D}_0^{\gamma}\xi_0\right)(\zeta) = 0, \quad \zeta \in \nabla,$$

$$\left(\mathfrak{D}_0^{\gamma}\xi_{n+1}\right)(\zeta) = \aleph\left(\zeta, \xi_n(\zeta), \mathfrak{D}_0^{\gamma}\xi_n(\zeta)\right), \quad \zeta \in \nabla.$$

And since $\xi_n \in \digamma$, then there exist two constants $\delta_1, \delta_2 > 0$ such that

$$\|\xi_n\|_\infty \leq \delta_1 \text{ and } \|\mathfrak{D}_0^{\gamma}\xi_n\|_\infty \leq \delta_2.$$

Let $\zeta_1, \zeta_2 \in \nabla$, $\zeta_1 < \zeta_2$. Then,

$$\begin{aligned}
|\xi_n(\zeta_2) - \xi_n(\zeta_1)| \leq & \left| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_2}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^{\varpi} G(\zeta_2, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^{\gamma}\xi_{n-1}(s)\right) ds \right. \\
& \left. - \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_1}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} + \int_0^{\varpi} G(\zeta_1, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^{\gamma}\xi_{n-1}(s)\right) ds \right| \\
\leq & \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}\right] \right| \\
& + \left| \int_0^{\varpi} G(\zeta_2, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^{\gamma}\xi_{n-1}(s)\right) ds \right. \\
& \left. - \int_0^{\varpi} G(\zeta_1, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^{\gamma}\xi_{n-1}(s)\right) ds \right| \\
\leq & \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}\right] \right| \\
& + \sup_{(\zeta,\xi,\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} |\aleph(\zeta, \xi, \Im)| \int_0^{\varpi} |G(\zeta_2, s) - G(\zeta_1, s)| \, ds.
\end{aligned}$$

As $\zeta_1 \longrightarrow \zeta_2$ the right hand side of the above inequality tends to zero. On the other hand, we have

$$\begin{aligned}
& \left| \left(\mathfrak{D}_0^{\gamma}\xi_n\right)(\zeta_2) - \left(\mathfrak{D}_0^{\gamma}\xi_n\right)(\zeta_1) \right| \\
& \quad \leq \left| \aleph\left(\zeta_2, \xi_{n-1}(\zeta_2), \mathfrak{D}_0^{\gamma}\xi_{n-1}(\zeta_2)\right) - \aleph\left(\zeta_1, \xi_{n-1}(\zeta_1), \mathfrak{D}_0^{\gamma}\xi_{n-1}(\zeta_1)\right) \right| \\
& \quad \longrightarrow 0, \text{ as } \zeta_1 \longrightarrow \zeta_2.
\end{aligned}$$

Thus,

$$\left| \left(\mathfrak{D}_0^{\gamma}\xi_n\right)(\zeta_2) - \left(\mathfrak{D}_0^{\gamma}\xi_n\right)(\zeta_1) \right| \longrightarrow 0, \text{ as } \zeta_1 \longrightarrow \zeta_2.$$

As a result, the sequences $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ and $\{\left(\mathfrak{D}_0^{\gamma}\xi_n\right)(\zeta);\ n \in \mathbb{N}\}$ are equicontinuous on ∇.

Let

$$\vartheta := \sup\Big\{\lambda \in [0,1] : \{\xi_n(\zeta);\ n \in \mathbb{N}\} \text{ converges uniformly on } \nabla_\lambda\Big\}.$$

If $\vartheta = 1$, then we have the global convergence of successive approximations. Suppose that $\vartheta < 1$, then the sequence $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ converges uniformly on ∇_{ϑ}. As this sequence is equicontinuous, it converges uniformly to a continuous function $\tilde{\xi}(\zeta)$. In the case that we prove that there exists $\theta \in (\vartheta, 1]$ that $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ converges uniformly on ∇_{θ}, this will yield a contradiction.

Put $\xi(\zeta) = \tilde{\xi}(\zeta)$ for $\zeta \in \nabla_{\vartheta}$. From (H_2), there exist a constant $\varkappa > 0$ and a continuous function $h : \nabla \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$ ensuring inequality (4.16). Also, there exist $\theta \in [\vartheta, 1]$ and $n_0 \in \mathbb{N}$, such that for all $\zeta \in \nabla_{\theta}$ and $n, m > n_0$, we have

$$|\xi_n(\zeta) - \xi_m(\zeta)| \leq \varkappa,$$

and

$$|(\mathfrak{D}_0^{\gamma}\xi_n)(\zeta) - (\mathfrak{D}_0^{\gamma}\xi_m)(\zeta)| \leq \varkappa.$$

For all $\zeta \in \nabla_{\theta}$, put

$$R^{(n,m)}(\zeta) = |\xi_n(\zeta) - \xi_m(\zeta)|,$$

$$R_{\jmath}(\zeta) = \sup_{n,m \geq \jmath} R^{(n,m)}(\zeta),$$

$$\left(\mathfrak{D}_0^{\gamma}R^{(n,m)}\right)(\zeta) = \left|(\mathfrak{D}_0^{\gamma}\xi_n)(\zeta) - (\mathfrak{D}_0^{\gamma}\xi_m)(\zeta)\right|,$$

and

$$(\mathfrak{D}_0^{\gamma}R_{\jmath})(\zeta) = \sup_{n,m \geq \jmath} \left(\mathfrak{D}_0^{\gamma}R^{(n,m)}\right)(\zeta),$$

Since the sequence $R_{\jmath}(\zeta)$ is non-increasing, it is convergent to a function $R(\zeta)$ for each $\zeta \in \nabla_{\theta}$. From the equi-continuity of $\{R_{\jmath}(\zeta)\}$, it follows that $\lim_{\jmath \to \infty} R_{\jmath}(\zeta) = R(\zeta)$ uniformly on ∇_{θ}. Furthermore, for $\zeta \in \nabla_{\theta}$ and $n, m \geq \jmath$, we have

$$\begin{aligned} R^{(n,m)}(\zeta) &= |\xi_n(\zeta) - \xi_m(\zeta)| \\ &\leq \sup_{s \in [0,\zeta]} |\xi_n(\zeta) - \xi_m(\zeta)| \\ &\leq \int_0^{\varpi} |G(\zeta, s)| \Big| \aleph(s, \xi_{n-1}(s), (\mathfrak{D}_0^{\gamma}\xi_{n-1})(s)) \\ &\quad - \aleph(s, \xi_{m-1}(s), (\mathfrak{D}_0^{\gamma}\xi_{m-1})(s)) \Big| ds. \end{aligned}$$

Then, by inequality (3.9), we have

$$R^{(n,m)}(\zeta) \leq$$
$$\int_0^{\varpi} |G(\zeta,s) \times |h\left(s, |\xi_{n-1}(s)-\xi_{m-1}(s)|, |\left(\mathfrak{D}_0^{\gamma}\xi_{n-1}\right)(s) - \left(\mathfrak{D}_0^{\gamma}\xi_{m-1}\right)(s)|\right) ds$$
$$\leq \int_0^{\varpi} |G(\zeta,s) \times |h\left(s, R^{(n-1,m-1)}(s), \left(\mathfrak{D}_0^{\gamma}R^{(n-1,m-1)}\right)(s)\right) ds.$$

Thus,

$$R_\jmath(\zeta) \leq \int_0^{\varpi} |G(\zeta,s)| h\left(s, R_{\jmath-1}(s), \left(\mathfrak{D}_0^{\gamma}R_{\jmath-1}\right)(s)\right) ds.$$

By the Lebesgue dominated convergence theorem we have

$$R(\zeta) \leq \int_0^{\varpi} |G(\zeta,s)| h\left(s, R(s), \left(\mathfrak{D}_0^{\gamma}R\right)(s)\right) ds.$$

Then, by (H_3) we get $R \equiv 0$ on ∇_θ, which yields that $\lim_{\jmath\to\infty} R_\jmath(\zeta) = 0$ uniformly on ∇_θ. Thus, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^{\infty}$ is a Cauchy sequence on ∇_θ. Thus, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^{\infty}$ is uniformly convergent on ∇_θ, which yields the contradiction.

Also, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^{\infty}$ converges uniformly on ∇ to a continuous function $\xi_*(\zeta)$. We get

$$\lim_{\jmath\to\infty} \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^{\varpi} G(\zeta,s) h\left(s, \xi_\jmath(s), \left(\mathfrak{D}_0^{\gamma}\xi_\jmath\right)(s)\right) ds$$
$$= \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^{\varpi} G(\zeta,s) h\left(s, \xi_*(s), \left(\mathfrak{D}_0^{\gamma}\xi_*\right)(s)\right) ds,$$

for all $\zeta \in \nabla$. This means that ξ_* is a solution of the problem (3.1).

Let us now prove the uniqueness result of the problem (3.1). Let ξ_1 and ξ_2 be two solutions of (3.1). As above, put

$$\widehat{\vartheta} := \sup\left\{\lambda \in [0,1];\ \xi_1(\zeta) = \xi_2(\zeta) \text{ for } \zeta \in \nabla_\lambda\right\},$$

and suppose that $\widehat{\vartheta} < 1$. There exist a constant $\varkappa > 0$ and a comparison function $h : \nabla_{\widehat{\vartheta}} \times [0,\varkappa] \times [0,\varkappa] \longrightarrow \mathbb{R}_+$ verifying inequality (3.9). We take $\theta \in (\lambda, 1)$ such that

$$|\xi_1(\zeta) - \xi_2(\zeta)| \leq \varkappa,$$

and

$$|(\mathfrak{D}_0^\gamma \xi_1)(\zeta) - (\mathfrak{D}_0^\gamma \xi_2)(\zeta)| \leq \varkappa.$$

for $\zeta \in \nabla_\theta$. Then, for all $\zeta \in \nabla_\theta$, we have

$$\begin{aligned} |\xi_1(\zeta) - \xi_2(\zeta)| &\leq \int_0^\varpi |G(\zeta, s)| \, |\aleph(s, \xi_1(s), (\mathfrak{D}_0^\gamma \xi_1)(s)) - \aleph(s, \xi_2(s), (\mathfrak{D}_0^\gamma \xi_2)(s))| \, ds \\ &\leq \int_0^\varpi |G(\zeta, s)| h\left(s, |\xi_1(s) - \xi_2(s)|, |(\mathfrak{D}_0^\gamma \xi_1)(s) - (\mathfrak{D}_0^\gamma \xi_2)(s)|\right) ds. \end{aligned}$$

Again, by (H_3) we get $\xi_1 - \xi_2 \equiv 0$ on ∇_θ. This gives us $\xi_1 = \xi_2$ on ∇_θ, which gives a contradiction. Consequently, $\widehat{\vartheta} = 1$ and the solution of the problem (3.1) is unique on ∇.

□

3.5 Some Examples

We give now some examples that illustrate our obtained results.

Example 3.5.1. *Consider the following problem:*

$$\begin{cases} (\mathfrak{D}_0^{\frac{1}{2}} \xi)(\zeta) = \dfrac{1}{90\,(1 + |\xi|)} + \dfrac{1}{30\left(1 + |(\mathfrak{D}_0^{\frac{1}{2}} \xi)(\zeta)|\right)}; \ \zeta \in [0, 1], \\ \xi(0) + \xi(1) = 0. \end{cases} \tag{3.10}$$

Set

$$\aleph(\zeta, \xi, \Im) = \frac{1}{90\,(1 + |\xi|)} + \frac{1}{30\,(1 + |\Im|)}; \ \zeta \in [0, 1], \ \xi, \Im \in \mathbb{R}.$$

For any $\xi, \widetilde{\xi}, \xi, \widetilde{\xi} \in \mathbb{R}$, *and* $\zeta \in [0, 1]$, *we have*

$$|\aleph(\zeta, \xi, \Im) - \aleph(\zeta, \widetilde{\xi}, \widetilde{\Im})| \leq \frac{1}{90}|\xi - \widetilde{\xi}| + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_1) *is satisfied with*

$$\omega_1 = \frac{1}{90} \quad \textit{and} \quad \omega_2 = \frac{1}{30}.$$

Next, the condition (3.6) is verifies with $\chi = \frac{1}{2}$ *and* $\gamma = \frac{1}{2}$. *Indeed,*

$$\frac{(\imath e^{\frac{\chi}{\gamma}\varpi} + 2\jmath)\omega_1}{\chi(\imath e^{\frac{\chi}{\gamma}\varpi} + \jmath)(1-\omega_2)} = \frac{(e^2+2)\frac{1}{90}}{(e^2+1)(1-\frac{1}{30})} < 1.$$

Some calculations indicate that all of the requirements of Theorem 3.3.1 are verified. Thus, (3.10) has at least a solution.

Example 3.5.2. *We consider the following problem involving the improved Caputo-type conformable fractional derivative:*

$$\begin{cases} (\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta) = \dfrac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}(1+|\xi(\zeta)|+|(\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)|}, \quad \zeta \in [0,\pi], \\ \\ \xi(0)+\xi(\pi)=0. \end{cases} \tag{3.11}$$

Set

$$\aleph(\zeta, \xi(\zeta), (\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)) = \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}(1+|\xi(\zeta)|+|(\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)|},$$

where $\gamma = \frac{1}{3}$.
For each $\xi_1, \bar{\xi_1}, \xi_2, \bar{\xi_2} \in \mathbb{R}$ *and* $\zeta \in [0,\pi]$, *we have*

$$|\aleph(\zeta,\xi_1,\xi_2) - \aleph(\zeta,\bar{\xi_1},\bar{\xi_2})| \leq \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}}\left[|\xi_1-\bar{\xi_1}| + |\xi_2-\bar{\xi_2}|\right].$$

Therefore, (H_2) *is verified for all* $\zeta \in [0,\pi]$, $\varkappa > 0$ *and the comparison function* $h : \nabla \times [0,\varkappa] \times [0,\varkappa] \longrightarrow \mathbb{R}_+$ *is defined by:*

$$h(\zeta,\xi_1,\xi_2) = \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}}(\xi_1+\xi_2).$$

Moreover, we have

$$\lim_{\zeta_1 \longrightarrow \zeta_2} \left(\aleph\left(\zeta_2,\xi_1,\xi_2\right) - \aleph\left(\zeta_1,\xi_1,\xi_2\right)\right) = 0.$$

Thus, the hypothesis (H_4) *is verified. Consequently, Theorem 3.4.1 means that the successive approximations* ξ_n; $n \in \mathbb{N}$, *defined by*

$$\xi_0(\zeta) = 0, \quad \zeta \in [0, \pi],$$

$$\xi_{n+1}(\zeta) = -\int_0^{\pi} \frac{G(\zeta, s)(8e^s + 3s^3 + 1)}{83e^{s+1}(1 + |\xi_n(s)| + |(\mathfrak{D}_0^{\frac{1}{3}}\xi_n)(s)|} ds,$$

converges uniformly on $[0, \pi]$ *to the unique solution of the problem (3.11).*

Chapter 4

Implicit Improved Conformable Fractional Differential Equations[3]

4.1 Introduction

In this chapter, we will treat the existence, the Ulam stability, results and successive approximations for the initial value problem with nonlinear Implicit fractional differential equation involving improved Caputo-type conformable fractional diffferentive.

$$ {}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}y(t) = f\left(t, y(t), {}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}y(t)\right), \ t \in [0, T_f], \tag{4.1} $$

$$ y(0) = 0, \tag{4.2} $$

where $0 < \vartheta < 1$, ${}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}$ is the improved Caputo-type conformable fractional derivative of order ϑ defined in [69], $I := [0, T_f]$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function such that $f(t, 0, 0) \neq 0$ for all $t \in I$.
We shall make use of Schauder's fixed point theorem and Banach's contraction principle.

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[3] [37] A. Benchaib, S. Krim, A. Salim and M. Benchohra, Existence, Ulam Stability Results and Successive Approximations for Implicit Improved Conformable Fractional Differential Equations, submitted.

4.2 Preliminaries

We denote by $\mathcal{C} := C(I, \mathbb{R})$ the Banach space of all continuous functions from I into $\mathbb{R}$ with the following norm

$$\|y\|_{\mathcal{C}} = \sup_{t \in I} |y(t)|.$$

$AC(I, \mathbb{R})$ is the space of absolutely continuous functions on I, and

$$AC^1(I) := \{y : I \longrightarrow \mathbb{R} : y' \in AC(I)\},$$

where

$$y'(t) = \frac{d}{dt} y(t), \;\; t \in I.$$

Consider the space $X_b^p(0, T_f)$, $(b \in \mathbb{R}, \; 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions f on $[0, T]$ for which $\|f\|_{X_b^p} < \infty$, with:

$$\|f\|_{X_b^p} = \left(\int_0^{T_f} |t^b f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \;\; (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 4.2.1 ([92]). *The conformable fractional derivative of a given function $\psi : [0, +\infty) \longrightarrow \mathbb{R}$ of order ϑ is defined by:*

$$\mathcal{T}_\vartheta(\psi)(t) = \lim_{\varepsilon \to 0} \frac{\psi\left(t + \varepsilon t^{1-\vartheta}\right) - \psi(t)}{\varepsilon},$$

for $t > 0$ and $\vartheta \in (0, 1]$. If ψ is ϑ-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0+} \mathcal{T}_{\{a}(\psi)(t)$ exists, then define $\mathcal{T}_\vartheta(\psi)(0) = \lim_{t \to 0+} \mathcal{T}_\vartheta(\psi)(t)$. If the conformable fractional derivative of ψ of order ϑ exists, then we simply say that ψ is ϑ-differentiable. It is easy to see that if ψ is differentiable, then $\mathcal{T}_\vartheta(\psi)(t) = t^{1-\vartheta}\psi'(t)$.

Definition 4.2.2 (The improved Caputo-type conformable fractional derivative [69]). *The improved Caputo-type conformable fractional derivative of a given function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ of order ϑ is defined by*

$${}_a^C\tilde{\mathcal{T}}_\vartheta(\psi)(t) = \lim_{\varepsilon \to 0} \left[(1 - \vartheta)(\psi(t) - \psi(a)) + \vartheta \frac{\psi\left(t + \varepsilon(t - a)^{1-\vartheta}\right) - \psi(t)}{\varepsilon} \right],$$

where $-\infty < a < t < +\infty$, a is a given number and $\vartheta \in [0, 1]$.

Definition 4.2.3 (The improved Riemann-Liouville-type conformable fractional derivative [69]). *The improved Riemann-Liouville-type conformable fractional derivative of a given function* $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ *of order* ϑ *is defined by*

$$ {}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(\psi)(t) = \lim_{\varepsilon \to 0} \left[(1-\vartheta)\psi(t) + \vartheta \frac{\psi\left(t + \varepsilon(t-a)^{1-\vartheta}\right) - \psi(t)}{\varepsilon} \right], $$

where $-\infty < a < t < +\infty$, a *is a given number and* $\vartheta \in [0,1]$.

Lemma 4.2.1 ([69]). *If* $\vartheta \in [0,1]$, f *and* g *are two* ϑ*-differentiable functions at a point* t *and* m, n *are two given numbers, then the improved conformable fractional derivatives satisfy the following properties:*

- ${}_{a}^{C}\tilde{\mathcal{T}}_{\vartheta}(mf+ng) = m{}_{a}^{C}\tilde{\mathcal{T}}_{\vartheta}(f) + n{}_{a}^{C}\tilde{\mathcal{T}}_{\{\vartheta}(g)$;
- ${}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(mf+ng) = m{}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(f) + n{}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(g)$;
- ${}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(fg) = (1-\vartheta){}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(f)g + f{}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(g) - (1-\vartheta)fg$;
- ${}_{a}^{RL}\tilde{\mathcal{T}}_{\vartheta}(f(g(t))) = (1-\vartheta)f(g(t)) + \vartheta f'(g(t))\mathcal{T}_{\vartheta}(g(t))$.

Definition 4.2.4 (The ϑ-fractional integral [69]). *For* $\vartheta \in (0,1]$ *and a continuous function* f*, let*

$$ \left(\mathcal{I}_{\vartheta} f\right)(t) = \frac{1}{\vartheta} \int_0^t \frac{f(s)}{s^{1-\vartheta}} e^{\left(1-\vartheta/\vartheta^2\right)\left(s^{\vartheta}-t^{\vartheta}\right)} ds. $$

When $\vartheta = 1$, $\mathcal{I}_1(f) = \int_0^t f(s)ds$*, the usual Riemann integral.*

Lemma 4.2.2 ([69]). *If* $\vartheta \in [0,1]$, ψ *is* ϑ*-differentiable function at a point* t *and* $\psi(0) = 0$*, then we have:*

- $\left(\mathcal{I}_{\vartheta}\, {}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}(\psi)\right)(t) = {}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}\left(\mathcal{I}_{\vartheta}\psi\right)(t) = \psi(t)$;
- $\left(\mathcal{I}_{\vartheta}\, {}_{0}^{RL}\tilde{\mathcal{T}}_{\vartheta}(\psi)\right)(t) = {}_{0}^{RL}\tilde{\mathcal{T}}_{\vartheta}\left(\mathcal{I}_{\vartheta}\psi\right)(t) = \psi(t)$.

4.3 Main Results

4.3.1 Existence and uniqueness of solutions

Lemma 4.3.1. *Let $0 < \vartheta < 1$ and $h : I \to \mathbb{R}$ be a continuous function. Then, the problem*

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = h(t), \quad t \in I := [0, T_f], \tag{4.3}$$

$$ y(0) = 0, \tag{4.4}$$

has a unique solution given by:

$$ y(t) = \frac{1}{\vartheta}\int_0^t \frac{h(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds, \quad t \in I. \tag{4.5}$$

Proof. *To obtain the integral equation (4.5), we apply the ϑ-fractional integral to both sides of (4.3), and by Lemma 4.2.2 we get*

$$ y(t) = \frac{1}{\vartheta}\int_0^t \frac{h(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds. \tag{4.6}$$

Now, we apply the improved Caputo-type conformable fractional derivative of order ϑ to both sides of (4.6)*, for $t \in I$ we obtain*

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = h(t). $$

Also, it is clear that $y(0) = 0$.

□

Definition 4.3.1. *By a solution of problem* (4.1)-(4.2) *we mean a function $y \in C(I, \mathbb{R})$ that satisfies the equation* (4.1) *and the condition* (4.2).

Lemma 4.3.2. *Let $f : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then, the problem* (4.1)-(4.2) *is equivalent to the following integral equation:*

$$ y(t) = \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} f\left(s, y(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y(s)\right) ds, \quad t \in I. $$

In the sequel, the following hypotheses are used:

(H_1) The function $f : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H_2) There exist continuous functions $p_1, p_2 : I \longrightarrow \mathbb{R}_+$, such that

$$|f(t,\beta_1,\bar{\beta}_1) - f(t,\beta_2,\bar{\beta}_2)| \leq p_1(t)|\beta_1 - \beta_2| + p_2(t)|\bar{\beta}_1 - \bar{\beta}_2|,$$

for $t \in I$ and $\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2 \in \mathbb{R}$, with

$$p_1^* = \sup_{t\in I} p(t) \text{ and } p_2^* = \sup_{t\in I} p_2(t) < 1.$$

Now we declare and demonstrate our first existence result for problem (4.1)-(4.2) based on the Banach contraction principle [71].

Theorem 4.3.1. *Assume that* (H_1)-(H_2) *hold. If*

$$\frac{p_1^*\left(1 - e^{\frac{(\vartheta-1)T_f^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-p_2^*)} < 1, \tag{4.7}$$

then the problem (4.1)-(4.2) has a unique solution.

Proof. Let $T : \mathcal{C} \longmapsto \mathcal{C}$ be the operator defined by

$$(Tx)(t) = \frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}}\varrho(s)ds, \quad t \in I, \tag{4.8}$$

where ϱ is a function satisfying the following functional equation

$$\varrho(t) = f\left(t, x(t), \varrho(t)\right).$$

According to Lemma 4.3.2, the fixed points of T are solutions of problem (4.1)-(4.2).

Let $x_1, x_2 \in \mathcal{C}$. For $t \in I$, we have

$$|(Tx_1)(t) - (Tx_2)(t)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}}|\varrho_1(s) - \varrho_2(s)|ds, \tag{4.9}$$

where ϱ_1, ϱ_2 are the functions satisfying the following functional equations:

$$\varrho_1(t) = f\left(t, x_1(t), \varrho_1(t)\right),$$
$$\varrho_2(t) = f\left(t, x_2(t), \varrho_2(t)\right).$$

By (H_2), we have

$$\begin{aligned} |\varrho_1(t)-\varrho_2(t)| &= |f(t,x_1(t),\varrho_1(t))-f(t,x_2(t),\varrho_2(t))| \\ &\leq p_1(t)|x_1(t)-x_2(t)|+p_2(t)|\varrho_1(t)-\varrho_2(t)| \\ &\leq p_1^*\|x_1-x_2\|_{\mathcal{C}}+p_2^*|\varrho_1(t)-\varrho_2(t)|. \end{aligned}$$

Then,

$$|\varrho_1(t)-\varrho_2(t)| \leq \frac{p_1^*}{1-p_2^*}\|x_1-x_2\|_{\mathcal{C}}.$$

Therefore, for each $t \in I$, we get

$$\begin{aligned} |(Tx_1)(t)-(Tx_2)(t)| &\leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\frac{p_1^*}{1-p_2^*}\|x_1-x_2\|_{\mathcal{C}}ds \\ &\leq \left[\frac{1-e^{\frac{(\vartheta-1)t^\vartheta}{\vartheta^2}}}{1-\vartheta}\right]\frac{p_1^*}{1-p_2^*}\|x_1-x_2\|_{\mathcal{C}}. \end{aligned}$$

Thus,

$$\|Tx_1-Tx_2\|_{\mathcal{C}} \leq \frac{p_1^*\left(1-e^{\frac{(\vartheta-1)T_f^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-p_2^*)}\|x_1-x_2\|_{\mathcal{C}}.$$

Hence, by the Banach contraction principle, T has a unique fixed point which is a unique solution of the problem (4.1)-(4.2).

□

Our second existence result for (4.1)-(4.2) is based on the fixed point theorem of Schauder [71].

Remark 4.3.1. *Let us put*

$$k_1(t)=|f(t,0,0)|,\ k_2(t)=p_1(t),\ k_3(t)=p_2(t).$$

Then, the assumption (H_2) *implies that*

$$|f(t,\beta,\bar{\beta})| \leq k_1(t)+k_2(t)|\beta|+k_3(t)|\bar{\beta}|,$$

for $t \in I$ *and* $\beta, \bar{\beta} \in \mathbb{R}$. *Set*

$$k_1^*=\sup_{t\in I}k_1(t),\quad k_2^*=\sup_{t\in I}k_2(t) \text{ and } k_3^*=\sup_{t\in I}k_3(t)<1.$$

Theorem 4.3.2. *Assume that* (H_1)*-*(H_2) *hold. If*

$$\eta = \frac{k_2^* \left(1 - e^{\frac{(\vartheta-1)T_f^{\vartheta}}{\vartheta^2}}\right)}{(1-k_3^*)(1-\vartheta)} < 1. \tag{4.10}$$

then problem (4.1)-(4.2) has at least one solution.

Proof. We will establish the proof in various steps.

Step 1. T is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{C}$. For $t \in I$, we have

$$|(Tx_n)(t) - (Tx)(t)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} |h_n(s) - h(s)| ds, \tag{4.11}$$

where

$$h_n(t) = f\left(t, x_n(t), h_n(t)\right),$$

and

$$h(t) = f\left(t, x(t), h(t)\right).$$

Since $x_n \longrightarrow x$, and by (H_1), we get $h_n(t) \longrightarrow h(t)$ as $n \longrightarrow \infty$ for each $t \in I$.
Then, by Lebesgue dominated convergence theorem and (H_1), equation (4.11) implies

$$|(Tx_n)(t) - (Tx)(t)| \longrightarrow 0 \; as \; n \longrightarrow \infty,$$

and hence

$$\|T(x_n) - T(x)\|_{\mathcal{C}} \longrightarrow 0 \; as \; n \longrightarrow \infty.$$

As a result, T is continuous.

Let the constant $R > 0$, such that

$$R \geq \frac{k_1^* \eta}{k_2^*(1-\eta)}, \tag{4.12}$$

with

$$\eta = \frac{k_2^* \left(1 - e^{\frac{(\vartheta-1)T_f^{\vartheta}}{\vartheta^2}}\right)}{(1-k_3^*)(1-\vartheta)} < 1.$$

And, we define the following ball

$$B_R = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \le R\}.$$

Then, B_R is a convex, closed and bounded subset of $\mathcal{C}$.

Step 2. $T(B_R) \subset B_R$.
Let $x \in B_R$. We show that $Tx \in B_R$. For $t \in I$, we have

$$|(Tx)(t)| \le \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \left| f\left(s, y(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y(s)\right)\right| ds. \tag{4.13}$$

By Remark 4.3.1, for $t \in I$, we have

$$\begin{aligned}|h(t)| &= |f(t, x(t), h(t))| \\ &\le k_1(t) + k_2(t)|x(t)| + k_3(t)|h(t)|.\end{aligned}$$

That means that

$$|h(t)| \le k_1^* + k_2^*\|x\|_{\mathcal{C}} + k_3^*(\alpha)|h(t)|.$$

Then,

$$|h(t)| \le \frac{k_1^* + k_2^* R}{1 - k_3^*} := \Lambda.$$

Thus, for $t \in I$ and from (4.13), we obtain

$$\begin{aligned}|(Tx)(t)| &\le \frac{\Lambda\left(1 - e^{\frac{(\vartheta-1)T_f^\vartheta}{\vartheta^2}}\right)}{1-\vartheta} \\ &\le R,\end{aligned}$$

which implies that $\|Tx\|_{\mathcal{C}} \le R$. Consequently,

$$T(B_R) \subset B_R.$$

Step 3: $T(B_R)$ is equicontinuous and bounded.
By Step 2 we have $T(B_R)$ is bounded. Let $\gamma_1, \gamma_2 \in I = [0, T_f], \gamma_1 < \gamma_2,$ *and* $x \in B_R$. Then,

$$\begin{aligned}&|(Tx)(\gamma_2) - (Tx)(\gamma_1)| \\ &\le \left|\frac{1}{\vartheta}\int_0^{\gamma_2} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta-\gamma_2^\vartheta)}{\vartheta^2}} h(s)\, ds - \frac{1}{\vartheta}\int_0^{\gamma_1} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta-\gamma_1^\vartheta)}{\vartheta^2}} h(s)\, ds\right| \\ &\le \frac{\Lambda}{1-\vartheta}\left[2 - 2e^{\frac{(1-\vartheta)(\gamma_1^\vartheta-\gamma_2^\vartheta)}{\vartheta^2}} + e^{\frac{(\vartheta-1)\gamma_1^\vartheta}{\vartheta^2}} - e^{\frac{(\vartheta-1)\gamma_2^\vartheta}{\vartheta^2}}\right].\end{aligned}$$

As $\gamma_1 \longrightarrow \gamma_2$ the right hand side of the above inequality tends to zero. As a result of Step 1 to Step 3, together with the Arzela-Ascoli theorem, we can say that T is continuous and completely continuous. From Schauder's theorem, we conclude that T has a fixed point wich is a solution of the problem (4.1)-(4.2).

□

4.3.2 Ulam-Hyers-Rassias stability

Considering now the Ulam stability for problem (4.1)-(4.2). Let $x \in \mathcal{C}$, $\epsilon > 0$ and $v : I \longmapsto [0, \infty)$ be a continuous function. For $t \in I$, we have the following inequality:

$$\left| {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) - f\left(t, y(t), {}_0^C\tilde{\mathcal{T}}_\vartheta y(t)\right) \right| \leq \epsilon v(t). \tag{4.14}$$

Definition 4.3.2 ([5]). *Problem* (4.1)-(4.2) *is Ulam-Hyers-Rassias (U-H-R) stable with respect to* v *if there exists a real number* $a_{f,v} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $x \in \mathcal{C}$ *of inequality (4.14) there exists a solution* $y \in \mathcal{C}$ *of* (4.1)-(4.2) *with*

$$|x(t) - y(t)| \leq \epsilon a_{f,v} v(t), \qquad t \in I,$$

Remark 4.3.2. *A function* $x \in \mathcal{C}$ *is a solution of inequality (4.14) if and only if there exist* $\sigma \in \mathcal{C}$ *such that*

1. $|\sigma(t)| \leq \epsilon v(t)$, $t \in I$,
2. ${}_0^C\tilde{\mathcal{T}}_\vartheta x(t) = f\left(t, x(t), {}_0^C\tilde{\mathcal{T}}_\vartheta x(t)\right) + \sigma(t)$.

Theorem 4.3.3. *Assume that in addition to* (H_1)-(H_2)*, the following hypothesis hold.*

(H_3) *There exist a nondecreasing function* $v(\cdot) \in \mathcal{C}$ *and* $\kappa_v > 0$*, such that for* $t \in I$*, we have*

$$\mathcal{I}_\vartheta v(t) \leq \kappa_v v(t),$$

(H_4) *There exist continuous functions* $q, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3 : I \longrightarrow \mathbb{R}_+$*, such that for* $t \in I$*, we have*

$$|f(t, \beta, \bar{\beta})| \leq \tilde{k}_1(t) + \tilde{k}_2(t)\frac{|\beta|}{1 + |\beta|} + \tilde{k}_3(t)|\bar{\beta}|,$$

and

$$\frac{\tilde{k}_1(t)+\tilde{k}_2(t)}{1-\tilde{k}_3(t)} \leq q(t)v(t).$$

for $t \in I$ *and* $\beta, \bar{\beta} \in \mathbb{R}$.

Then, problem (4.1)-(4.2) *is U-H-R stable.*

Set $q^* = \sup_{t\in I} q(t)$.

Proof. Let $x \in \mathcal{C}$ be a solution if inequality (4.14), and assume that y is the unique solution of the problem

$${}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = f\left(t, y(t), {}_0^C\tilde{\mathcal{T}}_\vartheta y(t)\right), \ t \in I.$$

By Lemma 4.3.2, we obtain

$$y(t) = \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} f\left(s, y(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y(s)\right) ds, \text{ if } \ t \in I.$$

Since x is a solution of the inequality (4.14), by Remark 4.3.2, for $t \in I$, we have

$${}_0^C\tilde{\mathcal{T}}_\vartheta x(t) = f\left(t, x(t), {}_0^C\tilde{\mathcal{T}}_\vartheta x(t)\right) + \sigma(t). \tag{4.15}$$

Clearly, the solution of (4.15) is given by

$$x(t) = \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \left(f\left(s, x(s), {}_0^C\tilde{\mathcal{T}}_\vartheta x(s)\right) + \sigma(s)\right) ds, \text{ if } \ t \in I.$$

For each $t \in I$, we have

$$\begin{aligned}|x(t) - y(t)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Big| f\left(s, x(s), {}_0^C\tilde{\mathcal{T}}_\vartheta x(s)\right) \\ - f\left(s, y(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y(s)\right)\Big| ds + \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} |\sigma(s)| ds.\end{aligned}$$

By the hypothesis (H_4), for $t \in I$, we have

$$|f(t, x(t), h(t))| \leq \tilde{k}_1(t) + \tilde{k}_2(t) + \tilde{k}_3(t)\,|f(t, x(t), h(t))|,$$

which implies that

$$|f(t, x(t), h(t))| \leq \frac{\tilde{k}_1(t) + \tilde{k}_2(t)}{1 - \tilde{k}_3(t)}.$$

Then, for each $t \in I$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq \epsilon \kappa_v v(t) + \frac{2}{\vartheta} \int_0^t \frac{\tilde{k}_1(t) + \tilde{k}_2(t)}{1 - \tilde{k}_3(t)} s^{\vartheta - 1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds \\ &\leq v(t) \left(\epsilon \kappa_v + \frac{2q^* \left(1 - e^{\frac{(\vartheta - 1)T_f^\vartheta}{\vartheta^2}}\right)}{1 - \vartheta} \right). \end{aligned}$$

Then,

$$|x(t) - y(t)| \leq a_{f,v} \epsilon v(t),$$

where

$$a_{f,v} = \kappa_v + \frac{2q^* \left(1 - e^{\frac{(\vartheta - 1)T_f^\vartheta}{\vartheta^2}}\right)}{\epsilon(1 - \vartheta)}.$$

Hence, problem (4.1)-(4.2) is U-H-R stable.

□

4.3.3 Successive approximations and uniqueness results

This section is devoted to giving the main result of the global convergence of successive approximations of our problem (4.1)-(4.2).

Set $I_\lambda := [0, \lambda T_f]$ for any $\lambda \in [0, 1]$. In what follows, we need the following hypotheses.

(H_5) There exist a constant $\varkappa > 0$ and a continuous function $\Psi : I \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$, such that $\Psi(t, \cdot, \cdot)$ is nondecreasing for all $t \in I$ and the inequality

$$|f(t, \beta_1, \bar{\beta}_1) - f(t, \beta_2, \bar{\beta}_2)| \leq \Psi\left(t, |\beta_1 - \beta_2|, |\bar{\beta}_1 - \bar{\beta}_2|\right) \tag{4.16}$$

holds for $t \in I$ and $\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2 \in \mathbb{R}$, with $|\beta_1 - \beta_2| \leq \varkappa$ and $|\bar{\beta}_1 - \bar{\beta}_2| \leq \varkappa$.

(H_6) $R \equiv 0$ is the only function in $C(I_\xi, [0, \varkappa])$ which satisfies the integral inequality

$$R(t) \leq \frac{1}{\vartheta} \int_0^{\xi T_f} s^{\vartheta - 1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Psi\left(s, R(s), \left({}_0^C\tilde{\mathcal{T}}_\vartheta R\right)(s)\right) ds,$$

with $\lambda \leq \xi \leq 1$.

We define the successive approximations of the problem (4.1)-(4.2) as follows:

$$y_0(t) = 0, \quad t \in I,$$

$$y_{n+1}(t) = \frac{1}{\vartheta} \int_0^t s^{\vartheta - 1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} f\left(s, y_n(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_n(s)\right) ds, \quad t \in I.$$

Theorem 4.3.4. *Assume that the hypotheses (H_1)-(H_2), (H_5) and (H_6) hold. Then, the successive approximations y_n; $n \in \mathbb{N}$ are well defined and converge to the unique solution of the problem (4.1)-(4.2) uniformly on I.*

Proof. From (H_1), the successive approximations are well defined. Differentiating the two sides of the successive approximations y_n; $n \in \mathbb{N}$ by using the improved Caputo conformable fractional derivative of order ϑ, by Lemma 4.2.2, we have

$$\left({}_0^C\tilde{\mathcal{T}}_\vartheta y_0\right)(t) = 0, \quad t \in I,$$

$$\left({}_0^C\tilde{\mathcal{T}}_\vartheta y_{n+1}\right)(t) = f\left(t, y_n(t), {}_0^C\tilde{\mathcal{T}}_\vartheta y_n(t)\right), \quad t \in I.$$

And since $y \in \mathcal{C}$, then there exist two constants $\delta_1, \delta_2 > 0$ such that

$$\|y_n\|_{\mathcal{C}} \leq \delta_1 \text{ and } \|{}_0^C\tilde{\mathcal{T}}_\vartheta y_n\|_{\mathcal{C}} \leq \delta_2.$$

Let $\gamma_1, \gamma_2 \in I = [0, T_f], \gamma_1 < \gamma_2$. Then,

$$
\begin{aligned}
&|y_n(\gamma_2) - y_n(\gamma_1)| \\
&\leq \left| \frac{1}{\vartheta} \int_0^{\gamma_2} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) ds \right. \\
&\quad \left. - \frac{1}{\vartheta} \int_0^{\gamma_1} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_1^\vartheta)}{\vartheta^2}} f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) ds \right| \\
&\leq \frac{1}{\vartheta} \int_0^{\gamma_1} s^{\vartheta-1} \left| e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} - e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_1^\vartheta)}{\vartheta^2}} \right| \left| f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) \right| ds \\
&\quad + \left| \frac{1}{\vartheta} \int_{\gamma_1}^{\gamma_2} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) ds \right| \\
&\leq \frac{1}{\vartheta} \sup_{(t,y,z)\in I\times[0,\delta_1]\times[0,\delta_2]} |f(t,y,z)| \int_0^{\gamma_1} s^{\vartheta-1} \left| e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} - e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_1^\vartheta)}{\vartheta^2}} \right| ds \\
&\quad + \frac{1}{\vartheta} \sup_{(t,y,z)\in I\times[0,\delta_1]\times[0,\delta_2]} |f(t,y,z)| \int_{\gamma_1}^{\gamma_2} s^{\vartheta-1} \left| e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} \right| ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|y_n(\gamma_2) - y_n(\gamma_1)| \\
&\leq \frac{1}{1-\vartheta} \sup_{(t,y,z)\in I\times[0,\delta_1]\times[0,\delta_2]} |f(t,y,z)| \left[2 - 2e^{\frac{(1-\vartheta)(\gamma_1^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} + e^{\frac{(\vartheta-1)\gamma_1^\vartheta}{\vartheta^2}} - e^{\frac{(\vartheta-1)\gamma_2^\vartheta}{\vartheta^2}} \right].
\end{aligned}
$$

As $\gamma_1 \longrightarrow \gamma_2$ the right hand side of the above inequality tends to zero. As a result, the sequence $\{y_n(t);\ n \in \mathbb{N}\}$ is equicontinuous on I.

Let

$$\tau := \sup \Big\{ \lambda \in [0,1];\ \{y_n(t);\ n \in \mathbb{N}\} \text{ converges uniformly on } I_\lambda \Big\}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose that $\tau < 1$, then the sequence $\{y_n(t);\ n \in \mathbb{N}\}$ converges uniformly on I_τ. As this sequence is equicontinuous, it converges uniformly to a continuous function $\tilde{y}(t)$. In the case that we prove that there exists $\xi \in (\tau, 1]$ that $\{y_n(t);\ n \in \mathbb{N}\}$ converges uniformly on I_ξ, this will yield a contradiction.

Put $y(t) = \tilde{y}(t)$ for $t \in I_\tau$. From (H_5), there exist a constant $\varkappa > 0$ and a continuous function $\Psi : I \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$ ensuring inequality (4.16). Also, there exist $\xi \in [\tau, 1]$ and $n_0 \in \mathbb{N}$, such that for all $t \in I_\xi$ and $n, m > n_0$, we have

$$|y_n(t) - y_m(t)| \leq \varkappa,$$

and

$$\left| \left({}_0^C\tilde{\mathcal{T}}_\vartheta y_n \right)(t) - \left({}_0^C\tilde{\mathcal{T}}_\vartheta y_m \right)(t) \right| \leq \varkappa.$$

For all $t \in I_\xi$, put

$$R^{(n,m)}(t) = |y_n(t) - y_m(t)|,$$

$$R_k(t) = \sup_{n,m \geq k} R^{(n,m)}(t),$$

$$\left({}_0^C\tilde{\mathcal{T}}_\vartheta R^{(n,m)} \right)(t) = \left| \left({}_0^C\tilde{\mathcal{T}}_\vartheta y_n \right)(t) - \left({}_0^C\tilde{\mathcal{T}}_\vartheta y_m \right)(t) \right|,$$

and

$$\left({}_0^C\tilde{\mathcal{T}}_\vartheta R_k \right)(t) = \sup_{n,m \geq k} \left({}_0^C\tilde{\mathcal{T}}_\vartheta R^{(n,m)} \right)(t),$$

Since the sequence $R_k(t)$ is non-increasing, it is convergent to a function $R(t)$ for each $t \in I_\xi$. From the equi-continuity of $\{R_k(t)\}$, it follows that $\lim_{k \to \infty} R_k(t) = R(t)$ uniformly on I_ξ. Furthermore, for $t \in I_\xi$ and $n, m \geq k$, we have

$$\begin{aligned}
R^{(n,m)}(t) &= |y_n(t) - y_m(t)| \\
&\leq \sup_{s \in [0,t]} |y_n(s) - y_m(s)| \\
&\leq \frac{1}{\vartheta} \int_0^t s^{\vartheta - 1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Big| f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) \\
&\quad - f\left(s, y_{m-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{m-1}(s)\right) \Big| ds \\
&\leq \frac{1}{\vartheta} \int_0^{\xi T} s^{\vartheta - 1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Big| f\left(s, y_{n-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)\right) \\
&\quad - f\left(s, y_{m-1}(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_{m-1}(s)\right) \Big| ds.
\end{aligned}$$

Then, by equality (4.16), we have

$$R^{(n,m)}(t)$$
$$\leq \frac{1}{\vartheta}\int_0^{\xi T_f} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\Psi\left(s,\left|y_{n-1}(s)-y_{m-1}(s)\right|,\left|{}_0^C\tilde{\mathcal{T}}_\vartheta y_{n-1}(s)-{}_0^C\tilde{\mathcal{T}}_\vartheta y_{m-1}(s)\right|\right)ds$$
$$\leq \frac{1}{\vartheta}\int_0^{\xi T_f} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\Psi\left(s,R^{(n-1,m-1)}(s),\left({}_0^C\tilde{\mathcal{T}}_\vartheta R^{(n-1,m-1)}\right)(s)\right)ds.$$

Thus,

$$R_k(t)\leq \frac{1}{\vartheta}\int_0^{\xi T} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\Psi\left(s,R_{k-1}(s),\left({}_0^C\tilde{\mathcal{T}}_\vartheta R_{k-1}\right)(s)\right)ds.$$

By the Lebesgue dominated convergence theorem we have

$$R(t)\leq \frac{1}{\vartheta}\int_0^{\xi T} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\Psi\left(s,R(s),\left({}_0^C\tilde{\mathcal{T}}_\vartheta R\right)(s)\right)ds.$$

Then, by (H_1) and (H_6) we get $R\equiv 0$ on I_ξ, which yields that $\lim_{k\to\infty} R_k(t)=0$ uniformly on I_ξ. Thus, $\{y_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on I_ξ. Consequently, $\{y_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on I_ξ, which yields the contradiction.

Also, $\{y_k(t)\}_{k=1}^{\infty}$ converges uniformly on I to a continuous function $y_*(t)$. By the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty}\frac{1}{\vartheta}\int_0^{t} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}f\left(s,y_k(s),{}_0^C\tilde{\mathcal{T}}_\vartheta y_k(s)\right)ds$$
$$=\frac{1}{\vartheta}\int_0^{t} s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}f\left(s,y_*(s),{}_0^C\tilde{\mathcal{T}}_\vartheta y_*(s)\right)ds,$$

for all $t\in I$. This means that y_* is a solution of the problem (4.1)-(4.2).

Let us now prove the uniqueness result of the problem (4.1)-(4.2). Let y_1 and y_2 be two solutions of (4.1)-(4.2). As above, put

$$\widehat{\tau}:=\sup\left\{\lambda\in[0,1];\ y_1(t)=y_2(t)\text{ for }t\in I_\lambda\right\},$$

and suppose that $\widehat{\tau}<1$. There exist a constant $\varkappa>0$ and a comparison function $\Psi: I_{\widehat{\tau}}\times[0,\varkappa]\times[0,\varkappa]\longrightarrow\mathbb{R}_+$ verifying inequality (4.16). We take $\xi\in(\lambda,1)$ such that

$$|y_1(t)-y_2(t)|\leq\varkappa,$$

and

$$\left|\left({}_0^C\tilde{\mathcal{T}}_\vartheta y_1\right)(t) - \left({}_0^C\tilde{\mathcal{T}}_\vartheta y_2\right)(t)\right| \leq \varkappa.$$

for $t \in I_\xi$. Then, for all $t \in I_\xi$, we have

$$\begin{aligned} |y_1(t) - y_2(t)| &\leq \frac{1}{\vartheta}\int_0^{\xi T_f} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Big| f\left(s, y_0(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_0(s)\right) \\ &\quad - f\left(s, y_1(s), {}_0^C\tilde{\mathcal{T}}_\vartheta y_1(s)\right)\Big| ds \\ &\leq \frac{1}{\vartheta}\int_0^{\xi T_f} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Psi\left(s, |y_0(s) - y_1(s)|, \left|{}_0^C\tilde{\mathcal{T}}_\vartheta y_0(s) - {}_0^C\tilde{\mathcal{T}}_\vartheta y_1(s)\right|\right). \end{aligned}$$

Again, by (H_1) and (H_6) we get $y_1 - y_2 \equiv 0$ on I_ξ. This gives us $y_1 = y_2$ on I_ξ, which gives a contradiction. Consequently, $\widehat{\tau} = 1$ and the solution of the problem (4.1)-(4.2) is unique on I.

□

4.4 Examples

Example 4.4.1. *We consider the following problem involving the improved Caputo-type conformable fractional derivative:*

$$\begin{cases} {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t) = \dfrac{\sin(t) + t^2 + 1}{163e^{t+5}(1 + |x(t)| + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t)|)}, \quad t \in [0,1], \\ \\ x(0) = 0. \end{cases} \tag{4.17}$$

Set

$$f(t, x(t), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t)) = \frac{\sin(t) + t^2 + 1}{163e^{t+5}(1 + |x(t)| + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t)|)},$$

where $\vartheta = \frac{1}{2}$.
For each $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \in \mathbb{R}$ *and* $t \in [0,1]$, *we have*

$$|f(t, \beta_1, \beta_2) - f(t, \bar{\beta}_1, \bar{\beta}_2)| \leq \frac{\sin(t) + t^2 + 1}{163e^{t+5}}\left[|\beta_1 - \bar{\beta}_1| + |\beta_2 - \bar{\beta}_2|\right].$$

Therefore, (H_2) *is verified with*

$$p_1(t) = p_2(t) = \frac{\sin(t) + t^2 + 1}{163e^{t+5}},$$

and

$$p_1^* = p_2^* = \frac{3}{163e^5}.$$

Also, for $t \in I$ *we have*

$$\begin{aligned}\frac{p_1^*\left(1 - e^{\frac{(\vartheta-1)T^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-p_2^*)} &= \frac{6 - 6e^{-2}}{163e^5 - 3} \\ &\approx 0.000214482979914345 \\ &< 1.\end{aligned}$$

Then, the condition (4.7) is satisfied. Hence, as all conditions of Theorem 4.3.1 are met, the problem (4.17) admit a unique solution.

Example 4.4.2. *Consider the following problem:*

$$\begin{cases} {}_0^C\tilde{\mathcal{T}}_{\frac{1}{4}} x(t) = f(t, x(t), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{4}} x(t)), \quad t \in I = [0,3], \\ \\ x(0) = 0, \end{cases} \tag{4.18}$$

where

$$f(t, x, \bar{x}) = \frac{1}{122 + 22e^{3-t}}\left[1 + \frac{|x|}{1+|x|} + \frac{|\bar{x}|}{1+|\bar{x}|}\right],$$

for $t \in [0,3]$, $x, \bar{x} \in \mathbb{R}$ *and* $\vartheta = \frac{1}{4}$.
All conditions of Theorem 4.3.2 are satisfied with

$$p_1(t) = p_2(t) = \frac{1}{122 + 22e^{3-t}},$$

$$p_1^* = p_2^* = \frac{1}{144},$$

and

$$\begin{aligned}\eta &= \frac{k_2^*\left(1 - e^{\frac{(\vartheta-1)T^\vartheta}{\vartheta^2}}\right)}{(1-k_3^*)(1-\vartheta)} \\ &= \frac{4 - 4e^{-12(3)^{\frac{1}{4}}}}{429} \\ &\approx 0.00932400803327006 \\ &< 1.\end{aligned}$$

Then, it follows that the problem (4.18) admit at least one solution. Also, the hypothesis (H_3) and (H_4) are satisfied with

$$\tilde{k}_1(t) = \tilde{k}_2(t) = \tilde{k}_3(t) = \frac{1}{122 + 22e^{3-t}},$$

$$v(t) = 2 \text{ and } q(t) = \frac{1}{121 + 22e^{3-t}}.$$

Hence, Theorem 4.3.3 implies that problem (4.18) is U-H-R stable.

Example 4.4.3. *We consider the following problem involving the improved Caputo-type conformable fractional derivative:*

$$\begin{cases} {}_0^C\tilde{\mathcal{T}}_{\frac{1}{3}}x(t) = \dfrac{5e^t + 2t^3 + 1}{73e^{t+1}(1 + |x(t)| + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{3}}x(t)|)}, \quad t \in [0, \pi], \\ \\ x(0) = 0. \end{cases} \tag{4.19}$$

Set

$$f(t, x(t), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}}x(t)) = \frac{5e^t + 2t^3 + 1}{73e^{t+1}(1 + |x(t)| + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{3}}x(t)|)},$$

where $\vartheta = \frac{1}{3}$.
For each $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \in \mathbb{R}$ *and* $t \in [0, \pi]$*, we have*

$$|f(t, \beta_1, \beta_2) - f(t, \bar{\beta}_1, \bar{\beta}_2)| \leq \frac{5e^t + 2t^3 + 1}{73e^{t+1}} \left[|\beta_1 - \bar{\beta}_1| + |\beta_2 - \bar{\beta}_2| \right].$$

Therefore, (H_2) *and* (H_5) *are verified for all* $t \in [0, \pi]$*,* $\varkappa > 0$ *and the comparison function* $\Psi : I \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$ *is defined by:*

$$\Psi(t, \beta_1, \beta_2) = \frac{5e^t + 2t^3 + 1}{73e^{t+1}}(\beta_1 + \beta_2).$$

Consequently, Theorem 4.3.4 means that the successive approximations y_n*;* $n \in \mathbb{N}$*, defined by*

$$y_0(t) = 0, \quad t \in [0, \pi],$$

$$y_{n+1}(t) = 3\int_0^t \frac{\left(s^{-\frac{2}{3}} e^{6(s^{\frac{1}{3}} - t^{\frac{1}{3}})}\right)(5e^s + 2s^3 + 1)}{73e^{s+1}(1 + |y_n(s)| + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{3}}y_n(s)|)} ds, \quad t \in [0, \pi].$$

converges uniformly on $[0, \pi]$ *to the unique solution of the problem (4.19).*

Chapter 5

Abstract Fractional Differential Equations with Delay and non Instantaneous Impulses[4]

5.1 Introduction

In this chapter, in the first section we will treat the uniqueness and ulam-hayers-rassias stability of abstract fractional differential equations with finite delay , with infinite delay, with state-dependent delay and in the second section we will treat the existence of mild solutions for a class of impulsive fractional equations with infinite delay.

$$\begin{cases} {}^cD^{\zeta}_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_\vartheta); \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 0, \dots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta)); \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \dots, \omega, \\ \chi(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in [-\kappa_2, 0], \end{cases} \tag{5.1}$$

where $\Im_0 := [0, \vartheta_1],\ \widehat{\Im}_\jmath := (\vartheta_\jmath, \delta_\jmath],\ \Im_\jmath := (\delta_\jmath, \vartheta_{\jmath+1}];\ \jmath = 1, \dots, \omega,\ {}^cD^{\zeta}_{\delta_\jmath}$ is the fractional Caputo derivative of order $\zeta \in (0, 1],\ 0 = \delta_0 < \vartheta_1 \leq \delta_1 \leq \vartheta_2 <$

(4) [38] A. Benchaib, A. Salim, S. Abbas and M. Benchohra, New Stability Results for Abstract Fractional Differential Equations with Delay and non Instantaneous Impulses. Mathematics **2023**, 11, 3490.

$\cdots < \delta_{\omega-1} \leq \vartheta_\omega \leq \delta_\omega \leq \vartheta_{\omega+1} = \kappa_1,\ \kappa_2, \kappa_1 > 0,\ \aleph : \Im_\jmath \times \mathcal{C} \to \Xi;\ \jmath = 0, \ldots, \omega,\ \widehat{\aleph}_\jmath : \widehat{\Im}_\jmath \times \Xi \to \Xi;\ \jmath = 1, \ldots, \omega,\ \wp : [-\kappa_2, 0] \to \Xi$ are continuous functions, Ξ is a Banach space, Θ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{\mathfrak{H}(\vartheta);\ \vartheta > 0\}$ in Ξ and $\mathcal{C}$ is the Banach space defined by

$$\mathcal{C} = C_{\kappa_2} = \{\chi : [-\kappa_2, 0] \to \Xi : \text{ continuous and there exist } \varepsilon_\jmath \in (-\kappa_2, 0); \\ \jmath = 1, \ldots, \omega, \text{ such that } \chi(\varepsilon_\jmath^-) \text{ and } \chi(\varepsilon_\jmath^+) \text{ exist with } \chi(\varepsilon_\jmath^-) = \chi(\varepsilon_\jmath)\big\},$$

with the norm

$$\|\chi\|_{\mathcal{C}} = \sup_{\vartheta \in [-\kappa_2, 0]} \|\chi(\vartheta)\|_\Xi.$$

We denote by χ_ϑ the element of $\mathcal{C}$ defined by

$$\chi_\vartheta(\varepsilon) = \chi(\vartheta + \varepsilon);\ \varepsilon \in [-\kappa_2, 0],$$

here $\chi_\vartheta(\cdot)$ represents the history of the state from time $\vartheta - \kappa_2$ up to the present time ϑ.

In section 5.5, we consider the following abstract impulsive fractional differential equations with infinite delay of the form:

$$\begin{cases} {}^cD^\zeta_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_\vartheta);\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases} \tag{5.2}$$

where Θ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.1), $\aleph : \Im_\jmath \times \Bbbk \to \Xi;\ \jmath = 0, \ldots, \omega,\ \wp : \mathbb{R}_- \to \Xi$ are given continuous functions, and $\Bbbk$ is called a phase space that will be specified in Section 5.4.

The third problem is the abstract impulsive fractional differential equations with state-dependent delay of the form

$$\begin{cases} {}^cD^\zeta_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_{\rho(\vartheta,\chi_\vartheta)});\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in [-\kappa_2, 0], \end{cases} \tag{5.3}$$

where $\Theta,\ \aleph,\ \wp$ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.1) and $\rho : \Im_\jmath \times \mathcal{C} \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is a given continuous function.

The fourth problem is in section 5.6, where we consider the following abstract impulsive fractional differential equations with state-dependent delay of the form:

$$\begin{cases} {}^cD^{\zeta}_{\delta_\jmath}\chi(\vartheta) = \Theta\chi(\vartheta) + \aleph(\vartheta, \chi_{\rho(\vartheta,\chi_\vartheta)}); \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 0, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta)); \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-, \end{cases} \tag{5.4}$$

where $\Theta, \aleph, \wp$ and $\widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (5.2) and $\rho : \Im_\jmath \times \Bbbk \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is a given continuous function. Let us define some definitions and notations.

5.2 Preliminaries

Let $\Im = [0, \kappa_1];\ \kappa_1 > 0$, denote $L^1(\Im)$ the space of Bochner-integrable functions $\chi : \Im \to \Xi$ with the norm

$$\|\chi\|_{L^1} = \int_0^{\kappa_1} \|\chi(\vartheta)\|_\Xi d\vartheta,$$

where $\|\cdot\|_\Xi$ denotes a norm on Ξ.
As usual, by $AC(\Im)$ we denote the space of absolutely continuous functions from $\Im$ into Ξ, and $\mathcal{C} := C(\Im)$ is the Banach space of all continuous functions from $\Im$ into Ξ with the norm $\|.\|_\infty$ defined by

$$\|\chi\|_\infty = \sup_{\vartheta \in \Im} \|\chi(\vartheta)\|_\Xi.$$

Consider the Banach space

$$\begin{aligned} PC =& \big\{\chi : [-\kappa_2, \kappa_1] \to \Xi : \chi|_{[-\kappa_2,0]} = \wp,\ \chi|_{\widehat{\Im}_\jmath} = \widehat{\aleph}_\jmath;\ \jmath = 1, \ldots, \omega,\ \chi|_{\Im_\jmath};\ \jmath = 1, \ldots, \omega \\ &\text{is continuous and there exist } \chi(\delta_\jmath^-),\ \chi(\delta_\jmath^+),\ \chi(\vartheta_\jmath^-) \text{ and } \chi(\vartheta_\jmath^+) \\ &\text{with } \chi(\delta_\jmath^+) = \widehat{\aleph}_\jmath(\delta_\jmath, \chi(\delta_\jmath)) \text{ and } \chi(\vartheta_\jmath^-) = \widehat{\aleph}_\jmath(\vartheta_\jmath, \chi(\vartheta_\jmath))\big\}, \end{aligned}$$

with the norm

$$\|\chi\|_{PC} = \sup_{\vartheta \in [-\kappa_2, \kappa_1]} \|\chi(\vartheta)\|_\Xi.$$

Let $\zeta > 0$, for $\chi \in L^1(\Im)$, the expression

$$(I_0^\zeta \chi)(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_0^\vartheta (\vartheta - \varepsilon)^{\zeta-1} \chi(\varepsilon) d\varepsilon,$$

is called the left-sided mixed Riemann-Liouville integral of order ζ, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty \vartheta^{\varsigma-1} e^{-\vartheta} d\vartheta;\ \varsigma > 0$.

In particular,

$$(I_0^0 \chi)(\vartheta) = \chi(\vartheta),\ (I_0^1 \chi)(\vartheta) = \int_0^\vartheta \chi(\varepsilon) d\varepsilon; \text{ for almost all } \vartheta \in \Im.$$

For instance, $I_0^\zeta \chi$ exists for all $\zeta \in (0, \infty)$, when $\chi \in L^1(\Im)$. Note also that when $\chi \in C(\Im)$, then $(I_0^\zeta \chi) \in C(\Im)$.

Definition 5.2.1 ([6, 122]). *Let $\zeta \in (0,1]$ and $\chi \in AC(\Im)$. The Caputo fractional-order derivative of order ζ of χ is given by*

$${}^c D_0^\zeta \chi(\vartheta) = (I_0^{1-\zeta} \frac{d}{d\vartheta} \chi)(\vartheta) = \frac{1}{\Gamma(1-\zeta)} \int_0^\vartheta (\vartheta - \varepsilon)^{-\zeta} \frac{d}{d\varepsilon} \chi(\varepsilon) d\varepsilon.$$

Example 5.2.1. *Let $\varpi \in (-1, 0) \cup (0, \infty)$ and $\zeta \in (0, 1]$, then*

$${}^c D_0^\zeta \frac{\vartheta^\varpi}{\Gamma(1+\varpi)} = \frac{\vartheta^{\varpi-\zeta}}{\Gamma(1+\varpi-\zeta)}; \text{ for almost all } \vartheta \in \Im.$$

Let $a_1 \in [0, \kappa_1],\ \widehat{\Im}_1 = (a_1, \kappa_1],\ \zeta > 0$. For $\chi \in L^1(\widehat{\Im}_1)$, the expression

$$(I_{\kappa_1^+}^\zeta \chi)(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_{a_1^+}^\vartheta (\vartheta - \varepsilon)^{\zeta-1} \chi(\varepsilon) d\varepsilon,$$

is called the left-sided mixed Riemann-Liouville integral of order ζ of χ.

Definition 5.2.2. *[6, 122] For $\chi \in L^1(\widehat{\Im}_1)$ where $\frac{d}{d\vartheta}\chi$ is Bochner integrable on $\widehat{\Im}_1$, the Caputo fractional order derivative of order ζ of χ is defined by the expression*

$$({}^c D_{\kappa_1^+}^\zeta \chi)(\vartheta) = (I_{\kappa_1^+}^{1-\zeta} \frac{d}{d\vartheta} \chi)(\vartheta).$$

Definition 5.2.3 ([133]). *A function $\chi : [-\kappa_2, \kappa_1] \to \Xi$ is said to be a mild solution of (5.1) if χ satisfies*

$$\begin{cases} \chi(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon; \;\; \text{if } \vartheta \in [0, \vartheta_1], \\ \chi(\vartheta) = \mathfrak{F}_\zeta(\vartheta - \delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \chi(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon; \;\; \text{if } \vartheta \in \Im_\jmath, \; \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta)); \; \text{if } \vartheta \in \widehat{\Im}_\jmath, \; \jmath = 1, \ldots, \omega, \\ \chi(\vartheta) = \wp(\vartheta); \; \text{if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

where

$$\mathfrak{F}_\zeta(\vartheta) = \int_0^\infty \mu_\zeta(\eta)\mathfrak{H}(\vartheta^\zeta\eta)d\eta, \; \mathfrak{H}_\zeta(\vartheta) = \zeta\int_0^\infty \eta\mu_\zeta(\eta)\mathfrak{H}(\vartheta^\zeta\eta)d\eta, \; \mu_\zeta(\eta) = \frac{1}{\zeta}\eta^{-1-\frac{1}{\zeta}}\overline{\tau}_\zeta(\eta^{-\frac{1}{\zeta}}) \geq 0,$$

and

$$\overline{\tau}_\zeta(\eta) = \frac{1}{\pi}\sum_{\imath=0}^{\infty}(-1)^{\imath-1}\eta^{-\imath\zeta-1}\frac{\Gamma(\imath\zeta+1)}{\imath!}\sin(\imath\zeta\pi); \; \eta \in (0, \infty).$$

μ_ζ is a probability density function on $(0, \infty)$, that is $\int_0^\infty \mu_\zeta(\eta)d\eta = 1$.

Remark 5.2.1. *We can deduce that for $\varkappa \in [0, 1]$, we have*

$$\int_0^\infty \eta^\varkappa\mu_\zeta(\eta)d\eta = \int_0^\infty \eta^{-\zeta\varkappa}\overline{\tau}_\zeta(\eta)d\eta = \frac{\Gamma(1+\varkappa)}{\Gamma(1+\zeta\varkappa)}.$$

Lemma 5.2.1 ([133]). *For any $\vartheta \geq 0$, the operators $\mathfrak{F}_\zeta(\vartheta)$ and $\mathfrak{H}_\zeta(\vartheta)$ have the following properties:*

(a) *For $\vartheta \geq 0$, $\mathfrak{F}_\zeta$ and $\mathfrak{H}_\zeta$ are linear and bounded operators, ie., for any $\chi \in \Xi$,*

$$\|\mathfrak{F}_\zeta(\vartheta)\chi\|_\Xi \leq \Delta\|\chi\|_\Xi, \;\; \|\mathfrak{H}_\zeta(\vartheta)\chi\|_\Xi \leq \frac{\Delta}{\Gamma(\zeta)}\|\chi\|_\Xi.$$

(b) *$\{\mathfrak{F}_\zeta(\vartheta); \; \vartheta \geq 0\}$ and $\{\mathfrak{H}_\zeta(\vartheta); \; \vartheta \geq 0\}$ are strongly continuous.*

(c) *For every $\vartheta \geq 0$, $\mathfrak{F}_\zeta(\vartheta)$ and $\mathfrak{H}_\zeta(\vartheta)$ are also compact operators.*

Now, we consider the Ulam stability for (5.1). Let $\upsilon > 0,\ \mathcal{Y} \geq 0$ and $\mathcal{Z} : \Im \to [0, \infty)$ be a continuous function. Let

$$\begin{cases} \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta)\wp(0) - \int_0^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \upsilon;\ \text{ if } \vartheta \in [0, \vartheta_1], \\ \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta - \delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \chi(\delta_\jmath)) \\ - \int_{\delta_\jmath}^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \upsilon;\ \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|\chi(\vartheta) - \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta))\|_\Xi \leq \upsilon;\ \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases} \tag{5.5}$$

$$\begin{cases} \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta)\wp(0) - \int_0^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \mathcal{Z}(\vartheta);\ \text{ if } \vartheta \in [0, \vartheta_1], \\ \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta - \delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \chi(\delta_\jmath)) \\ - \int_{\delta_\jmath}^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \mathcal{Z}(\vartheta);\ \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|\chi(\vartheta) - \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta))\|_\Xi \leq \mathcal{Y};\ \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases} \tag{5.6}$$

$$\begin{cases} \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta)\wp(0) - \int_0^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \upsilon\mathcal{Z}(\vartheta);\ \text{ if } \vartheta \in [0, \vartheta_1], \\ \|\chi(\vartheta) - \mathfrak{F}_\zeta(\vartheta - \delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \chi(\delta_\jmath)) \\ - \int_{\delta_\jmath}^\vartheta (\vartheta - \varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta - \varepsilon)\aleph(\varepsilon, \chi_\varepsilon)d\varepsilon\|_\Xi \leq \upsilon\mathcal{Z}(\vartheta);\ \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|\chi(\vartheta) - \widehat{\aleph}_\jmath(\vartheta, \chi(\vartheta))\|_\Xi \leq \upsilon\mathcal{Y};\ \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases} \tag{5.7}$$

Definition 5.2.4. *[8, 121, 122] Problem (5.1) is Ulam-Hyers stable if there exists a real number $c_{\aleph,\widehat{\aleph}_\jmath} > 0$ such that for each $\upsilon > 0$ and for each solution $\chi \in PC$ of the inequalities (2.6) there exists a mild solution $\varkappa \in PC$ of problem (5.1) with*

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_\Xi \leq \upsilon c_{\aleph,\widehat{\aleph}_\jmath};\ \vartheta \in \Im.$$

Definition 5.2.5. *[8, 121] Problem (5.1) is generalized Ulam-Hyers stable if there exists $\eta_{\aleph,\widehat{\aleph}_\jmath} : C([0, \infty), [0, \infty))$ with $\eta_{\aleph,\widehat{\aleph}_\jmath}(0) = 0$ such that for each $\upsilon > 0$ and for each solution $\chi \in PC$ of the inequalities (5.5) there exists a mild solution $\varkappa \in PC$ of problem (5.1) with*

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_\Xi \leq \eta_{\aleph,\widehat{\aleph}_\jmath}(\upsilon);\ \vartheta \in \Im.$$

Definition 5.2.6. *[8,121] Problem (5.1) is Ulam-Hyers-Rassias stable with respect to* $(\mathcal{Z},\mathcal{Y})$ *if there exists a real number* $c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}} > 0$ *such that for each* $\upsilon > 0$ *and for each solution* $\chi \in PC$ *of the inequalities (5.7) there exists a mild solution* $\varkappa \in PC$ *of problem (5.1) with*

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_\Xi \leq \upsilon c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta));\ \vartheta \in \Im.$$

Definition 5.2.7. *[8,121] Problem (5.1) is generalized Ulam-Hyers-Rassias stable with respect to* $(\mathcal{Z},\mathcal{Y})$ *if there exists a real number* $c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}} > 0$ *such that for each solution* $\chi \in PC$ *of the inequalities (5.6) there exists a mild solution* $\varkappa \in PC$ *of problem (5.1) with*

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_\Xi \leq c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta));\ \vartheta \in \Im.$$

Remark 5.2.2. *It is clear that: (i) Definition 2.2.8 ⇒ Definition 2.2.9, (ii) Definition 2.2.10 ⇒ Definition 2.2.11, (iii) Definition 2.2.10 for* $\mathcal{Z}(\cdot) = \mathcal{Y} = 1$ *⇒ Definition 2.2.8.*

Remark 5.2.3. *A function* $\chi \in PC$ *is a solution of the inequalities (5.6) if and only if there exist a function* $G \in PC$ *and a sequence* $\{G_\jmath\}_{J=1\cdots\omega};\subset \Xi$ *(which depend on* χ*) such that*

(i) $\|G(\vartheta)\|_\Xi \leq \mathcal{Z}(\vartheta)$ and $\|G_\jmath\|_\Xi \leq \mathcal{Y};\ \jmath = 1,\ldots,\omega,$

(ii) *the function* $\chi \in PC$ *satisfies*

$$\begin{cases} \chi(\vartheta) = G(\vartheta) + \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi_\varepsilon)d\varepsilon;\ \ \textit{if}\ \vartheta \in [0,\vartheta_1], \\ \chi(\vartheta) = G(\vartheta) + \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath,\chi(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi_\varepsilon)d\varepsilon;\ \ \textit{if}\ \vartheta \in \Im_\jmath,\ \jmath = 1,\ldots,\omega, \\ \chi(\vartheta) = G_\jmath + \widehat{\aleph}_\jmath(\vartheta,\chi(\vartheta));\ \ \textit{if}\ \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Lemma 5.2.2 ([131]). *Suppose* $\beta > 0,\ a(\vartheta)$ *is a nonnegative function locally integrable on* $0 \leq \vartheta < T$ *(some* $T \leq +\infty$*) and* $\widehat{\aleph}(\vartheta)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \leq \vartheta < T,\ \widehat{\aleph}(\vartheta) \leq \Delta$ *(constant), and suppose* $\chi(\vartheta)$ *is nonnegative and locally integrable on* $0 \leq \vartheta < T$ *with*

$$\chi(\vartheta) \leq a(\vartheta) + \widehat{\aleph}(\vartheta)\int_0^\vartheta (\vartheta-\delta)^{\beta-1}\chi(\delta)d\delta$$

on this interval. Then

$$\chi(\vartheta) \leq a(\vartheta) + \int_0^\vartheta \left[\sum_{\imath=1}^{\infty} \frac{(\widehat{\aleph}(\vartheta)\Gamma(\beta))^\imath}{\Gamma(\imath\beta)} (\vartheta-\delta)^{\imath\beta-1} a(\delta) \right] d\delta, \ 0 \leq \vartheta < T.$$

5.3 Uniqueness and Ulam stabilities results with finite delay

In this section, we discuss the uniqueness of mild solutions and we present conditions for the Ulam stability for the problem (5.1).

Theorem 5.3.1. *Assume that the following hypotheses hold:*

(H_1) *The semigroup $\mathfrak{H}(\vartheta)$ is compact for $\vartheta > 0$,*

(H_2) *For each $\vartheta \in \Im_\jmath;\ \jmath = 0, \dots, \omega$, the function $\aleph(\vartheta, \cdot) : \Xi \to \Xi$ is continuous and for each $\varkappa \in \mathcal{C}$, the function $\aleph(\cdot, \varkappa) : \Im_\jmath \to \Xi$ is measurable,*

(H_3) *There exists a constant $l_\aleph > 0$ such that*

$$\|\aleph(\vartheta, \chi) - \aleph(\vartheta, \overline{\chi})\|_\Xi \leq l_\aleph \|\chi - \overline{\chi}\|_{\mathcal{C}}, \text{ for each } \vartheta \in \Im_\jmath;\ \jmath = 0, \dots, \omega,$$

and each $\chi, \overline{\chi} \in \mathcal{C}$,

(H_4) *There exist constants $0 < l_{\widehat{\aleph}_\jmath} < 1;\ \jmath = 1, \dots, \omega$, such that*

$$\|\widehat{\aleph}_\jmath(\vartheta, \chi) - \widehat{\aleph}_\jmath(\vartheta, \overline{\chi})\|_\Xi \leq l_{\widehat{\aleph}_\jmath} \|\chi - \overline{\chi}\|_\Xi,$$

for each $\vartheta \in \widehat{\Im}_\jmath$, and each $\chi, \overline{\chi} \in \Xi,\ \jmath = 1, \dots, \omega$.

If

$$\ell := \Delta l_{\widehat{\aleph}} + \frac{\Delta l_\aleph {\kappa_1}^\zeta}{\Gamma(\zeta)} < 1, \tag{5.8}$$

where $l_{\widehat{\aleph}} = \max_{\jmath=1,\dots,\omega} l_{\widehat{\aleph}_\jmath}$, then the problem (5.1) has a unique mild solution on $[-\kappa_2, \kappa_1]$.

Furthermore, if the following hypothesis

(H_5) *There exists $\varpi_{\mathcal{Z}} > 0$ such that for each $\vartheta \in \Im$, we have*

$$\int_{\delta_\jmath}^{\vartheta} \left[\sum_{\imath=1}^{\infty} \frac{(\Delta l_{\aleph})^{\imath}}{(1-\Delta l_{\widehat{\aleph}})^{\imath}\Gamma(\imath\zeta)} (\vartheta-\varepsilon)^{\imath\zeta-1}\mathcal{Z}(\varepsilon) \right] d\varepsilon \leq \varpi_{\mathcal{Z}}\mathcal{Z}(\vartheta);\ \jmath = 0,\dots,\omega,$$

holds, then the problem (5.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator $\digamma : PC \to PC$ defined by

$$\begin{cases} (\digamma\chi)(\vartheta) = \mathfrak{F}_{\zeta}(\vartheta)\wp(0) + \int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon;\ \text{ if } \vartheta \in [0,\vartheta_1], \\ (\digamma\chi)(\vartheta) = \mathfrak{F}_{\zeta}(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath,\chi(\delta_\jmath)) \\ +\int_{\delta_\jmath}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon;\ \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ (\digamma\chi)(\vartheta) = \widehat{\aleph}_\jmath(\vartheta,\chi(\vartheta));\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega, \\ (\digamma\chi)(\vartheta) = \wp(\vartheta);\ \text{if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

Clearly, the fixed points of the operator $\digamma$ are solution of the problem (5.1).

Let $\chi, \varkappa \in PC$, then, for each $\vartheta \in \Im$, we have

$$\begin{cases} \|(\digamma\chi)(\vartheta) - (\digamma\varkappa)(\vartheta)\|_{\Xi} \leq \| \int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon) \\ \times[\aleph(\varepsilon,\chi_\varepsilon) - \aleph(\varepsilon,\varkappa_\varepsilon)]d\varepsilon\|_{\Xi};\ \text{ if } \vartheta \in [0,\vartheta_1], \\ \|(\digamma\chi)(\vartheta) - (\digamma\varkappa)(\vartheta)\|_{\Xi} \leq \|\mathfrak{F}_{\zeta}(\vartheta-\delta_\jmath)(\widehat{\aleph}_\jmath(\delta_\jmath,\chi(\delta_\jmath)) - \widehat{\aleph}_\jmath(\delta_\jmath,\varkappa(\delta_\jmath))\|_{\Xi} \\ +\| \int_{\delta_\jmath}^{\vartheta}(\vartheta-\varepsilon)^{r_1-1}\mathfrak{H}_{\zeta}(\vartheta-\delta_\jmath)[\aleph(\varepsilon,\chi_\varepsilon) - \aleph(\varepsilon,\varkappa_\varepsilon)]d\varepsilon\|_{\Xi};\ \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ \|(\digamma\chi)(\vartheta) - (\digamma\varkappa)(\vartheta)\|_{\Xi} = \|\widehat{\aleph}_\jmath(\vartheta,\chi(\vartheta)) - \widehat{\aleph}_\jmath(\vartheta,\varkappa(\vartheta))\|_{\Xi};\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega. \end{cases}$$

Thus, we get

$$\begin{cases} \|(\digamma\chi)(\vartheta)-(\digamma\varkappa)(\vartheta)\|_{\Xi}\leq\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}l_{\aleph}\|\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)(\chi_{\varepsilon}-\varkappa_{\varepsilon})\|_{\mathcal{C}}d\varepsilon;\\ \leq\frac{\Delta l_{\aleph}\kappa_1{}^{\zeta}}{\Gamma(\zeta)}\|\chi-\varkappa\|_{PC};\text{ if }\vartheta\in[0,\vartheta_1],\\ \|(\digamma\chi)\vartheta-(\digamma\varkappa)\vartheta\|_{\Xi}\leq l_{\widehat{\aleph}}\|\mathfrak{F}_{\zeta}(\vartheta-\delta_{\jmath})(\chi(\vartheta)-\varkappa(\vartheta))\|_{\Xi}\\ +\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}l_{\aleph}\|\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)(\chi_{\varepsilon}-\varkappa_{\varepsilon})\|_{\mathcal{C}}d\varepsilon\\ \leq\left(\Delta l_{\widehat{\aleph}}+\frac{\Delta l_{\aleph}\kappa_1{}^{\zeta}}{\Gamma(\zeta)}\right)\|\chi-\varkappa\|_{PC};\text{ if }\vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega,\\ \|(\digamma\chi)(\vartheta)-(\digamma\varkappa)(\vartheta)\|_{\Xi}\leq l_{\widehat{\aleph}}\|\chi-\varkappa\|_{PC};\text{if }\vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega. \end{cases}$$

Hence

$$\|\digamma(\chi)-\digamma(\varkappa)\|_{PC}\leq\ell\|\chi-\varkappa\|_{PC}.$$

By the condition (5.8), we conclude that $\digamma$ is a contraction. As a consequence of the Banach fixed point theorem, we deduce that $\digamma$ has a unique fixed point $\varkappa$ which is the unique mild solution of (5.1). Then we have

$$\begin{cases} \varkappa(\vartheta)=\mathfrak{F}_{\zeta}(\vartheta)\wp(0)+\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\varkappa_{\varepsilon})d\varepsilon;\ \text{ if }\vartheta\in[0,\vartheta_1],\\ \varkappa(\vartheta)=\mathfrak{F}_{\zeta}(\vartheta-\delta_{\jmath})\widehat{\aleph}_{\jmath}(\delta_{\jmath},\varkappa(\delta_{\jmath}))\\ +\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\varkappa_{\varepsilon})d\varepsilon;\ \text{ if }\vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega,\\ \varkappa(\vartheta)=\widehat{\aleph}_{\jmath}(\vartheta,\varkappa(\vartheta));\ \text{ if }\vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega,\\ \varkappa(\vartheta)=\wp(\vartheta);\ \text{if }\vartheta\in[-\kappa_2,0]. \end{cases}$$

Let $\chi\in PC$ be a solution of the inequality (5.6). By Remark 5.2.3, (ii) and (H_5) for each $\vartheta\in\Im$, we get

$$\begin{cases} \|\chi(\vartheta)-\mathfrak{F}_{\zeta}(\vartheta)\wp(0)-\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\chi_{\varepsilon})d\varepsilon\|_{\Xi}\\ \leq\mathcal{Z}(\vartheta);\ \text{ if }\vartheta\in[0,\vartheta_1],\\ \|\chi(\vartheta)-\mathfrak{F}_{\zeta}(\vartheta-\delta_{\jmath})\widehat{\aleph}_{\jmath}(\delta_{\jmath},\chi(\delta_{\jmath}))-\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)\aleph(\varepsilon,\chi_{\varepsilon})d\varepsilon;\\ \leq\mathcal{Z}(\vartheta);\ \text{ if }\vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega,\\ \|\chi(\vartheta)-\widehat{\aleph}_{\jmath}(\vartheta,\chi(\vartheta))\|_{\Xi}\leq\mathcal{Y};\ \text{if }\vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega. \end{cases}$$

Thus,

$$\begin{cases} \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Z}(\vartheta)+\| \int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_{\zeta}(\vartheta-\varepsilon) \\ \times[\aleph(\varepsilon,\chi_{\varepsilon})-\aleph(\varepsilon,\varkappa_{\varepsilon})]d\varepsilon\|_{\Xi};\ \text{if}\ \ \vartheta\in[0,\vartheta_1], \\ \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Z}(\vartheta)+\Delta\|\widehat{\aleph}_{\jmath}(\delta_{\jmath},\chi(\delta_{\jmath}))-\widehat{\aleph}_{\jmath}(\delta_{\jmath},\varkappa(\delta_{\jmath})\|_{\Xi} \\ +\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{r_1-1}\|\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)(\aleph(\varepsilon,\chi_{\varepsilon})-\aleph(\varepsilon,\varkappa_{\varepsilon}))\|_{\Xi}d\varepsilon; \\ \text{if}\ \ \vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega, \\ \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Y}+\|\widehat{\aleph}_{\jmath}(\vartheta,\chi(\vartheta))-\widehat{\aleph}_{\jmath}(\vartheta,\varkappa(\vartheta))\|_{\Xi};\ \text{if}\ \vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega. \end{cases}$$

Hence

$$\begin{cases} \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Z}(\vartheta)+\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}l_{\aleph}\|\mathfrak{H}_{\zeta}(\vartheta-\varepsilon)(\chi_{\varepsilon}-\varkappa_{\varepsilon})\|_{\mathcal{C}}d\varepsilon \\ \le \mathcal{Z}(\vartheta)+\frac{\Delta l_{\aleph}}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\|\chi_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon;\ \ \text{if}\ \ \vartheta\in[0,\vartheta_1]\times[0,b], \\ \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Z}(\vartheta)+\Delta l_{\widehat{\aleph}}\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \\ +\frac{\Delta l_{\aleph}}{\Gamma(\zeta)}\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{r_1-1}\|\chi_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon;\ \ \text{if}\ \ \vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega, \\ \|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Y}+l_{\widehat{\aleph}}\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi};\ \text{if}\ \vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega. \end{cases}$$

For each $\vartheta\in[0,\vartheta_1]$, we have

$$\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} \le \mathcal{Z}(\vartheta)+\frac{\Delta l_{\aleph}}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\|\chi_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon.$$

We consider the function ϱ defined by

$$\varrho(\vartheta)=\sup\{\|\chi(\varepsilon)-\varkappa(\varepsilon)\|:-\kappa_2\le\varepsilon\le\vartheta\};\ \vartheta\in\Im.$$

Let $\vartheta^*\in[-\kappa_2,\vartheta]$ be such that $\varrho(\vartheta)=\|\chi(\vartheta^*)-\varkappa(\vartheta^*)\|_{\Xi}$. If $\vartheta^*\in[-\kappa_2,0]$, then $\varrho(\vartheta)=0$. Now, if $\vartheta^*\in\Im$, then by the previous inequality, we have for $\vartheta\in\Im$, we have

$$\varrho(\vartheta)\le\mathcal{Z}(\vartheta)+\frac{\Delta l_{\aleph}}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\varrho(\vartheta)d\varepsilon.$$

From Lemma 5.2.2, we have

$$\begin{aligned}\varrho(\vartheta) &\le \mathcal{Z}(\vartheta)+\int_0^{\vartheta}\left[\sum_{\imath=1}^{\infty}\frac{(\Delta l_{\aleph})^{\imath}}{\Gamma(\imath\zeta)}(\vartheta-\varepsilon)^{\imath\zeta-1}\mathcal{Z}(\varepsilon)\right]d\varepsilon,\\ &\le (1+\varpi_{\mathcal{Z}})\mathcal{Z}(\vartheta)\\ &:= c_{1,\aleph,\widehat{\aleph}_{\jmath},\mathcal{Z}}\mathcal{Z}(\vartheta).\end{aligned}$$

Since for every $\vartheta\in[0,\vartheta_1]$, $\|\chi_{\vartheta}\|_{\mathcal{C}}\le\varrho(\vartheta)$, then we get

$$\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\le c_{1,\aleph,\widehat{\aleph}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Now, for each $\vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega$, we have

$$\begin{aligned}&\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\le\mathcal{Z}(\vartheta)+\Delta l_{\widehat{\aleph}}\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\\ &+\frac{\Delta l_{\aleph}}{\Gamma(\zeta)}\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\|\chi_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\le\frac{1}{1-\Delta l_{\widehat{\aleph}}}\mathcal{Z}(\vartheta)\\ &+\frac{\Delta l_{\aleph}}{(1-\Delta l_{\widehat{\aleph}})\Gamma(\zeta)}\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\zeta-1}\|\chi_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon.\end{aligned}$$

Again, from Lemma 5.2.2, we have

$$\begin{aligned}\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi} &\le \frac{1}{1-\Delta l_{\widehat{\aleph}}}\left(\mathcal{Z}(\vartheta)+\int_0^{\vartheta}\left[\sum_{\imath=1}^{\infty}\frac{(\Delta l_{\aleph})^{\imath}}{(1-\Delta l_{\widehat{\aleph}})^{\imath}\Gamma(\imath\zeta)}(\vartheta-\varepsilon)^{\imath\zeta-1}\mathcal{Z}(\varepsilon)\right]d\varepsilon\right)\\ &\le \frac{1}{1-\Delta l_{\widehat{\aleph}}}(1+\varpi_{\mathcal{Z}})\mathcal{Z}(\vartheta)\\ &:= c_{2,\aleph,\widehat{\aleph}_{\jmath},\mathcal{Z}}\mathcal{Z}(\vartheta).\end{aligned}$$

Hence, for each $\vartheta\in\Im_{\jmath},\ \jmath=1,\ldots,\omega$, we get

$$\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\le c_{2,\aleph,\widehat{\aleph}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Now, for each $\vartheta\in\widehat{\Im}_{\jmath},\ \jmath=1,\ldots,\omega$, we have

$$\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}\le\mathcal{Y}+l_{\widehat{\aleph}}\|\chi(\vartheta)-\varkappa(\vartheta)\|_{\Xi}.$$

This gives,

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_{\Xi} \leq \frac{\mathcal{Y}}{1 - l_{\widehat{\aleph}}} := c_{3,\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}\mathcal{Y}.$$

Thus, for each $\vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1, \ldots, \omega$, we get

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_{\Xi} \leq c_{3,\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta)).$$

Set $c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}} := \max_{i\in\{1,2,3\}} c_{i,\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}$. Hence, for each $\vartheta \in \Im$, we obtain

$$\|\chi(\vartheta) - \varkappa(\vartheta)\|_{PC} \leq c_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta)).$$

Consequently, problem (5.1) is generalized Ulam-Hyers-Rassias stable.

5.4 The phase space $\Bbbk$

The notation of the phase space $\Bbbk$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [73]. More precisely, $\Bbbk$ will denote the vector space of functions defined from $\mathbb{R}_-$ into Ξ endowed with a semi norm denoted $\|\cdot\|_{\Bbbk}$ and such that the following axioms hold.

- (A_1) If $\xi : (-\infty, b) \to \Xi$, is continuous on $[0, b]$ and $\xi_0 \in \Bbbk$, then for $\vartheta \in [0, b)$ the following conditions hold
  - (i) $\xi_\vartheta \in \Bbbk$
  - (ii)$\|\xi_\vartheta\|_{\Bbbk} \leq \widehat{\Delta}(\vartheta) \sup\{|\xi(\delta)| : 0 \leq \delta \leq \vartheta\} + \Delta(\vartheta)\|\xi_0\|_{\Bbbk}$,
  - (iii)$|\xi(\vartheta)| \leq H\|\xi_\vartheta\|_{\Bbbk}$
    where $H \geq 0$ is a constant, $\widehat{\Delta} : [0, b) \to [0, +\infty)$,
    $\Delta : [0, +\infty) \to [0, +\infty)$ with $\widehat{\Delta}$ continuous and Δ locally bounded and H, $\widehat{\Delta}$ and Δ are independent of $\xi(.)$.
- (A_2) For the function ξ in (A_1), the function $\vartheta \to \xi_\vartheta$ is a $\Bbbk$-valued continuous function on $[0, b]$.
- (A_3) The space $\Bbbk$ is complete.

Denote $\widehat{\Delta}_b = \sup\{\widehat{\Delta}(\vartheta) : \vartheta \in [0, b]\}$ and $\Delta_b = \sup\{\Delta(\vartheta) : \vartheta \in [0, b]\}$.

Remark 5.4.1. 1. $[(iii)]$ *is equivalent to* $|\wp(0)| \leq H\|\wp\|_{\Bbbk}$ *for every* $\wp \in \Bbbk$.

2. *Since* $\|\cdot\|_{\Bbbk}$ *is a semi norm, two elements* $\wp, \psi \in \Bbbk$ *can verify* $\|\wp-\psi\|_{\Bbbk} = 0$ *without necessarily* $\wp(\eta) = \psi(\eta)$ *for all* $\eta \leq 0$.

3. *From the equivalence of in the first remark, we can see that for all* $\wp, \psi \in \Bbbk$ *such that* $\|\wp-\psi\|_{\Bbbk} = 0$*, we necessarily have that* $\wp(0) = \psi(0)$.

Example 5.4.1 ([90])**.** *Let:*

BC *the space of bounded continuous functions defined from* $\mathbb{R}_-$ *to* Ξ;

BUC *the space of bounded uniformly continuous functions defined from* $\mathbb{R}_-$ *to* Ξ;

$C^\infty := \{\wp \in BC : \lim_{\eta\to-\infty} \wp(\eta)$ *exist in* $\Xi\}$;

$C^0 := \{\wp \in BC : \lim_{\eta\to-\infty} \wp(\eta) = 0\}$, *endowed with the uniform norm*

$$\|\wp\| = \sup\{|\wp(\eta)| : \eta \leq 0\}.$$

We have that the spaces $BUC,\ C^\infty$ *and* C^0 *satisfy conditions* $(A_1) - (A_3)$. *However,* BC *satisfies* $(A_1), (A_3)$ *but* (A_2) *is not satisfied.*

Example 5.4.2 ([90])**.** *The spaces* $C_{\widehat{\aleph}},\ UC_{\widehat{\aleph}},\ C^\infty_{\widehat{\aleph}}$ *and* $C^0_{\widehat{\aleph}}$.
Let $\widehat{\aleph}$ *be a positive continuous function on* $(-\infty, 0]$. *We define:*

$C_{\widehat{\aleph}} := \left\{\wp \in C(\mathbb{R}_-, \Xi) : \frac{\wp(\eta)}{\widehat{\aleph}(\eta)} \text{ is bounded on } \mathbb{R}_-\right\}$;

$C^0_{\widehat{\aleph}} := \left\{\wp \in C_{\widehat{\aleph}} : \lim_{\eta\to-\infty} \frac{\wp(\eta)}{\widehat{\aleph}(\eta)} = 0\right\}$, *endowed with the uniform norm*

$$\|\wp\| = \sup\left\{\frac{|\wp(\eta)|}{\widehat{\aleph}(\eta)} :\ \eta \leq 0\right\}.$$

Then we have that the spaces $C_{\widehat{\aleph}}$ *and* $C^0_{\widehat{\aleph}}$ *satisfy conditions* $(A_1) - (A_3)$. *We consider the following condition on the function* $\widehat{\aleph}$.

(g_1) *For all* $\kappa_1 > 0,\ \sup_{0\leq\vartheta\leq\kappa_1} \sup\left\{\frac{\widehat{\aleph}(\vartheta+\eta)}{\widehat{\aleph}(\eta)} : -\infty < \eta \leq -\vartheta\right\} < \infty.$

They satisfy conditions (A_1) *and* (A_2) *if* $(\widehat{\aleph}_1)$ *holds.*

Example 5.4.3 ([90]). *The space C_ϱ. For any real constant ϱ, we define the functional space C_ϱ by*

$$C_\varrho := \left\{\wp \in C(\mathbb{R}_-, \Xi) : \lim_{\eta \to -\infty} e^{\varrho\eta}\wp(\eta) \text{ exists in } \Xi\right\}$$

endowed with the following norm

$$\|\wp\| = \sup\{e^{\varrho\eta}|\wp(\eta)| : \ \eta \leq 0\}.$$

Then C_ϱ satisfies axioms $(A_1) - (A_3)$.

5.5 Uniqueness and Ulam stabilities results with infinite delay

In this section, we present conditions for the Ulam stability of problem (5.2). Consider the space

$$\Omega := \{\chi : (-\infty, \kappa_1] \to \Xi : \ \chi_\vartheta \in \Bbbk \text{ for } \vartheta \in \mathbb{R}_- \ \text{ and } \chi|_{\Im} \in PC\}.$$

Theorem 5.5.1. *Assume that (H_1), (H_4) and the following hypotheses hold:*

(H_6) *For each $\vartheta \in \Im_\jmath$; $\jmath = 0, \dots, \omega$, the function $\aleph(\vartheta, \cdot) : \Xi \to \Xi$ is continuous and for each $\varkappa \in \Bbbk$, the function $\aleph(\cdot, \varkappa) : \Im_\jmath \to \Xi$ is measurable,*

(H_7) *There exists a constant $l'_\aleph > 0$ such that*

$$\|\aleph(\vartheta, \chi) - \aleph(\vartheta, \overline{\chi})\|_\Xi \leq l'_\aleph\|\chi - \overline{\chi}\|_\Bbbk, \text{ for each } \vartheta \in \Im_\jmath;\ \jmath = 0, \dots, \omega, \text{ and each } \chi, \overline{\chi} \in \Bbbk.$$

If

$$\ell' := \Delta l_{\widehat{\aleph}} + \frac{\Delta\widehat{\Delta} l'_\aleph \kappa_1{}^\zeta}{\Gamma(\zeta)} < 1, \tag{5.9}$$

then the problem (5.2) has a unique mild solution on $(-\infty, \kappa_1]$. Furthermore, if the hypothesis (H_5) holds, then the problem (5.2) is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator $F' : \Omega \to \Omega$ defined by,

$$\begin{cases} (F'\chi)(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ (F'\chi)(\vartheta) = \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath,\chi(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon; \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ (F'\chi)(\vartheta) = \widehat{\aleph}_\jmath(\vartheta,\chi(\vartheta)); \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega, \\ (F'\chi)(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-, \end{cases}$$

Clearly, the fixed points of the operator F' are mild solutions of the problem (5.2). Consider the function $\varkappa(\cdot) : (-\infty,\kappa_1] \to \Xi$ defined by,

$$\begin{cases} \varkappa(\vartheta) = 0; \text{ if } \vartheta \in \Im, \\ \varkappa(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-. \end{cases}$$

Then $\varkappa_0 = \wp$. For each $\tau \in \mathcal{C}(\Im)$ with $\tau(0) = 0$, we denote by $\overline{\tau}$ the function defined by

$$\begin{cases} \overline{\tau}(\vartheta) = \tau(\vartheta) \text{ if } \vartheta \in \Im, \\ \overline{\tau}(\vartheta) = 0, \text{ if } \vartheta \in \tilde{\Im}'. \end{cases}$$

If $\chi(\cdot)$ satisfies

$$\begin{cases} \chi(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ \chi(\vartheta) = \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath,\chi(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon; \text{ if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ \chi(\vartheta) = \widehat{\aleph}_\jmath(\vartheta,\chi(\vartheta)); \text{ if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega, \\ \chi(\vartheta) = \wp(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-, \end{cases}$$

we decompose $\chi(\vartheta)$ as $\chi(\vartheta) = \tau(\vartheta) + \varkappa(\vartheta);\ \vartheta \in \Im$, witch implies $\chi_\vartheta = \tau_\vartheta + \varkappa_\vartheta;\ \vartheta \in \Im$ and the function τ satisfies $\tau_0 = 0$ and for $\vartheta \in \Im$, we get

$$\begin{cases} \tau(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon;\ \text{if } \vartheta \in [0,\vartheta_1], \\ \tau(\vartheta) = \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \overline{\tau}_{\delta_\jmath} + \varkappa_{\delta_\jmath}) \\ +\int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon;\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ \tau(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \overline{\tau}_\vartheta + \varkappa_\vartheta);\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega. \end{cases}$$

Set

$$C_0 = \{\tau \in PC :\ \tau(0) = 0\},$$

and let $\|\cdot\|_a$ be the seminorm in C_0 defined by

$$\|\tau\|_a = \|\tau_0\|_{\Bbbk} + \sup_{\vartheta\in\Im}\|\tau(\vartheta)\| = \sup_{\vartheta\in\Im}\|\tau(\vartheta)\|;\ \tau \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_a$. Let the operator $P:\ C_0 \to C_0$ be defined by

$$\begin{cases} (Pw)(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon;\ \text{if } \vartheta \in [0,\vartheta_1], \\ (Pw)(\vartheta) = \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath, \overline{\tau}_{\delta_\jmath} + \varkappa_{\delta_\jmath}) \\ +\int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon;\ \text{if } \vartheta \in \Im_\jmath,\ \jmath = 1,\dots,\omega, \\ (Pw)(\vartheta) = \widehat{\aleph}_\jmath(\vartheta, \overline{\tau}_\vartheta + \varkappa_\vartheta);\ \text{if } \vartheta \in \widehat{\Im}_\jmath,\ \jmath = 1,\dots,\omega. \end{cases}$$

Obviously the operator F' has a fixed point is equivalent to P has one. We shall use the Banach contraction principle to prove that P has a fixed point. Indeed, consider $\tau, \tau^* \in C_0$. Then, for each $\vartheta \in \Im$, we get

$$\|P(\tau) - P(\tau^*)\|_a \le \ell'\|\overline{\tau} - \overline{\tau}^*\|_a.$$

By the condition (5.9), we conclude that P is a contraction. As a consequence of Banach fixed point theorem, we deduce that P has a unique fixed point τ^*. Then we have

$$\begin{cases} \tau^*(\vartheta) = \mathfrak{F}_\zeta(\vartheta)\wp(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\overline{\tau}_\varepsilon+\varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta\in[0,\vartheta_1], \\ \tau^*(\vartheta) = \mathfrak{F}_\zeta(\vartheta-\delta_\jmath)\widehat{\aleph}_\jmath(\delta_\jmath,\overline{\tau}^*_{\delta_\jmath}+\varkappa_{\delta_\jmath}) \\ +\int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\zeta-1}\mathfrak{H}_\zeta(\vartheta-\varepsilon)\aleph(\varepsilon,\overline{\tau}^*_\varepsilon+\varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta\in\Im_\jmath,\ \jmath=1,\ldots,\omega, \\ \tau^*(\vartheta) = \widehat{\aleph}_\jmath(\vartheta,\overline{\tau}^*_\vartheta+\varkappa_\vartheta); \text{ if } \vartheta\in\widehat{\Im}_\jmath,\ \jmath=1,\ldots,\omega. \end{cases}$$

Let $\tau\in C_0$ be a solution of the inequality (5.6). Thus, by (H_5) and Lemma 5.2.2 and as in the proof of Theorem 5.3.1, we can show that; for each $\vartheta\in\Im$,

$$\|\tau(\vartheta,\xi)-\tau^*(\vartheta,\xi)\|_\Xi \le c'_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta,\xi)),$$

for some $c'_{\aleph,\widehat{\aleph}_\jmath,\mathcal{Z}} > 0$, which gives that the problem (5.2) is generalized Ulam-Hyers-Rassias stable.

5.6 Uniqueness and Ulam stabilities results with state-dependent delay

In this section, we present (without proof) uniqueness and Ulam stability results for problems (5.3) and (5.4).

Set

$$\mathcal{R} := \{\rho(\delta,\chi) : (\delta,\chi)\in\Im_\jmath\times\mathcal{D},\ \rho(\delta,\chi)\le 0,\ \jmath=0,\ldots,\omega\},$$

where $\mathcal{D}\in\{\mathcal{C},\Bbbk\}$. We always assume that $\rho:\Im_\jmath\times\mathcal{D}\to\mathbb{R};\ \jmath=0,\ldots,\omega$ is continuous and the function $\delta\longmapsto\chi_\delta$ is continuous from $\mathcal{R}$ into $\mathcal{D}$.

Theorem 5.6.1. *Assume that* $(H_1),\ (H_2),\ (H_4)$ *and the following hypothesis hold:*

(H_8) *There exists a constant* $l''_\aleph > 0$ *such that*

$$\|\aleph(\vartheta,\chi_{\rho(\vartheta,\chi_\vartheta)})-\aleph(\vartheta,\overline{\chi}_{\rho(\vartheta,\overline{\chi}_\vartheta)})\|_\Xi \le l''_\aleph\|\chi_{\rho(\vartheta,\chi_\vartheta)}-\overline{\chi}_{\rho(\vartheta,\overline{\chi}_\vartheta)}\|_\mathcal{C};$$

for each $\vartheta\in\Im_\jmath;\ \jmath=0,\ldots,\omega,$ *and each* $\chi,\overline{\chi}\in\mathcal{C}$.

If

$$\ell'' := \Delta l_{\widehat{\aleph}} + \frac{\Delta l''_{\aleph} \kappa_1{}^{\zeta}}{\Gamma(\zeta)} < 1, \tag{5.10}$$

then the problem (5.3) has a unique mild solution on $[-\kappa_2, \kappa_1]$. *Furthermore, if the hypothesis* (H_5) *holds, then the problem (5.3) is generalized Ulam-Hyers-Rassias stable.*

Theorem 5.6.2. *Assume that* (H_1), (H_4), (H_6) *and the following hypothesis hold:*

(H_9) *There exists a constant* $l'''_{\aleph} > 0$ *such that*

$$\|\aleph(\vartheta, \chi_{\rho(\vartheta,\chi_\vartheta)}) - \aleph(\vartheta, \overline{\chi}_{\rho(\vartheta,\overline{\chi}_\vartheta)})\|_{\Xi} \leq l'_{\aleph}\|\chi_{\rho(\vartheta,\chi_\vartheta)} - \overline{\chi}_{\rho(\vartheta,\overline{\chi}_\vartheta)}\|_{\Bbbk};$$

for each $\vartheta \in \Im_\jmath$; $\jmath = 0, \ldots, \omega$, *and each* $\chi, \overline{\chi} \in \Bbbk$.

If

$$\ell''' := \Delta l_{\widehat{\aleph}} + \frac{\Delta\widehat{\Delta} l'''_{\aleph} \kappa_1{}^{\zeta}}{\Gamma(\zeta)} < 1, \tag{5.11}$$

then the problem (5.4) has a unique mild solution on $(-\infty, \kappa_1]$. *Furthermore, if the hypothesis* (H_5) *holds, then the problem (5.4) is generalized Ulam-Hyers-Rassias stable.*

5.7 Examples

As applications of our results, we present two examples.

Example 5.7.1. *Consider the functional abstract fractional differential equations with not instantaneous impulses of the form*

$$\begin{cases} D^{\zeta}_{0,\vartheta}\lambda(\vartheta,\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi) + \beth(\vartheta, \lambda(\vartheta-1,\xi)); & \vartheta \in [0,1]\cup(2,3], \quad \xi \in [0,\pi], \\ \lambda(\vartheta,\xi) = \widehat{\aleph}(\vartheta, \lambda(\vartheta,\xi)); & \vartheta \in (1,2], \quad \xi \in [0,\pi], \\ \lambda(\vartheta,0) = \lambda(\vartheta,\pi) = 0; & \vartheta \in [0,1]\cup(2,3], \\ \lambda(\vartheta,\xi) = \wp(\vartheta,\xi); & \vartheta \in [-1,0], \quad \xi \in [0,\pi], \end{cases} \tag{5.12}$$

where $D^{\zeta}_{0,\vartheta} := \frac{\partial^\zeta}{\partial\vartheta^\zeta}$ *is the Caputo fractional partial derivative of order* $\zeta \in (0,1]$ *with respect to* ϑ. *It is defined by the expression*

$${}^{c}D^{\vartheta}_{0,\vartheta}\lambda(\vartheta,\xi) = \frac{1}{\Gamma(1-\zeta)}\int_0^{\vartheta}(\vartheta-\varepsilon)^{-\zeta}\frac{\partial}{\partial\varepsilon}\lambda(\varepsilon,\xi)d\varepsilon,$$

$\mathcal{C} := C_1,\ \gimel : ([0,1] \cup (2,3]) \times \mathcal{C} \to \mathbb{R}$ *and* $\widehat{\aleph} : (1,2] \times \mathbb{R} \to \mathbb{R}$ *are given by*

$$\gimel(\vartheta, \lambda(\vartheta-1,\xi)) = \frac{1}{(1+110e^{\vartheta})(1+|\lambda(\vartheta-1,\xi)|)};\ \vartheta \in [0,1] \cup (2,3],\ \xi \in [0,\pi],$$

$$\widehat{\aleph}(\vartheta, \lambda(\vartheta,\xi)) = \frac{1}{1+110e^{\vartheta+\xi}} \ln(1+\vartheta^2+|\lambda(\vartheta,\xi)|);\ \vartheta \in (1,2],\ \xi \in [0,\pi],$$

and $\wp : [-1,0] \times [0,\pi] \to \mathbb{R}$ *is a continuous function.*

Let $\Xi = L^2([0,\pi],\mathbb{R})$ *and define* $\Theta : D(\Theta) \subset \Xi \to \Xi$ *by* $\Theta\tau = \tau''$ *with domain*

$$D(\Theta) = \{\tau \in \Xi : \tau,\ \tau' \text{ are absolutely continuous},\ \tau'' \in \Xi,\ \tau(0) = \tau(\pi) = 0\}.$$

It is well known that Θ *is the infinitesimal generator of an analytic semigroup on* Ξ *(see [115]). Then*

$$\Theta\tau = -\sum_{\imath=1}^{\infty} \imath^2 < \tau, e_\imath > e_\imath; \tau \in D(\Theta),$$

where

$$e_\imath(\xi) = \sqrt{\frac{2}{\pi}} \sin(\imath\xi);\ \xi \in [0,\pi],\ \imath = 1,2,3,\ldots.$$

The semigroup $\mathfrak{H}(\vartheta);\ \vartheta \geq 0$ *is given by*

$$\mathfrak{H}(\vartheta)\tau = \sum_{\imath=1}^{\infty} e^{-\imath^2\vartheta} < \tau, e_\imath > e_\imath;\ \tau \in \Xi.$$

Hence the assumptions (H_1) *and* (H_2) *are satisfied.*

For $\xi \in [0,\pi]$, *set* $\chi(\vartheta)(\xi) = \lambda(\vartheta,\xi); \quad \vartheta \in [0,3],\ \wp(\vartheta)(\xi) = \wp(\vartheta,\xi); \quad \vartheta \in [-1,0]$,

$$\Theta\chi(\vartheta)(\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi); \quad \vartheta \in [0,1] \cup (2,3],$$

$$\aleph(\vartheta,\chi(\vartheta))(\xi) = \gimel(\vartheta,\lambda(\vartheta,\xi)); \quad \vartheta \in [0,1] \cup (2,3],$$

and

$$\widehat{\aleph}(\vartheta,\chi(\vartheta))(\xi) = \widehat{\aleph}(\vartheta,\lambda(\vartheta,\xi)); \quad \vartheta \in (1,2].$$

Consequently, employing the given definitions of $\wp,\ \Theta,\ \aleph$, *and* $\widehat{\aleph}$, *the system (5.12) can be equivalently expressed as the functional abstract problem*

(5.1).

For each $\lambda, \overline{\lambda}, \in \mathcal{C},\ \vartheta \in [0,1] \cup (2,3]$ *and* $\xi \in [0,\pi]$, *we have*

$$|\aleph(\vartheta, \lambda_{\vartheta})(\xi) - \aleph(\vartheta, \overline{\lambda}_{\vartheta})(\xi)| \leq \frac{1}{111}|\lambda(\vartheta,\xi) - \overline{\lambda}(\vartheta,\xi)|,$$

then, we obtain

$$\|\aleph(\vartheta, \lambda) - \aleph(\vartheta, \overline{\lambda})\|_{\Xi} \leq \frac{1}{111}\|\lambda - \overline{\lambda}\|_{\mathcal{C}}.$$

Also, for each $\lambda, \overline{\lambda}, \in \Xi,\ \vartheta \in (1,2]$ *and* $\xi \in [0,\pi]$, *we can easily get*

$$\|\widehat{\aleph}(\vartheta, \lambda) - \widehat{\aleph}(\vartheta, \overline{\lambda})\|_{\Xi} \leq \frac{1}{111}\|\lambda - \overline{\lambda}\|_{\Xi}.$$

Thus, (H_3) *and* (H_4) *are verified with* $l_{\aleph} = l_{\widehat{\aleph}} = \frac{1}{111}$. *We shall show that condition (5.8) holds with* $\kappa_1 = 3$ *and* $\Delta = 1$. *Indeed, for each* $\zeta \in (0,1]$ *we get*

$$\begin{aligned} \ell &= \Delta l_{\widehat{\aleph}} + \frac{\Delta l_{\aleph} {\kappa_1}^{\zeta}}{\Gamma(\zeta)} \\ &= \frac{1}{111} + \frac{3^{\zeta}}{111\Gamma(\zeta)} \\ &< \frac{7}{111} \\ &< 1. \end{aligned}$$

Therefore, we guarantee the existence of a distinct mild solution defined on the interval $[-1,3]$ *for the given problem (2.17). In conclusion, the condition* (H_5) *is fulfilled by* $\mathcal{Z}(\vartheta) = 1$ *and*

$$\varpi_{\mathcal{Z}} = \sum_{\imath=1}^{\infty} \frac{1}{(110)^{\imath}\Gamma(1+\imath\zeta)} 3^{\imath\zeta}.$$

Consequently, Theorem 5.3.1 implies that the problem (2.17) is generalized Ulam-Hyers-Rassias stable.

Example 5.7.2. *Consider now the functional abstract fractional differential equations with state-dependent delay and not instantaneous impulses of the form*

$$\begin{cases} D^{\zeta}_{0,\vartheta}\lambda(\vartheta,\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi) \\ +\daleth(\vartheta,\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)); & \vartheta\in[0,1]\cup(2,3], \quad \xi\in[0,\pi], \\ \lambda(\vartheta,\xi) = \widehat{\aleph}(\vartheta,\lambda(\vartheta,\xi)); & \vartheta\in(1,2], \quad \xi\in[0,\pi], \\ \lambda(\vartheta,0) = \lambda(\vartheta,\pi) = 0; & \vartheta\in[0,1]\cup(2,3], \\ \lambda(\vartheta,\xi) = \wp(\vartheta,\xi); & \vartheta\in(-\infty,0], \quad \xi\in[0,\pi], \end{cases} \tag{5.13}$$

where $D^{\zeta}_{0,\vartheta} := \frac{\partial^\zeta}{\partial\vartheta^\zeta}$ *is the Caputo fractional partial derivative of order* $\zeta\in(0,1]$ *with respect to* ϑ, $\sigma\in C(\mathbb{R},[0,\infty))$, $\daleth:([0,1]\cup(2,3])\times\Bbbk\to\mathbb{R}$ *and* $\widehat{\aleph}:(1,2]\times\mathbb{R}\to\mathbb{R}$ *are given by*

$$\daleth(\vartheta,\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)) = \frac{1}{111(1+|\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)|)};\ \vartheta\in[0,1]\cup(2,3],\ \xi\in[0,\pi],$$

$$\widehat{\aleph}(\vartheta,\lambda(\vartheta,\xi)) = \frac{\arctan(\vartheta^2+|\lambda(\vartheta,\xi)|)}{1+110e^{\vartheta+\xi}};\ \vartheta\in(1,2],\ \xi\in[0,\pi],$$

and $\wp:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ *is a continuous function, we choose* $\Bbbk=\Bbbk_\varrho$ *the phase space defined by*

$$\Bbbk_\varrho := \left\{\wp\in C((-\infty,0],\Xi) : \lim_{\eta\to-\infty} e^{\varrho\eta}\wp(\eta) \text{ exists in } \Xi\right\}$$

endowed with the norm

$$\|\wp\| = \sup\{e^{\varrho\eta}|\wp(\eta)| :\ \eta\le 0\}.$$

Let $\Xi = L^2([0,\pi],\mathbb{R})$ *and* Θ *is the operator defined in the Example 1. For* $\xi\in[0,\pi]$, *set* $\chi(\vartheta)(\xi) = \lambda(\vartheta,\xi);\quad \vartheta\in[0,3],\quad \wp(\vartheta)(\xi) = \wp(\vartheta,\xi);\quad \vartheta\in(-\infty,0]$,

$$\Theta\chi(\vartheta)(\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi);\quad \vartheta\in[0,1]\cup(2,3],$$

$$\aleph(\vartheta,\chi(\vartheta-\sigma(\lambda(\vartheta,\xi))))(\xi) = \daleth(\vartheta,\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi));\quad \vartheta\in[0,1]\cup(2,3],$$

and

$$\widehat{\aleph}(\vartheta,\chi(\vartheta))(\xi) = \widehat{\aleph}(\vartheta,\lambda(\vartheta,\xi));\quad \vartheta\in(1,2].$$

Thus, under the above definitions of $\wp$, Θ, $\aleph$ *and* $\widehat{\aleph}$, *the system (5.13) can be represented by the functional abstract problem (5.4). We can see that all hypotheses of Theorem 5.6.2 are fulfilled. Consequently, problem (5.13) has a unique mild solution defined on* $(-\infty,3]$. *Moreover, problem (5.13) is generalized Ulam-Hyers-Rassias stable.*

Chapter 6

Controllability Results for Second-Order Integro-differential Equations with State-Dependent Delay

6.1 Introduction

In this chapter, we discuss the approximate controllability and complete controllability for second-order Integro-differential equations with state-dependent delay described by

$$\begin{cases} \vartheta''(\varsigma) = A(\varsigma)\vartheta(\varsigma) + \mathcal{K}\left(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (\Psi\vartheta)(\varsigma)\right) + \int_0^\varsigma \Upsilon(\varsigma,s)\vartheta(s)ds + \mathcal{P}u(\varsigma), \text{ if } \varsigma \in J, \\ \vartheta'(0) = \zeta_0 \in E, \;\; \vartheta(\varsigma) = \Phi(\varsigma), \text{ if } \varsigma \in \mathbb{R}_-, \end{cases} \tag{6.1}$$

where $J = [0,T]$, $A(\varsigma) : D(A(\varsigma)) \subset E \to E$, $\Upsilon(\varsigma, s)$ are closed linear operators on E, with dense domain $D(A(\varsigma))$, which is independent of t, and $D(A(s)) \subset D(\Upsilon(\varsigma, s))$, the operator Ψ is defined by

$$(\Psi\vartheta)(\varsigma) = \int_0^T \Xi(\varsigma, s, \vartheta(s))ds, \;\; a > 0,$$

the nonlinear terms $\Xi : J \times J \times E \to E,\ \mathcal{K} : J \times \mathcal{B} \times E \to E,\ \Phi : \mathbb{R}_- \to E,$ $\rho : J \times \mathcal{B} \to (-\infty, \infty)$, are a given functions, the control function u is give function in $L^2(J,U)$ Banach space of admissible control with U as a Banach space. $\mathcal{P}$ is a bounded linear operator from U into E, and $(E, \|\cdot\|)$ is a Banach space.

6.2 Preliminaries

In this section, we will go through the essential concepts, notations, and mathematical tools that will be utilized throughout the article. This covers definitions, fixed point theorems, and significant results that form the basis of our study.

Let $C(J,E)$ be the Banach space of continuous functions y mapping J into E.
Next, we consider the second-order integro-differential systems

$$\begin{aligned} z''(\varsigma) &= A(\varsigma)z(\varsigma) + \int_0^{\varsigma} \Upsilon(\varsigma,\tau)z(\tau)d\tau, \quad 0 \le \varsigma \le T, \\ z(0) &= 0, \quad z'(0) = x \in E, \end{aligned} \tag{6.2}$$

This problem was discussed in [65]. We denote $\Delta = D_{\Xi} = \{(\varsigma, s) : 0 \le s \le \varsigma \le T\}$. We now introduce some conditions fulfilling the operator Υ:

(B1) For each $0 \le s \le \varsigma \le T,\ \Upsilon(\varsigma,s) : D(A(\varsigma)) \to E$ is a bounded linear operator, for every $z \in D(A), \Upsilon(\cdot,\cdot)z$ is continuous and

$$\|\Upsilon(\varsigma,s)z\| \le b\|z\|_{[D(A)]},$$

for $b > 0$ which is a constant independent of $(s,\varsigma) \in \Delta$.

(B2) There exists $L_{\Upsilon} > 0$ such that

$$\|\Upsilon(\varsigma_2,s)z - \Upsilon(\varsigma_1,s)z\| \le L_{\Upsilon}|\varsigma_2 - \varsigma_1|\,\|z\|_{[D(A)]},$$

for all $z \in D(A), 0 \le s \le \varsigma_1 \le \varsigma_2 \le T$.

(B3) There exists $b_1 > 0$ such that

$$\left\| \int_{\sigma}^{\varsigma} S(\varsigma,s)\Upsilon(s,\sigma)z ds \right\| \le b_1\|z\|, \text{ for all } z \in D(A).$$

Under these conditions, it has been established that there exists a resolvent operator $(\mathcal{Q}(\varsigma, s))_{\varsigma \geq s}$ associated with the systems (6.2). From now on, we are going to consider that such a resolvent operator exists, and we adopt its properties as a definition.

Definition 6.2.1 ([65]). *A family of bounded linear operators $(\mathcal{Q}(\varsigma, s))_{\varsigma \geq s}$ on E is said to be a resolvent operator for the systems (6.2) if it satisfies:*

(a) The map $\mathcal{Q} : \Delta \to \mathcal{L}(E)$ is strongly continuous, $\mathcal{Q}(\varsigma, \cdot)z$ is continuously differentiable for all $z \in E$, $\mathcal{Q}(s, s) = 0$, $\frac{\partial}{\partial \varsigma}\mathcal{Q}(\varsigma, s)\big|_{\varsigma = s} = I$ and $\frac{\partial}{\partial s}\mathcal{Q}(\varsigma, s)\big|_{s=\varsigma} = -I$.

(b) Assume $x \in D(A)$. The function $\mathcal{Q}(\cdot, s)x$ is a solution for the systems (6) and (7). This means that

$$\frac{\partial^2}{\partial \varsigma^2}\mathcal{Q}(\varsigma, s)x = A(\varsigma)\mathcal{Q}(\varsigma, s)x + \int_s^\varsigma \Upsilon(\varsigma, \tau)\mathcal{Q}(\tau, s)x d\tau,$$

for all $0 \leq s \leq \varsigma \leq T$.

It follows from condition (a) that there are constants $M_{\mathcal{Q}} > 0$ and $\widetilde{M_{\mathcal{Q}}} > 0$ such that

$$\|\mathcal{Q}(\varsigma, s)\| \leq M_{\mathcal{Q}}, \quad \left\|\frac{\partial}{\partial s}\mathcal{Q}(\varsigma, s)\right\| \leq \widetilde{M_{\mathcal{Q}}}, \quad (\varsigma, s) \in \Delta.$$

Moreover, the linear operator

$$G(\varsigma, \tau)x = \int_\tau^\varsigma \Upsilon(\varsigma, s)\mathcal{Q}(s, \tau)x ds, x \in D(A), 0 \leq \tau \leq \varsigma \leq T,$$

can be extended to E. Portraying this expansion by the similar notation $G(\varsigma, \tau), G : \Delta \to \mathcal{L}(E)$ is strongly continuous, and it is verified that

$$\mathcal{Q}(\varsigma, \tau)x = S(\varsigma, \tau) + \int_\tau^\varsigma S(\varsigma, s)G(s, \tau)x ds, \text{ for all } x \in E.$$

It follows from this property that $\mathcal{Q}(\cdot)$ is uniformly Lipschitz continuous, that is, there exists a constant $L_{\mathcal{Q}} > 0$ such that

$$\|\mathcal{Q}(\varsigma + h, \tau) - \mathcal{Q}(\varsigma, \tau)\| \leq L_{\mathcal{Q}}|h|, \text{ for all } \varsigma,\ \varsigma + h,\ \tau \in [0, T].$$

We assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were introduced by Hale and Kato in [73].

(A_1) If $y \in C$ and $y_0 \in \mathcal{B}$, then for every $\varsigma \in J$, the following conditions hold:

- (i) $y_\varsigma \in \mathcal{B}$,
- (ii) There exists a positive constant H such that $|y(\varsigma)| \leq H \|y_\varsigma\|_{\mathcal{B}}$,
- (iii) There exist two functions $L(\cdot)$ and $M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$independent of y with L continuous and bounded and M locally bounded such that:

$$\|y_\varsigma\|_{\mathcal{B}} \leq L(\varsigma) \sup\{|y(s)| : 0 \leq s \leq t\} + M(\varsigma) \|y_0\|_{\mathcal{B}} .$$

(A_2) For the function y in (A_1) , y_ς is a $\mathcal{B}$ - valued continuous function on $\mathbb{R}^+$.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$L_* = \sup\{L(\varsigma) : \varsigma \in J\},$$
$$M_* = \sup\{M(\varsigma) : \varsigma \in J\},$$

and

$$\aleph = \max\{L_*, M_*\}.$$

We define the space

$$C_\theta := \{\phi \in C(\mathbb{R}^-, E) : \lim_{\tau \to -\infty} \phi(\tau) \text{ exist in } E\},$$

endowed with the norm

$$\|\phi\|_\theta = \sup\{|\phi(\tau)| : \tau \leq 0\}.$$

Then, the axioms $(A_1) - (A_3)$ are satisfied in the space C_θ. So in all what follows, we consider the phase space $\mathcal{B} = C_\theta$, and let

$$\mathcal{X} = C(\widetilde{J}, E) = \left\{ y : \widetilde{J} \to E \ : \ y|_{\mathbb{R}^-} \in \mathcal{B}, \ y|_J \in C(J, E) \right\},$$

such that

$$\|y\|_{\mathcal{X}} = \sup_{\varsigma \in \widetilde{J}} \{\|y(\varsigma)\|\} .$$

6.3 Existence of mild solutions

In this part, we prove the existence of mild solutions system of the problem:

$$\begin{cases} \vartheta''(\varsigma) = A\vartheta(\varsigma) + \mathcal{K}\left(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (\Psi\vartheta)(\varsigma)\right) + \int_0^\varsigma \Upsilon(\varsigma, s)\vartheta(s)ds, \text{ if } \varsigma \in J, \\ \\ \vartheta'(0) = \zeta_0 \in E, \ \ \vartheta(\varsigma) = \Phi(\varsigma), \text{ if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{6.3}$$

In [120], the authors have investigated the existence of mild solution of system (6.3) and they used the Leray-Schauder's alternative theorem and Krasnoselskii's theorem. So we will weaken the conditions (in particular the compactness property) by using Darbo fixed point theorem.

Definition 6.3.1. *A function $\vartheta \in \mathcal{X}$ is called a mild solution of problem (6.3), if it satisfies*

$$\vartheta(\varsigma) = \begin{cases} -\frac{\partial \mathcal{Q}(\varsigma,s)\Phi(0)}{\partial s}\Big|_{s=0} + \mathcal{Q}(\varsigma,0)\zeta_0 + \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s, \vartheta_{\rho(s,\vartheta_s)}, (\Psi\vartheta)(s))ds; \ \textit{if } \varsigma \in J, \\ \\ \Phi(\varsigma); \ \textit{if } \varsigma \in \mathbb{R}_-. \end{cases}$$

The following assumption will be needed throughout the paper:

$(C1)$ $\mathcal{K} : J \times \mathcal{B} \times E \to E$ is a Carathéodory function and there exist positive constants $\xi_1, \ \xi_2$ and continuous nondecreasing functions $\psi_{\mathcal{K}}^1, \psi_{\mathcal{K}}^2 : J \to (0, +\infty)$ such that:

$$||\mathcal{K}(\varsigma, \vartheta_1, \vartheta_2)|| \le \xi_1\psi_{\mathcal{K}}^1(\|\vartheta_1\|_{\mathcal{B}}) + \xi_2\psi_{\mathcal{K}}^2(\|\vartheta_2\|), \quad \text{for } \vartheta_1 \in \mathcal{B}, \ \vartheta_2 \in E.$$

And there exists a positive constant $l_{\mathcal{K}}$, such that for any bounded set $B \subset E$, and $B_\varsigma \in \mathcal{B}$ and each $\varsigma \in \mathbb{R}$, we have

$$\mu(\mathcal{K}(\varsigma, B_\varsigma, \Psi(B(\varsigma)))) \le l_{\mathcal{K}}\mu(B).$$

$(C2)$ The function $\Xi : D_\Xi \times E \to E$ is continuous and there exists $\Xi_{c_1} > 0$, such that

$$\|\Xi(\varsigma, s, \vartheta_1) - \Xi(\varsigma, s, \vartheta_2)\| \le \Xi_{c_1}\|\vartheta_1 - \vartheta_2\|,$$

for each $(\varsigma, s) \in D_\Xi$ and $\vartheta_1, \ \vartheta_2 \in E$, where

$$\sup_{D_\Xi}\{\|\Xi(\varsigma, s, 0)\|\} = \Xi^* < \infty.$$

$(C3)$ Assume that $(B1)-(B3)$ hold, and there exist $M_{\mathcal{Q}}, \widetilde{M_{\mathcal{Q}}} \geq 1$ and $\mu \geq 0$, such that

$$\|\mathcal{Q}(\varsigma, s)\|_{\Upsilon(E)} \leq M_{\mathcal{Q}} e^{-\mu\varsigma},$$

and

$$\left\|\frac{\partial \mathcal{Q}(\varsigma, s)}{\partial s}\right\|_{\Upsilon(E)} \leq \widetilde{M_{\mathcal{Q}}} e^{-\mu\varsigma}.$$

(C_H) Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$. We assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Moreover we assume the following assumption and hypothesis:

• (H_Φ) The function $t \to \Phi_\varsigma$ is continuous from $\mathcal{R}(\rho^-)$into $\mathcal{B}$ and there exists a continuous and bounded function $L^\Phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\Phi_\varsigma\|_{\mathcal{B}} \leq L^\Phi(\varsigma)\|\Phi\|_{\mathcal{B}}, \quad \text{for every } \varsigma \in \mathcal{R}\left(\rho^-\right).$$

Remark 6.3.1. *The condition (H_Φ), is frequently verified by continuous and bounded functions. For more details, see for instance [90].*

Lemma 6.3.1 ([88]). *If $y : (-\infty, +\infty) \to E$ is a function such that $y_0 = \Phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq \left(M + \mathcal{L}^\Phi\right)\|\Phi\|_{\mathcal{B}} + l \sup\{|y(\theta)| \; ; \; \theta \in [0, \max\{0, s\}]\}, \;\; s \in \mathcal{R}\left(\rho^-\right) \cup J,$$

where $\mathcal{L}^\Phi = \sup_{\varsigma \in \mathcal{R}(\rho^-)} \mathcal{L}^\Phi(\varsigma)$.

Theorem 6.3.1. *Assume that the conditions $(C1)-(C3)$ and (C_H) are satisfied. Then, the system (6.3) has at least one mild solution.*

Proof. Firstly we define on $\mathcal{X}$ measures of non compactness by

$$\mu_C(S) = \omega_0(S) + \sup\left\{e^{-\tau\Sigma(\varsigma)}\mu(S(\varsigma))\right\},$$

with $\tau > 1,\ \Sigma(\varsigma) = 4M_{\mathcal{Q}} l_{\mathcal{K}} \varsigma$, $S(\varsigma) = \{v(\varsigma) \in E \; ; \; v \in S\}$, and $\omega^T(v,\epsilon)$ denotes the modulus of continuity of the function v on the interval $[-T,T]$, namely,

$$\begin{array}{lcl} \omega^T(v,\epsilon) & = & \sup\{\|e^{-\kappa_1}v(\kappa_1) - e^{-\kappa_2}v(\kappa_2)\| \; ; \; \kappa_1,\kappa_2 \in [-T,T], \;\text{ with } |\kappa_1-\kappa_2| \leq \epsilon\}, \\ \omega^T(S,\epsilon) & = & \sup\{\omega^T(v,\epsilon) \; ; \; v \in S\}, \\ \omega_0(S) & = & \lim_{\epsilon\to 0}\{\omega^T(S,\epsilon)\}. \end{array}$$

Notice that if the set S is equicontinuous, then $\omega_0(S) = 0$.
Now, transform the problem (6.3) into a fixed point problem and define the operator $\Theta_1 : \mathcal{X} \to \mathcal{X}$ by:

$$\Theta_1\vartheta(\varsigma) = \begin{cases} -\frac{\partial \mathcal{Q}(\varsigma,s)\Phi(0)}{\partial s}\Big|_{s=0} + \mathcal{Q}(\varsigma,0)\zeta_0 \\ +\int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))ds; \text{ if } \varsigma \in J, \\ \\ \Phi(\varsigma), \text{ if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{6.4}$$

Let $x(\cdot) : (-\infty, T] \to E$ be the function defined by:

$$x(\varsigma) = \begin{cases} -\frac{\partial \mathcal{Q}(\varsigma,s)\Phi(0)}{\partial s}\Big|_{s=0} + \mathcal{Q}(\varsigma,0)\zeta_0, & \text{if } \varsigma \in J, \\ \\ \Phi(\varsigma), & \text{if } \varsigma \in \mathbb{R}_-. \end{cases}$$

Then, $x_0 = \Phi$, and for each $w \in \mathcal{X}$, with $w(0) = 0$, we denote by $\overline{w}$ the function

$$\overline{w}(\varsigma) = \begin{cases} w(\varsigma), & \text{if } \varsigma \in \mathbb{R}^+, \\ \\ 0, & \text{if } \varsigma \in \mathbb{R}_-. \end{cases}$$

If ϑ satisfies (6.4), we can decompose it as $\vartheta(\varsigma) = w(\varsigma) + x(\varsigma)$, which implies $\vartheta_\varsigma = w_\varsigma + x_\varsigma$, and the function $w(\cdot)$ satisfies

$$w(\varsigma) = \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s, w_{\rho(s,w_s+x_s)} + x_{\rho(s,w_s+x_s)}, \Psi(w+x)(s))ds; \text{ if } \varsigma \in J.$$

Set

$$\Omega = \{w \in \mathcal{X} \ : \ w(0) = 0\}.$$

Let the operator $\widetilde{\Theta}_1 : \Omega \to \Omega$ defined by

$$\widetilde{\Theta}_1 w(\varsigma) = \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s, w_{\rho(s,w_s+x_s)} + x_{\rho(s,w_s+x_s)}, \Psi(w+x)(s))ds, \text{ if } \varsigma \in J.$$

The operator Θ_1 has a fixed point is equivalent to say that $\widetilde{\Theta}_1$ has one, so it turns to prove that $\widetilde{\Theta}_1$ has a fixed point. We shall check that operator $\widetilde{\Theta}_1$ satisfies all conditions of Darbo's theorem.

Let $\Pi_{\theta'} = \{w \in \Omega : \|w\|_\Omega \leq \theta'\}$, with

$$M_{\mathcal{Q}}\big(\xi_1 \psi^1_{\mathcal{K}}(\eta^*_{\theta'}) + \xi_2 \psi^2_{\mathcal{K}}(\overline{\eta}^*)\big)T \leq \theta',$$

such that $\eta^*_{\theta'},\ \overline{\eta}^*$ are constants, they will be specific later.
The set $\Pi_{\theta'}$ is bounded, closed and convex. We have divided the proof into four steps.

Step 1 : $\widetilde{\Theta}_1(\Pi_{\theta'}) \subset \Pi_{\theta'}$.
For $w \in \Pi_{\theta'}$, $\varsigma \in J$ and by $(C1)-(C3)$, we have

$$\begin{aligned}
\big\|w_{\rho(s,w_s+x_s)} + x_{\rho(s,w_s+x_s)}\big\|_{\mathcal{B}} &\leq \big\|w_{\rho(s,w_s+x_s)}\big\|_{\mathcal{B}} + \big\|x_{\rho(s,w_s+x_s)}\big\|_{\mathcal{B}} \\
&\leq L(\varsigma)\sup_{[0,s]}|w(\varsigma)| + \big(M(\varsigma)+\mathcal{L}^{\Phi}\big)\|\Phi\|_{\mathcal{B}} \ + L(\varsigma)\sup_{[0,s]}\|x(\theta)\| \\
&\leq L_*\theta' + \big(M_* + \mathcal{L}^{\Phi}\big)\|\Phi\|_{\mathcal{B}} \\
&\quad + L_*\big(\widetilde{M_{\mathcal{Q}}}\|\Phi_0\| + M_{\mathcal{Q}}\|\zeta_0\|\big)H\|\Phi\|_{\mathcal{B}} \\
&\leq L_*\theta' + \Big[M_* + \mathcal{L}^{\Phi} + L_*\big(\widetilde{M_{\mathcal{Q}}}\|\Phi_0\| + M_{\mathcal{Q}}\|\zeta_0\|\big)H\Big]\|\Phi\|_{\mathcal{B}} \\
&= \eta^*_{\theta'},
\end{aligned}$$

and

$$\|\Psi(w+x)(s)\| \leq a\Xi_{c_1}\big(\theta' + \widetilde{M_{\mathcal{Q}}}\|\Phi_0\| + M_{\mathcal{Q}}\|\zeta_0\|\big) + a\Xi^* = \overline{\eta}^*.$$

Then,

$$\|\widetilde{\Theta}_1 w(\varsigma)\| \leq M_{\mathcal{Q}}\Big[\psi^1_{\mathcal{K}}(\eta^*_{\theta'})\xi_1 + \psi^2_{\mathcal{K}}(\overline{\eta}^*)\xi_2\Big]T.$$

Thus,

$$\|\widetilde{\Theta}_1 w\|_\Omega \leq \theta'.$$

Therefore $\widetilde{\Theta}_1(\Pi_{\theta'}) \subset \Pi_{\theta'}$, implies that $\widetilde{\Theta}_1(\Pi_{\theta'})$ is bounded.

Step 2: $\widetilde{\Theta}_1$ is continuous.
Let $\{w_m\}_{m\in\mathbb{N}}$ be a sequence such that $w_m \to w^*$ in $\Pi_{\theta'}$. At the first, we study the convergence of the sequences $\left(w^m_{\rho(s,w^m_s)}\right)_{m\in\mathbb{N}}$, $s \in J$. If $s \in J$ is such that $\rho\,(s, w_s) > 0$, then we have

$$\begin{aligned}
\big\|w^m_{\rho(s,w^m_s)} - w^*_{\rho(s,w^*_s)}\big\|_{\mathcal{B}} &\leq \big\|w^m_{\rho(s,w^n_s)} - w^*_{\rho(s,w^m_s)}\big\|_{\mathcal{B}} + \big\|w^*_{\rho(s,w^m_s)} - w^*_{\rho(s,w^*_s)}\big\|_{\mathcal{B}} \\
&\leq L\,\|w_m - w^*\| + \big\|w^*_{\rho(s,w^n_s)} - w^*_{\rho(s,w^*_s)}\big\|_{\mathcal{B}},
\end{aligned}$$

which proves that $w^m_{\rho(s,w^m_s)} \to w^*_{\rho(s,w_s)}$ in $\mathcal{B}$, as $m \to \infty$, for every $s \in J$ such that $\rho\,(s, w_s) > 0$. Similarly, if $\rho\,(s, w_s) < 0$, we get

$$\left\| w^m_{\rho(s,w^m_s)} - w^*_{\rho(s,w_s)} \right\|_{\mathcal{B}} = \left\| \Phi^m_{\rho(s,w^m_s)} - \Phi_{\rho(s,w^*_s)} \right\|_{\mathcal{B}} = 0,$$

which also shows that $w^m_{\rho(s,w^m_s)} \to w^*_{\rho(s,w_s)}$ in $\mathcal{B}$, as $m \to \infty$, for every $s \in J$ such that $\rho\,(s, w_s) < 0$. Then for $\varsigma \in J$, we have

$$\begin{aligned} \|(\widetilde{\Theta}_1 w^m)(\varsigma) - (\widetilde{\Theta}_1 w^*)(\varsigma)\| \le M_{\mathcal{Q}} \int_0^{\varsigma} &\|\mathcal{K}(s, w^m_{\rho(s,w^m_s)} + x_{\rho(s,w^m_s+x_s)}, H(w^m + x)(s)) \\ &- \mathcal{K}(s, (w^*_{\rho(s,w^*_s)} + x_{\rho(s,w^*_s+x_s)}), H(w^* + x)(s))\| ds. \end{aligned}$$

Since Ξ and $\mathcal{K}$ are continuous, we obtain that

$$\Xi(\varsigma, s, (w^m + x)(s)) \to \Xi(\varsigma, s, (w^* + x)(s)), \quad as \;\; m \to +\infty,$$

and

$$\|\Xi(\varsigma, s, (w^m + x)(s)) - \Xi(\varsigma, s, (w^* + x)(s))\| \le \Xi^*_{c_1} \|w^m(s) - w^*(s)\|.$$

By the Lebesgue dominated convergence theorem, we have

$$\int_0^{\varsigma} \Xi(\varsigma, s, (w^m + x)(s))ds \xrightarrow[m\to+\infty]{} \int_0^{\varsigma} \Xi(\varsigma, s, (w^* + x)(s))ds.$$

Then, by $(C1)$, we get

$$\mathcal{K}(s, w^m_{\rho(s,w^m_s)} + x_{\rho(s,w^m_s+x_s)}, \Psi(w^m + x)(s)) \xrightarrow[m\to+\infty]{} \mathcal{K}(s, (w^*_{\rho(s,w^*_s)} + x_{\rho(s,w^*_s+x_s)}), \Psi(w^* + x)(s)).$$

By Lebesgue dominated convergence theorem, we obtain

$$\|(\widetilde{\Theta}_1 w^m)(\varsigma) - (\widetilde{\Theta}_1 w^*)(\varsigma)\| \to 0, \quad as \;\; m \to +\infty.$$

Thus, $\widetilde{\Theta}_1$ is continuous.

Step 3: $\widetilde{\Theta}_1$ is μ_C-contraction.
Let Π be a bounded equicontinuous subset of $\Pi_{\theta'}$, $w \in \Pi$, and $\kappa_1, \kappa_2 \in J$, with $\kappa_2 > \kappa_1$, we have

$$
\begin{aligned}
&\left\|\widetilde{\Theta}_1 w(\kappa_1) - \widetilde{\Theta}_1 w(\kappa_2)\right\| \\
&\leq \int_{\kappa_1}^{\kappa_2} \|\mathcal{Q}(\kappa_2, s)\| \left\|\mathcal{K}(s, w_{\rho(s,w_s+x_s)} + x_{\rho(s,w_s+x_s)}, \Psi(w+x)(s))\right\| ds \\
&\quad + \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_2, s) - \mathcal{Q}(\kappa_1, s)\| \left\|\mathcal{K}(s, w_{\rho(s,w_s+x_s)} + x_{\rho(s,w_s+x_s)}, \Psi(w+x)(s))\right\| ds \\
&\leq \left[\psi^1_{\mathcal{K}}(\eta^*_{\theta'})\xi_1 + \psi^2_{\mathcal{K}}(\overline{\eta}^*)\xi_2\right] \left(M_{\mathcal{Q}}|\kappa_2 - \kappa_1| + \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_2, s) - \mathcal{Q}(\kappa_1, s)\| ds \right).
\end{aligned}
$$

By the strong continuity of $\mathcal{Q}(\cdot)$, we get

$$
\left\|\widetilde{\Theta}_1 w(\kappa_1) - \widetilde{\Theta}_1 w(\kappa_2)\right\| \to 0, \ as\ \kappa_1 \to \kappa_2.
$$

Thus $\widetilde{\Theta}_1 (\Pi)$ is equicontinuous, then $\omega_0 \left(\widetilde{\Theta}_1(\Pi)\right) = 0$.
Now, for $w \in \Pi$, and for any $\varrho > 0$, there exist a sequence $\{w^k\}_{k=0}^{\infty} \subset \Pi$ such that for $\varsigma \in J$. We have

$$
\begin{aligned}
\mu(\widetilde{\Theta}_1(\Pi)(\varsigma)) &\leq \mu\left(\left\{ \int_0^{\varsigma} \mathcal{Q}(\varsigma, s)\mathcal{K}(s, w_{\rho(s,w_s)} + x_{\rho(s,w_s+x_s)}, \Psi(w+x)(s))ds \ ;\ w \in \Pi \right\} \right) \\
&\leq 2\mu \left(\left\{ \int_0^{\varsigma} \mathcal{Q}(\varsigma, s)\mathcal{K}(s, w^k_{\rho(s,w^k_s)} + x_{\rho(s,w^k_s+x_s)}, \Psi(w^k+x)(s))ds \ ;\ w \in \Pi \right\} \right) \\
&\quad + \varrho \\
&\leq \int_0^{\varsigma} 4M_{\mathcal{Q}} l_{\mathcal{K}} \mu(\{\Pi(s)\})ds + \varrho \\
&\leq \int_0^{\varsigma} e^{4\tau M_{\mathcal{Q}} l_{\mathcal{K}} s} e^{-4\tau M_{\mathcal{Q}} l_{\mathcal{K}} s} 4M_{\mathcal{Q}} l_{\mathcal{K}} \mu(\Pi(s))ds + \varrho \\
&\leq \int_0^{\varsigma} 4M_{\mathcal{Q}} l_{\mathcal{K}} e^{4\tau M_{\mathcal{Q}} l_{\mathcal{K}} s} \sup_{s\in[0,\varsigma]} e^{-4\tau M_{\mathcal{Q}} l_{\mathcal{K}} s} \mu(\Pi(s))ds + \varrho \\
&\leq \mu_C(\Pi) \int_0^{\varsigma} \left(\frac{e^{4\tau M_{\mathcal{Q}} l_{\mathcal{K}} s}}{\tau} \right)' ds + \varrho \\
&\leq \frac{e^{4\tau M_{\mathcal{Q}} l_{\mathcal{K}} t}}{\tau} \mu_C(\Pi) + \varrho.
\end{aligned}
$$

Since ϱ is arbitrary, we get

$$
\mu(\widetilde{\Theta}_1(\Pi)(\varsigma)) \quad \leq \quad \frac{e^{4\tau M_{\mathcal{Q}} l_{\mathcal{K}} t}}{\tau} \mu_C(\Pi).
$$

Thus,

$$\mu_C(\widetilde{\Theta}_1(\Pi)) \ \leq \ \frac{1}{\tau}\mu_C(\Pi).$$

As a consequence of Theorem 1.3.4, we deduce that $\widetilde{\Theta}_1$ has at least one fixed point w^*. Then $\vartheta^* = w^* + x$ is a fixed point of the operator Θ_1, which is a mild solution of problem (6.3).

□

6.4 Controllability results

6.4.1 Complete controllability

Definition 6.4.1. *The system (6.1) is said to be exactly controllable on the interval J, if for every function $\Phi \in \mathcal{B}$ and $\zeta_0,\ \hat{v} \in E$, there is some control $u \in L^2(J,E)$ such that the mild solution v of this problem satisfies the terminal condition $v(T) = \hat{v}$.*

We will need to introduce the following hypotheses:

$(C4)$ (i) The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^T \mathcal{Q}(T,s)\mathcal{P}u(s)ds,$$

has a pseudo-inverse operator W^{-1}, which takes values in $L^2(J,U)\diagdown Ker(W)$,

(ii) There exist positive constants m_1, m_2, such that

$$\|\mathcal{P}\| \leq m_1 \ \ and \ \ \|W^{-1}\| \leq m_2.$$

(iii) There exist $q_w > 0,\ m_{\mathcal{P}} > 0$, such that for any bounded sets $\widetilde{M_1} \subset E,\ \widetilde{M_2} \subset U$,

$$\mu((W^{-1}\widetilde{M_1})(\varsigma)) \leq q_w\mu(\widetilde{M_1}), \quad \mu((\mathcal{P}\widetilde{M_2})(\varsigma)) \leq m_{\mathcal{P}}\mu(\widetilde{M_2}(\varsigma)).$$

$(C5)$ There exists a positive constant ρ, such that $\varphi_1^\rho \leq \rho$, with

$$\varphi_1^\rho = M_{\mathcal{Q}}\Bigg[\psi_{\mathcal{K}}^1(\eta_\rho^*)\xi_1 + \psi_{\mathcal{K}}^2(\widetilde{\eta}^*)\xi_2 + m_1m_2\Bigg(\rho + \widetilde{M_{\mathcal{Q}}}\|\Phi_0\| + M_{\mathcal{Q}}\|\zeta_0\| + M_{\mathcal{Q}}\psi_{\mathcal{K}}^1(\eta_\rho^*)\xi_1 + M_{\mathcal{Q}}\psi_{\mathcal{K}}^2(\widetilde{\eta}^*)\xi_2\Bigg)\Bigg],$$

$$\eta_\rho^* = L_*\rho + \left[M_* + \mathcal{L}^\Phi + L_*\big(\widetilde{M_\mathcal{Q}}\|\Phi_0\| + M_\mathcal{Q}\|\zeta_0\|\big)H\right]\|\Phi\|_\mathcal{B},$$

and

$$\widetilde{\eta}^* = a\Xi_{c_1}^*\big(\rho + \widetilde{M_\mathcal{Q}}\|\Phi_0\| + M_\mathcal{Q}\|\zeta_0\|\big) + a\Xi^*.$$

Theorem 6.4.1. *Suppose that the hypotheses $(C1)-(C5)$ and (C_H) are valid. Then the problem (6.1) is exactly controllable.*

Proof. Since the calculating techniques were covered in-depth in the previous proofs, the steps of the proof won't be described in detail. We define in $\mathcal{X}$ measures of noncompactness as in Section 4, but we change Σ by $\varkappa$, such that

$$\varkappa(\varsigma) = 4M_\mathcal{Q}\left(l_\mathcal{K} + m_\mathcal{P}(M_\mathcal{Q}l_\mathcal{K}T)q_w\right)\varsigma.$$

Now, using $(C4)$ we define the control:

$$u_\vartheta(\varsigma) = W^{-1}\Bigg(\vartheta(T) + \frac{\partial\mathcal{Q}(T,s)\Phi(0)}{\partial s}\bigg|_{s=0} - \mathcal{Q}(T,0)\zeta_0 - \int_0^T \mathcal{Q}(T,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))ds\Bigg).$$

We shall show that when using the control $u(\cdot)$, the operator $\Upsilon_3' : \mathcal{X} \to \mathcal{X}$ defined by:

$$\Upsilon_3'\vartheta(\varsigma) = -\frac{\partial\mathcal{Q}(\varsigma,s)\Phi(0)}{\partial s}\bigg|_{s=0} + \mathcal{Q}(\varsigma,0)\zeta_0 + \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))ds + \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{P}u_\vartheta(s)ds;\ \text{if } \varsigma\in J,$$

has fixed point, this fixed point is a mild solution of system (6.1), and this implies that the system is controllable.

If ϑ is a fixed point of Υ_3', then similar transformation to that in the Proof of Theorem 6.3.1, give the following decomposition $\vartheta(\varsigma) = y(\varsigma) + x(\varsigma)$, which implies $\vartheta_\varsigma = y_\varsigma + x_\varsigma$.
Let the operator $\Upsilon_3 : \Omega \to \Omega$ defined by

$$\Upsilon_3\vartheta(\varsigma) = \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))ds + \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{P}u_\vartheta(s)ds;\ \text{if } \varsigma\in J.$$

It thus becomes necessary to demonstrate that Υ_3 has a fixed point since the operator Υ_3' having a fixed point is similar to saying that Υ_3 has one. We will make sure operator Υ_3' satisfies all of the conditions of Darbo's theorem.
Let $B_\rho = B(0,\rho) = \{y \in \Omega \ : \ \|y\|_\Omega \leq \rho\}$, then the set B_ρ is closed, bounded and convex.

Step 1 : $\Upsilon_3(B_\rho) \subset B_\rho$.
For $\varsigma \in J$ and $y \in B_\rho$, we have

$$\begin{aligned}
\|\Upsilon_3 y(\varsigma)\| &\leq \int_0^\varsigma \|\mathcal{Q}(\varsigma,s)\| \left\|\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))\right\| ds + \int_0^\varsigma \|\mathcal{Q}(\varsigma,s)\| \, \|\mathcal{P}u_\vartheta(s)\| \, ds \\
&\leq M_{\mathcal{Q}}\Bigg(\psi^1_{\mathcal{K}}(\eta^*_\rho)\xi_1 + \psi^2_{\mathcal{K}}(\widetilde{\eta}^*)\xi_2 \\
&\quad + m_1 m_2 \left(\rho + \widetilde{M_{\mathcal{Q}}}\|\Phi_0\| + M_{\mathcal{Q}}\|\zeta_0\| + M_{\mathcal{Q}}\psi^1_{\mathcal{K}}(\eta^*_\rho)\xi_1 + M_{\mathcal{Q}}\psi^2_{\mathcal{K}}(\widetilde{\eta}^*)\xi_2 \right) \Bigg).
\end{aligned}$$

Thus, we deduce from $(C5)$ that $\Upsilon_3(B_\rho) \subset B_\rho$ and $\Upsilon_3(B_\rho)$ is bounded.

Step 2: Υ_3 is continuous.
Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y_*$ in B_ρ.
Since $\mathcal{K}$, Ξ, $\mathcal{P}$ are continuous, and by the Lebegue dominated convergence theorem, we have

$$\int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{P}u_{y_n+x}(s)ds \xrightarrow[n\to+\infty]{} \int_0^\varsigma \mathcal{Q}(\varsigma,s)\mathcal{P}u_{y_*+x}(s)ds.$$

Then, similar to Step 2 in Proof of Theorem 6.3.1, we get

$$\|(\Upsilon_3 y_n)(\varsigma) - (\Upsilon_3 y_*)(\varsigma)\| \to 0, \quad as \ \ n \to +\infty.$$

Consequently, Υ_3 is continuous.

Step 3: Υ_3 is μ_C-contraction operator.
Let Π be a bounded equicontinuous subset of B_ρ, $y \in \Pi$, and $\kappa_1, \kappa_2 \in J$, with $\kappa_2 > \kappa_1$, we have

$$\begin{aligned}
&\left\| \int_0^{\kappa_2} \mathcal{Q}(\kappa_2, s)\mathcal{P}u_{y_n+x}(s)ds - \int_0^{\kappa_1} \mathcal{Q}(\kappa_1, s)\mathcal{P}u_{y_n+x}(s)ds \right\| \\
\leq\ & \int_{\kappa_1}^{\kappa_2} \|\mathcal{Q}(\kappa_2, s)\| \, \|\mathcal{P}u_{y_n+x}(s)\| \, ds + \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_2, s) - \mathcal{Q}(\kappa_1, s)\| \, \|\mathcal{P}u_{y_n+x}(s)\| \, ds \\
\leq\ & m_1 m_2 \left(\rho + \widetilde{M_{\mathcal{Q}}} \|\Phi_0\| + M_{\mathcal{Q}} \|\zeta_0\| + M_{\mathcal{Q}} \psi^1_{\mathcal{K}}(\eta^*_\rho)\xi_1 + M_{\mathcal{Q}} \psi^2_{\mathcal{K}}(\widetilde{\eta}^*)\xi_2 \right) \\
& \times \left(M_{\mathcal{Q}} |\kappa_2 - \kappa_1| + \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_2, s) - \mathcal{Q}(\kappa_1, s)\| \, ds \right) \xrightarrow[\kappa_1 \to \kappa_2]{} 0.
\end{aligned}$$

Thus $\{\Upsilon_3(\Pi)\}$ is equicontinuous, then $\omega_0 (\Upsilon_3(\Pi)) = 0$. Now for any $\varrho > 0$ there exist a sequence $\{y_k\}_{k=0}^{\infty} \subset \Pi$, such that for $\varsigma \in J$, we get

$$\begin{aligned}
\mu(\Upsilon_3(\Pi)(\varsigma)) &\leq 4 \int_0^{\varsigma} M_{\mathcal{Q}}(l_{\mathcal{K}} + m_{\mathcal{P}}(M_{\mathcal{Q}} l_{\mathcal{K}} T) q_y) \mu(\{\Pi(s)\}) ds + \varrho \\
&\leq \frac{e^{\tau \varkappa(\varsigma)}}{\tau} \mu_C(\Pi) + \varrho.
\end{aligned}$$

Therefore,

$$\mu_C(\Upsilon_3(\Pi)) \leq \frac{1}{\tau} \mu_C(\Pi).$$

We come to the conclusion that Υ_3 has at least one fixed point y^* according to Darbo's fixed point theorem. Consequently, $\vartheta^* = y^* + x$ is a fixed point of the operator Υ_3', implies that the system is exactly controllable.

□

6.4.2 Approximate Controllability

Definition 6.4.2. *For $(\Phi, \zeta_0) \in \mathcal{B} \times E$, system (6.1) is said to be approximately controllable on the interval $J = [0, T]$ if $\mathcal{R}(T, \Phi, \zeta_0)$ is dense in E, i.e. $\overline{\mathcal{R}(T, \Phi, \zeta_0)} = E$, where $\mathcal{R}(T, \Phi, \zeta_0) = \{x(T, \Phi, \zeta_0, u), u(\cdot) \in L^2(J; U)\}$.*

As mentioned in Section 1, we shall study the approximate controllability by using a so-called resolvent operator condition. For this purpose, we introduce the following controllability operator $\Gamma_0^T : E \to E$ and resolvent operator $\mathcal{W}\left(\lambda, \Gamma_0^T\right) : E \to E$ defined by

$$\Gamma_0^T = \int_0^T \mathcal{Q}(T, s)\mathcal{P}\mathcal{P}^* \mathcal{Q}^*(T, s) ds, \quad \mathcal{W}\left(\lambda, \Gamma_0^T\right) = \left(\lambda I + \Gamma_0^T\right)^{-1},$$

where $\mathcal{P}^*$ and $\mathcal{Q}^*$ denote the adjoints of the operators $\mathcal{P}$ and $\mathcal{Q}$ respectively, It is straightforward to see that the operator Γ_0^T is a linear bounded operator. So we assume that the operator $\mathcal{W}\left(\lambda, \Gamma_0^T\right)$ satisfies

(C_0) $\lambda\mathcal{W}\left(\lambda, \Gamma_0^T\right) \longrightarrow 0$ as $\lambda \longrightarrow 0^+$ in the strong operator topology.

From [54], hypothesis (C_0) is equivalent to the fact that the linear control system corresponding to system (6.1) is approximately controllable on $[0, T]$.

Theorem 6.4.2. *The following statements are equivalent:*

- (i) *The linear control system corresponding to system (6.1) is approximately controllable on* $[0, T]$.
- (ii) *If* $\mathcal{W}^*\mathcal{Q}^*(\varsigma, s)z = 0$ *for all* $s,\ \varsigma \in [0, T]$, *with* $s \leq \varsigma$, *then* $z = 0$.
- (iii) *The condition* (C_0) *holds.*

The proof of this theorem is similar to that of ([33], Theorem 2) and ([54], Theorem 4.4.17), so we omit it here. Right now, we can demonstrate that the system (6.1) is approximately controllable.
For any given $\delta^T \in E$, $\lambda \in (0, 1]$, we take the control function $u^\lambda(\varsigma)$ as follows:

$$u^\lambda(\varsigma) = \mathcal{P}^*\mathcal{Q}^*(T, s)\mathcal{W}\left(\lambda, \Gamma_0^T\right)\Delta(\delta^T, \varsigma),$$

where

$$\Delta(\delta^T, \varsigma) = \delta^T + \left.\frac{\partial\mathcal{Q}(\varsigma, s)\Phi(0)}{\partial s}\right|_{s=0} - \mathcal{Q}(\varsigma, 0)\zeta_0 - \int_0^\varsigma \mathcal{Q}(\varsigma, s)\mathcal{K}(s, \vartheta_{\rho(s,\vartheta_s)}, (\Psi\vartheta)(s))ds.$$

Theorem 6.4.3. *Assume that the hypotheses* $(C0) - (C3)$ *and* (C_H) *are satisfied, in addition, the function* f *is uniformly bounded. Then, equation (6.1) is approximately controllable on* $[0, T]$.

Proof. We can observe that system (6.1) has at least one mild solution ρ^λ, based on Theorem 6.3.1. Then, we have

$$\begin{aligned}
\rho^{\lambda}(T) &= -\frac{\partial \mathcal{Q}(T,s)\Phi(0)}{\partial s}\Big|_{s=0} + \mathcal{Q}(T,0)\zeta_0 \\
&\quad + \int_0^T \big(\mathcal{Q}(T,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s)) + \mathcal{P}u(s)\big)\,ds \\
&= -\frac{\partial \mathcal{Q}(T,s)\Phi(0)}{\partial s}\Big|_{s=0} + \mathcal{Q}(T,0)\zeta_0 + \int_0^T \big(\mathcal{Q}(T,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))\big)\,ds \\
&\quad + \int_0^T \mathcal{Q}(T,s)\left(\mathcal{P}^*\mathcal{Q}^*(T,s)\mathcal{W}\left(\lambda,\Gamma_0^T\right)\Delta(\delta^T,T)\right)ds \\
&= \delta^T + (\Gamma_0^T\mathcal{W}\left(\lambda,\Gamma_0^T\right) - I)\Delta(\delta^T,T) \\
&= \delta^T + \lambda\mathcal{W}\left(\lambda,\Gamma_0^T\right)\Delta(\delta^T,T).
\end{aligned}$$

Furthermore, we infer from the uniform boundedness of $\mathcal{K}(\cdot,\cdot,\cdot)$ that there exists $M_{\mathcal{K}} > 0$, such that

$$\int_0^T \|\mathcal{K}(s,\vartheta^{\lambda}_{\rho(s,\vartheta_s)},(\Psi\vartheta^{\lambda})(s))\|^2 ds \le T(M_{\mathcal{K}})^2.$$

Therefore, the sequence $\left\{\mathcal{K}(s,\vartheta^{\lambda}_{\rho(s,\vartheta_s)},(\Psi\vartheta^{\lambda})(s))\right\}_{\lambda}$ is bounded in $L^2(J,E)$, then there exists a subsequence still indicated by $\left\{\mathcal{K}(s,\vartheta^{\widetilde{\lambda}}_{\rho(s,\vartheta_s)},(\Psi\vartheta^{\lambda})(s))\right\}_{\lambda}$ that weakly converge to the limit $\widetilde{\mathcal{K}}(s)$ in $L^2(J,E)$. Then, we have

$$\int_0^T \|\mathcal{K}(s,\vartheta^{\lambda}_{\rho(s,\vartheta_s)},(\Psi\vartheta^{\lambda})(s)) - \widetilde{\mathcal{K}}(s)\| ds \xrightarrow[\lambda\to 0]{} 0.$$

Thus,

$$\begin{aligned}
\|\rho^{\lambda}(T) - \delta^T\| &\le \left\|\mathcal{W}\left(\lambda,\Gamma_0^T\right)\left[\delta^T + \frac{\partial \mathcal{Q}(T,s)\Phi(0)}{\partial s}\Big|_{s=0} - \mathcal{Q}(T,0)\zeta_0\right]\right\| \\
&\quad + \left\|\mathcal{W}\left(\lambda,\Gamma_0^T\right)\left[+\int_0^T \big(\mathcal{Q}(T,s)\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s))\big)\,ds\right]\right\| \\
&\le \left\|\mathcal{W}\left(\lambda,\Gamma_0^T\right)\left[\delta^T + \frac{\partial \mathcal{Q}(T,s)\Phi(0)}{\partial s}\Big|_{s=0} - \mathcal{Q}(T,0)\zeta_0\right]\right\| \\
&\quad + \left\|\mathcal{W}\left(\lambda,\Gamma_0^T\right)\left[\int_0^T \mathcal{Q}(T,s)\left(\mathcal{K}(s,\vartheta_{\rho(s,\vartheta_s)},(\Psi\vartheta)(s)) - \widetilde{\mathcal{K}}(s)\right)ds\right]\right\| \\
&\quad + \left\|\mathcal{W}\left(\lambda,\Gamma_0^T\right)\left[\int_0^T \mathcal{Q}(T,s)\widetilde{\mathcal{K}}(s)ds\right]\right\| \xrightarrow[\lambda\to 0]{} 0.
\end{aligned}$$

Thus, $\rho^{\lambda}(\varsigma) \to \delta^{T}$ holds, and consequently system (6.1) is approximately controllable on J.

□

6.5 An Example

Consider the following class of partial integro-differential system:

$$\begin{cases} \frac{\partial^2\zeta(\varsigma,x)}{\partial^2 t} = \dfrac{\partial^2\zeta(\varsigma,x)}{\partial^2 x} - \displaystyle\int_0^{\varsigma}\Gamma(\varsigma-s)\dfrac{\partial^2\zeta(s,x)}{\partial^2 x}ds \\ \qquad + \int_{-\infty}^{-t}\dfrac{e^{-8\tau}\|\zeta(\varsigma+\sigma(\varsigma,\zeta(\varsigma+\tau,x)),x)\|_{L^2}}{83\left((\varsigma+\tau)^2+2\varsigma+1\right)}d\tau \\ \qquad - \dfrac{1-e^{-16\pi}}{332\left(\varsigma+1\right)^2} + \displaystyle\int_0^{a}\dfrac{\cos(\varsigma)\ln(1+e^{-\varsigma^2})(1+\zeta(s,x))}{177(1+2\varsigma^2+s^2)e^{4\varsigma}}ds + \widetilde{\sigma}(t)\zeta(\varsigma,x) \\ \qquad + \mathcal{L}(\varsigma,x), \text{ if } \varsigma \in I \text{ and } x \in (0,\pi), \\ \zeta(\varsigma,0) = \zeta(\varsigma,1) = 0, \quad \text{for } \varsigma \in I, \\ \left.\frac{\partial\zeta(\varsigma,x)}{\partial t}\right|_{\varsigma=0} = \zeta_1(x), \ \zeta(\varsigma,x) = \Phi(\varsigma,x), \ \text{if } \varsigma \in \mathbb{R}_- \text{ and } x \in (0,\pi), \end{cases} \tag{6.5}$$

where $I=[0,1], \sigma : J\times\mathbb{R}\to\mathbb{R}$, $\mathcal{L} : [0,1]\times[0,\pi]\to[0,\pi]$.
Let

$$\mathcal{H} := L^2(0,\pi) = \left\{u : (0,\pi) \longrightarrow \mathbb{R} : \int_0^{\pi}|u(x)|^2dx < \infty\right\},$$

be the Hilbert space with the scalar product $\langle u,v\rangle = \int_0^{\pi} u\,(x)v(x)dx$, and the norm

$$\|u\|_2 = \left(\int_0^{\pi}|u(x)|^2dx\right)^{1/2}.$$

Let the phase space $\mathcal{B}$ be $BUC\left(\mathbb{R}^{-},\mathcal{H}\right)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\psi\|_{\mathcal{B}} = \sup_{-\infty<\tau\leq 0}\|\psi(\tau)\|_{L^2}, \psi\in\mathcal{B}.$$

It is well known that $\mathcal{B}$ satisfies the axioms (A_1) and (A_2) with $K=1$ and $L(\varsigma) = M(\varsigma) = 1$, (see [90]). We define the operator $\widehat{A}$ induced on $\mathcal{H}$ as follows:

$$\widehat{A}z = z'', \text{ and } D(A) = \{z \in H^2(0,\pi) \,:\, z(0) = z(\pi) = 0\}.$$

Then $\widehat{A}$ is the infinitesimal generator of a cosine function of operators $(C_0(\varsigma))_{\varsigma\in\mathbb{R}}$ on H associated with sine function $(S_0(\varsigma))_{\varsigma\in\mathbb{R}}$. Additionally, $\widehat{A}$ has discrete spectrum which consists of eigenvalues $-n^2$ for $n \in \mathbb{N}$, with corresponding eigenvectors

$$w_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{N}.$$

The set $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of H. Applying this idea, we can write

$$\widehat{A}z = \sum_{n=1}^{\infty} -n^2 \langle z, w_n \rangle w_n,$$

for $z \in D(A)$, $(C_0(\varsigma))_{\varsigma\in\mathbb{R}}$ is given by

$$C_0(\varsigma)z = \sum_{n=1}^{\infty} \cos(n\varsigma) \langle z, w_n \rangle w_n, \quad \varsigma \in \mathbb{R},$$

and the sine function is given by

$$S_0(\varsigma)z = \sum_{n=1}^{\infty} \frac{\sin(n\varsigma)}{n} \langle z, w_n \rangle w_n, \quad \varsigma \in \mathbb{R}.$$

It is immediate from these representations that $\|C_0(\varsigma)\| \leq 1$ and that $S_0(\varsigma)$ is compact for all $\varsigma \in \mathbb{R}$. We define $A(\varsigma)z = \widehat{A}z + \widetilde{\sigma}(\varsigma)z$ on $D(A)$. Clearly, $A(\varsigma)$ is a closed linear operator. Therefore, $A(\varsigma)$ generates $(S(\varsigma, s))_{(\varsigma,s)\in\Delta}$ such that $S(\varsigma, s)$ is compact and self-adjoint for all $(\varsigma, s) \in \Delta = \{(\varsigma, s) : 0 \leq s \leq \varsigma \leq 1\}$, (see [65]).
We define the operators $\Lambda(\varsigma, s) : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$ as follows:

$$\Lambda(\varsigma, s)z = \Gamma(\varsigma - s)\widehat{A}z, \ for\ 0 \leq s \leq \varsigma \leq 1, \ z \in D(A).$$

The assumption $(C4)$ holds under more suitable conditions on the operator B. Furthermore, it is not difficult to see that conditions $(B1)-(B3)$ are fulfilled, which in turn implies that there exists a resolvent operator and

it's a compact operator. More details about these facts can be seen from the monograph [65,78,115].
Now let $\mathcal{P}: U \to \mathcal{H}$ be defined by $\mathcal{P}u(\varsigma)(x) = \mathcal{L}(\varsigma, x),\ x \in [0,\pi], u \in U$, where $\mathcal{L}: [0,1]\times[0,\pi] \to \mathcal{H}$ is linear continuous and for $\Phi \in BUC(\mathbb{R}^-, H)$, we put $\rho(t,\Phi)(\zeta) = \sigma(t, \zeta(t+\tau, x))$, such that (C_Φ) hold, and let $t \to \Phi_t$ be continuous on $\mathcal{R}(\rho^-)$.
We put $\zeta(\varsigma)(x) = \zeta(\varsigma, x)$, for $\varsigma \in [0,1]$, and define

$$\begin{aligned}\mathcal{K}(\varsigma, \vartheta_1, \vartheta_2)(x) = &\int_{-\infty}^{-t} \frac{e^{-8\tau}\|\vartheta_1(\varsigma + \sigma(\varsigma, \zeta(\varsigma+\tau, x)), x)\|_{L^2}}{83\left((t+\tau)^2 + 2\varsigma + 1\right)} d\tau \\ &- \frac{1-e^{-16\pi}}{332\left(\varsigma+1\right)^2} + \frac{\cos(\varsigma)\vartheta_2(\varsigma)(x)}{e^{-4\varsigma}},\end{aligned}$$

and

$$\vartheta_2(\varsigma)(x) = \Psi(\vartheta_1)(x) = \int_0^a \frac{\ln(1+e^{-\varsigma^2})(1+\vartheta_1(s,x))}{177(1+2\varsigma^2+s^2)} ds.$$

These definitions allow us to depict the system (6.5) in the abstract form (6.1).
Now, for $\varsigma \in [0,1]$, we have

$$\|\mathcal{K}(\varsigma, \varkappa_{1(\varsigma)}, \varkappa_2(\varsigma))\| \leq \frac{1-e^{-16\pi}}{332\left(\varsigma+1\right)^2}\left(1 + \|\varkappa_1\|_{\mathcal{B}}\right) + \cos(\varsigma)e^{-4\varsigma}\left(\|\varkappa_2(\varsigma)\|\right).$$

So, $\psi_{i+1}(\varsigma) = t + i;\ i = 0, 1$ are continuous nondecreasing functions, and we have

$$\xi_1 = \frac{\left(1-e^{-16\pi}\right)\left(1-(1+\pi)^{-3}\right)}{332\sqrt{3}}, \text{ and } \xi_2 = \frac{1}{4}\sqrt{\frac{33}{17}(1-e^{-8\pi})}.$$

And for any bounded set $\Pi \subset \mathcal{H}$, and $\Pi_\varsigma \in \mathcal{B}$, we get

$$\chi(\mathcal{K}(\varsigma, \Pi_\varsigma, \Psi(\Pi(\varsigma)))) \leq (\xi_1 + \xi_2)\,\chi(\Pi).$$

Now, about Ξ, we obtain

$$\|\Xi(\varsigma, s, \varkappa_1) - \Xi(\varsigma, s, \varkappa_2)\|_2 \quad \leq \quad \frac{\ln(2)}{177}\|\varkappa_1 - \varkappa_2\|_2.$$

Now, similar reasoning as in [124], if the corresponding linear system is approximately controllable, then from Theorem 6.4.2 we obtain

$$\lambda\left(\lambda I + \int_0^1 \mathcal{Q}(1,s)\mathcal{L}(s,x)\mathcal{L}(\varsigma,x)^*\mathcal{Q}^*(1,s)ds\right)^{-1} \xrightarrow[\lambda\to 0^+]{} 0.$$

And for $p_3 = \|\varkappa_1\|_{\mathcal{B}},\ p_4 = \|\varkappa_2\|_2$, for all $\varkappa_1 \in \mathcal{B},\ \varkappa_2 \in H$, we get

$$\|\mathcal{K}(\cdot, \varkappa_{1(\cdot)}, \varkappa_2(\cdot))\|_2 \leq \frac{1}{332\sqrt{3}}\left(1 - e^{-16\pi}\right)\left(1 - (1+\pi)^{-3}\right)\left(1 + p_3 + p_4\right).$$

Thus, all the assumptions of Theorem 6.4.3 are fulfilled. Consequently, the problem (6.5) is approximately controllable on $[0, 1]$.

Remark 6.5.1. *We can take the same example but we change the operator* $A(t)$ *by another operator such that* $(S(\varsigma, s))_{(\varsigma,s)\in\Delta}$ *will be not compact. On the other hand, from [102] the operator* W *given by*

$$Wu = \int_0^1 \mathcal{Q}(1, s)\mathcal{P}u(s)ds,$$

is a bounded linear operator but not necessarily one-to-one. Let

$$\operatorname{Ker} W = \left\{u \in L^2([0, 1], U), Wu = 0\right\}$$

be the null space of W *and* $[\operatorname{Ker} W]^\perp$ *be its orthogonal complement in* $L^2([0, 1], U)$*. Let* $\widetilde{W} : [\operatorname{Ker} W]^\perp \longrightarrow \operatorname{Range}(W)$ *be the restriction of* W *to* $[\operatorname{Ker} W]^\perp, \widetilde{W}$ *is necessarily one-to-one operator. The inverse mapping theorem says that* $\widetilde{W}^{-1}$ *is bounded since* $[\operatorname{Ker} W]^\perp$ *and Range* (W) *are Banach spaces. So that* W^{-1} *is bounded and takes values in* $L^2([0, 1], U) \setminus \operatorname{Ker} W$*, hypothesis* $(C4)$ *is satisfied. Then, all the assumptions given in Theorem (6.4.1) are verified. Therefore, the problem (6.5) is exactly controllable on* $[0, 1]$.

Chapter 7

Conclusion and Perspective

In this thesis, we have presented some results on the existence, Ulam stability and controllability of solutions of some classes of fractional differential equations with delay in finite and infinite dimensional Banach spaces. Some equations are subject to impulses which are instantaneous as well as noninstantaneous.The delay may be bounded or unbounded or depending on the state. The presented results are based on the semigroup theory, the notion of measure of noncompactness, Picard process and the fixed point approach. In particular we have used the Banach contraction principle, Schauder's theorem, Burton-Kirk's theorem and Darbo's fixed point theorem.

It would be interesting, for a future research, to look for the complete controllability and approximate controllability of such problems in the case of nondensely defined linear operators.

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