Elliptic problems with nonlinear terms depending on the gradient and singular on the boundary

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Abstract:

In this paper we consider the problem

$$
\begin{cases}
-\Delta u = u^{q\alpha}|\nabla u|^q + \lambda f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < q \leq 2$, $\alpha \in \mathbb{R}$ and $f \geq 0$. We prove that:

(1) If $q\alpha < -1$, then problem (P) has a distributional solution for all $f \in L^1(\Omega)$, and all $\lambda > 0$.

(2) If $-1 \leq q\alpha < 0$, then problem (P) has a solution for all $f \in L^s(\Omega)$, where $s > \frac{N}{q}$ if $N \geq 2$, and without any restriction on $\lambda$.

(3) If $q = 2$ and $-1 \leq q\alpha < 0$ then problem (P) has infinitely many solutions under suitable hypotheses on $f$.

(4) If $0 \leq q\alpha$ and $f \in L^1(\Omega)$ satisfies

$$
\lambda_1(f) = \inf_{\phi \in W^{1,2}_0(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^2 \, dx}{\int_{\Omega} f \phi^2 \, dx} > 0,
$$

then problem (P) has a positive solution if $0 < \lambda < \lambda_1(f)$ and no positive solution for large $\lambda$.

Keywords: Elliptic equations; Singular nonlinearities; Dependence on the gradient.