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Exponential stability for the damped wave equation

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Contents

Chapter 1

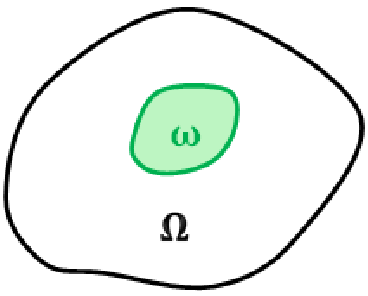
Introduction

In this work, we are going to consider the damped wave equation, which can be written as follow:

$$\begin{cases} u'' - \Delta u + a(.)g(u') = 0 & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(0) = u^0 \ , \ u'(0) = u^1. \end{cases}$$

Where, (u^0, u^1) initial data are taken in the phase space $H_0^1(\Omega) \times L^2(\Omega)$, Ω is a C^2 bounded domain of $\mathbb{R}^N$, $g \ : \ \mathbb{R} \longrightarrow \mathbb{R}$ a C^1 function and $a : \quad \overline{\Omega} \to \mathbb{R}$ a continuous non-negative function that satisfies for some positive constant $a_0 \, : \, a \geq a_0$ on some non-empty open set of Ω that we are going to denote later by ω.

The term $a(x)g(u')$ reprents the feedback, and it is a damping in our case, the function a is what we call the damping coefficient and it defines the region ω in which the feedback is active.



The goal is to study the stability of our problem, under which conditions (on g and a) we have the convergence of tragectories starting from (u^0, u^1) to 0 (the unique equilibrium state of the problem).
To do so, we start by defining the energy of the system by:

$$E(t) = \int_\Omega (|\nabla u|^2 + u'^2) dx. \tag{1.1}$$

We will be studying exponential stability through exponential decay of the energy.
Dafermos [3] and Haraux [4] proved the strong stabilization in the case of an increasing g, i.e.

$$\lim_{t\to+\infty} E(t) = 0, \tag{1.2}$$

the proof is based on Lasalle invariance principle and we can refer to [1] for its details.
Hence, based on their work we know that the energy goes to 0 as t goes to infinity, the goal now is to obtain more information about the type of the decay, particularly, the exponential decay.
Two methods have been developed to study exponential decay of the damped wave equation:

The method of geometric optics: based on microlocal analysis, gives necessary and sufficient conditions on the damping domain ω for exponential stability, these conditions are not explicit but they allow us to get energy decay estimates under very general hypotheses, we are not interested in this method here as it is more adapted to the linear problems.

The multiplier method: it is based on energy estimates and Gronwall's inequalities, it gives explicit sufficient geometric conditions on ω, it consists on taking the main equation of the problem and multiplying it by different quantities called multipliers. Each one of these multipliers will play a role and they will lead when combined in the end, to energy estimates which will prove potential exponential stability through Gronwall's inequalities. There exists a generalization of this method called the piecewise multiplier method which requires weaker geometrical conditions on the damping domain. In chapter 3, we will discuss the multiplier method as well as the geometrical conditions in more details. (see 3.1)

First, we study the case of a linear damping (case $g = Id$) with $a = 1$ on Ω, which is treated by introducing a perturbed energy functional.
Then, we treat the localized linear damping case using the multiplier method (cf. [1] and [7]) in higher dimensions and redo the proof of exponential stability in the one dimensional case with the possible simplifications.
Next, we move to the nonlinear case with a unitary damping coefficient $a = 1$, exponential stability along strong solutions has been proved by P. Martinez and J. Vancostenoble [8] using only one multiplier with Gagliardo-Nirenberg inequality. Finally, we try to generalize their results based on the remark in [9] by adapting their proof to a localized nonlinear damping, we try to treat the nonlinearity their way, and the localization the way used in the linear case.

Chapter 2

Background Material

Theorem 2.1 ***Density result*** *[2]*

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, $u \in W^{1,p}(\Omega)$. Then there exists a sequence $(u_n)_n$ of $C^\infty(\bar{\Omega})$ such that :

$$\lim_{n\to\infty} ||u_n - u||_{W^{1,p}(\Omega)} = 0. \tag{2.1}$$

Theorem 2.2 ***Green's Formula*** *[2]*

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$ of boundary Γ and $u, v : \Omega \longrightarrow \mathbb{R}$ such that : $u \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$ then we have the following Green's Formula :

$$\int_\Omega (\Delta u) v dx = \int_\Gamma v(\nabla u).\nu d\Gamma - \int_\Omega \nabla u.\nabla v dx, \tag{2.2}$$

where ν is the unit outward normal vector for Γ and $d\Gamma$ is the surface measure on Γ.

We prove using the density result in theorem 2.1that the formula stays valid for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ and we have :

$$\int_\Omega (\Delta u) v dx = \int_\Gamma \gamma_0 v(\nabla \gamma_0 u).\nu d\Gamma - \int_\Omega \nabla u.\nabla v dx, \tag{2.3}$$

where γ_0 is the trace linear continuous function and is defined as :

$$\begin{aligned} \gamma_0 : \quad & H^1(\Omega) \longrightarrow L^2(\Gamma) \\ & u \longmapsto \gamma_0 u = \lim_{n\to\infty} u_{n/\Gamma}, \end{aligned} \tag{2.4}$$

where (u_n) is the density sequence from (2.1)

Theorem 2.3 ***Fubini-Tonelli***

if X and Y are two open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, and if $f : X \times Y \longrightarrow \mathbb{R}$ is a measurable function, *then :*

$$\int_X \left(\int_Y |f(x,y)|\, dy \right) dx = \int_Y \left(\int_X |f(x,y)|\, dx \right) dy = \int_{X\times Y} |f(x,y)|\, d(x,y). \tag{2.5}$$

Besides if any one of the three integrals :

$$\int_X \left(\int_Y |f(x,y)|\, dy \right) dx,$$
$$\int_Y \left(\int_X |f(x,y)|\, dx \right) dy,$$
$$\int_{X\times Y} |f(x,y)|\, d(x,y),$$

is finite; then :

$$\int_X \left(\int_Y f(x,y)\, dy \right) dx = \int_Y \left(\int_X f(x,y)\, dx \right) dy = \int_{X\times Y} f(x,y)\, d(x,y). \tag{2.6}$$

Theorem 2.4 *(Rellich-Kondrachov) [2]*
Suppose Ω a C^1 bounded openset of R^N $(N \geq 1)$, then we have :

- ***if*** $\mathbf{p} < \mathbf{N}$ *then* $W_p^1(\Omega) \subset L^q(\Omega)$ *for* $q \in [1, p^*[$,
- ***if*** $\mathbf{p} = \mathbf{N}$ *then* $W_p^1(\Omega) \subset L^q(\Omega)$ *for* $q \in [p, +\infty[$,
- ***if*** $\mathbf{p} > \mathbf{N}$ *then* $W_p^1(\Omega) \subset C(\overline{\Omega})$.

Which compact injections.

Where

$$p^* = \frac{pN}{N-p}$$

Some important inequalities :

Theorem 2.5 ***Young's inequality*** *[2]*
If a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then :

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.7}$$

Theorem 2.6 ***Hölder's inequality*** *[2]*
If Ω is an open subset of $\mathbb{R}^N$, $f, g : \Omega \longrightarrow \mathbb{R}$ such that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $p, q \in [1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$; then we have the Hölder's inequality :

$$\int_\Omega \big|f(x)g(x)\big|\, \mathrm{d}x \leq \left(\int_\Omega |f(x)|^p\, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\, \mathrm{d}x \right)^{\frac{1}{q}}. \tag{2.8}$$

Theorem 2.7 ***Cauchy-Schwarz inequality***
Under the hypotheses of Hölder's inequality (theorem 2.6) and in the special case of $p = q = 2$ we obtain the Cauchy-Schwarz inequality:

$$\int_{\Omega} |f(x)g(x)| \, \mathrm{d}x \le \left(\int_{\Omega} |f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\Omega} |g(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.9}$$

Theorem 2.8 ***Poincaré inequality*** *[2]*
Let p, so that $1 \le p < \infty$ and Ω a subset with at least one bound. Then there exists a constant $C(\Omega)$, depending only on Ω and p, so that, for every function u of the Sobolev space $W_0^{1,p}(\Omega)$ of zero-trace functions we have :

$$\|u\|_{L^p(\Omega)} \le C(\Omega) \|\nabla u\|_{L^p(\Omega)}. \tag{2.10}$$

Theorem 2.9 ***Gagliardo–Nirenberg interpolation inequality*** *[10]*
Let $1 \le r < p \le \infty, 1 \le q \le p$ and $m \ge 0$. Then the inequality:

$$\| v \|_p \le C \| v \|_{m,q}^{\theta} \| v \|_r^{1-\theta} \qquad for \quad v \in W^{m,q} \cap L^r \tag{2.11}$$

holds for some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q} \right)^{-1} \tag{2.12}$$

Where $0 < \theta \le 1 (0 < \theta < 1$ if $p = \infty$ and $mq = N)$ and $\| \, . \, \|_p$ denotes the usual $L^p(\Omega)$ norm and $\| \, . \, \|_{m,q}$ the norm in $W^{m,q}(\Omega)$

Maximal monotone operators and some properties :

Consider the following evolutionary problem :

$$\begin{cases} U' + AU = 0 \qquad in \ \ \mathbb{R}_+, \\ U(0) = U^0. \end{cases} \tag{2.13}$$

Where $A \ : \ D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is an operator (non necessarily linear) and $\mathcal{H}$ is a real Hilbert space.

Definition 2.1 ***Maximal monotone operator***
We say that A is a maximal monotone operator if the following two properties are satisfied :
• *A is monotone :*

$$\langle AU - AV, U - V \rangle_{\mathcal{H}} \ge 0 \ , \quad \forall \ U, V \ \in D(A) \tag{2.14}$$

• *$I + A$ is surjective :*

$$Im(I + A) = \mathcal{H} \tag{2.15}$$

Theorem 2.10 *[6]*
Let A be a maximal monotone operator in a Hilbert space $\mathcal{H}$. Then for every $U^0 \in \overline{D(A)}$, the problem (2.13) *has a unique solution :*

$$U \in C(\mathbb{R}_+; \mathcal{H}) \tag{2.16}$$

Moreover, if $U^0 \in D(A)$, then the regularity of the solution is higher and we have :

$$U \in W^{1,\infty}(\mathbb{R}_+; \mathcal{H}) \tag{2.17}$$

Chapter 3

Statement of the problem

We consider the following problem, which is the nonlinear damped wave equation with Dirichlet boundary conditions :

$$\begin{cases} u'' - \Delta u + a(.)g(u') = 0 & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(0) = u^0 \ , \ u'(0) = u^1. \end{cases} \tag{3.1}$$

Where, Ω is a C^2 bounded domain of $\mathbb{R}^N$,
$g \ : \ \mathbb{R} \longrightarrow \mathbb{R}$ an increasing C^1 function,
$a : \quad \overline{\Omega} \rightarrow \mathbb{R}$ a continuous function that satisfies :

$$a \geq 0 \ \ on \ \ \Omega \ \ and \ \ a \geq a_0 > 0 \ \ on \ \ \omega, \tag{3.2}$$

where a_0 is a real constant.

Remark 3.1
When $\omega = \Omega$, the damping is effective everywhere in Ω and it's said to be globally distributed.
When $\omega \subsetneq \Omega$, the damping is localized in ω and it's said to be locally distributed.

Definition 3.1 *ϵ-neighborhood*
Let O be a subset of $\overline{\Omega}$, the ϵ-neighborhood of O (denoted by $\mathcal{N}_\epsilon(O)$) is defined by:

$$\mathcal{N}_\epsilon(O) = \{x \in \Omega \ : \ dist(x, O) \leq \epsilon.\} \tag{3.3}$$

Where $dist(x, O) = \inf_{y \in O} |x - y|$.

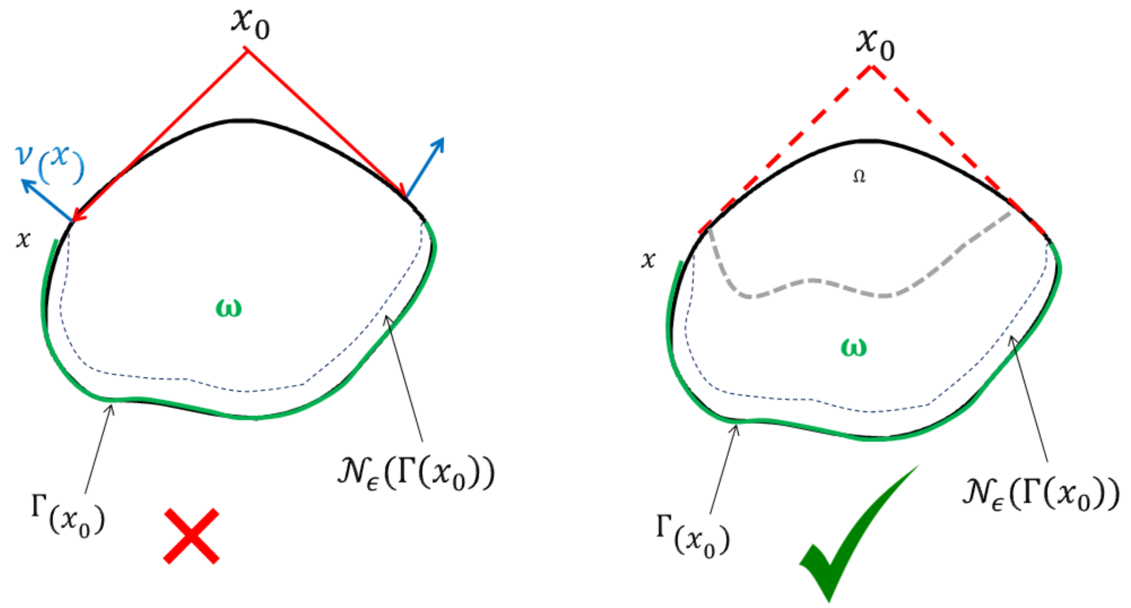
3.1 Geometrical conditions

We should keep in mind that the geometry of the domain Ω is very imporant in our study, thus, its size and localization play a significant role in the stabilization and the control of the wave-like equations. In our case, and for the damped wave equation, what really makes a difference is the region in which the feedback is effective, the region which we denoted ω when we introduced the problem, and to be able to prove the stability, the exponential one in particular, we need to impose some geometrical conditions on ω, previous results ([7] and [1] for instance) proved exponential energy decay in the case of ω satisfying the following condition which we will denote $(GC1)$:

There exists an observation point $x_0 \in \mathbb{R}^N$ for which ω contains an ϵ-neighborhood of :

$$\Gamma(x_0) = \{x \in \partial\Omega, (x - x_0).\nu(x) \geq 0\}, \tag{3.4}$$

where ν is the unit outward normal vector for $\partial\Omega$



The problem here is handled by the multiplier method. But in the case where $(GC1)$ doesn't hold (an example would be when a vanishes in the two neighbohoods of the two poles of a ball) we introduce a piecewise multiplier method to treat such a case, using a weaker geometrical condition $(GC2)$ where we consider multiple distinct observation points $x^j \in \mathbb{R}^N, j = 1, ..., J$ and disjoint Ω_j of $\Omega, j = 1, ..., J$ and we define :

$$\gamma_j(x^j) = \{x \in \partial\Omega_j, (x - x^j).\nu_j(x) \geq 0\}, \tag{3.5}$$

where ν_j stands for the unit outward normal vector to $\partial\Omega_j$
The $(GC2)$ is defined as follows :

$$\omega \supset \mathcal{N}_\epsilon\left(\bigcup_{j=1}^{J} \gamma_j(x^j) \cup (\Omega \setminus \bigcup_{j=1}^{J} \Omega_j)\right). \tag{3.6}$$

3.2 Well-posedness

Theorem 3.1 *Given* $((u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the probelm* (4.35) *has a unique solution :*

$$u \in C(\mathbb{R}_+, H_0^1(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)). \tag{3.7}$$

Given $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$*, the probelm* (3.1) *has a unique solution :*

$$u \in W^{2,\infty}(\mathbb{R}_+, L^2(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+, H_0^1(\Omega)). \tag{3.8}$$

We recall that the energy of the solution is given by :

$$E(t) = \frac{1}{2}\int_\Omega (u'^2 + |\nabla u|^2)dx, \tag{3.9}$$

and it defines a natural norm on the space $H_0^1(\Omega) \times L^2(\Omega)$.

proof.

We rewrite the problem as an evolutionary problem.

Define the operator :

$$\begin{aligned} A : H_0^1(\Omega) \times L^2(\Omega) &\longrightarrow H_0^1(\Omega) \times L^2(\Omega) \\ (u, v) &\longmapsto (v, \Delta u - ag(v)) \end{aligned}$$

With domain :

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \tag{3.10}$$

$$U = \begin{pmatrix} u \\ u_t \end{pmatrix}$$

The problem (3.1) becomes :

$$\begin{cases} U' &= AU. \\ U(0) &= \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}. \end{cases} \tag{3.11}$$

We apply theorem 2.10, we prove that $-A$ is a maximal monotone operator on $H_0^1(\Omega) \times L^2(\Omega)$.

We prove the two points of definition 2.1 of a maximal monotone operator:

• Let $U_1,\ U_2\ \in D(-A) = D(A)$ and we prove that :

$$\langle (-A)U_1 - (-A)U_2\ ,\ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)} \ \geq 0. \tag{3.12}$$

Set $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, then

$$\langle (-A)U_1 - (-A)U_2 \ , \ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)} = \langle AU_2 - AU_1 \ , \ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)}$$

We have :

$$\begin{aligned}
&\langle AU_2 - AU_1 \ , \ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)} \\
&= \left\langle \begin{pmatrix} v_2 - v_1 \\ \Delta(u_2 - u_1) + a(x)(g(v_1) - g(v_2)) \end{pmatrix} , \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\rangle_{H_0^1(\Omega)\times L^2(\Omega)} \\
&= \langle v_2 - v_1 \ , u_1 - u_2 \ \rangle_{H_0^1(\Omega)} + \langle \Delta(u_2 - u_1) + a(x)(g(v_1) - g(v_2)) \ , v_1 - v_2 \ \rangle_{L^2(\Omega)} \\
&= \int_\Omega \nabla(v_2 - v_1)\nabla(u_1 - u_2)dx + \int_\Omega \Delta(u_2 - u_1)(v_1 - v_2)dx + \int_\Omega a(x)(g(v_1) - g(v_2))(v_1 - v_2)dx
\end{aligned} \quad (3.13)$$

On another hand, using an integration by part, we obtain :

$$\int_\Omega \Delta(u_2 - u_1)(v_1 - v_2)dx = -\int_\Omega \nabla(u_2 - u_1)\nabla(v_1 - v_2)dx. \quad (3.14)$$

(3.13) with (3.14) gives :

$$\langle AU_2 - AU_1 \ , \ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)} = \int_\Omega a(x)(g(v_1) - g(v_2))(v_1 - v_2)dx. \quad (3.15)$$

And since a is non-negative, and g is increasing (which means $(g(v_1) - g(v_2))(v_1 - v_2) \geq 0$) :

$$\langle AU_2 - AU_1 \ , \ U_1 - U_2 \rangle_{H_0^1(\Omega)\times L^2(\Omega)} = \int_\Omega a(x)(g(v_1) - g(v_2))(v_1 - v_2)dx \geq 0. \quad (3.16)$$

Hence, A is monotone.

• Now we prove that $-A + I$ is surjective:

Let $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$, we prove that there exists $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(A)$, such that :

$$(-A + I)X \ = \ Y. \quad (3.17)$$

Which means:

$$\begin{pmatrix} -x_2 + x_1 \\ -\Delta x_1 + ag(x_2) + x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (3.18)$$

We pose $v = x_2$, and we replace x_1 by $y_1 + x_2$ in the second line, (3.18) becomes:

$$\begin{cases} v \ = \ x_1 - y_1, \\ -\Delta v + v + ag(v) \ = \ y_2 + \Delta y_1 \end{cases} \quad (3.19)$$

We start by finding $x_2 = v$, and then x_1 is determined by $x_1 = x_2 + y_1$. Since $v \in H_0^1(\Omega)$ it satisfies then the following system :

$$\begin{cases} -\Delta v + v + ag(v) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

where $f = y_2 + \Delta y_1$. The goal is to prove the existence of the solution of (3.20) which means the existence of x_2. We are going to use a variational approach:

We multiply the first equation of (3.20) by a test function $\phi \in H_0^1(\Omega)$, we integrate on Ω and then we integrate by part, we obtain the weak formulation of the problem:

$$\int_\Omega v\phi dx + \int_\Omega \nabla u \nabla \phi dx + \int_\Omega ag(v)\phi dx - \int_\Omega fvdx = 0 \quad \forall v \in H_0^1(\Omega). \tag{3.21}$$

Define:

$$G(v) = \int_0^v g(s)ds \tag{3.22}$$

The energy functional is given by:

$$J(v) = \frac{1}{2}\left(\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx\right) + \int_\Omega aG(v)dx - \int_\Omega fvdx \tag{3.23}$$

We prove that J has a minimum, and this minimum will be the solution of the variational formulation. (we refer to [5] for more background material about minimization of energy functional)

First of all, we have:

$$J(v) \geq \int_\Omega |\nabla v|^2 dx + \int_\Omega aG(v)dx - ||f||_{H^{-1}(\Omega)} ||v||_{L^2(\Omega)}. \tag{3.24}$$

Since g is increasing, G is non-negative.

By that and Poincaré inequality

$$J(v) \geq \int_\Omega |\nabla u|^2 dx - C||\nabla v||_{L^2(\Omega)}. \tag{3.25}$$

where $C = C(\Omega)||f||_{H^{-1}(\Omega)}$ We pose $||\nabla v||_{L^2(\Omega)} = X$ and we study the function:

$$x \longmapsto x^2 - Cx, \tag{3.26}$$

and we find that it is bounded from below, which means that J is bounded from below and therefore the existence of an infimum m.

By passing to the limit in (3.25) we deduce that J is coercive.

Remark 3.2 *By 'coercive' here, we mean that* $\lim J(v)_{||v||_{H_0^1(\Omega)} \longrightarrow +\infty} = +\infty$.

Now we prove that m is actually a minimum. Let $(v_n)_n \subset H_0^1(\Omega)$ be a minimizing sequence of J i.e.

$$\lim_n J(v_n) = m = \inf_{v \in H_0^1(\Omega)} J(v) \tag{3.27}$$

We recall:

$$J(v) = \frac{1}{2}\left(\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx\right) + \int_\Omega aG(v)dx - \int_\Omega fvdx \tag{3.28}$$

Since J is coercive $(v_n)_n$ is forced to be bounded in $H_0^1(\Omega)$ (if we suppose it's not, then it has a subsequence also denoted by $(v_n)_n$ that goes to infinity in $H_0^1(\Omega)$, we apply J to v_n, and we pass to the limit, on one hand, the limit is m because of (3.27), on the other hand it's infinity because of (3.25), which is absurd, and hence $(v_n)_n$ is bounded in $H_0^1(\Omega)$.
Due to reflexivity of $H_0^1(\Omega)$, there exists a subsequence of $(v_n)_n$ such that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ where $\rightharpoonup$ denotes the weak convergence.
Now since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, we have:

$$\int_\Omega |\nabla v|^2 dx \leq \liminf_n \int_\Omega |\nabla v|^2 dx \tag{3.29}$$

Remark 3.3 (3.29) *is a consequence of Banach-Steinhaus theorem (see [2] for proof)*

From Rellich-Kondrachov theorem (theorem 2.4):

$$H_0^1(\Omega) \subset L^2(\Omega), \tag{3.30}$$

with compact injection.
Since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ then it converges strongly in $L^2(\Omega)$, which means :

$$\int_\Omega v^2 dx = \lim_n \int_\Omega |\nabla v|^2 dx \tag{3.31}$$

And also it has a subsequence that converges to v almost everywhere in Ω.
Then, Fatou's lemma and the continuity of G give:

$$\int_\Omega aG(v)dx = \int_\Omega \liminf_n aG(v_n)dx \leq \liminf_n \int_\Omega aG(v)dx \tag{3.32}$$

On another hand, since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$ then:

$$\int_\Omega fvdx = \lim_n \int_\Omega fv_n dx \tag{3.33}$$

(3.29), (3.31) (3.33) and (3.32) imply:

$$m = \lim_n J(v_n) = \lim_n \left(\frac{1}{2}\left(\int_\Omega v_n^2 dx + \int_\Omega |\nabla v_n|^2 dx\right) + \int_\Omega aG(v_n)dx - \int_\Omega fv_n dx\right) \geq J(v) \tag{3.34}$$

On another hand:

$$m = \inf_{z \in H_0^1(\Omega)} J(z) \leq J(v) \tag{3.35}$$

From (3.34) and (3.35) we obtain that $J(v) = m$ Which proves the existence of the solution of (3.20) and hence, the existence of x_1 and x_2, which means that $-A + I$ is surjective.
Finally, $-A$ is a maximal monotone operator and theorem 3.1 is proved.

3.3 Some stability tools

Definition 3.2 ***Strong stability***
(3.1) *is said to be strongly stable if*

$$E(t) \longrightarrow 0 \quad as \quad t \longrightarrow +\infty \tag{3.36}$$

Definition 3.3 ***Exponential stability***
(3.1) *is said to be exponentially stable if there exists two constants* $\gamma, C > 0$ *such that:*

$$E(t) \leq CE(0)e^{-\gamma t} \quad \forall \ t \geq 0. \tag{3.37}$$

Definition 3.4 ***Uniform exponential stability***
(3.1) *is said to be uniformly exponentially stable if there exists two constants* $\gamma, C > 0$ *such that:*

$$E(t) \leq CE(0)e^{-\gamma t} \quad \forall \ t \geq 0. \tag{3.38}$$

Where C *is independent of initial data* u^0 *and* u^1

Theorem 3.2 ***Gronwall*** *[1]*
Let E $[0, +\infty) \rightarrow [0, +\infty)$ *be a non-increasing function satisfying; for some* $T > 0$*, the linear Gronwall inequality :*

$$\int_t^{+\infty} E(s)ds \leq TE(t), \quad \forall \ t \geq 0. \tag{3.39}$$

Then, E *satisfies*

$$E(t) \leq E(0)e^{1-\frac{t}{T}}, \quad \forall \ \geq 0. \tag{3.40}$$

Proof.
Define :

$$f(t) = \int_t^{+\infty} E(s)ds, \quad \forall \ t \geq 0. \tag{3.41}$$

f *as defined above satisfies :*

$$Tf'(t) + f(t) \leq 0 \quad \forall \ t \geq 0. \tag{3.42}$$

Which gives after variables separation and integration :

$$f(t)\exp\left(\frac{t}{T}\right) \leq f(0) = \int_0^{+\infty} E(s)ds \leq TE(0) \quad \forall\ t \geq 0. \tag{3.43}$$

Hence, we have :

$$\int_t^{+\infty} E(s)ds \leq TE(0)e^{-\frac{t}{T}} \quad \forall\ t \geq 0. \tag{3.44}$$

And since E is nonnegative and nonincreasing we get :

$$TE(t) \leq \int_{t-T}^{t} E(s)ds \leq \int_{t-T}^{+\infty} E(s)ds \leq TE(0)e^{-\frac{t-T}{T}} \quad \forall\ t \geq 0. \tag{3.45}$$

■

Chapter 4

Case of the linear damped wave equation

4.1 Case of a globally distributed damping with unitary damping coefficient

The problem becomes :

$$\begin{cases} u'' - \Delta u + u' = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(0) = u^0, u'(0) = u^1. \end{cases} \tag{4.1}$$

4.1.1 An equivalent norm

We consider the following quantity :

$$E_\epsilon(t) = E(t) + \epsilon \int_\Omega uu' dx. \tag{4.2}$$

We are going to prove that $E_\epsilon(t)$ is an equivalent norm to the natural norm on $H_0^1(\Omega) \times L^2(\Omega)$, i.e. $E(t)$.

$$\begin{aligned} E_\epsilon(t) &= \frac{1}{2}(||\nabla u||_{L^2}^2 + ||u'||_{L^2}^2) + \epsilon \int_\Omega uu' dx, \\ &\leq E(t) + \epsilon ||u||_{L^2} ||u'||_{L^2} \quad \text{(Cauchy-Schwarz inequality)}, && (4.3) \\ &\leq E(t) + \epsilon C(\Omega) ||\nabla u||_{L^2} ||u'||_{L^2} \quad \text{(Poincaré inequality)}, && (4.4) \\ &\leq E(t) + \epsilon C(\Omega) \left(\frac{1}{2} ||\nabla u||_{L^2}^2 + \frac{1}{2} ||u'||_{L^2}^2 \right) \quad \text{(Young inequality)} , && (4.5) \\ &\leq \frac{1}{2} ||\nabla u||_{L^2}^2 + \frac{1}{2} ||u'||_{L^2}^2 + \frac{1}{2} \epsilon C(\Omega) ||\nabla u||_{L^2}^2 + \frac{1}{2} \epsilon C(\Omega) ||u'||_{L^2}^2, && (4.6) \\ &\leq \frac{1}{2}(1 + \epsilon C(\Omega)) \left(||\nabla u||_{L^2}^2 + ||u'||_{L^2}^2 \right). && (4.7) \end{aligned}$$

Hence,

$$E_\epsilon(t) \leq \frac{1}{2}(1+\epsilon C(\Omega))E(t). \tag{4.8}$$

That's one inequality, and for the other we have :

$$\begin{aligned}
E_\epsilon(t) &\geq \frac{1}{2}(||\nabla u||_{L^2}^2 + ||u'||_{L^2}^2) - \epsilon \int_\Omega |uu'|dx, && (4.9)\\
&\geq E(t) - \epsilon ||u||_{L^2}||u'||_{L^2} \quad \text{(Cauchy-Schwarz inequality)}, && (4.10)\\
&\geq E(t) - \epsilon C(\Omega)||\nabla u||_{L^2}||u'||_{L^2} \quad \text{(Poincaré inequality)}, && (4.11)\\
&\geq E(t) - \epsilon C(\Omega)(\frac{1}{2}||\nabla u||_{L^2}^2 + \frac{1}{2}||u'||_{L^2}^2) \quad \text{(Young inequality)}, && (4.12)\\
&\geq \frac{1}{2}||\nabla u||_{L^2}^2 + \frac{1}{2}||u'||_{L^2}^2 - \frac{1}{2}\epsilon C(\Omega)||\nabla u||_{L^2}^2 - \frac{1}{2}\epsilon C(\Omega)||u'||_{L^2}^2, && (4.13)\\
E_\epsilon(t) &\geq \frac{1}{2}(1-\epsilon C(\Omega))E(t). && (4.14)
\end{aligned}$$

We can see clearly that $(1-\epsilon C(\Omega)) > 0$ for ϵ small enough. Hence, the equivalence between the two norms with $\epsilon < 1/C(\Omega)$.

4.1.2 Exponential stability

We start by proving the following inequality :

$$E'_\epsilon(t) \leq -CE_\epsilon(t), \tag{4.15}$$

where C is a constant to be defined later on.
We have :

$$E'(t) = \int_\Omega \nabla u' \nabla u \; dx + \int_\Omega u'u'' \; . \tag{4.16}$$

We multiply the first equation of (4.1) by u_t and we intergrate, we obtain :

$$\int_\Omega u''u'dx - \int_\Omega \Delta u u'dx + \int_\Omega u'^2 dx = 0. \tag{4.17}$$

An integration by part and using the fact that $u \in H_0^1(\Omega)$, we obtain :

$$\int_\Omega u''u'dx - \int_\Omega \nabla u' \nabla u \; dx = -\int_\Omega u'^2 dx. \tag{4.18}$$

Combining (4.16) and (4.18), we obtain :

$$E'(t) = -\int_\Omega u'^2 dx$$

On another hand we have :

$$E'_\epsilon(t) = E'(t) + \epsilon \frac{d}{dt}(\int_\Omega uu'dx). \tag{4.19}$$

Starting by the term $\frac{d}{dt}(\int_\Omega uu'dx)$:

$$\frac{d}{dt}(\int_\Omega uu'dx) = \int_\Omega u'^2 dx + \int_\Omega uu''dx. \tag{4.20}$$

We multiply the first equation of (4.1) by u and we intergrate, we obtain :

$$\int_\Omega u''udx - \int_\Omega \Delta uudx + \int_\Omega u'udx = 0. \tag{4.21}$$

An integration by part and using the fact that $u \in H_0^1(\Omega)$, we obtain :

$$\int_\Omega u''udx = -\int_\Omega |\nabla u|^2 dx - \int_\Omega u'udx. \tag{4.22}$$

Going back to (4.20) and using the results we just proved, we obtain :

$$\frac{d}{dt}(\int_\Omega uu'dx) = \int_\Omega u'^2 dx + - \int_\Omega |\nabla u|^2 dx - \int_\Omega u'udx. \tag{4.23}$$

Going back to (4.19) now :

$$\begin{aligned}
E'_\epsilon(t) &= E'(t) + \epsilon \frac{d}{dt}\left(\int_\Omega uu'dx\right), \\
&= -\int_\Omega u'^2 dx + \epsilon \int_\Omega u'^2 dx - \epsilon \int_\Omega |\nabla u|^2 dx - \epsilon \int_\Omega u'udx, \\
&\leq -\int_\Omega u'^2 dx + \epsilon \int_\Omega u'^2 dx - \epsilon \int_\Omega |\nabla u|^2 dx + \epsilon ||u||_{L^2} ||u'||_{L^2}, \\
&\leq -\int_\Omega u'^2 dx + \epsilon \int_\Omega u'^2 dx - \epsilon \int_\Omega |\nabla u|^2 dx + \epsilon C(\Omega) ||\nabla u||_{L^2} ||u'||_{L^2}, \\
&\leq (\epsilon - 1) \int_\Omega u'^2 dx - \epsilon \int_\Omega |\nabla u|^2 dx + \frac{\epsilon}{2} ||\nabla u||^2_{L^2} + \epsilon \frac{C(\Omega)^2}{2} ||u'||^2_{L^2}, \\
&\leq \left(\epsilon - 1 + \epsilon \frac{C(\Omega)^2}{2}\right) \int_\Omega u'^2 dx - \frac{\epsilon}{2} \int_\Omega |\nabla u|^2 dx, && (4.24) \\
&\leq -min\left(1 - \epsilon - \epsilon \frac{C(\Omega)^2}{2}, \frac{\epsilon}{2}\right) E(t), && (4.25) \\
E'_\epsilon(t) &\leq -\frac{2}{1 + \epsilon C(\Omega)} min\left(1 - \epsilon - \epsilon \frac{C(\Omega)^2}{2}, \frac{\epsilon}{2}\right) E_\epsilon(t). && (4.26)
\end{aligned}$$

Hence,

$$E'_\epsilon(t) \leq -CE_\epsilon(t), \tag{4.27}$$

With $C = C(\epsilon) = min\left(1 - \epsilon - \epsilon \frac{C(\Omega)^2}{2}, \frac{\epsilon}{2}\right)$

This estimate is valid when choosing ϵ small enough such that :

$$1-\epsilon\left(1+\frac{C(\Omega)^2}{2}\right)>0 \tag{4.28}$$

which means :

$$\epsilon<\frac{2}{2+C(\Omega)^2} \tag{4.29}$$

Combinining this estimate with the previous estimate of ϵ we obtain :

$$\epsilon<min\left(\frac{2}{2+C(\Omega)^2},\frac{1}{C(\Omega)}\right)=\epsilon_0 \tag{4.30}$$

We choose $\epsilon\in]0,\epsilon_0[$, for example $\epsilon=\frac{\epsilon_0}{2}$ Then,

$$E'_\epsilon(t)\leq -CE_\epsilon(t) \tag{4.31}$$

By Gronwall's lemma applied to (4.31) we obtain :

$$E_\epsilon(t)\leq E_\epsilon(0)\exp(-Ct) \tag{4.32}$$

And then using the equivalence between $E(t)$ and $E_\epsilon(t)$ we obtain :

$$\frac{1}{2}(1-\epsilon C(\Omega))E(t)\leq E_\epsilon(t)\leq\frac{1}{2}(1+\epsilon C(\Omega))E(0)\exp(-Ct) \tag{4.33}$$

Hence,

$$E(t)\leq\frac{1+\epsilon C(\Omega)}{1-\epsilon C(\Omega)}E(0)\exp(-Ct) \tag{4.34}$$

Which proves the exponential decay of the energy of the solutions of (4.1).

4.2 Case of a locally distributed damping

$$\begin{cases} u''-\Delta u+a(.)u'=0 & \text{in}\Omega\times\mathbb{R} \\ u=0 & \text{on }\partial\Omega\times\mathbb{R}_+ \\ u(0)=u^0,u'(0)=u^1 \end{cases} \tag{4.35}$$

4.2.1 Case of one observation point

Theorem 4.1 *Suppose that the geometrical condition (GC1) holds, then the energy E of a solution u of (4.35) with $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ satisfies the following estimate :*

$$\int_t^T E(s)ds\leq C_1E(t)+C_2\int_t^T(\int_\Omega|\rho(.,u')|^2+\int_\omega|u'|^2)ds\quad t\geq 0, \tag{4.36}$$

where $C_1,\ C_2$ are positive constants.

Proof.

We are going to prove the theorem for $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and then with a density argument we will conclude the result for all initial conditions in $H_0^1 \times L^2(\Omega)$.

Let $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

We start by proving the following important lemma :

Lemma 4.1 *$t \longmapsto E(t)$ is nonincreasing on $\mathbb{R}_+$.*

Proof.

Multiplying the first equation of (4.35) by u' and we integrate on Ω we obtain :

$$\int_\Omega (u'' - \Delta u + a(x)u')u'\ dx = 0. \tag{4.37}$$

An integration by part using Green's formula given by (2.3) and the fact that $u \in H_0^1(\Omega)$ give :

$$-\int_\Omega \Delta u u'\ dx = \int_\Omega \nabla u \nabla u' dx. \tag{4.38}$$

(4.37) and (4.38) with some changes, we obtain the following dissipation relation:

$$\frac{1}{2}\int_\Omega ((|u'|^2)' + (|\nabla u|^2)')dx = E' = -\int_\Omega a(x)u'^2 dx. \tag{4.39}$$

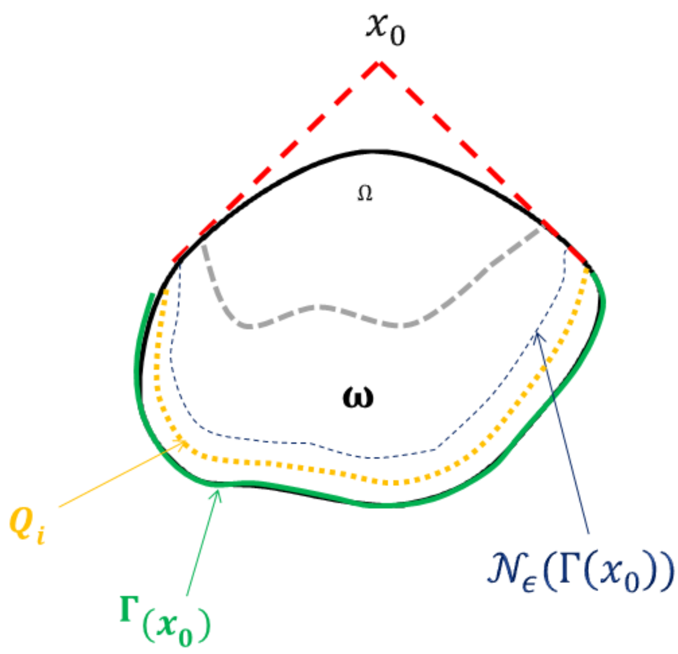
Integrating between some arbitrary $T, S \in \mathbb{R}_+$ such that $S < T$, we obtain :

$$E(T) - E(S) = -\int_S^T \int_\Omega a(x)u'^2 dxdt \leq 0. \tag{4.40}$$

Hence, the proof of the lemma.

Let $x_0 \in \mathbb{R}^N$ be an observation point and $\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$ where ϵ is the same defined in 3.1 and let us define Q_i for $i = 0, 1, 2.$ as follows :

$$Q_i = \mathcal{N}_{\epsilon_i}[\Gamma(x_0)]. \tag{4.41}$$



Since $\overline{\Omega} \setminus Q_1 \cap \overline{Q_0} = \emptyset$ we can define a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that :

$$\begin{cases} 0 \leq \psi \leq 1 \\ \psi = 1 \ on \ \bar{\Omega} \setminus Q_1. \\ \psi = 0 \ on \ Q_0. \end{cases} \tag{4.42}$$

And we define the C^1 vector field h on Ω as :

$$h(x) = \psi(x)(x - x_0). \tag{4.43}$$

• **First multiplier : h.∇u**

This multiplier is going to help us to treat the boundary terms, as ψ is null on Q_0 we are sure that we are away from $\Gamma(x_0)$ and this will lead us to inequality (4.70), and through this multiplier, the geometrical conditions start to make sense and we see why we needed ω to contain that part of the boundary.

Lemma 4.2 *Under the hypotheses of theorem* (4.1) *we have the following identity :*

$$\begin{aligned} \int_S^T \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} h.\nabla u + \frac{1}{2}(h.\nu)(u'^2 - |\nabla u|^2) \right) d\Gamma dt = \left[\int_\Omega u' h.\nabla u dx \right]_S^T \\ + \int_S^T \int_\Omega \left(\frac{1}{2} div h(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} + \rho(.,u')h.\nabla u \right) dx dt \end{aligned} \tag{4.44}$$

Where ν is the unit outward normal vector for Γ and $\rho : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function

proof.

We multiply the first equation of 4.45 by the multiplier $h.\nabla u$ and we integrate :

$$\int_S^T \int_\Omega h.\nabla u(u'' - \Delta u + \rho(.,u'))dxdt = 0 \tag{4.45}$$

we start treating each term seperately :

Using an integration by part with respect to t :

$$\int_S^T h.\nabla uu''dt = [h.\nabla uu']_S^T - \int_S^T h.\nabla u'u'dt. \tag{4.46}$$

We integrate on Ω we get :

$$\int_\Omega \int_S^T h.\nabla uu''dtdx = \int_\Omega [h.\nabla uu']_S^T dx - \int_\Omega \int_S^T h.\nabla u'u'dtdx. \tag{4.47}$$

Fubini-Tonelli Theorem (Theorem 2.3 allows us to exchange the integrals.

We obtain,

$$\int_S^T \int_\Omega h.\nabla uu''dxdt = \left[\int_\Omega h.\nabla uu'dx\right]_S^T - \int_\Omega \int_S^T h.\nabla u'u'dtdx. \tag{4.48}$$

On another hand we have :

$$\begin{aligned}\int_\Omega h.\nabla u'u'dx &= \sum_{i=1}^N \int_\Omega h_i \frac{\partial u'}{\partial x_i} u'dx = \frac{1}{2}\sum_{i=1}^N \int_\Omega h_i \frac{\partial |u'|^2}{\partial x_i} dx,\\ &= \frac{1}{2}\sum_{i=1}^N \int_\Gamma h_i \nu_i |u'|^2 d\Gamma - \frac{1}{2}\sum_{i=1}^N \int_\Omega \frac{\partial h_i}{\partial x_i}|u'|^2 dx,\\ &= \frac{1}{2}\int_\Gamma (h.\nu)|u'|^2 d\Gamma - \frac{1}{2}\int_\Omega divh|u'|^2 dx. \end{aligned} \tag{4.49}$$

Combining (4.48) and (4.49) we get :

$$\int_S^T \int_\Omega h.\nabla uu''dxdt = \left[\int_\Omega h.\nabla uu'dx\right]_S^T - \frac{1}{2}\int_S^T \int_\Gamma (h.\nu)|u'|^2 d\Gamma dt + \frac{1}{2}\int_S^T \int_\Omega divh|u'|^2 dxdt. \tag{4.50}$$

Moving now to the term $-\int_S^T \int_\Omega h.\nabla u\Delta udxdt$, we have:

$$-\int_\Omega h.\nabla u\Delta udx = -\sum_{i,k=1}^N \int_\Omega h_i \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_k}\right) dx.$$

An integration by part with respect to x justified by Green's formula (2.3) gives :

$$\begin{aligned}-\int_\Omega h.\nabla u\Delta udx &= -\sum_{i,k=1}^N \int_\Gamma h_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\nu_k \, d\Gamma + \sum_{i,k=1}^N \int_\Omega \frac{\partial}{\partial x_k}\left(h_i \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_k} dx,\\ &= -\int_\Gamma h.\nabla u \frac{\partial u}{\partial \nu} d\Gamma + \sum_{i,k=1}^N \int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} dx + \sum_{i,k=1}^N \int_\Omega h_i \frac{\partial^2 u}{\partial x_k \partial x_i}\frac{\partial u}{\partial x_k} dx. \end{aligned} \tag{4.51}$$

On another hand we have :

$$\sum_{i,k=1}^{N}\int_{\Omega}h_i\frac{\partial^2 u}{\partial x_k\partial x_i}\frac{\partial u}{\partial x_k}dx = \frac{1}{2}\sum_{i,k=1}^{N}\int_{\Omega}h_i\frac{\partial}{\partial x_i}\left|\frac{\partial u}{\partial x_k}\right|^2 dx,$$
$$= \frac{1}{2}\sum_{i,k=1}^{N}\int_{\Gamma}h_i\left|\frac{\partial u}{\partial x_k}\right|^2\nu_i d\Gamma - \frac{1}{2}\sum_{i,k=1}^{N}\int_{\Omega}\frac{\partial h_i}{\partial x_i}\left|\frac{\partial u}{\partial x_k}\right|^2 dx,$$
$$= \frac{1}{2}\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma dt - \frac{1}{2}\int_{\Omega}divh|\nabla u|^2 dx. \tag{4.52}$$

Combining (4.51) and (4.52) we get :

$$-\int_{\Omega}h.\nabla u\Delta u dx = -\int_{\Gamma}h.\nabla u\frac{\partial u}{\partial\nu}d\Gamma + \sum_{i,k=1}^{N}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx + \frac{1}{2}\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma - \frac{1}{2}\int_{\Omega}divh|\nabla u|^2 dx. \tag{4.53}$$

Combining (4.50),(4.53) and plugging them into (4.45) we obtain (4.44)

Hence, the proof of the lemma. ■

Since ψ vanishes on Q_0 then only the term on $\Gamma\setminus\Gamma(x_0)$ is non-vanishing we have :

$$\int_{\Gamma}\left(\frac{\partial u}{\partial\nu}h.\nabla u + \frac{1}{2}(h.\nu)(u'^2 - |\nabla u|^2)\right)d\Gamma = \int_{\Gamma\setminus\Gamma(x_0)}\left(\frac{\partial u}{\partial\nu}h.\nabla u + \frac{1}{2}(h.\nu)(u'^2 - |\nabla u|^2))\right)d\Gamma. \tag{4.54}$$

On another hand and since $u = 0$ on Γ we have : $\nabla u = \frac{\partial u}{\partial\nu}\nu$ and $u' = 0$ on Γ

Hence,

$$\int_{\Gamma\setminus\Gamma(x_0)}\left(\frac{\partial u}{\partial\nu}h.\nabla u + \frac{1}{2}(h.\nu)(u'^2 - |\nabla u|^2))\right)d\Gamma = \int_{\Gamma\setminus\Gamma(x_0)}\left(\frac{\partial u}{\partial\nu}h.\frac{\partial u}{\partial\nu}\nu - \frac{1}{2}(h.\nu)\left|\frac{\partial u}{\partial\nu}\right|^2\right)d\Gamma$$
$$= \int_{\Gamma\setminus\Gamma(x_0)}\left((h.\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 - \frac{1}{2}(h.\nu)\left|\frac{\partial u}{\partial\nu}\right|^2\right)d\Gamma$$
$$= \int_{\Gamma\setminus\Gamma(x_0)}(\psi.(x-x_0).\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma \tag{4.55}$$

And since $(x-x_0).\nu(x)\geq 0 \Leftrightarrow x\in\Gamma(x_0)$ then $(x-x_0).\nu < 0$ on $\Gamma\setminus\Gamma(x_0)$, therefore :

$$\int_{\Gamma\setminus\Gamma(x_0)}(\psi.(x-x_0).\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma \leq 0. \tag{4.56}$$

Using this result on (4.44) we obtain :

$$\left[\int_{\Omega}u'h.\nabla u dx\right]_S^T + \int_S^T\int_{\Omega}\left(\frac{1}{2}divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^{N}\frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + \rho(.,u')h.\nabla u\right)dxdt \leq 0. \tag{4.57}$$

And again, using the fact that ψ vanishes on Q_0 :

$$\left[\int_\Omega u'h.\nabla u dx\right]_S^T + \int_S^T \int_{\Omega\setminus Q_0} \left(\frac{1}{2}divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + \rho(.,u')h.\nabla u\right) dxdt \leq 0. \tag{4.58}$$

Using now the fact that $\psi = 1$ on $\Omega \setminus Q_1$ as well as the identity :

$$div(\psi(x - x_0)) = \nabla\psi.(x - x_0) + \psi div(x - x_0). \tag{4.59}$$

With :

$$div(x - x_0) = \sum_{i=1}^N \frac{\partial(x_i - x_0)}{\partial x_i} = N. \tag{4.60}$$

we obtain:

$$\int_{\Omega\setminus Q_0} divh(u'^2 - |\nabla u|^2)dx \quad = \int_{\Omega\setminus Q_1} N(u'^2 - |\nabla u|^2)dx + \int_{Q_1\setminus Q_0} divh(u'^2 - |\nabla u|^2)dx. \tag{4.61}$$

On another hand we have, also using the fact that $\psi = 1$ on $\Omega \setminus Q_1$:

$$\int_{\Omega\setminus Q_0} \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx = \int_{\Omega\setminus Q_1} \sum_{i,k=1}^N \frac{\partial x_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx + \int_{Q_1\setminus Q_0} \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx. \tag{4.62}$$

And,

$$\int_{\Omega\setminus Q_1} \sum_{i,k=1}^N \frac{\partial x_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx = \int_{\Omega\setminus Q_1} \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}dx = \int_{\Omega\setminus Q_1} |\nabla u|^2 dx. \tag{4.63}$$

Combining (4.61), (4.62), (4.63) with (4.58) we obtain :

$$\begin{aligned}&\left[\int_\Omega u'h.\nabla u dx\right]_S^T + \int_S^T \int_{\Omega\setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2)\right) dxdt + \int_S^T \int_\Omega \rho(.,u')h.\nabla u dxdt \\ &\leq -\int_S^T \int_{Q_1\setminus Q_0} \left(\frac{1}{2}divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) dxdt.\end{aligned} \tag{4.64}$$

And because h vanishes on Q_0 we can write :

$$\begin{aligned}&\left[\int_\Omega u'h.\nabla u dx\right]_S^T + \int_S^T \int_{\Omega\setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2)\right) dxdt + \int_S^T \int_\Omega \rho(.,u')h.\nabla u dxdt \\ &-\int_S^T \int_{Q_1\cap\Omega} \left(\frac{1}{2}divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^N \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) dxdt.\end{aligned} \tag{4.65}$$

Since h is at least a C^1 vector field and has clearly a compact support then its components are bounded as well as their first partial derivitives, we shall then find a constant C_h such that :

$$|divh| \leq C_h \quad and \quad ||\frac{\partial h_k}{\partial x_i}||_\infty < C_h \quad for\ all \quad i,k = 1,...,N \tag{4.66}$$

We can take for example $C_h = N \sup_{i,k} ||\frac{\partial h_k}{\partial x_i}||_\infty$. Then we have :

$$\int_{Q_1 \cap \Omega} \left(\frac{1}{2} divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^{N} \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \right) dx$$
$$\leq \int_{Q_1 \cap \Omega} \left(\frac{C_h}{2} |(u'^2 - |\nabla u|^2)| + C_h \sum_{i,k=1}^{N} \left| \frac{\partial u}{\partial x_i} \right| \left| \frac{\partial u}{\partial x_k} \right| \right) dx. \tag{4.67}$$

Young inequality implies :

$$C_h \int_{Q_1 \cap \Omega} \sum_{i,k=1}^{N} \left| \frac{\partial u}{\partial x_i} \right| \left| \frac{\partial u}{\partial x_k} \right| dx \leq \frac{C_h}{2} \int_{Q_1 \cap \Omega} \sum_{i,k=1}^{N} \left(\left| \frac{\partial u}{\partial x_i} \right|^2 + \left| \frac{\partial u}{\partial x_k} \right|^2 \right) dx = \int_{Q_1 \cap \Omega} C_h N |\nabla u|^2 dx. \tag{4.68}$$

Combining (4.68) and (4.67) we get :

$$-\int_{Q_1 \cap \Omega} \left(\frac{1}{2} divh(u'^2 - |\nabla u|^2) + \sum_{i,k=1}^{N} \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \right) dx \leq \frac{C_h}{2}(2N+1) \int_{Q_1 \cap \Omega} \left(u'^2 + |\nabla u|^2\right) dx. \tag{4.69}$$

Hence, there exists a positive constant $C_3 = \frac{C_h}{2}(2N+1)$ depending on h only (therefore on ψ and $(x - x_0)$ such that :

$$\left[\int_\Omega u' h.\nabla u dx \right]_S^T + \int_S^T \int_{\Omega \setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2) \right) dxdt + \int_S^T \int_\Omega \rho(.,u') h.\nabla u dxdt$$
$$\leq C_3 \int_S^T \int_{Q_1 \cap \Omega} \left(u'^2 + |\nabla u|^2\right) dxdt. \tag{4.70}$$

• **second multiplier :** $\mathbf{\frac{(N-1)u}{2}}$

This multiplier is gonna help us to absorb the negativity of $\int_S^T \int_{\Omega \setminus Q_1} (2-N)|\nabla u|^2$ in (4.70) since we are going to combine the results of this multiplier with the previous one. As a result, we will get the expression of the energy on one side and move the other terms on the other side and this way, we will get rid of every integral on $\Omega \setminus Q_1$ which will be useful since we want to localize things on ω where we can handle things better and make estimations since $a > a_0$ on ω.

Lemma 4.3 *Under the hypotheses of theorem* (4.1) *we have the following identity :*

$$\frac{N-1}{2} \left[\int_\Omega uu' dx \right]_S^T + \frac{N-1}{2} \int_S^T \int_\Omega \left(|\nabla u|^2 - u'^2 + \rho(.,u')u) \right) dxdt = 0. \tag{4.71}$$

Proof.

We multiply the first equation of (4.35) by the multiplier $\frac{(N-1)u}{2}$ and we integrate :

$$\int_S^T \int_\Omega \frac{(N-1)u}{2}(u'' - \Delta u + \rho(.,u'))dxdt = 0. \tag{4.72}$$

An integration by part with respect to t gives :

$$\int_S^T \int_\Omega \frac{N-1}{2} u'' u dxdt = \frac{N-1}{2}\left[\int_\Omega uu'dx\right]_S^T - \frac{N-1}{2}\int_S^T \int_\Omega u'^2 dxdt \tag{4.73}$$

And an integration by part with respect to x taking in consideration the fact that $u = 0$ on Γ gives :

$$-\int_\Omega \frac{N-1}{2}\Delta u u dx = \int_\Omega \frac{N-1}{2}|\nabla u|^2 dx. \tag{4.74}$$

Combining the two results and plugging them into (4.72) we obtain (4.71) .

Now going back to the proof of the theorem :

• **Combining the results of the first two multipliers :**

Setting $M(u) = h.\nabla u + \frac{N-1}{2}u$ and adding (4.71) to (4.70) we obtain :

$$\left[\int_\Omega u'h.\nabla u dx\right]_S^T + \int_S^T \int_{\Omega \setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2)\right) dxdt + \int_S^T \int_\Omega \rho(.,u')h.\nabla u dxdt$$
$$+\frac{N-1}{2}\left[\int_\Omega uu'dx\right]_S^T + \frac{N-1}{2}\int_S^T \int_\Omega \left(|\nabla u|^2 - u'^2 + \rho(.,u')u)\right) dxdt \leq C_3 \int_S^T \int_{Q_1 \cap \Omega} \left(u'^2 + |\nabla u|^2\right) dxdt. \tag{4.75}$$

which gives :

$$\left[\int_\Omega u'M(u)dx\right]_S^T + \int_S^T \int_{\Omega \setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2)\right) dxdt + \int_S^T \int_\Omega \rho(.,u')M(u)dxdt +$$
$$+ \frac{N-1}{2}\int_S^T \int_\Omega \left(|\nabla u|^2 - u'^2)\right) dxdt \leq C_3 \int_S^T \int_{Q_1 \cap \Omega} \left(u'^2 + |\nabla u|^2\right) dxdt. \tag{4.76}$$

$$\int_\Omega \left(|\nabla u|^2 - u'^2)\right) dx = \int_{\Omega \setminus Q_1} \left(|\nabla u|^2 - u'^2)\right) dx + \int_{\Omega \cap Q_1} \left(|\nabla u|^2 - u'^2)\right) dx. \tag{4.77}$$

(4.77) implies :

$$\int_{\Omega \setminus Q_1} \left(\frac{1}{2}(Nu'^2 + (2-N)|\nabla u|^2)\right) dx + \frac{N-1}{2}\int_{\Omega \cap Q_1} \left(|\nabla u|^2 - u'^2)\right) dx = \frac{1}{2}\int_\Omega (u'^2 + |\nabla u|^2)dx$$
$$+ \frac{N}{2}\int_{\Omega \cap Q_1} u'^2 dx + (\frac{N}{2} - 1)\int_{\Omega \cap Q_1} |\nabla u|^2 dx. \tag{4.78}$$

(4.78) with (4.76) give :

$$\left[\int_\Omega u'M(u)dx\right]_S^T + \frac{1}{2}\int_S^T\int_\Omega (u'^2+|\nabla u|^2)dxdt + \int_S^T\int_\Omega \rho(.,u')M(u)dxdt +$$
$$\leq -C_3\int_S^T\int_{Q_1\cap\Omega}\left((\frac{N}{2}-1)u'^2+(\frac{N}{2}-2)|\nabla u|^2\right)dxdt.$$

Hence, there exists a constant C_3 such that :

$$\int_S^T E(t)dt \leq C_3\int_S^T\int_{Q_1\cap\Omega}\left(u'^2+|\nabla u|^2\right)dxdt - \left[\int_\Omega u'M(u)dx\right]_S^T - \int_S^T\int_\Omega \rho(.,u')M(u)dxdt. \quad (4.79)$$

Where $\rho(x,u') = a(x)u'$ in this case.

• **Estimating the right side terms of** (4.79) :

Now we try to estimate the right terms of (4.79),

Estimating $[\int_\Omega \mathbf{u'M(u)dx}]_\mathbf{S}^\mathbf{T}$:

$$\begin{aligned}\left|\int_\Omega u'M(u)dx\right| &= \left|\int_\Omega u'(h.\nabla u+\frac{N-1}{2}u)dx\right| \\ &= \left|\int_\Omega u'h.\nabla u dx + \int_\Omega \frac{N-1}{2}uu'dx\right| \\ &\leq \sum_{i=1}^N\int_\Omega |u'||h_i|\left|\frac{\partial u}{\partial x_i}\right|dx + \int_\Omega\frac{N-1}{2}|u||u'|dx \quad (4.80) \\ &\leq \sum_{i=1}^N\int_\Omega |u'||h_i|\left|\frac{\partial u}{\partial x_i}\right|dx + \int_\Omega\frac{N-1}{2}|u||u'|dx \quad (4.81)\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega u'M(u)dx\right| &\leq \frac{A}{2}\sum_{i=1}^N\int_\Omega(|u'|^2+\left|\frac{\partial u}{\partial x_i}\right|^2)dx + \int_\Omega\frac{N-1}{2}(|u|^2+|u'|^2)dx \ \text{(Young inequality)} \quad (4.82) \\ &\leq \frac{A\ N}{2}\int_\Omega|u'|^2 + \frac{A}{2}\int_\Omega|\nabla u|^2)dx + \frac{N-1}{2}C(\Omega)\int_\Omega|\nabla u|^2dx + \frac{N-1}{2}|u'|^2dx \quad (4.83) \\ &\leq \frac{N(A+1)-1}{2}\int_\Omega|u'|^2dx + \frac{(N-1)C(\Omega)+A}{2}\int_\Omega|\nabla u|^2dx, \quad (4.84)\end{aligned}$$

where $A = sup_i||h_i||_\infty$. Hence, there exists a positive constant $C_4 = max\,(N(A+1)-1, (N-1)C(\Omega)+A)$ such that :

$$\left|\left(\int_\Omega u'M(u)dx\right)(t)\right| \leq \frac{C_4}{2}E(t). \quad (4.85)$$

Therefore,

$$\begin{aligned}\left|\left[\left(\int_\Omega u'M(u)dx\right)(t)\right]_S^T\right| &= \left|\left(\int_\Omega u'M(u)dx\right)(T) - \left(\int_\Omega u'M(u)dx\right)(S)\right| \\ &\leq \left|\left(\int_\Omega u'M(u)dx\right)(T)\right| + \left|\left(\int_\Omega u'M(u)dx\right)(S)\right| \\ &\leq \frac{C_4}{2}E(T) + \frac{C_4}{2}E(S). \quad (4.86)\end{aligned}$$

Using the result of lemma (4.40) which states that the energy is nonincreasing, we have $E(T) \leq E(S)$ and then:

$$\left| \left[\int_\Omega u' M(u) dx \right]_S^T \right| \leq C_4 E(S). \tag{4.87}$$

Estimating the term $\int_{\mathbf{S}}^{\mathbf{T}} \int_{\mathbf{\Omega}} \rho(.,\mathbf{u}')\mathbf{M}(\mathbf{u})\mathbf{dxdt}$:

We have:

$$\begin{aligned}\left| \int_S^T \int_\Omega \rho(.,u')M(u)dxdt \right| &= \left| \int_S^T \int_\Omega \rho(.,u') \left(\nabla u.h + \frac{N-1}{2} u \right) dxdt \right| \\ &\leq \int_S^T \int_\Omega |\rho(.,u')\nabla u.h|\, dxdt + \int_S^T \int_\Omega \left| \frac{N-1}{2}\rho(.,u')u \right| dxdt.\end{aligned}$$

Using Young inequality :

$$\begin{aligned}\int_\Omega |\rho(.,u')\nabla u.h|\, dxdt + \int_\Omega \left| \frac{N-1}{2}\rho(.,u')u \right| dx &\leq \int_\Omega \left(\frac{A}{\sqrt{\delta}} |\rho(.,u')| \right) \left(\sqrt{\delta} |\nabla u| \right) dx \\ &\quad + \int_\Omega \left(\frac{(N-1)C(\Omega)^2}{2\sqrt{\delta}} |\rho(.,u')| \right) \left(\frac{\sqrt{\delta}}{C(\Omega)^2} |u| \right) dx \\ &\leq \frac{A^2}{2\delta} \int_\Omega |\rho(.,u')|^2 dx + \frac{\delta}{2} \int_\Omega |\nabla u|^2 dx \\ &\quad + \frac{(N-1)^2 C(\Omega)^2}{8\delta} \int_\Omega |\rho(.,u')|^2 dx + \frac{\delta}{2C(\Omega)^2} \int_\Omega |u|^2 dx.\end{aligned} \tag{4.88}$$

Since $u = 0$ on Γ, Poincare inequality and (4.88) imply that :

$$\begin{aligned}\left| \int_\Omega \rho(.,u')M(u)dx \right| &\leq \left(\frac{A^2}{2\delta} + \frac{(N-1)^2 C(\Omega)^2}{8\delta} \right) \int_\Omega |\rho(.,u')|^2 dx + \delta \int_\Omega |\nabla u|^2 dx \\ &\leq \frac{1}{\delta} \left(\frac{A^2}{2} + \frac{(N-1)^2 C(\Omega)^2}{8} \right) \int_\Omega |\rho(.,u')|^2 dx + \delta E \end{aligned} \tag{4.89}$$

Therefore there exists a positive constant $C_5 = \frac{A^2}{2} + \frac{(N-1)^2 C(\Omega)^2}{8}$ such that:

$$\left| \int_\Omega \rho(.,u')M(u)dx \right| \leq \frac{C_5}{\delta} \int_\Omega |\rho(.,u')|^2 dx + \delta E. \tag{4.90}$$

Hence,

$$\left| \int_S^T \int_\Omega \rho(.,u')M(u)dxdt \right| \leq \frac{C_5}{\delta} \int_S^T \int_\Omega |\rho(.,u')|^2 dxdt + \delta \int_S^T E dt. \tag{4.91}$$

Where $\delta > 0$ is an arbitrary positive real number to be chosen later.

Left to estimate the term $\int_{\mathbf{S}}^{\mathbf{T}} \int_{\mathbf{\Omega \cap Q_1}} |\nabla \mathbf{u}|^{\mathbf{2}} \mathbf{dxdt}$:

To estimate this term we are going to use two multipliers.

Since $\overline{\mathbb{R}^N \setminus Q_2} \cap \overline{Q_2} = \emptyset$ there exists a function $\xi \in C_0^\infty(\mathbb{R}^N)$ such that:

$$\begin{cases} 0 \leq \xi \leq 1. \\ \xi = 1 \ on \ Q_1. \\ \xi = 0 \ on \ \mathbb{R}^N \setminus Q_2. \end{cases} \tag{4.92}$$

• Third multiplier : ξu

To estimate the term $\int_S^T \int_{\Omega\cap Q_1} |\nabla u|^2 dxdt$, it is natural to think of using this multiplier, u is going to lead to ∇u after integration by part, and ξ being null outside Q_2 is going to keep us localized in ω. However, this multiplier will not fully estimate the term we want to estimate, and it will create another term to be estimated, which will require another multiplier.

Lemma 4.4 *Under the hypotheses of theorem* (4.1) *with* ξ *as defined above we have the following identity:*

$$\begin{aligned} \int_S^T \int_\Omega \xi |\nabla u|^2 dxdt = \int_S^T \int_\Omega \xi |u'|^2 dxdt + \frac{1}{2} \int_S^T \int_\Omega \Delta\xi u^2 dxdt \\ - \left[\int_\Omega \xi u u' dx \right]_S^T - \int_S^T \int_\Omega \xi u \rho(.,u') dxdt. \end{aligned} \tag{4.93}$$

Proof.

We multiply the first equation of (4.35) by ξu and we integrate, we obtain:

$$\int_S^T \int_\Omega \xi u(u'' - \Delta u + \rho(.,u'))dxdt = 0 \tag{4.94}$$

An integration by part with respect to t gives:

$$\int_S^T \xi u u'' dx = [\xi u u']_S^T - \int_S^T \xi |u'|^2 dxdt. \tag{4.95}$$

An integration by part with respect to x gives:

$$\begin{aligned} -\int_\Omega \xi u \Delta u dx &= \int_\Omega \nabla(\xi u).\nabla u dx \\ &= \int_\Omega u \nabla\xi.\nabla u dx + \int_S^T \int_\Omega \xi |\nabla u|^2 dxdt \\ &= \frac{1}{2}\int_\Omega \nabla\xi.\nabla(|u|^2)dx + \int_\Omega \xi |\nabla u|^2 dx. \end{aligned} \tag{4.96}$$

And another integration by part with respect to x gives:

$$\frac{1}{2}\int_\Omega \nabla\xi.\nabla(|u|^2)dx = -\frac{1}{2}\int_\Omega \Delta\xi(|u|^2)dx. \tag{4.97}$$

Plugging (4.97) into (4.96), we obtain:

$$-\int_S^T \int_\Omega \xi u \Delta u dxdt = -\frac{1}{2}\int_S^T \int_\Omega \Delta\xi(|u|^2)dxdt + \int_S^T \int_\Omega \xi|\nabla u|^2 dxdt. \tag{4.98}$$

And by plugging (4.98) and (4.95) into (4.94) we obtain (4.93):

$$\begin{aligned}\int_S^T \int_\Omega \xi|\nabla u|^2 dxdt &= \int_S^T \int_\Omega \xi|u'|^2 dxdt + \frac{1}{2}\int_S^T \int_\Omega \Delta\xi u^2 dxdt \\ &\quad - \left[\int_\Omega \xi u u' dx\right]_S^T - \int_S^T \int_\Omega \xi u \rho(.,u')dxdt.\end{aligned}$$

Hence, the proof of the lemma.

■

Now since ξ vanishes on $\mathbb{R}^N \setminus Q_2$ and $\xi = 1$ on Q_1 with $\xi \leq 1$, (4.93) becomes:

$$\begin{aligned}\int_S^T \int_{\Omega\cap Q_1} |\nabla u|^2 dxdt &\leq \int_S^T \int_{\Omega\cap Q_2} |u'|^2 dxdt + \frac{1}{2}\int_S^T \int_{\Omega\cap Q_2} |\Delta\xi| u^2 dxdt \\ &\quad + \left|\left[\int_{\Omega\cap Q_2} \xi u u' dx\right]_S^T\right| + \int_S^T \int_{\Omega\cap Q_2} |u\rho(.,u')|dxdt.\end{aligned} \tag{4.99}$$

We have

$$\int_{\Omega\cap Q_2} |u\rho(.,u')|dx \leq \frac{1}{2}\int_{\Omega\cap Q_2} |u|^2 dx + \frac{1}{2}\int_{\Omega\cap Q_2} |\rho(.,u')|^2 dx, \tag{4.100}$$

and

$$\begin{aligned}\left|\int_{\Omega\cap Q_2} \xi u u' dx\right| &\leq \int_{\Omega\cap Q_2} |uu'|dx \\ &\leq \frac{1}{2}\int_{\Omega\cap Q_2} |u|^2 dx + \frac{1}{2}\int_{\Omega\cap Q_2} |u'|^2 dx \\ &\leq \max(C(\Omega)^2, 1)E.\end{aligned} \tag{4.101}$$

(4.99), (4.100), (4.101) with (4.40) we obtain:

$$\left|\left[\int_{\Omega\cap Q_2} \xi u u' dx\right]_S^T\right| \leq 2\max(C(\Omega)^2, 1)E(S). \tag{4.102}$$

Since, $\xi \in C_0^\infty(\mathbb{R}^N$ there exists a constant $B > 0$ such that

$$|\Delta\xi| \leq B. \tag{4.103}$$

$$\int_S^T \int_{\Omega\cap Q_2} |\Delta\xi| u^2 dxdt \leq \int_S^T \int_{\Omega\cap Q_2} B u^2 dxdt$$

Which gives :

$$\int_S^T\int_{\Omega\cap Q_1}|\nabla u|^2dxdt \leq \int_S^T\int_{\Omega\cap Q_2}|u'|^2dxdt+\frac{1}{2}(B+1)\int_S^T\int_{\Omega\cap Q_2}u^2dxdt \\ +2\max(C(\Omega)^2,1)E(S)+\int_S^T\int_{\Omega\cap Q_2}|\rho(.,u')|^2dxdt. \tag{4.104}$$

Hence, there exists two constants $C_6 = max\left(1,\frac{1}{2}(B+1)\right), C_7 = 2\max(C(\Omega)^2,1)$ such that :

$$\int_S^T\int_{\Omega\cap Q_1}|\nabla u|^2dxdt \leq C_6\int_S^T\int_{\Omega\cap Q_2}\left(|u'|^2+u^2+|\rho(.,u')|^2\right)dxdt+C_7E(S). \tag{4.105}$$

Estimating the term $\int_S^T\int_{\Omega\cap Q_2}u^2dxdt$ inside (4.105)
Since $\overline{\mathbb{R}^N\setminus\omega}\cap\overline{Q_2}=\emptyset$, there exists a function $\beta\in C_0^\infty(\mathbb{R}^N)$ such that :

$$\begin{cases} 0\leq\beta\leq 1. \\ \beta = 1 \quad on \quad Q_2. \\ \beta = 0 \quad on \quad \mathbb{R}^N\setminus\omega. \end{cases} \tag{4.106}$$

We fix $t\in\mathbb{R}_+$ and we consider the solution z of the following problem :

$$\begin{cases} \Delta z = \beta u & \text{in } \Omega. \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.107}$$

To do our estimations we are going to use the last multiplier:

• **Fourth multiplier :** **z** This multiplier is used to treat the term $\int_S^T\int_{\Omega\cap Q_2}u^2dxdt$, it is the solution to the above elliptic equation, which will make it satisfy the properties bellow (inequalities that will estimate its norm using the norm of u and u', no term of the type that contains ∇u will appear in the calculus.

Lemma 4.5 *z as defined above, satisfies the following estimates :*

$$||z||_{L^2(\Omega)}\leq C'||u||_{L^2(\Omega)}. \tag{4.108}$$

$$||z'||^2_{L^2(\Omega)}\leq C''\int_\Omega\beta|u'|^2dx. \tag{4.109}$$

Proof.
We first multiply the first equation of (4.107) by z and we integrate on Ω we obtain :

$$\int_\Omega z\Delta zdx = \int_\Omega\beta uzdx. \tag{4.110}$$

After integration by part

$$-\int_{\Omega} |\nabla z|^2 dx = \int_{\Omega} \beta u z dx. \tag{4.111}$$

Which means

$$\int_{\Omega} |\nabla z|^2 dx \leq \int_{\Omega} |\beta u z| dx. \tag{4.112}$$

Using Cauchy-Schwarz inequality and the fact that $0 \leq \beta \leq 1$

$$\int_{\Omega} |\nabla z|^2 dx \leq \| u \|_{L^2(\Omega)} \| z \|_{L^2(\Omega)} . \tag{4.113}$$

Now using Poincaré inequality

$$\frac{1}{C(\Omega)^2} \| z \|^2_{L^2(\Omega)} \leq \int_{\Omega} |\nabla z|^2 dx \leq \| u \|_{L^2(\Omega)} \| z \|_{L^2(\Omega)} . \tag{4.114}$$

Now by simplifying we get

$$\| z \|_{L^2(\Omega)} \leq C(\Omega)^2 \| u \|_{L^2(\Omega)} . \tag{4.115}$$

Which proves (4.108) with $C' = C(\Omega)^2$.

Now deriving the two equations of (4.107) with respect to t we see that z' satisfies :

$$\begin{cases} \Delta z' = \beta u' & \text{in } \Omega. \\ z' = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.116}$$

Multiplying the first equation of (4.116) by z', integrating on Ω and then inegrating by part

$$-\int_{\Omega} |\nabla z'|^2 dx = \int_{\Omega} \beta u' z' dx. \tag{4.117}$$

Which means

$$\int_{\Omega} |\nabla z'|^2 dx \leq \int_{\Omega} |\beta u' z'| dx. \tag{4.118}$$

Cauchy-Schwarz inequality gives

$$\int_{\Omega} |\nabla z'|^2 dx \leq ||\beta u'||_{L^2(\Omega)} ||z'||_{L^2(\Omega)}. \tag{4.119}$$

Using Poincaré inequality

$$\| z' \|_{L^2(\Omega)} \leq C(\Omega)^2 \| \beta u' \|_{L^2(\Omega)} . \tag{4.120}$$

Which gives using the fact that $0 \leq \beta \leq 1$:

$$\int_{\Omega} z'^2 dx \leq C(\Omega)^4 \int_{\Omega} \beta u'^2 dx. \tag{4.121}$$

Which proves (4.109) with $C'' = C(\Omega)^4$

■

Lemma 4.6 *Under the hypotheses of theorem* (4.1) *and with* z *as defined above, we have the following identity :*

$$\int_S^T \int_\Omega \beta u^2 dxdt = \left[\int_\Omega zu'dx\right]_S^T + \int_S^T \int_\Omega \left(-z'u' + z\rho(.,u')\right) dxdt. \tag{4.122}$$

Proof.

Multiplying the first equation of (4.35) by z and we integrate, we obtain :

$$\int_S^T \int_\Omega z(u'' - \Delta u + \rho(.,u'))dxdt = 0. \tag{4.123}$$

An integration by part with respect to t gives :

$$\int_S^T zu''dt = [zu']_S^T - \int_S^T z'u'dt. \tag{4.124}$$

And now two integrations by part with respect to x and using first the fact that $z = 0$ on Γ and then the fact that $u = 0$ on Γ give :

$$\begin{aligned} -\int_\Omega z\Delta udx &= \int_\Omega \nabla z.\nabla udx \\ &= -\int_\Omega \Delta zudx. \end{aligned}$$

And since $\Delta z = \beta u$ we have :

$$-\int_\Omega z\Delta udx = -\int_\Omega \beta u^2 dx. \tag{4.125}$$

Plugging (4.124) and (4.125) into (4.123) we get :

$$\left[\int_\Omega zu'dx\right]_S^T - \int_S^T \int_\Omega z'u'dxdt - \int_S^T \int_\Omega \beta u^2 dxdt + \int_S^T \int z\rho(.,u')dxdt = 0.$$

Which proves the lemma and gives (4.122)

∎

Using the fact that $\beta = 1$ on Q_2 and 0 on $\mathbb{R}^N \setminus \omega$, we have :

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt + \int_S^T \int_{\omega\setminus Q_2} \beta u^2 dxdt = \left[\int_\Omega zu'dx\right]_S^T + \int_S^T \int_\Omega \left(-z'u' + z\rho(.,u')\right) dxdt. \tag{4.126}$$

Which gives, since $\beta \geq 0$:

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \leq \left[\int_\Omega zu'dx\right]_S^T + \int_S^T \int_\Omega \left(-z'u' + z\rho(.,u')\right) dxdt. \tag{4.127}$$

Now we estimate the terms on the right :

Using Cauchy-Schwarz inequality we have :

$$\left|\int_\Omega zu'dx\right| \leq ||z||_{L^2(\Omega}||u'||_{L^2(\Omega}. \tag{4.128}$$

— Using the estimate(4.108) we get :

$$\left|\int_\Omega zu'dx\right| \leq C'||u||_{L^2(\Omega}||u'||_{L^2(\Omega}. \tag{4.129}$$

Using Young inequality:

$$\left|\int_\Omega zu'dx\right| \leq C'\frac{1}{2}\left(||u||^2_{L^2(\Omega} + ||u'||^2_{L^2(\Omega)}\right) = C'E. \tag{4.130}$$

Which gives :

$$\begin{aligned}\left|\left[\int_\Omega zu'dx\right]_S^T\right| &\leq \left|\left(\int_\Omega zu'dx\right)(T) - \left(\int_\Omega zu'dx\right)(S)\right| \\ &\leq \left|\left(\int_\Omega zu'dx\right)(T)\right| - \left|\left(\int_\Omega zu'dx\right)(S)\right| \\ &\leq C'E(T) + C'E(S).\end{aligned}$$

And using (4.40) which is the fact that the energy is nonincreasing we get :

$$\left|\left[\int_\Omega zu'dx\right]_S^T\right| \leq 2C'E(S). \tag{4.131}$$

On another hand we have :

$$\begin{aligned}\left|\int_S^T\int_\Omega(-z'u' + z\rho(.,u'))\,dxdt\right| &\leq \int_S^T\int_\Omega |z'u'|dxdt + \int_S^T\int_\Omega |z\rho(.,u')|dxdt \\ &\leq \int_S^T\int_\Omega \left(\frac{1}{\sqrt{\eta}}|z'|\right)(\sqrt{\eta}|u'|)dxdt \\ &+ \int_S^T\int_\Omega\left(\frac{\sqrt{\eta}}{C'C(\Omega)}|z|\right)\left(\frac{C'C(\Omega)}{\sqrt{\eta}}|\rho(.,u')|\right)dxdt\end{aligned} \tag{4.132}$$

Using Young inequality :

$$\begin{aligned}\left|\int_S^T\int_\Omega(-z'u' + z\rho(.,u'))\,dxdt\right| &\leq \int_S^T\int_\Omega \frac{1}{2\eta}|z'|^2dxdt + \int_S^T\int_\Omega \frac{\eta}{2}|u'|^2dxdt \\ &+ \int_S^T\int_\Omega \frac{\eta}{2C(\Omega)^2C'^2}|z|^2dxdt + \int_S^T\int_\Omega \frac{C(\Omega)^2C'^2}{2\eta}|\rho(.,u')|^2dxdt.\end{aligned} \tag{4.133}$$

Now using the estimates (4.108) and (4.109) we obtain with Poincare inequality :

$$\left|\int_S^T \int_\Omega (-z'u' + z\rho(.,u'))\, dxdt\right| \leq \frac{C''}{2\eta}\int_S^T \int_\Omega \beta|u'|^2 dxdt + \frac{\eta}{2}\int_S^T \int_\Omega |u'|^2 dxdt$$
$$+ \frac{C(\Omega)^2 C'^2 \eta}{2C'^2 C(\Omega)^2}\int_S^T \int_\Omega |\nabla u|^2 dxdt + \frac{C(\Omega)^2 C'^2}{2\eta}\int_S^T \int_\Omega |\rho(.,u')|^2 dxdt. \tag{4.134}$$

Since $\beta = 0$ on $\mathbb{R}^N \setminus \omega$ and $0 \leq \beta \leq 1$ we have :

$$\int_\Omega \beta|u'|^2 dxdt = \int_\omega \beta|u'|^2 dxdt \leq \int_\omega |u'|^2 dxdt.$$

Then,

$$\left|\int_S^T \int_\Omega (-z'u' + z\rho(.,u'))\, dxdt\right| \leq \frac{C''}{2\eta}\int_S^T \int_\omega |u'|^2 dxdt + \frac{\eta}{2}\int_S^T \int_\Omega (|u'|^2 + |\nabla u|^2) dxdt$$
$$+ \frac{C(\Omega)^2 C'^2}{2\eta}\int_S^T \int_\Omega |\rho(.,u')|^2 dxdt. \tag{4.135}$$

Plugging (4.131) and (4.135) into (4.127) we get :

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \leq \frac{C''}{2\eta}\int_S^T \int_\omega |u'|^2 dxdt + \eta\int_S^T E dt + \frac{C(\Omega)^2 C'^2}{2\eta}\int_S^T \int_\Omega |\rho(.,u')|^2 dxdt + 2C'E(S). \tag{4.136}$$

Hence, there exists constants $C_8 = \frac{C''}{2}$, $C_9 = \frac{C(\Omega)^2 C'^2}{2}$ such that :

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \leq \frac{C_8}{\eta}\int_S^T \int_\omega |u'|^2 dxdt + \frac{C_9}{\eta}\int_S^T \int_\Omega |\rho(.,u')|^2 dxdt + \eta\int_S^T E dt + 2C'E(S). \tag{4.137}$$

Where η is an arbitrary positive number.

• **Combining our estimations :**

Now combining (4.137),(4.105),(4.79),(4.98) , with the fact that $\int_{\Omega\cap Q_2}|u'|^2 dx \leq \int_\omega |u'|^2$ and $\int_{\Omega\cap Q_2}|\rho(.,u')|^2 dx \leq \int_\Omega |\rho(.,u')|^2$ we obtain :

$$\int_S^T E dt \leq (C_4 + C_7 C_6 + 2C' C_6 C_3)E(S) + (\eta C_3 C_6 + \delta)\int_S^T E dt$$
$$+ \left(\frac{C_5}{\delta} + \frac{C_9 C_6 C_3}{\eta} + C_6 C_3\right)\int_S^T \int_\Omega |\rho(.,u')|^2 dxdt + \left(\frac{C_3 C_6 C_8}{\eta} + C_3 C_6 + C_3\right)\int_S^T \int_\omega |u'|^2 dxdt. \tag{4.138}$$

We choose η and δ small enough so that $(1 - (\eta C_3 C_6 + \delta)) > 0$

Which gives the existence of two positive constants :

$$C_1 = \frac{C_4 + C_7C_6 + 2C'C_6C_3}{1 - \eta C_3C_6 + \delta} \quad , \quad C_2 = \max\left(\frac{\frac{C_9C_6C_3}{\eta} + C_6C_3 + C_3}{1 - \eta C_3C_6 + \delta}, \frac{\frac{C_3C_6C_8}{\eta} + C_3C_6}{1 - \eta C_3C_6 + \delta}.\right) \tag{4.139}$$

Such that :

$$\int_S^T Edt \leq C_1E(S) + C_2\left(\int_S^T \int_\Omega |\rho(.,u')|^2dxdt + \int_S^T \int_\omega |u'|^2dxdt\right). \tag{4.140}$$

And since, S and T are taken to be arbitrary in $\mathbb{R}_+$, fixing T and taking $S = t$ prove the theorem (4.1) and give (4.36) for $(u^0, u^1) \in (H^2 \cap H_0^1) \times H_0^1$.
Now we conclude for all initial data in $(u^0, u^1) \in H_0^1 \times L^2$ using the density of $(H^2 \cap H_0^1) \times H_0^1$ in $H_0^1 \times L^2$
.

■

Theorem 4.2 ***exponential stability***
Under the hypothesis of theorem 4.1, E satisfies :

$$E(t) \leq CE(0)e^{-\gamma t}, \tag{4.141}$$

where C and γ are positive constants.

Proof.
Using the dissipation relation (4.39), we get:

$$\int_t^T \int_\Omega a|u_t|^2dxds = -\int_t^T E'(s)ds \leq E(t), \quad 0 \leq t \leq T. \tag{4.142}$$

On another hand, using (3.2) we get:

$$\int_t^T \int_\omega |u_t|^2dxds \leq \int_t^T \int_\omega \frac{a}{a_0}|u_t|^2dxds \leq \frac{1}{a_0}E(t), \quad 0 \leq t \leq T. \tag{4.143}$$

Hence, using the result of theorem 4.1 as well as (4.2.1) and (4.142)

$$\int_t^T E(s)ds \leq C_1E(t) + C_2\left(\int_S^T \int_\Omega au'^2dxdt + \int_S^T \int_\omega u'^2dxdt\right) \leq \frac{C_2}{a_0}E(t) + C_2E(t) + C_1E(t) \tag{4.144}$$

Which means:

$$\int_t^T E(s)ds \leq CE(t), \tag{4.145}$$

where $C = C_1 + C_2 + \frac{C_2}{a_0}$.
Now by applying Gronwall inequality (theorem (3.2)) we obtain :

$$E(t) \leq E(0)e^{-\frac{t-C}{C}} \quad \forall\, t \geq 0. \tag{4.146}$$

Which proves the theorem and give the exponential stability of the problem (4.35)

4.2.2 One dimensional case

The problem to one dimensional case reduces to :

$$\begin{cases} u'' - u_{xx} + a(x)u' = 0 & \text{in } \Omega \times \mathbb{R}. \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+. \\ u(0) = u^0, u'(0) = u^1. \end{cases} \tag{4.147}$$

Where $\Omega =]0,1[$ with $(a < b)$ represents here an open interval of $\mathbb{R}$

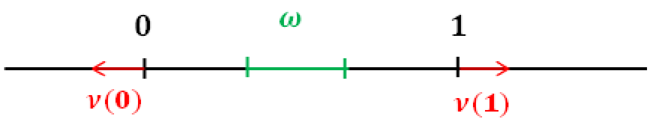
The geometrical condition imposed in the damping domain ω gets simplified.

Indeed we can see that for every observation point x_0 we choose in $\mathbb{R}$ the only possibilities of $\Gamma(x_0)$ (defined in (3.4)) are :

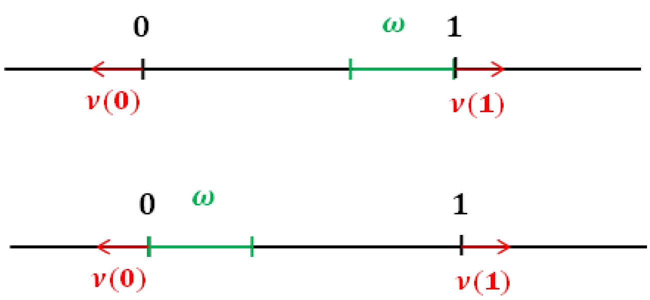
$$\Gamma(x_0) = 0 \;\text{ if }\; \in (-\infty, 0[, \tag{4.148}$$

$$\Gamma(x_0) = 1 \;\text{ if }\; \in]1, +\infty), \tag{4.149}$$

$$\Gamma(x_0) = \{0, 1\} \;\text{ if }\; \in [0, 1]. \tag{4.150}$$



Hence, the geometrical condition $(GC1)$ is reduced to ω containing an ϵ-neighborhood of a or b or the whole boundary.



Now we try to adapt the proof of theorem 4.1 in the one dimensional case and see the simplifications that can be made:

Let $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, let x_0 be an observation point and $0 < \epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$ where ϵ is the one defined in 3.1

$$Q_i = \mathcal{N}_{\epsilon_i}[\Gamma(x_0)] \quad for \;\; i = 0, 1, 2 \tag{4.151}$$

We define $\psi \in C_0^\infty(\mathbb{R})$ as defined in (4.42) And we define h on Ω as :

$$h(x) = \psi(x)(x - x_0). \tag{4.152}$$

We multiply the first equation of (4.147) by hu_x and we integrate, we get :

$$\int_S^T \int_\Omega hu_x(u'' - u_{xx} + au')dxdt = 0. \tag{4.153}$$

An integration by part with respect to t then to x gives:

$$\int_S^T hu_x u'' dt = -\int_S^T hu'_x u' dt + [hu_x u']_S^T . \tag{4.154}$$

On another hand we have :

$$-\int_\Omega hu'_x u' dx = -\int_\Omega h \left(\frac{|u'|^2}{2}\right)_x dx = \int_\Omega h' \frac{|u'|^2}{2} dx. \tag{4.155}$$

An integration by part with respect to x gives:

$$-\int_\Omega hu_x u_{xx} dx = -\int_\Omega h \left(\frac{u_x^2}{2}\right)_x dx = -\left[h\frac{u_x^2}{2}\right]_a^b + \int_\Omega h' \frac{u_x^2}{2} dx. \tag{4.156}$$

On another hand we have :

$$\left[h|u_x^2|\right]_a^b = (1 - x_0)|u_x^2|(1) - (0 - x_0)(|u_x^2|)(0) \leq 0. \tag{4.157}$$

Combining (4.153),(4.154),(4.155), (4.156) and (4.157) we get :

$$\int_S^T \int_\Omega h' \left(\frac{|u_x|^2}{2} + \frac{|u'|^2}{2}\right) dxdt + \left[\int_\Omega hu_x u' dx\right]_S^T + \int_S^T \int_\Omega hau' u_x \leq 0. \tag{4.158}$$

Which gives using (4.42) :

$$\begin{aligned}\int_S^T \int_{\Omega \setminus Q_1} \left(\frac{|u_x|^2}{2} + \frac{|u'|^2}{2}\right) dxdt + \left[\int_\Omega hu_x u' dx\right]_S^T \leq -\int_S^T \int_{\Omega \cap Q_1} h' \left(\frac{|u_x|^2}{2} + \frac{|u'|^2}{2}\right) dxdt \\ -\int_S^T \int_\Omega hau' u_x dxdt,\end{aligned} \tag{4.159}$$

and then,

$$\int_S^T E dt \leq \int_S^T \int_{\Omega \cap Q_1} (1 - h') \left(\frac{|u_x|^2}{2} + \frac{|u'|^2}{2}\right) dxdt - \int_S^T \int_\Omega hau' u_x dxdt - \left[\int_\Omega hu_x u' dx\right]_S^T . \tag{4.160}$$

Now we estimate each term of the right side, we obtain :

$$\left|\int_\Omega h u_x u' dx\right| \leq \int_\Omega |h||u_x||u'|dx \leq E. \quad (4.161)$$

Hence using (4.40) we get :

$$\left|\left[\int_\Omega h u_x u' dx\right]_S^T\right| \leq 2E(S). \quad (4.162)$$

And,

$$\begin{aligned}\left|\int_\Omega hau'u_x dx\right| &\leq \int_\Omega a\frac{C(\Omega)}{C(\Omega)}|u'||u_x|dx \\ &\leq \int_\Omega \frac{C(\Omega)^2}{2\delta}|au'|^2dx + \int_\Omega \frac{\delta C(\Omega)^2}{2}|u_x|dx \\ &\leq \int_\Omega \frac{C(\Omega)^2}{2\delta}|au'|^2dx + \delta E. \end{aligned} \quad (4.163)$$

Therefore :

$$\left|\int_S^T\int_\Omega hau'u_x dxdt\right| \leq \frac{C(\Omega)^2}{2\delta}\int_S^T\int_\Omega |au'|^2dxdt + \delta\int_S^T Edt. \quad (4.164)$$

Left to estimate the term $\int_S^T\int_{\Omega\cap Q_1}|u_x|dxdt$ for that we are going to define

$$\begin{cases} 0 \leq \xi \leq 1. \\ \xi = 1 \ on \ Q_1. \\ \xi = 0 \ on \ \mathbb{R}^N \setminus Q_2. \end{cases} \quad (4.165)$$

We multiply the first equation of (4.147) by ξu and we integrate by part, we obtain :

$$\begin{aligned}\int_S^T\int_\Omega \xi|u_x|^2dxdt = \int_S^T\int_\Omega \xi|u'|^2dxdt + \frac{1}{2}\int_S^T\int_\Omega \xi'' u^2dxdt \\ -\left[\int_\Omega \xi u u' dx\right]_S^T - \int_S^T\int_\Omega \xi u a u' dxdt.\end{aligned} \quad (4.166)$$

Since, $\xi \in C_0^\infty(\mathbb{R}^N$ there exists a constant $B > 0$ such that :

$$|\xi''| \leq B. \quad (4.167)$$

Proceeding the same way we did in the multidimensional case, we obtain :

$$\int_S^T\int_{\Omega\cap Q_1}|u_x|^2dxdt \leq \left(\frac{B}{2}+1\right)\int_S^T\int_{\Omega\cap Q_2}\left(|u'|^2+u^2+|au'|^2\right)dxdt + 2\max(C(\Omega)^2,1)E(S). \quad (4.168)$$

We define the function $\beta \in C_c^\infty(\mathbb{R})$ as in (4.106), we fix $t \in \mathbb{R}_+$ and we consider the solution z of the following problem :

$$\begin{cases} z_{xx} = \beta u & \text{in } \Omega. \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.169}$$

z as defined above, satisfies the following estimates :

$$||z||_{L^2(\Omega)} \leq C_1' ||u||_{L^2(\Omega)}. \tag{4.170}$$

$$||z'||^2_{L^2(\Omega)} \leq C_1'' \int_\Omega \beta |u'|^2 dx. \tag{4.171}$$

Multiplying the first equation of (4.147) by z and we integrate, we obtain :

$$\int_S^T \int_\Omega z(u'' - u_{xx} + a(.)u')dxdt = 0. \tag{4.172}$$

After integrations by part with respect to t and x we get:

$$\left[\int_\Omega zu'dx\right]_S^T - \int_S^T \int_\Omega z'u'dxdt - \int_S^T \int_\Omega \beta u^2 dxdt + \int_S^T \int za(.)u'dxdt = 0.$$

Now using the same steps we used in the multidimensional case, we get :

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \leq \frac{C_1''}{2\eta} \int_S^T \int_\omega |u'|^2 dxdt + \frac{C(\Omega)^2 C_1'^2}{2\eta} \int_S^T \int_\Omega |au'|^2 dxdt + \eta \int_S^T Edt + 2C'E(S). \tag{4.173}$$

Where η is an arbitrary positive number.
Now combining (4.160), (4.162),(4.164), (4.173), (4.168) and choosing *eta* and *delta* small enough we get the existence of two positive constants :

$$C_{1,1} \quad , \quad C_{1,2} \tag{4.174}$$

Such that :

$$\int_S^T Edt \leq C_{1,1}E(S) + C_{1,2}\left(\int_S^T \int_\Omega |au'|^2 dxdt + \int_S^T \int_\omega |u'|^2 dxdt\right). \tag{4.175}$$

And then using Gronwall's inequality we obtain the exponential decay of the energy.

Chapter 5

Case of the nonlinear damped wave equation

5.1 Case of a globally distributed damping with unitary damping coefficient

In this section we take $a = 1$ on Ω The problem becomes :

$$\begin{cases} u'' - \Delta u + g(u') = 0 & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(0) = u^0, u'(0) = u^1 \end{cases} \tag{5.1}$$

Where Ω is a C^2 bounded domain of $\mathbb{R}^2$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 an increasing function such that :

$$\forall\, x \in \mathbb{R} \quad g(x)x \geq 0. \tag{5.2}$$

5.1.1 Exponential stability

Theorem 5.1 *(theorem 2. [8])*
Assume $N = 2$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function of class C^1 , such that $g(0) = 0$, $g'(0) \neq 0$, and :

$$\forall |x| \geq 1, \qquad |g(x)| \leq c|x|^q, \tag{5.3}$$

With $c \geq 0$ and $q \geq 0$.
Given $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1$, the energy of the solution $u(t)$ of (5.1) decays exponentially and there exists an explicit constant $\gamma > 0$ depending on u^0, u^1 such that :

$$\forall t \geq 0, \quad E(t) \leq E(0) \exp(1 - \gamma t) \tag{5.4}$$

We prove first some key lemmas :

Lemma 5.1

Ω is a C^2 bounded domain of $\mathbb{R}^2$ and $g\ :\mathbb{R}\longrightarrow\mathbb{R}$ such that(5.2) is satisfied. Then we have :

$$\forall\ 0\leq S<T<+\infty,\qquad E(T)-E(S)=-\int_S^T\int_\Omega u'g(u')dxdt\leq 0. \tag{5.5}$$

Proof.

Multiplying the first equation of (3.1) by u' and we integrate on Ω we obtain :

$$\int_\Omega (u''-\Delta u+g(u'))u'\ dx=0. \tag{5.6}$$

An integration by part using Green's formula given by (2.3) and the fact that $u\in H_0^1(\Omega)$ we obtain :

$$-\int_\Omega \Delta u u'\ dx=\int_\Omega \nabla u\nabla u' dx. \tag{5.7}$$

(5.6) and (5.7) with some changes, we obtain the following dissipation relation:

$$\frac{1}{2}\int_\Omega ((|u'|^2)'+(|\nabla u|^2)')dx=E'=-\int_\Omega u'g(u')dx \tag{5.8}$$

Integrating between some arbitrary $T,S\in\mathbb{R}_+$ such that $S<T$, we obtain :

$$E(T)-E(S)=-\int_S^T\int_\Omega u'g(u')dxdt. \tag{5.9}$$

(5.2) gives :

$$E(T)-E(S)=-\int_S^T\int_\Omega u'g(u')dxdt\leq 0 \tag{5.10}$$

Hence, the proof of the lemma.

■

Lemma 5.2 *Consider u the solution of the problem (5.1) with initial conditions $(u^0,u^1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1$. Denote :*

$$C(u^0,u^1):=\parallel -\Delta u^0+g(u^1)\parallel^2_{L^2(\Omega)}+\parallel u^1\parallel^2_{H_0^1(\Omega)} \tag{5.11}$$

Then,

$$\forall t\geq 0,\parallel -\Delta u(t)+g(u'(t))\parallel^2_{L^2(\Omega)}+\parallel u'(t)\parallel^2_{H_0^1(\Omega)}\leq C(u^0,u^1) \tag{5.12}$$

Proof.

Denote $w := u'$. Differentiating the three equations of (3.1) with respect to t, we obtain :

$$w'' - \Delta w + g'(w)w' = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{5.13}$$

$$w = 0 \quad \text{in } \partial\Omega \times \mathbb{R}_+, \tag{5.14}$$

$$w(0) = u^1, w'(0) = \Delta u^0 - g(u^1). \tag{5.15}$$

We multiply (5.13) by w' and we integrate on $\Omega \times [0,t]$ where $t \in \mathbb{R}_+$:

$$\int_0^t \int_\Omega (w'w'' - w'\Delta w + g'(w)w'^2) dx d\tau = 0 \tag{5.16}$$

On one hand we have,

$$\int_0^t w'w'' dx d\tau = \frac{1}{2}\int_0^t (w'^2)' d\tau = \frac{1}{2}\left[w'^2\right]_0^t \tag{5.17}$$

On another hand using an integration by part with respect to t we have :

$$-\int_\Omega w'\Delta w dx = \int_\Omega \nabla w' \nabla w dx = \frac{1}{2}\int_\Omega (|\nabla w|^2)' dx = \frac{1}{2}\int_\Omega (|\nabla w|^2)' dx. \tag{5.18}$$

Combining (5.16), (5.17) and (5.18) we obtain :

$$\begin{aligned} -\int_0^t \int_\Omega a(.)g'(w)w'^2 dx d\tau &= \int_0^t \int_\Omega w'w'' dx d\tau - \int_0^t \int_\Omega \Delta w dx d\tau \\ &= \frac{1}{2}\left[\int_\Omega w'^2 dx\right]_S^T + \frac{1}{2}\int_0^t \left(\int_\Omega |\nabla w|^2 dx\right)' d\tau \\ &= \frac{1}{2}\left[\int_\Omega (w'^2 + |\nabla w|^2) dx\right]_0^t. \end{aligned} \tag{5.19}$$

Since $a \geq 0$ and $g' \geq 0$ (since g is increasing) :

$$\left[\int_\Omega (w'^2 + |\nabla w|^2) dx\right]_0^t \leq 0. \tag{5.20}$$

On another hand we have :

$$\begin{aligned} \left[\int_\Omega (w'^2 + |\nabla w|^2) dx\right]_0^t &= \int_\Omega (w'(t)^2 + |\nabla w(t)|^2) dx - \int_\Omega (w'(0)^2 + |\nabla w(0)|^2) dx \\ &= \int_\Omega (u''(t)^2 + |\nabla u'(t)|^2) dx - \int_\Omega ((\Delta u^0 - g(u^1))^2 + |\nabla u^1|^2) dx \\ &= \int_\Omega (\Delta u - g(u'))^2 dx + \int_\Omega |\nabla u'(t)|^2 dx - C(u^0, u^1) \end{aligned} \tag{5.21}$$

From (5.21) and (5.20) we deduce (5.12):

$$\| -\Delta u(t) + g(u'(t)) \|^2_{L^2(\Omega)} + \| u'(t) \|^2_{H^1_0(\Omega)} \leq C(u^0, u^1)$$

■

Lemma 5.3 *Suppose g satisfies for some positive constant C:*

$$g(x) \leq C|x|^q \quad \text{with } 1 \leq q < +\infty, \tag{5.22}$$

and

$$g(0) = 0 \ , \ g'(0) \neq 0, \tag{5.23}$$

then the energy E of the solution u of (3.1) with $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ satisfies the following estimate :

$$\int_S^T E dt \leq CE(S) + C \int_S^T \int_\Omega |u'|^2 dx dt, \tag{5.24}$$

where C are positive constants.

Proof.

We multiply the first equation of (5.1) by u and we integrate on $[S, T] \times \Omega$:

$$\int_S^T \int_\Omega u(u'' - \Delta u + g(u')) dx dt = 0. \tag{5.25}$$

An integration by part with respect to x gives :

$$\int_S^T \int_\Omega u u'' dx dt + \int_S^T \int_\Omega |\nabla u|^2 dx dt + \int_S^T \int_\Omega g(u') u dx dt = 0. \tag{5.26}$$

Treating each term seperately :

• **Estimating $\int_S^T \int_\Omega uu''dxdt$**

An integration by part with respect to t gives

$$\int_S^T \int_\Omega u u'' dx = \left[\int_\Omega u u' dx \right]_S^T - \int_S^T \int_\Omega u'^2 dx dt. \tag{5.27}$$

Using Young and Poincaré inequalities we have

$$\int_\Omega |uu'| dx \leq CE. \tag{5.28}$$

Hence,

$$\left| \left[\int_\Omega u u' dx \right]_S^T \right| \leq CE(S). \tag{5.29}$$

• **Estimating $\int_S^T \int_\Omega ug(u')dxdt$**

$$\left| \int_\Omega g(u') u dx \right| \leq \int_\Omega |g(u') u| \, dx. \tag{5.30}$$

We have

$$\int_S^T \int_\Omega |g(u')u|\,dxdt = \int_S^T \int_{|u'|\le 1} |g(u')u|\,dxdt + \int_S^T \int_{|u'|>1} |g(u')u|\,dxdt \tag{5.31}$$

First we look at the part of Ω where $|u'| \le 1$:

Using Young's inequality

$$\int_S^T \int_{|u'|\le 1} |g(u')u|\,dxdt \le \int_S^T \int_{|u'|\le 1} \left(\frac{\eta}{2}|u|^2 + \frac{1}{2\eta}|g(u')|^2\right) dxdt \tag{5.32}$$

Using Poincaré inequality :

$$\int_S^T \int_{|u'|\le 1} \frac{\eta}{2}|u|^2 dxdt \le \frac{C\eta}{2}\int_S^T \int_{|u'|\le 1} \left(|\nabla u|^2 + |u'|^2\right) dxdt \le C\eta \int_S^T E dt \tag{5.33}$$

$g(0) = 0$ and $g'(0) \neq 0$ implies :

There exists a constant $C > 0$ such that

$$|g(x)| \le C|x| \quad if \;\; |x| \le 1. \tag{5.34}$$

Using (5.34)

$$\frac{1}{2\eta}\int_S^T \int_{|u'|\le 1} |g(u')|^2 dxdt \le \frac{C}{2\eta}\int_S^T \int_{|u'|\le 1} |u'|^2 dxdt \tag{5.35}$$

Combining (5.32), (5.33) and (5.35) we obtain :

$$\int_S^T \int_{|u'|\le 1} |g(u')u|\,dxdt \le C\eta \int_S^T E dt + \frac{C}{2\eta}\int_S^T \int_\Omega |u'|^2 dxdt \tag{5.36}$$

Next, we look at the part of Ω where $|u'| > 1$:

Since $N = 2$ and $1 \le q < \infty$, theorem 2.4 with $p = 2$, $N = p$ and $q + 1 < +\infty$ gives :

$$H^1(\Omega) \subset L^{q+1}(\Omega). \tag{5.37}$$

Which gives :

$$\| u \|_{L^{q+1}(\Omega)} \le C \| u \|_{H^1(\Omega)} \le C\sqrt{E}. \tag{5.38}$$

Now using Holder's inequality,

$$\int_S^T \int_{|u'|>1} |g(u')u|\,dxdt$$

$$\le \int_S^T \left(\int_{|u'|>1} |u|^{q+1}\,dx\right)^{\frac{1}{q+1}} \left(\int_{|u'|>1} |g(u')|^{\frac{q+1}{q}}\,dx\right)^{\frac{q}{q+1}} dt \tag{5.39}$$

Using (5.38) and the fact that a is bounded :

$$\int_S^T \int_{|u'|>1} |g(u')u|\, dxdt \leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |g(u')|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.40}$$

$$\leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |g(u')||g(u')|^{\frac{1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.41}$$

Now using the hypothesis (5.22) we obtain :

$$\int_S^T \int_{|u'|>1} |g(u')u|\, dxdt \leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} g(u')u' dx \right)^{\frac{q}{q+1}} dt$$

$$\leq \int_S^T C\sqrt{E} \left(\int_\Omega g(u')u' dx \right)^{\frac{q}{q+1}} dt \tag{5.42}$$

Using the dissipation relation (5.8)

$$\int_S^T \int_{|u'|>1} |g(u')u|\, dxdt \leq \int_S^T C\sqrt{E}\, (-E')^{\frac{q}{q+1}}\, dt$$

$$\leq C \int_S^T E^{\frac{1}{2}}\, (-E')^{\frac{q}{q+1}}\, dt \tag{5.43}$$

Using Young's inequality

$$\int_S^T E^{\frac{1}{2}}\, (-E')^{\frac{q}{q+1}}\, dt \leq C\eta^{q+1} \int_S^T E^{\frac{q+1}{2}} dt + \frac{C}{\eta^{\frac{q+1}{q}}} \int_S^T (-E')dt \tag{5.44}$$

Using the non-increasingness of E with (5.43) and (5.44), we obtain :

$$\int_S^T \int_{|u'|>1} |g(u')u|\, dxdt \leq C\eta^{q+1} E(0)^{\frac{q-1}{2}} \int_S^T E dt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) \tag{5.45}$$

Combining (5.36) and (5.45) we have :

$$\left| \int_\Omega g(u')u dx \right| \leq \eta \left(C + \eta^q E(0)^{\frac{q-1}{2}} \right) \int_S^T E dt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) + \frac{C}{2\eta} \int_S^T \int_\Omega |u'|^2 dxdt \tag{5.46}$$

From (5.26) and (5.27) we have

$$\int_S^T \int_\Omega (|\nabla u|^2 + u')^2 dxdt = -\left[\int_\Omega uu' dx \right]_S^T - \int_S^T \int_\Omega g(u')u dxdt + 2\int_S^T \int_\Omega u'^2 dxdt \tag{5.47}$$

Now taking the absolute value of each side of (5.47) and using (5.46) and (5.29), we obtain :

$$\int_S^T E dt \leq \eta \left(C + \eta^q E(0)^{\frac{q-1}{2}} \right) \int_S^T E dt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) + \frac{C}{2\eta} \int_S^T \int_\Omega |u'|^2 dxdt + CE(S) \tag{5.48}$$

Now choosing η small enough we get to prove (5.24) :

$$\int_S^T Edt \leq CE(S) + C\int_S^T\int_\Omega |u'|^2 dxdt,$$

■

Proof of theorem (5.1) :

From lemma 5.3 we have :

$$\int_S^T Edt \leq CE(S) + C\int_S^T\int_\Omega |u'|^2 dxdt,$$

the goal is to estimate

$$\int_S^T\int_\Omega |u'|^2 dxdt.$$

Set $R > 0$ and fix $t \geq 0$.

Define

$$\Omega_1^t := \{x \in \Omega : \ |u'| \leq R\} \tag{5.49}$$

$$\Omega_2^t := \{x \in \Omega : \ |u'| > R\} \tag{5.50}$$

First, we look at the part Ω_2^t of Ω :

Using theorem 2.9 (Gagliardo-Nirenberg interpolation inequality) with $p = 3, N = 2$ and $\theta = \frac{1}{3}$, there exists a constant $C > 0$ that depends on Ω such that :

$$\forall v \in H^1(\Omega), \quad \| v \|_{L^3(\Omega)} \leq C \| v \|_{H^1(\Omega)}^{\frac{1}{3}} \| v \|_{L^2(\Omega)}^{\frac{2}{3}} \tag{5.51}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{\Omega_2^t} u'^2 dx = \int_{\Omega_2^t} u'^{\frac{1}{2}} u'^{\frac{3}{2}} dx &\leq \left(\int_{\Omega_2^t} |u'| dx\right)^{\frac{1}{2}} \left(\int_{\Omega_2^t} |u'|^3 dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\Omega_2^t} |u'| dx\right)^{\frac{1}{2}} \| u' \|_{L^3(\Omega)}^{\frac{3}{2}} . \end{aligned} \tag{5.52}$$

Which gives

$$\left(\int_{\Omega_2^t} u'^2 dx\right)^2 \leq \left(\int_{\Omega_2^t} |u'| dx\right) \| u' \|_{L^3(\Omega)}^3 . \tag{5.53}$$

Since we're on Ω_2^t we have :

$$\int_{\Omega_2^t} |u'|^2 dx > R\int_{\Omega_2^t} |u'| dx \tag{5.54}$$

Which means

$$\int_{\Omega_2^t} |u'| dx \leq \frac{1}{R}\int_{\Omega_2^t} |u'|^2 dx \tag{5.55}$$

On another hand, since we're dealing with a strong solution, we have $u' \in H^1(\Omega)$, we can apply (5.51) with $v = u'$ we get :

$$\forall v \in H^1(\Omega), \quad \| u' \|^3_{L^3(\Omega)} \leq C \| u' \|_{H^1(\Omega)} \| u' \|^2_{L^2(\Omega)} \leq \| u' \|_{H^1(\Omega)} E(t). \tag{5.56}$$

Combining (5.53), (5.55) and (5.56)

$$\int_{\Omega_2^t} u'^2 dx \leq \frac{C}{R} \| u' \|_{H^1(\Omega)} E(t). \tag{5.57}$$

From lemma 5.2 we get :

$$\| u' \|_{H^1(\Omega)} \leq \sqrt{C(u^0, u^1}. \tag{5.58}$$

Combining (5.58) with (5.57) and integrating between S and T we obtain

$$\int_S^T \int_{\Omega_2^t} u'^2 dxdt \leq \frac{C}{R} \sqrt{C(u^0, u^1)} \int_S^T E(t)dt. \tag{5.59}$$

From lemma 5.3 we have :

$$\int_S^T Edt \leq CE(S) + C \int_S^T \int_{\Omega_2^t} |u'|^2 dxdt + C \int_S^T \int_{\Omega_1^t} |u'|^2 dxdt,$$

using (5.59)

$$\int_S^T Edt \leq CE(S) + \frac{C}{R} \sqrt{C(u^0, u^1)} \int_S^T E(t)dt + C \int_S^T \int_{\Omega_1^t} |u'|^2 dxdt, \tag{5.60}$$

We choose $R > 0$ such that

$$\frac{C}{R} \sqrt{C(u^0, u^1)} \leq \frac{1}{2}, \tag{5.61}$$

which gives :

$$\int_S^T Edt \leq CE(S) + C \int_S^T \int_{\Omega_1^t} |u'|^2 dxdt. \tag{5.62}$$

Next, we look at the part Ω_1^t of Ω :

Since $g'(0) \neq 0$ we can choose $r > 0$ such that :

$$\forall v \in [-r, r], |g(v)| \geq \alpha_1 |v|, \tag{5.63}$$

is satisfied for some constant $\alpha_1 > 0$.

We define then :

$$\alpha_2 := \inf \left\{ \left| \frac{g(v)}{v} \right| \ : \ r \leq |v| \leq R \right\}, \tag{5.64}$$

where $\alpha := \min(\alpha_1, \alpha_2)$.

We have :

$$|g(v)| \geq \alpha |v| \quad if \quad |v| \leq R. \tag{5.65}$$

Then,

$$\begin{aligned}\int_S^T \int_{\Omega_1^t} u'^2 dxdt &= \int_S^T \int_{\Omega_1^t} u'g(u')\frac{u'}{g(u')}dxdt \\ &\leq \frac{1}{\alpha}\int_S^T \int_{\Omega_1^t} u'g(u')dxdt \end{aligned} \tag{5.66}$$

From (5.9) we have

$$\frac{1}{\alpha}\int_S^T \int_{\Omega_1^t} u'g(u')dxdt = \frac{1}{\alpha}\left(E(S) - E(T)\right) \leq \frac{1}{\alpha}E(S) \tag{5.67}$$

Combining (5.67) and (5.66)

$$\int_S^T \int_{\Omega_1^t} u'^2 dxdt \leq \frac{1}{\alpha}E(S). \tag{5.68}$$

Finally, combining (5.62) and (5.68) we have :

$$\int_S^T Edt \leq C\left(1 + \frac{1}{\alpha}\right)E(S), \tag{5.69}$$

letting T goes to infinity we get :

$$\int_S^\infty Edt \leq \frac{1}{\gamma}E(S), \tag{5.70}$$

where $\gamma = \frac{\alpha}{C(1+\alpha)} > 0$.
We apply now Gronwall's inequality (Theorem 3.2) we obtain exponential decay of the energy :

$$E(t) \leq E(0)e^{1-\gamma t}. \tag{5.71}$$

■

5.2 Case of a locally distributed damping

We finally get to our initial problem (problem (3.1)) with no particular cases:

$$\begin{cases} u'' - \Delta u + a(.)u' = 0 & \text{in}\Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(0) = u^0, u'(0) = u^1 \end{cases}$$

We recall the hypothesis under which we are working :

Ω is a C^2 bounded domain of $\mathbb{R}^N$,

$g\ :\ \mathbb{R} \longrightarrow \mathbb{R}$ an increasing C^1 function such that (5.2) is satisfied,

$a: \quad \overline{\Omega} \to \mathbb{R}$ a continuous function that satisfies (3.2), i.e.:

$$a \geq 0 \ \ on \ \ \Omega \ \ and \ \ a \geq a_0 > 0 \ \ on \ \ \omega,$$

where a_0 is a real constant.

The goal of this section is to prove that theorem 5.1 remains valid for a localized damping when $GC1$ holds.

Proposition 5.1

Theorem 5.1 remains true for problem (3.1), *i.e. for a localized damping when* $GC1$ *hold.*

Proof.

Remark 5.1 *We are going to keep the proof in the special case where* $N = 2$, *the lemmas are going to be considered in higher dimensions though.*

Lemma 5.4

Ω *is a* C^2 *bounded domain of* $\mathbb{R}^2$ *and* $g \ : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*(5.2) *is satisfied. Then we have :*

$$\forall \ 0 \leq S < T < +\infty, \qquad E(T) - E(S) = -\int_S^T \int_\Omega a u' g(u') dx dt \leq 0. \tag{5.72}$$

Proof.

Same as Lemma 5.1

Lemma 5.5 *Consider* u *the solution of the problem* (5.1) *with initial conditions* $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1$. *Denote :*

$$C(u^0, u^1) := \| -\Delta u^0 + a(.)g(u^1) \|_{L^2(\Omega)}^2 + \| u^1 \|_{H_0^1(\Omega)}^2 . \tag{5.73}$$

Then,

$$\forall t \geq 0, \quad \| -\Delta u(t) + a(.)g(u'(t)) \|_{L^2(\Omega)}^2 + \| u'(t) \|_{H_0^1(\Omega)}^2 \leq C(u^0, u^1). \tag{5.74}$$

Proof.

Same as lemma 5.1.

Lemma 5.6 *Suppose that the geometrical condition (GC1) holds and* g *satisfies for some positive constant* C:

$$g(x) \leq C|x|^q \qquad with \ 1 \leq q \leq \frac{N+2}{\max(0, N+2)}, \tag{5.75}$$

and

$$g(0) = 0 \ , \ g'(0) \neq 0, \tag{5.76}$$

then the energy E of the solution u of (3.1) *with* $(u^0, u^1) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$ *satisfies the following estimate:*

$$\int_S^T Edt \leq CE(S) + C\int_S^T \left(\int_\Omega a(.)|u'|^2dx + \int_\omega |u'|^2dx\right)dt, \tag{5.77}$$

where C are positive constants.

Proof.

We proceed the same way as (4.36) until we get to (4.79) :

$$E(t) \leq C\int_S^T\int_{Q_1\cap\Omega}\left(u'^2+|\nabla u|^2\right)dxdt - \left[\int_\Omega u'M(u)dx\right]_S^T - \int_S^T\int_\Omega \rho(.,u')M(u)dxdt.$$

Where ρ in our case is

$$\rho(x,u') = a(x)g(u') \tag{5.78}$$

Which means we are going to obtain:

$$E(t) \leq C\int_S^T\int_{Q_1\cap\Omega}\left(u'^2+|\nabla u|^2\right)dxdt - \left[\int_\Omega u'M(u)dx\right]_S^T - \int_S^T\int_\Omega a(.)g(u')M(u)dxdt. \tag{5.79}$$

Now, we estimate the right terms of (5.79)**:**

•$\mathbf{\left[\int_\Omega u'M(u)dx\right]_S^T}$

This term is estimated the exact same way as in (4.87), which means we have:

$$\left|\left[\int_\Omega u'M(u)dx\right]_S^T\right| \leq CE(S). \tag{5.80}$$

•$\mathbf{\int_S^T\int_{Q_1\cap\Omega}|\nabla u|^2dxdt}$:

From (4.99) we have :

$$\begin{aligned}\int_S^T\int_{\Omega\cap Q_1}|\nabla u|^2dxdt \leq& \int_S^T\int_{\Omega\cap Q_2}|u'|^2dxdt + \frac{1}{2}\int_S^T\int_{\Omega\cap Q_2}|\Delta\xi|u^2dxdt\\ &+\left|\left[\int_{\Omega\cap Q_2}\xi uu'dx\right]_S^T\right| + \int_S^T\int_{\Omega\cap Q_2}|ua(.)g(u)|dxdt.\end{aligned} \tag{5.81}$$

Estimating $\mathbf{\left[\int_{\Omega\cap Q_2}\xi uu'dx\right]_S^T}$ **:**

From (4.102) we have

$$\left|\left[\int_{\Omega\cap Q_2}\xi uu'dx\right]_S^T\right| \leq CE(S). \tag{5.82}$$

Estimating $\left[\int_{\Omega\cap Q_2}\xi u u' dx\right]_S^T$:
Since, $\xi \in C_0^\infty(\mathbb{R}^N$, $\Delta\xi$ is bounded, we have:

$$\int_S^T \int_{\Omega\cap Q_2} |\Delta\xi| u^2 dxdt \leq C \int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \tag{5.83}$$

Estimating $\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt$:
From (4.127) we have:

$$\int_S^T \int_{\Omega\cap Q_2} u^2 dxdt \leq \left[\int_\Omega zu' dx\right]_S^T + \int_S^T \int_\Omega (-z'u' + zag(u'))\, dxdt. \tag{5.84}$$

From (4.131) we have

$$\left|\left[\int_\Omega zu' dx\right]_S^T\right| \leq 2C'E(S). \tag{5.85}$$

On another hand we have:

$$\int_S^T \int_\Omega z'u' dxdt \leq \frac{C}{\eta} \int_S^T \int_\Omega z'^2 dxdt + C\eta \int_S^T \int_\Omega u'^2 dxdt \tag{5.86}$$

From lemma 4.5 we have

$$\int_S^T \int_\Omega z'u' dxdt \leq \frac{C}{\eta} \int_S^T \int_\Omega \beta u'^2 dxdt + C\eta \int_S^T \int_\Omega u'^2 dxdt \tag{5.87}$$

Which gives, from the definition of β:

$$\int_S^T \int_\Omega z'u' dxdt \leq \frac{C}{\eta} \int_S^T \int_\omega u'^2 dxdt + C\eta \int_S^T E dxdt \tag{5.88}$$

Left to estimate the term $\int_S^T \int_\Omega +a(.)zg(u')dxdt$:
We have

$$\int_S^T \int_\Omega |ag(u')z|\, dxdt = \int_S^T \int_{|u'|\leq 1} |ag(u')z|\, dxdt + \int_S^T \int_{|u'|>1} |ag(u')z|\, dxdt \tag{5.89}$$

First we look at the part of Ω where $|u'| \leq 1$:
Using Young's inequality

$$\int_S^T \int_{|u'|\leq 1} |ag(u')z|\, dxdt \leq \int_S^T \int_{|u'|\leq 1} \left(\frac{\eta}{2}|z|^2 + \frac{1}{2\eta}|ag(u')|^2\right) dxdt \tag{5.90}$$

Using lemma 4.5:

$$\int_S^T \int_{|u'|\leq 1} |ag(u')z|\, dxdt \leq \int_S^T \int_\Omega \frac{\eta}{2} C|u|^2 dxdt + \int_S^T \int_{|u'|\leq 1} \frac{1}{2\eta}|ag(u')|^2 dxdt \tag{5.91}$$

Using Poincaré inequality:

$$\int_S^T \int_\Omega C\frac{\eta}{2}|u|^2 dxdt \leq \frac{C\eta}{2}\int_S^T \int_\Omega \left(|\nabla u|^2 + |u'|^2\right) dxdt \leq C\eta \int_S^T Edt \tag{5.92}$$

$g(0) = 0$ and $g'(0) \neq 0$ implies :

There exists a constant $C > 0$ such that

$$|g(x)| \leq C|x| \quad if \quad |x| \leq 1. \tag{5.93}$$

Using (5.93)

$$\frac{1}{2\eta}\int_S^T \int_{|u'|\leq 1} |ag(u')|^2 dxdt \leq \frac{C}{2\eta}\int_S^T \int_{|u'|\leq 1} a|u'|^2 dxdt \tag{5.94}$$

Combining (5.91), (5.92) and (5.94) we obtain :

$$\int_S^T \int_{|u'|\leq 1} |ag(u')z|\, dxdt \leq C\eta \int_S^T Edt + \frac{C}{2\eta}\int_S^T \int_\Omega a|u'|^2 dxdt \tag{5.95}$$

Next, we look at the part of Ω where $|u'| > 1$:

We have:

$$H^1(\Omega) \subset L^{q+1}(\Omega). \tag{5.96}$$

Which gives using lemma (4.5) and Poincaré inequality:

$$\| z \|_{L^{q+1}(\Omega)} \leq C \| z \|_{H^1(\Omega)} = \left(\int_\Omega z^2 dx + \int_\Omega |\nabla z|^2 dx\right)^{\frac{1}{2}} \leq \left(C\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla z|^2 dx\right)^{\frac{1}{2}} \tag{5.97}$$

Let's recall the equation satisfied by z:

$$\begin{cases} \Delta z = \beta u & \text{in } \Omega. \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.98}$$

We multiply (5.98) by z, we integrate on Ω and then we integrate by part, we obtain:

$$\int_\Omega |\nabla z|^2 dx = \int_\Omega \beta uzdx \tag{5.99}$$

Using Cauchy-Schwarz, lemma (4.5), the boundedness of β and Poincaré:

$$\int_\Omega |\nabla z|^2 dx = \int_\Omega \beta uzdx \leq C||u||_{L^2(\Omega)}||z||_{L^2(\Omega)} \leq C||u||^2_{L^2(\Omega)} \leq C||\nabla u||^2_{L^2(\Omega)} \tag{5.100}$$

Combining (5.100) with (5.97) we obtain:

$$\| z \|_{L^{q+1}(\Omega)} \leq C \| z \|_{H^1(\Omega)} \leq C \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}} \leq C\sqrt{E} \tag{5.101}$$

Now using Holder's inequality,

$$\begin{aligned} &\int_S^T \int_{|u'|>1} |ag(u')z| \, dxdt \\ &\qquad \leq \int_S^T \left(\int_{|u'|>1} |z|^{q+1} \, dx \right)^{\frac{1}{q+1}} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} dt \\ &\qquad \leq \int_S^T \left(\int_\Omega |z|^{q+1} \, dx \right)^{\frac{1}{q+1}} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} dt \end{aligned} \tag{5.102}$$

Using (5.101) and the fact that a is bounded :

$$\int_S^T \int_{|u'|>1} |ag(u')z| \, dxdt \leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.103}$$

$$\leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |ag(u')||g(u')|^{\frac{1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.104}$$

Now using the hypothesis (5.75) we obtain :

$$\begin{aligned} \int_S^T \int_{|u'|>1} |ag(u')z| \, dxdt &\leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} ag(u')u' dx \right)^{\frac{q}{q+1}} dt \\ &\leq \int_S^T C\sqrt{E} \left(\int_\Omega ag(u')u' dx \right)^{\frac{q}{q+1}} dt \end{aligned} \tag{5.105}$$

Using the dissipation relation

$$\begin{aligned} \int_S^T \int_{|u'|>1} |ag(u')z| \, dxdt &\leq \int_S^T C\sqrt{E} \, (-E')^{\frac{q}{q+1}} \, dt \\ &\leq C \int_S^T E^{\frac{1}{2}} \, (-E')^{\frac{q}{q+1}} \, dt \end{aligned} \tag{5.106}$$

Using Young's inequality

$$\int_S^T E^{\frac{1}{2}} \, (-E')^{\frac{q}{q+1}} \, dt \leq C\eta^{q+1} \int_S^T E^{\frac{q+1}{2}} dt + \frac{C}{\eta^{\frac{q+1}{q}}} \int_S^T (-E') dt \tag{5.107}$$

Using the non-increasingness of E with (5.106) and (5.107), we obtain :

$$\int_S^T \int_{|u'|>1} |ag(u')z| \, dxdt \leq C\eta^{q+1} E(0)^{\frac{q-1}{2}} \int_S^T E dt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) \tag{5.108}$$

Combining (5.95) and (5.108) we have :

$$\left|\int_\Omega ag(u')zdx\right| \leq \eta\left(C+\eta^q E(0)^{\frac{q-1}{2}}\right)\int_S^T Edt + \frac{C}{\eta^{\frac{q+1}{q}}}E(S) + \frac{C}{2\eta}\int_S^T\int_\Omega |a(.)u'|^2dxdt \tag{5.109}$$

Estimating $\int_S^T\int_{\Omega\cap Q_2} a(x)ug(u)dxdt$:

$$\int_S^T\int_{\Omega\cap Q_2} a(x)ug(u)dxdt \leq \int_S^T\int_\Omega a(x)ug(u)dxdt \tag{5.110}$$

It's gonna be practically the same proof as (5.46) We have

$$\int_S^T\int_\Omega |ag(u')u|\,dxdt = \int_S^T\int_{|u'|\leq 1}|ag(u')u|\,dxdt + \int_S^T\int_{|u'|>1}|ag(u')u|\,dxdt \tag{5.111}$$

First we look at the part of Ω where $|u'|\leq 1$:
Using Young's inequality

$$\int_S^T\int_{|u'|\leq 1}|ag(u')u|\,dxdt \leq \int_S^T\int_{|u'|\leq 1}\left(\frac{\eta}{2}|u|^2+\frac{1}{2\eta}|ag(u')|^2\right)dxdt \tag{5.112}$$

Using Poincaré inequality :

$$\int_S^T\int_{|u'|\leq 1}\frac{\eta}{2}|u|^2dxdt \leq \frac{C\eta}{2}\int_S^T\int_{|u'|\leq 1}\left(|\nabla u|^2+|u'|^2\right)dxdt \leq C\eta\int_S^T Edt \tag{5.113}$$

$g(0)=0$ and $g'(0)\neq 0$ implies :
There exists a constant $C>0$ such that

$$|g(x)|\leq C|x| \quad if \quad |x|\leq 1. \tag{5.114}$$

Using (5.114)

$$\frac{1}{2\eta}\int_S^T\int_{|u'|\leq 1}|ag(u')|^2dxdt \leq \frac{C}{2\eta}\int_S^T\int_{|u'|\leq 1}a|u'|^2dxdt \tag{5.115}$$

Combining (5.112), (5.113) and (5.115) we obtain :

$$\int_S^T\int_{|u'|\leq 1}|ag(u')u|\,dxdt \leq C\eta\int_S^T Edt + \frac{C}{2\eta}\int_S^T\int_\Omega a|u'|^2dxdt \tag{5.116}$$

Next, we look at the part of Ω where $|u'|>1$:
We have:

$$H^1(\Omega)\subset L^{q+1}(\Omega). \tag{5.117}$$

Which gives :

$$\| u \|_{L^{q+1}(\Omega)} \leq C \| u \|_{H^1(\Omega)} \leq C\sqrt{E}. \tag{5.118}$$

Now using Holder's inequality,

$$\int_S^T \int_{|u'|>1} |ag(u')u| \, dxdt$$
$$\leq \int_S^T \left(\int_{|u'|>1} |u|^{q+1} \, dx \right)^{\frac{1}{q+1}} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} dt \tag{5.119}$$

Using (5.38) and the fact that a is bounded :

$$\int_S^T \int_{|u'|>1} |ag(u')u| \, dxdt \leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.120}$$
$$\leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} |ag(u')||g(u')|^{\frac{1}{q}} dx \right)^{\frac{q}{q+1}} dt \tag{5.121}$$

Now using the hypothesis (5.75) we obtain :

$$\int_S^T \int_{|u'|>1} |ag(u')u| \, dxdt \leq \int_S^T C\sqrt{E} \left(\int_{|u'|>1} ag(u')u'dx \right)^{\frac{q}{q+1}} dt$$
$$\leq \int_S^T C\sqrt{E} \left(\int_\Omega ag(u')u'dx \right)^{\frac{q}{q+1}} dt \tag{5.122}$$

Using the dissipation relation

$$\int_S^T \int_{|u'|>1} |ag(u')u| \, dxdt \leq \int_S^T C\sqrt{E} \, (-E')^{\frac{q}{q+1}} \, dt$$
$$\leq C \int_S^T E^{\frac{1}{2}} \, (-E')^{\frac{q}{q+1}} \, dt \tag{5.123}$$

Using Young's inequality

$$\int_S^T E^{\frac{1}{2}} \, (-E')^{\frac{q}{q+1}} \, dt \leq C\eta^{q+1} \int_S^T E^{\frac{q+1}{2}} dt + \frac{C}{\eta^{\frac{q+1}{q}}} \int_S^T (-E')dt \tag{5.124}$$

Using the non-increasingness of E with (5.124) and (5.43), we obtain :

$$\int_S^T \int_{|u'|>1} |ag(u')u| \, dxdt \leq C\eta^{q+1} E(0)^{\frac{q-1}{2}} \int_S^T Edt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) \tag{5.125}$$

Combining (5.116) and (5.125) we have :

$$\left| \int_\Omega ag(u')udx \right| \leq \eta \left(C + \eta^q E(0)^{\frac{q-1}{2}} \right) \int_S^T Edt + \frac{C}{\eta^{\frac{q+1}{q}}} E(S) + \frac{C}{2\eta} \int_S^T \int_\Omega |a(.)u'|^2 dxdt \tag{5.126}$$

• $\int_S^T \int_\Omega \mathbf{a(.)g(u')M(u)dxdt}$

$$\int_S^T \int_\Omega a(.)g(u')M(u)dxdt = \frac{N-1}{2} \int_S^T \int_\Omega a(.)g(u')udxdt + \int_S^T \int_\Omega a(.)g(u')\nabla u.hdxdt \tag{5.127}$$

We have already estimated the term $\int_S^T \int_\Omega a(.)g(u')udxdt$, left to estimate :

$\int_S^T \int_\Omega a(.)g(u')\nabla u.hdxdt$

Naturally, we try to proceed the same way as estimating $\int_S^T \int_\Omega a(.)g(u')udxdt$: First we look at the part of Ω where $|u'| \leq 1$:

Using Young's inequality and the fact that $|h|$ is bounded

$$\int_S^T \int_{|u'|\leq 1} |ag(u')\nabla u.h|\, dxdt \leq \int_S^T \int_{|u'|\leq 1} \left(\frac{\eta}{2}|\nabla u.h|^2 + \frac{1}{2\eta}|ag(u')|^2\right) dxdt \tag{5.128}$$

Using Poincaré inequality :

$$\int_S^T \int_{|u'|\leq 1} \frac{\eta}{2}|\nabla u|^2 dxdt \leq \frac{C\eta}{2} \int_S^T \int_{|u'|\leq 1} \left(|\nabla u|^2 + |u'|^2\right) dxdt \leq C\eta \int_S^T Edt \tag{5.129}$$

We have already proved that

$$\frac{1}{2\eta} \int_S^T \int_{|u'|\leq 1} |ag(u')|^2 dxdt \leq \frac{C}{2\eta} \int_S^T \int_{|u'|\leq 1} a|u'|^2 dxdt \tag{5.130}$$

Combining (5.128), (5.129) and (5.130) we obtain :

$$\int_S^T \int_{|u'|\leq 1} |ag(u')\nabla u|\, dxdt \leq C\eta \int_S^T Edt + \frac{C}{2\eta} \int_S^T \int_\Omega a|u'|^2 dxdt \tag{5.131}$$

Next, we look at the part of Ω where $|u'| > 1$:

We have:

$$H^1(\Omega) \subset L^{q+1}(\Omega). \tag{5.132}$$

Which gives :

$$\| u \|_{L^{q+1}(\Omega)} \leq C \| u \|_{H^1(\Omega)} \leq C\sqrt{E}. \tag{5.133}$$

Now using Holder's inequality,

$$\int_S^T \int_{|u'|>1} |ag(u')\nabla u.h|\, dxdt$$
$$\leq C \int_S^T \left(\int_{|u'|>1} |\nabla u|^{q+1}\, dx\right)^{\frac{1}{q+1}} \left(\int_{|u'|>1} |ag(u')|^{\frac{q+1}{q}}\, dx\right)^{\frac{q}{q+1}} dt \tag{5.134}$$

Now the thing is, we can't proceed the same way here, if we apply (5.133) to ∇u we are going to get:

$$\| \nabla u \|_{L^{q+1}(\Omega)} \leq C \| \nabla u \|_{H^1(\Omega)} \tag{5.135}$$

It's clear that if we get to estimate the term $\int_{\mathbf{S}}^{\mathbf{T}} \int_{\mathbf{\Omega}} \mathbf{a}(.)\mathbf{g}(\mathbf{u}')\nabla\mathbf{u}.\mathbf{hdxdt}$ the proof of lemma 5.6 is over, and we can proceed to prove exponential stability using the methods we used in the linear localized case as well as the nonlinear nonlocalized case :

•We treat the term $\int_\Omega a(x)u'^2dx$ as we did in section 5.1.

•And since $a(x) > a_0$ for $x \in \omega$ we have $\int_\omega u'^2dx \leq \int_\omega \frac{a(x)}{a_0}u'^2dx \leq \frac{1}{a_0}\int_\Omega a(x)u'^2dx$ and from here, we proceed the same way we did in section 5.1.

After the estimations we get an energy estimates to which we apply Gronwall's inequality to conclude stability.

Chapter 6

Conclusion and perspectives

Summary

In this work, we have studied the exponential stability of the damped wave equation, treated the linear case with both a global and local damping, we used a perturbed energy functional to treat the first case and we used the multiplier method to treat the second and we gave sufficient geometrical conditions on the damping domain for the stability to hold. Then, we briefly redid the main steps of the proof in the one dimensional case and as expected, it was significantly simplified (for instance, the use of only three multipliers instead of four). After that, we moved to the nonlinear case, we treated the equation with the global damping using the method introduced in [8], and then we tried to generalize it to the case of a localized damping, and there, we found some difficulties treating one of the terms, so up until now, what we concluded by looking at the problem is that the result will be obtained once we get to estimate the problematic term.

Perspectives

Since our last case is not finished yet, the first perspective is to finish the last proof and study the L^2 stability of the problem. Then this research work is going to be followed by a PhD thesis, directed and supervised by professors Frédéric Jean and Yacine Chitour. The thesis will be a continuity of what we have been working on during this internship, we will consider more general feedbacks, and work in more general frameworks, since the wave theory has always been developped in a hilbertian framework, we would like to change that, as it will be useful to consider frameworks like L^∞ to solve some arisen problems in the domain of automatic.

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