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Mr KEBIRI Omar

Sur le thème

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Mr. DIB Hacen	Professeur	Université de Tlemcen	Président
Mr. MOURID Tahar	Professeur	Université de Tlemcen	Directeur de thèse
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Mr. BOUKHARI Fakhreddine	MCA	Université de Tlemcen	Examinateur

Laboratoire de Statistiques et modélisation aléatoires
BP 119, 13000 Tlemcen - Algérie

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA. MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

FACULTY OF SCIENCES, MATHEMATICS DEPARTMENT
LABORATORY OF STATISTICS AND RANDOM MODELISATION

STOCHASTIC OPTIMAL CONTROL DRIVEN BY FORWARD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

By
Omar KEBIRI

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Dedication

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Abstract

The purpose of the present dissertation is to study existence of an optimal control given by a Forward Backward Stochastic Differential Equation (FBSDE, in short), the notion of backward stochastic differentail equations (BSDE, in short) and its applications.

In the first chapter some preliminaries, definitions, and theorems are presented.

In the next chapter the notion of existence of an optimal control for a system of fully coupled FBSDE in the degenerate case is given. The cost functional is defined by the first component of the solution of the controlled backward stochastic differential equation (BSDE in short) at the initial time. We study the case of degenerate diffusion coefficient σ in the forward equation. Our control problem is to find an optimal control holds the FBSDE and the optimization problem. This last is to minimize the cost functional in the set of the admissible controls. For that, we show first the existence of a relaxed control by constructing a sequence of approximating controlled system for which we show the existence of a sequence of feedback controls, and we prove that the approximating value function converges to the original one, the convergence is got at least along a subsequence , we suppose in addition some Filippov convexity conditions on the coefficients of the system to prove that the relaxed optimal control is strict.

Chapter 3 is devoted to another results of the thesis, it present existence of an optimal control whose dynamical system is driven by a coupled forward-backward stochastic differential equation in the non-degenerate case.

In Chapter 4 the thesis present the notion of the existence of the solution of one dimensional BSDEs with logarithmic growth, its present also some applications to PDEs.

In the last Chapter an application in high dimensional stochastic differential equations is given with numerical results, a real case of the Los Angeles University hospital is studied, a numerical analysis of fully coupled FBSDEs is also stated.

Introduction

The aim of this thesis is to studies some problems of stochastic optimal control analytically and numerically.

Due to their applications in Physics, mathematical finance and Molecular Dynamics, stochastic optimal control has been subject to extensive research during the last two decades.

Theory of stochastic differential equations has been developed quickly, K. Itô [85] and [86], L. E. Bertram and P. E. Sarachik [32] R. Z. Hasminskii [81], D. D. Bainov and V. B. Kolmanovskii [25], D. Q. Jiang and N. Z. Shi [89], D. Q. Jiang et al. [90] Y. Ouknine et al. [30].

General nonlinear BSDEs in the framework of Brownian motion were first introduced by Pardoux and Peng in [122], since then the theory of BSDEs develops very quickly, see El Karoui, Peng and Quenez [64], Peng [114], [115] , Ouknine et al. [76], and relation between stochastic optimal control and BSDEs (see for example [98] [117]).

Associated with the BSDEs theory, the field of fully coupled FBSDEs develops also very quickly, we refer to, Cvitanic and Ma [54], Delarue [59], Hu and Peng [82], Ma, Protter, and Yong [104] B. Mezerdi et al [106], Ma, Wu, Zhang, and Zhang [103], Ma and Yong [105], Pardoux and Tang [124], Peng and Wu [125], Yong [139], and Zhang [141], etc. For more details on fully coupled FBSDEs, the reader is referred to the book of Ma and Yong [105] ; also refer to Li and Wei [102] and the references therein. it have important applications in Mathematical fiance like in the pricing/hedging problem, in the stochastic control and game theory, we mention some works ([111] [45], [125], [135], [134], [137]), Optimal control ([125]

) and Molecular Dynamic simulations [79],[129] and [128].

The principal developments in this subject concern the existence of optimal control, Pontryagin's maximum principle (or necessary optimality conditions) and Bellman's principle (also called dynamic programming principle), etc., see e.g. [17, 40, 42, 41, 63, 72, 80, 93, 101, 102, 105, 115, 116].

Closer to our concern here, the existence of an optimal for a system driven by SDE-BSDE was established in [17] and [41] by different methods. In [17], the approach consists to directly show the existence of a relaxed control by using a compactness method and the Jakubowsky S-topology. In [41] the authors work on by the HJB equation associated the control problem. This allows them to construct a sequence of optimal feedback controls. After that, they analyse to the limit and use the result of [63] in order to get the existence of a relaxed optimal control. In both papers [17] and [41] the Filippov convexity condition is used in order to get the existence of a strict optimal control. It should be noted that in [17] and [41] the controlled system is driven by a decoupled system of SDE-BSDE.

The question of the existence of an optimal control in some appropriate sense is one of the important fields in control theory, and has been subject of large literature. We mention among them, Peng [116], Touzi [130] and Bahlali, Gherbal and Mezerdi, [17], application of optimal control has been subject of a large literature we mention some of them, [129], [79], [143], in molecular dynamics, and [36], [117], [58], in Mathematical finance.

One of the main goal of the thesis is to establish existence result on strict optimal control for the problem (1.1.1)–(1.1.3), for this we proceed as follows : we follow the method developed in [41], because our coefficients are not smooth enough to get strong solution of the corresponding HJB equation of our SOC we approximate our controlled FBSDE (1.1.1) by a sequence of FBSDEs with smooth data $b_\delta, \sigma_\delta, f_\delta$ and Φ_δ and consider a new value function V^δ which is associated to the FBSDE with these smooth data. This allows us to apply the result of Krylov [94] (Theorems 6.4.3 and 6.4.4), V^δ is sufficiently smooth and satisfies a Hamilton-Jacobi-Bellman equation. Since all admissible controls take their values in a compact set, we then deduce the existence of a feedback control u^δ. Next, we prove that the sequence V^δ converges uniformly to a function V which is the value function of our initial control problem. Comparing with [41], there are two main difficulties (see the next chapter for more details). We have to note that when the control enters the diffusion coefficient σ, we arrive to an SDE with measurable diffusion coefficient and, in this case, the uniqueness of solution fails. It is well known that when the diffusion coefficient is merely measurable then even the uniqueness in law fails in general for Itô's forward SDE in dimension strictly greater than 2, see [92] for more details. This explains why we consider the case when the control does not enter the diffusion coefficient, the idea behind this work (generally speaking) in the applications, is that we have the dynamics of two processes such that for the first one (the forward) we know the initial point of depart and the second one must end in a given position function of the end point of the first, and both are coupled along all the period of the dynamics, it means that not only they are related in the final time but also the solution of the first is in the coefficients of the equation of the second and conversely, the question is how to control the starting point of the backward dynamics.

The thesis is focused also on the existence and uniqueness of solution of one dimensional BSDEs with logarithmic growth. The problem is presented as follows : Let $f(t, \omega, y, z)$ be a real valued $\mathcal{F}_t$–progressively measurable process defined on $[0,\ T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Let ξ be an $\mathcal{F}_T$–measurable $\mathbb{R}$–valued random variable. The backward stochastic differential equations (BSDEs) under consideration is :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad t \in [0, T] \tag{0.0.1}$$

where the driver hold a lass regularity assumptions, which called logarithmic assumption see (4.1.2).

The previous equation will be denoted by $eq(\xi, f)$. The data ξ and f are respectively called the terminal condition and the coefficient or the generator of $eq(\xi, f)$. For $N \in \mathbb{N}^*$, we define

$$\rho_N(f) = E \int_0^T \sup_{|y|,|z| \leq N} |f(s, y, z)| ds, \tag{0.0.2}$$

The applications in reduction models is subjection of the last chapter, where we present the bridge between stochastic optimal control and FBSDEs, this end play an essential role in the model redaction technic of a high dimensional stochastic optimal control. The idea here is to write Hamilton-Jacobi-Bellman equation and to relate it to a FBSDE, and do homogenization to this end, finally back to the PDE form, to display the limiting PDE

0.1 Thesis Outline

Within the next chapter of this thesis we give some basic definitions and preliminaries, also some theorems that we will use in the next chapters, we present also . The next chapter is devoted to present results on the existence of an optimal control for a system of fully coupled FBSDE in the degenerate case, chapter 3 is concerned to the non-degenerate case. Chapter 4 is devoted on the studies of the existence and uniqueness of a one dimensional BSDEs with logarithmic growth and applications to PDEs. The last chapter is focused on the analysis of some real examples numerically, the new approach, was the reduction of such type of high dimensional problems and the difference in the scaling, linear and nonlinear quadratic optimal control was subject of this studies.

The high dimensionality came from a space desensitization of a time-space PDE or a molecular dynamic simulation.

Chapitre 1

Preliminaries

This chapter introduces basic notations and recalls results that will be used throughout this thesis.

First we present the form of the SOC subject to the first and second chapters

1.1 Controlled fully coupled FBSDE

Let $T > 0$ be a finite horizon and $t \in [0,\ T]$. Let $(\Omega,\ \mathcal{F},\ \mathbb{P},\ (\mathcal{F}_t))$ be a filtered probability space which satisfies the usual conditions. Let W be a d-dimensional Brownian motion with respect to the (not necessary Brownian) filtration $(\mathcal{F}_t)$. Let $\mathbb{U}$ be a compact metric space. We define the deterministic functions b, σ, f and Φ by

$$b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times \mathbb{R} \longmapsto \mathbb{R}^{d\times d},$$

$$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R},$$

$$\Phi : \mathbb{R}^d \longmapsto \mathbb{R}.$$

We consider the following controlled system of coupled [1] FBDSE define for $s \in [t,\ T]$ by :

$$\begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, Y_s^{t,x,u}, u_s)dW_s, \\ dY_s^{t,x,u} = -f(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s + dM_s^{t,x,u}, \\ \langle M^{t,x,u},\ W\rangle_s = 0, \\ X_t^{t,x,u} = x, \quad Y_T^{t,x,u} = \Phi(X_T^{t,x,u}), \quad M_t^{t,x,u} = 0, \end{cases} \tag{1.1.1}$$

1. The FBSDE considered here in coupled but not *TOTALY* fully coupled, this case is a project of a future work.

where, $X^{t,x,u}$, $Y^{t,x,u}$, $Z^{t,x,u}$ are $(\mathcal{F}_t)$-adapted square integrable processes and $M^{t,x,u}$ is an $(\mathcal{F}_t)$-adapted square integrable martingale which is orthogonal to W.

Remark 1. *Because the weak solution do not hold necessarily on the Brownian filtration, but on more larger filtration, then the Martingale Representation Theorem (MRT) do not hold, and hence the appearing of the orthogonal martingale M in the equation 1.1.1 is natural by the Kunuta-Watanabe (KW) representation, and not for other consideration like the reflected BSDEs which will be presented late in this thesis.*

The control variable u is an $\mathcal{F}_t$ adapted process with values in a given compact metric space $\mathbb{U}$. It should be noted that the filtered probability space and the Brownian motion may change with the control u.

On $\nu := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$, we define the following spaces of processes :

for $m \in \mathbb{N}^*$ and $t \in [0, T)$,

— $\mathcal{S}^2_\nu(t, T; \mathbb{R}^m)$ denote the set of $\mathbb{R}^m$-valued, $\mathbb{F}$-adapted, continuous processes $(X_s, s \in [t, T])$ which satisfy $\mathbb{E}[\sup_{t \leq s \leq T} |X_s|^2] < \infty$.

— $\mathcal{H}^2_\nu(t, T; \mathbb{R}^m)$ is the set of $\mathbb{R}^m$-valued, $\mathbb{F}$-predictable processes $(Z_s, s \in [t, T])$ which satisfy $\mathbb{E}[\int_t^T |Z_s|^2 ds] < \infty$.

— $\mathcal{M}^2_\nu(t, T; \mathbb{R}^m)$ denotes the set of all $\mathbb{R}^m$-valued, square integrable cadlag martingales $M = (M_s)_{s \in [t,T]}$ with respect to $\mathbb{F}$, with $M_t = 0$.

Now we present the meaning of a solution of a FBSDEs in a non necessary Brownian filtration in this :

Definition 2. *A solution of FBSDE (1.1.1) is a process* $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, M^{t,x,u}) \in$

$\mathcal{S}_\nu^2(t,T;\mathbb{R}^d)\times\mathcal{S}_\nu^2(t,T;\mathbb{R})\times\mathcal{H}_\nu^2(t,T;\mathbb{R}^d)\times\mathcal{M}_\nu^2(t,T;\mathbb{R}^d)$ *which satisfies equation (1.1.1).*

Let's define the following control spaces :

— $\mathcal{U}_\nu(t)$ denotes the set of admissible controls, i.e. the set of $\mathbb{F}$-progressively measurable processes $(u_s, s\in[t,T])$ with values in $\mathbb{U}$ and such that the FBSDE (1.1.1) has a unique solution in $\mathcal{S}_\nu^2(t,T;\mathbb{R}^d)\times\mathcal{S}_\nu^2(t,T;\mathbb{R})\times\mathcal{H}_\nu^2(t,T;\mathbb{R}^d)\times\mathcal{M}_\nu^2(t,T;\mathbb{R}^d)$.

— $\mathcal{R}_\nu(t)$ denotes the set of admissible relaxed controls.

The cost functional [2], which will be minimized, is defined for $u\in\mathcal{U}_\nu(t)$ by :

$$J(t,x,u) := Y_t^{t,x,u}. \tag{1.1.2}$$

An $\mathcal{F}_t$-adapted control $\widehat{u}$ is called optimal if it minimizes J, that is :

$$Y_t^{t,x,\widehat{u}} = \operatorname{essinf}\left\{Y_t^{t,x,u},\ \ u\in\mathcal{U}_\nu(t)\right\}\ ^3.$$

If moreover, $\widehat{u}$ belongs to $\mathcal{U}_\nu(t)$, we then say that $\widehat{u}$ is an optimal strict control.

The value function V is defined by :

$$V(t,x) := Y_t^{t,x,\hat{u}} = \operatorname{essinf}\left\{J(t,x,u),\ \ u\in\mathcal{U}_\nu(t)\right\}. \tag{1.1.3}$$

Next we present some definitions and theorems that we need in the our main results in the next two chapters, afterword we focused on the notion of FBSDEs where we present definitions and existence theorems of a fully coupled FBSDEs in the degenerate and non-degenerate

2. This is the non-linear case the special case is when the generator f is linear in its variables.
3. See Defintion 3 for the definition of *essinf*.

cases, the last section is devoted to present stochastic optimal control and Hamilton-Jacobie-Belman equation

1.1.1 Definitions

In this section we give some definitions that we need in the next chapters

The functional of our optimal control is defined by esssup in the following :

Definition 3. *Let $f : X \longrightarrow \mathbb{R}$ be a real valued function define on a measure space (X, Σ, μ), we suppose that f is measurable*[4] *A number a is called an* essential upper bound *of f if the measurable set $f^{-1}(a, \infty)$ is a set of measure zero, i.e., if $f(x) \leq a$ for almost all $x \in X$. Let*

$$U_f^{\text{ess}} = \{a \in \mathbb{R} : \mu(f^{-1}(a, \infty)) = 0\}$$

be the set of essential upper bounds. Then the essential supremum is defined similarly as

$$\operatorname{ess\,sup} f = \inf U_f^{\text{ess}}$$

if $U_f^{\text{ess}} \neq \emptyset$, and $esssup f = +\infty$ otherwise. we can define the essinf by the same way

$$\operatorname{ess\,inf} f = \sup\{b \in \mathbb{R} : \mu(\{x : f(x) < b\}) = 0\}$$

The notion of esssup and essinf are important tools in the field stochastic optimal control, when in many cases the supremum of a functional do not hold because it goes to infinity in a set of a measure zero in these cases the esssup play the role the following simple example explain more :

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\mathbb{R}$ endowed with the Lebesgue measure and its corresponding borealian

4. The definition of essential spermium can be in general case where f is not necessary measurable s-algebra S. such that :

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ \arctan x, & \text{else} \end{cases}$$

This function has no supremum and no infimum, However, from the point of view of the Lebesgue measure, the set of rational numbers is of measure zero; It follows that the essential supremum is $\frac{\pi}{2}$ while the essential infimum is $-\frac{\pi}{2}$.

Now we define the molllifier of a function by :

Let $\delta \in (0;1]$ and $\phi : \mathbb{R}^m \to \mathbb{R}$ be a function which satisfies : ϕ is a non-negative smooth function, $supp(\phi) \subset B_{\mathbb{R}^m}(0, 1)$ (the unit ball of $\mathbb{R}^m$), $\int_{\mathbb{R}^m} \phi(\xi)\, d\xi = 1$.

For a uniformly Lipschitz function $l : \mathbb{R}^m \to \mathbb{R}$, we define the mollifier of l by

$$l_\delta(\xi) = \delta^{-m} \int_{\mathbb{R}^m} l\left(\xi - \xi'\right) \phi\left(\delta^{-1}\xi'\right) d\xi'.$$

Let K_l denote the Lipschitz constant of l. Of course K_l is independent from δ.

Proposition 1.1.1. *For any $\xi, \xi' \in \mathbb{R}^m$ and $\delta, \delta' > 0$, we have*

1. $|l_\delta(\xi) - l(\xi)| \leq K_l \delta$
2. $|l_\delta(\xi) - l_{\delta'}(\xi)| \leq K_l |\delta - \delta'|$,
3. $|l_\delta(\xi) - l_\delta(\xi')| \leq K_l |\xi - \xi'|$,

1.1.2 Fully coupled FBSDEs and stochastic optimal control

Fully coupled FBSDEs

The Markovian case of a BSDE is a decoupled FBSDE (the solution of the forward equation appear in the backward one as a parameter) solving a decoupled FBSDE is easily done by solving the forward equation and then plug the solution of the forward X in the backward. A fully coupled FBSDE is a forward equation and a backward equation where the solution of the forward appear in the backward equation and conversely, it studied by several authors [21, 112, 3], finding the solution by the previous method does not work here, now let give the formal definition of a FBSDE.

Definition, existence and uniqueness of the solution of a FBSDEs in the degenerate and non degenerate cases

Let $T > 0$ be a finite horizon and $t \in [0,\ T]$. Let $(\Omega,\ \mathcal{F},\ \mathbb{P},\ (\mathcal{F}_t))$ be a filtered probability space which satisfies the usual conditions. Let W be a m-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_t)$. We define the deterministic functions b, σ, f and Φ by

$$b : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^{d\times m},$$

$$f : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^p,$$

$$\Phi : \mathbb{R}^d \longmapsto \mathbb{R}^p.$$

A FBDSE is define for $s \in [t,\ T]$ by :

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \sigma(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})dW_s, \\ X_t^{t,x} = x, \\ dY_s^{t,x} = -f(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x}dW_s, \\ Y_T^{t,x} = \Phi(X_T^{t,x}) \end{cases} \tag{1.1.4}$$

where, $X^{t,x}$, $Y^{t,x}$, $Z^{t,x}$ are $(\mathcal{F}_t)$-adapted square integrable processes, the notation $X^{t,x}$, $Y^{t,x}$, $Z^{t,x}$ is to show that the process X starts in x at the initial time t.

we note $\mathcal{K}_t^{d,k,p\times m} = \mathcal{S}^2_\nu(t,T;\mathbb{R}^d) \times \mathcal{S}^2_\nu(t,T;\mathbb{R}^k) \times \mathcal{H}^2_\nu(t,T;\mathbb{R}^{p\times m})$ [5]

Definition 4. *A solution of FBSDE (1.1.4) is a process* $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in \mathcal{K}_t^{d,p,p\times m}$ *which satisfies equation (1.1.4).*

In what follows we suppose that the diffusion σ is independent of Z.

Existence and uniqueness in the degenerate case :

For a given $1 \times d$ matrix G (with G^T be the transpose of G) and $\lambda := (x, y, z)$ we put

$$A(t,\lambda) := \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix} (t,\lambda),$$

Assumption (H). In this chapter, we assume that there exists a $1 \times d$ full rank matrix G such that the following assumptions are satisfied.

— (H1)

(i) $A(t,\lambda)$ is uniformly Lipschitz in λ uniformly on t, and for any λ, $A(\cdot,\lambda) \in \mathcal{H}^2(0,T;\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$.

(ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^d$, and for any $x \in \mathbb{R}^d$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$.

We denote by K the Lipschitz constant of A and Φ.

— (H2)

5. For the definitions of the spaces look the introduction

(i) $\langle A(t,\lambda) - A(t,\widehat{\lambda}), \lambda - \widehat{\lambda} \rangle \leq -\beta_1 |G\overline{x}|^2 - \beta_2(|G^T\overline{y}|^2 + |G^T\overline{z}|^2)$.

(ii) $\langle \Phi(x) - \Phi(\widehat{x}), G(x - \widehat{x}) \rangle \geq \mu_1 |G\overline{x}|^2, \quad \overline{x} = x - \widehat{x},\ \overline{y} = y - \widehat{y},\ \overline{z} = z - \widehat{z},$

where β_1, β_2, μ_1 are strictly positive constants.

Now we set an existence and uniqueness result for the fully coupled FBSDE (1.1.4)

Theorem 5. *[125] Let the condition (H) hold, we suppose that the diffusion σ is independent to Z. Then there exists a unique adapted solution (X, Y, Z) of the FBSDE (1.1.4).*

Existence and uniqueness of the solution of a FBSDE in a non-degenerate case :

Let the following hypothesis : There exists two constants K and $\lambda > 0$, such that the functions b, σ, f and Φ satisfy the following assumptions **(B)** :

— **(**B1)

1) For any (x, y, z) and $(x', y', z') \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$

$$|\sigma(x,y) - \sigma(x',y')|^2 \leq K^2(|x - x'|^2 + |y - y'|^2),$$

$$|\Phi(x) - \Phi(x')| \leq K|x - x'|,$$

$$|b(x,y,z) - b(x',y',z')| \leq K(|x - x'| + |y - y'| + |z - z'|),$$

$$|f(x,y,z) - f(x',y',z')| \leq K(|x - x'| + |y - y'| + |z - z'|).$$

2) The functions b, σ, f and Φ are bounded.

— (B2) For every $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$,

$$\forall \zeta \in \mathbb{R}^d \quad \langle \zeta, \sigma(t,x,y)\zeta \rangle \geq \lambda |\zeta|^2,$$

our conditions (B1) and (B2) are a special case of the result of [59], then the equation (1.1.4) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ in the space $\mathcal{S}^2_\nu(t,T;\mathbb{R}^d) \times \mathcal{S}^2_\nu(t,T;\mathbb{R}) \times \mathcal{H}^2_\nu(t,T;\mathbb{R}^d)$

Now let present some results on FBSDEs

1.1.3 Stochastic optimal control driven by a FBSDEs

We present here stochastic optimal control driven by a FBSDE and some results on it, in the next chapter we present an existence result of an optimal control of such type of SOC :

For some notation, let $T > 0$ be a finite horizon and $t \in [0,\ T]$. Let $(\Omega,\ \mathcal{F},\ \mathbb{P},\ (\mathcal{F}_t))$ be a filtered probability space which satisfies the usual conditions. Let W be a d-dimensional Brownian motion with respect to the (not necessary Brownian) filtration $(\mathcal{F}_t)$. Let $\mathbb{U}$ be a compact metric space. We define the deterministic functions b, σ, f and Φ by

$$b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^{d\times d},$$

$$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R},$$

$$\Phi : \mathbb{R}^d \longmapsto \mathbb{R}.$$

We consider the following controlled system of coupled FBDSE define for $s \in [t,\ T]$ by :

$$\begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)dW_s, \\ X_t^{t,x,u} = x, \\ dY_s^{t,x,u} = -f(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s, \\ Y_T^{t,x,u} = \Phi(X_T^{t,x,u}) \end{cases} \tag{1.1.5}$$

where, $X^{t,x,u}$, $Y^{t,x,u}$, $Z^{t,x,u}$ are $(\mathcal{F}_t)$-adapted square integrable processes The control variable u is an $\mathcal{F}_t$ adapted process with values in a given compact metric space $\mathbb{U}$

The cost functional, which will be minimized, is defined for all admissible control $u \in \mathcal{U}_\nu(t)$ as the first component of the solution of the BSDE :

$$J(t, x, u) := Y_t^{t,x,u}. \tag{1.1.6}$$

The objective is to optimize the cost functional (1.1.6) by an infrumum, supremum, essential inf or essential sup [6]

when 1.1.5 is decoupled i.e. when b and σ are independent of the solution of the BSDE (Y, Z) the existence of an optimal control is studied by [41], in their case the BSDE is add to an orthogonal Martingale to the Brownian mention because the filtration may change with the control u, for more details see the next chapter.

1.1.4 Hamilton Jacobi Bellman equation

The value function of a stochastic optimal control define in the last section should satisfy a certain partial differential equation called the Hamilton-Jacobi-Bellman equation (HJB in short), given in this

6. See Definition3 for the definition of essential sup and if and why is useful in the case of optimal control.

Definition

The HJB equations are second-order, possibly degenerate elliptic, fully nonlinear equations of the following form :

$$H(x, u, Du, D^2u) = 0, x \in \mathbb{R}^n.$$

The solution of the HJB equation is (under some conditions) the value function of an optimal control which gives the minimum cost for a given dynamical system with an associated cost function. H called the hamiltonian which supposed that is convex.

The HJB equation corresponding to the deterministic case is a first-order PDE. existence of solution of HJB equation are well studied by [48, 49, 47, 50, 51, 52, 53, 83, 84].

Viscosity solutions

Here we present an important type of weak solution of the HJB-equation.

Let where Ω an open subset of $\mathbb{R}^n$, consider nonlinear parabolic second-order partial differential equations :

$$F(t, x, w, \frac{\partial w}{\partial t}, D_x w, D^2_{xx} w) = 0, (t, x) \in [0, T) \times \Omega \tag{1.1.7}$$

Definition 6. *Let the PDE (1.1.7)*

1. *F is elliptic if* $\forall(t,x,r,p,q) \in [0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n and, M_1, M_2 \in \mathcal{S}_n$ [7] *we have :* $M_1 \leq M_2 \Rightarrow F(t,x,r,p,q,M_1) \geq F(t,x,r,p,q,M_2)$

2. *F is parabolic if* $\forall(t,x,r,q,M) \in [0,T) \times \mathbb{R} \times \mathcal{S}_n \times \mathbb{R}^n and, p_1, p_2 \in \mathbb{R}$ *we have :* $p_1 \leq p_2 \Rightarrow F(t,x,r,p_1,q,M_1) \geq F(t,x,r,p_2,q,M_2)$

suppose F is a continuous function of its arguments,elliptic and parabolic, Let a locally bounded function $w \in [0,T] \times \Omega$, we set the definition of an upper-semi continuous (USC), (resp. lower-semi continuous (LSC)) envelope w^*, (w_*) by :

$$w^*(t,x) = \lim_{t_1 \leq T \to t} \sup_{x_1 \to x} w(t_1,x_1) w_*(t,x) = \lim_{t_1 \leq T \to t} \inf_{x_1 \to x} w(t_1,x_1) \tag{1.1.8}$$

its clear that

$w_*(t,x) \leq w(t,x) \leq w^*(t,x)$

we have three cases :

— w is USC if $w = w^*$

— w is LSC if $w = w_*$

— w is continuous if $w = w^* = w_*$

Now let give the definition of a *viscosity solution*

Definition 7. *Let* $w : [0,T] \times \omega$ *be locally bounded and let* Π *be a smooth function on* $[0,T) \times \Omega$*. we have the following definitions*

1. *(viscosity supersolution)* w *is a viscosity supersolution of (1.1.7) on* $[0,T] \times \Omega$ *iif :*

7. *The set of symmetric square n matrices*

$$\forall (t_1,x_1)\in[0,T)\times\Omega, \forall \pi\in\mathcal{C}^{1,2} : \pi(t_1,x_1)=w(t_1,x_1) : F(t_1,x_1,\pi,\frac{\partial \pi}{\partial t_1},D_{x_1}\pi,D^2_{x_1x_1}\pi)\geq 0 \quad (1.1.9)$$

2. *(viscosity subsolution) w is a viscosity subsolution of (1.1.7) on* $[0,T]\times\Omega$ *iif :*

$$\forall (t_1,x_1)\in[0,T)\times\Omega, \forall \pi\in\mathcal{C}^{1,2} : \pi(t_1,x_1)=w(t_1,x_1) : F(t_1,x_1,\pi,\frac{\partial \pi}{\partial t_1},D_{x_1}\pi,D^2_{x_1x_1}\pi)\leq 0. \quad (1.1.10)$$

w is viscosity solution if it is supersolution and subsolution.

1.2 Notations and definitions for the third chapter

Here we present some notations that we will use for the chapter on existence and uniqueness of solution of one dimensional BSDEs with logarithmic growth.

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},P)$ be a probability space on which is defined a standard d-dimensional Brownian motion $W=(W_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}^0_t:=\sigma\{B_s,s\leq t\})_{0\leq t\leq T}$. Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}^0_t)_{0\leq t\leq T}$ with the P-null sets of $\mathcal{F}$. Let $f(t,\omega,y,z)$ be a real valued $\mathcal{F}_t$–progressively measurable process defined on $[0,\ T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$. Let ξ be an $\mathcal{F}_T$–measurable $\mathbb{R}$–valued random variable. The backward stochastic differential equations (BSDEs) under consideration is :

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_s dW_s, \qquad t\in[0,T] \quad (1.2.1)$$

The previous equation will be denoted by $eq(\xi, f)$. The data ξ and f are respectively called the terminal condition and the coefficient or the generator of $eq(\xi, f)$. For $N \in \mathbb{N}^*$, we define

$$\rho_N(f) = E\int_0^T \sup_{|y|,|z|\leq N} |f(s,y,z)|ds. \tag{1.2.2}$$

For $p \geq 1$, we denote by $Ł^p_{loc}(\mathbb{R})$ the space of (classes) of functions u defined on $\mathbb{R}$ which are p-integrable on bounded set of $\mathbb{R}$. We also define,

$\mathcal{C}$:= the space of continuous and $\mathcal{F}_t$ –adapted processes.

$\mathcal{S}^p$:= the space of continuous, $\mathcal{F}_t$ –adapted processes φ such that $\mathbb{E}\Big(\sup_{0\leq t\leq T}|\varphi_t|^p\Big) < \infty$.

$\mathcal{M}^p$:= the space of $\mathcal{F}_t$–adapted processes φ satisfying $\mathbb{E}\left[\left(\int_0^T |\varphi_s|^2 ds\right)^{\frac{p}{2}}\right] < +\infty$.

$\mathcal{L}^2$:= the space of $\mathcal{F}_t$ –adapted processes φ satisfying $\int_0^T |\varphi_s|^2 ds < +\infty$ $\mathbb{P}$–a.s.

For given real numbers a and b, we set $a \wedge b := \min(a,b)$, $a \vee b := \max(a,b)$, $a^- := \max(0,-a)$ and $a^+ := \max(0,a)$.

Definition 8. *A solution to $eq(\xi, f)$ is a process (Y,Z) which belongs to $\mathcal{C}\times\mathcal{L}^2$ such that (Y,Z) satisfies equation $eq(\xi, f)$ for each $t \in [0,T]$ and $\int_0^T |f(s,Y_s,Z_s)|ds < \infty$ a.s.*

1.3 Notion of stopping time and ergodicity

In the last chapter we study some SOC problems where the FSDE is bilinear, then the coefficients are not bounded in the entire space, therefor we suppose that the our dynamic lives in a bounded domain, which is the case in the most applications.

For this let give the notion of stopping time :

Definition 9. *A real value random variable* $\tau : \Omega \longrightarrow \mathbb{R}_+$ *is called an* $\mathcal{F}_t$*-stopping time if :*
$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ *for all* t *in* $\mathbb{R}_+$

Examples

1. Playing until the player either runs out of money or has played 100 games is a stopping rule.

2. If the filtration is complete, then a random time that is almost certainly a constant is also a stopping time.

3. One of the important examples of stopping time is the first time that the process hits a set of states define by :

$$\tau_D = \inf\{s \geq 0 : X_s \notin D\} ,$$

where X is a stochastic process in $\mathbb{R}^n$, and D is a subset of $\mathbb{R}^n$.

The notion of ergodicity is a very important tool in the homogenization technic, a dynamical system is said to be ergodic, if has the same behavior averaged over time as averaged over the space of all the system's states in its phase space, the formal definition is

Definition 10. *Let* $(X,\ \Sigma,\ \mathbb{P})$ *be a probability space, and* $T : X \to X$ *be a measure-preserving transformation. We say that* T *is ergodic with respect to* $\mathbb{P}$ *if :*
for every $E \in \Sigma$ *with* $T^{-1}(E) = E$ *either* $\mathbb{P}(E) = 0$ *or* $\mathbb{P}(E) = 1$ *.*

Now we give the notion of invariant measure

Definition 11. *Let* $(X, \mathcal{F})$ *be a measurable space and let* f *be a measurable function from* X *to itself. A measure* μ *on* $(X, \mathcal{F})$ *is said to be invariant under* f *if, for every measurable set* B *in* $\mathbb{F}$, $\mu\left(f^{-1}(B)\right) = \mu(B)$.

A reasons for some studies to work with an ergodic system is that the collection of ergodic measures, is a subset of the collection of invariant measures. In the most cases the periodicity is one hypothesis used for ensure the ergodicity and therefor the existence of an invariant measure, but in our case, and because the bilinear system that we study in the last chapter is no periodic, we use the Kalman condition to ensure the ergodicity.

Chapitre 2

Existence of an optimal control for a system of fully coupled FBSDE in the degenerate case

The aim of the present chapter is to extend the results of [17, 41], to a coupled FBSDE. Comparing with [17, 41], the first difficulty is related to the fact that the uniform Lipschitz condition on the coefficients is not sufficient to ensure the existence of a unique solution to equation (2.0.1) for an arbitrary duration. This fact is well explained in [2] where two illustrating examples are given. In order to ensure the existence and uniqueness of solutions for equation (2.0.1), we moreover assume the so-called G-monotony condition on the coefficients given in [125]. The second difficulty concerns the gradient estimate of the approximating value function. It turns out that the G-monotony condition combined with the comparison theorem of BSDEs play an important role to overcome this second difficulty. To begin, let us give a precise formulation of our problem.

In the second section, we give the assumptions and the main result. Section 3 is devoted to the proof. The later consists to construct an approximating sequence of controlled systems for which we prove the existence of a sequence of feedback controls u^{δ}. By passing to the limit, we show the existence of a feedback control to our initial system.

Consider the following SOC :

the dynamical system is defined for $s \in [t, T]$ by :

$$\begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, Y_s^{t,x,u}, u_s)dW_s, \\ dY_s^{t,x,u} = -f(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s + dM_s^{t,x,u}, \\ \langle M^{t,x,u},\ W\rangle_s = 0, \\ X_t^{t,x,u} = x, \quad Y_T^{t,x,u} = \Phi(X_T^{t,x,u}), \quad M_t^{t,x,u} = 0, \end{cases} \tag{2.0.1}$$

according to Theorem 5 there exist unique solution to the equation (2.0.1), we define the value function by :

$$J(t,x,u) := Y_t^{t,x,u}. \tag{2.0.2}$$

We say that $\hat{u}$ is called a strict optimal control, if it belongs to $\mathcal{U}_\nu(t)$ and satisfies

$$J(t,x,\hat{u}) = \text{essinf} \ \{ \ J(t,x,u), \quad u \in \mathcal{U}_\nu(t) \ \} \tag{2.0.3}$$

The value function of the control problem is given, for each $t \in [0,T]$ and $x \in \mathbb{R}^d$, by

$$V(t,x) := \text{essinf} \ \{ \ J(t,x,u), \quad u \in \mathcal{U}_\nu(t). \ \} \tag{2.0.4}$$

2.1 Assumptions and the main result

For a given $1 \times d$ matrix G (with G^T be the transpose of G) and $\lambda := (x,y,z)$ we put

$$A(t,\lambda,u) := \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix} (t,\lambda,u),$$

Assumption (B). Throughout this chapter, we assume that there exists a $1 \times d$ full rank matrix G such that the following assumptions are satisfied.

— (B1)

(i) $A(t,\lambda,u)$ is uniformly Lipschitz in λ uniformly on (t,u), and for any λ, $A(\cdot,\lambda,\cdot) \in \mathcal{H}^2(0,T;\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$;

(ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^d$, and for any $x \in \mathbb{R}^d$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$.

We denote by K the Lipschitz constant of A and Φ.

— (B2)

(i) $\langle A(t,\lambda,\cdot) - A(t,\widehat{\lambda},\cdot), \lambda - \widehat{\lambda}\rangle \leq -\beta_1|G\overline{x}|^2 - \beta_2(|G^T\overline{y}|^2 + |G^T\overline{z}|^2)$,

(ii) $\langle \Phi(x) - \Phi(\widehat{x}), G(x-\widehat{x})\rangle \geq \mu_1|G\overline{x}|^2, \quad \overline{x} = x - \widehat{x},\ \overline{y} = y - \widehat{y},\ \overline{z} = z - \widehat{z}$,

where β_1, β_2, μ_1 are strictly positive constants.

— (B3) the functions b, σ, f and Φ are bounded.

— (B4) for all $(x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ the functions $b(x,y,z,.)$, $\sigma(x,y,.)$ and $f(x,y,z,.)$ are continuous in $u \in U$.

Under assumptions (B1)–(B4), our controlled FBSDE has a unique solution. The proof can be performed as that of [125].

Let $\mathbb{S}^d$ denotes the space of symmetric matrices in $\mathbb{R}^{d\times d}$. Let H be the hamiltonian define on $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times U$ by

$$\begin{aligned} H(t,x,y,p,A,v) \ &= \frac{1}{2}\mathrm{tr}\left((\sigma\sigma^*)(t,x,y,v)A\right) + b(t,x,y,p\,\sigma(t,x,y,v),v)p \\ &\quad + f(t,x,y,p\,\sigma(t,x,y,v),v), \end{aligned} \tag{2.1.1}$$

Let $\nabla_x V$ and $\nabla_{xx} V$ respectively denotes gradient and the Hessian matrix of V.

According to Li and Wei [102] the value function $V(t,x)$ define by (2.0.4) is at most of linear growth and it is a viscosity solution of the following Hamilton-Jacobi-Bellman equation and it is deterministic

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) + \inf_{v\in U} H(t,x,V(t,x),\nabla_x V(t,x),\nabla_{xx}V(t,x),v) = 0,\ (t,x)\in[0,T]\times\mathbb{R}^d, \\ V(T,x) = \Phi(x),\ x\in\mathbb{R}^d, \end{cases} \tag{2.1.2}$$

We suppose also the following

2.1.1 Filippov's convexity condition

$$(\mathbf{H}) \qquad \begin{cases} \text{For all } (x,y)\in\mathbb{R}^d\times\mathbb{R} \text{ the following set is convex :} \\ \{((\sigma\sigma^*)(x,y,u), w(\sigma\sigma^*)(x,y,u), b(x,y,w\sigma(x,y,u),u), f(x,y,w\sigma(x,y,u),u)) \\ |(u,w)\in\mathbb{U}\times\bar{B}_C(0)\}\ , \end{cases}$$

where $\bar{B}_C(0) \subset \mathbb{R}^d$ is the closed ball around 0 with radius C.

The following lemma can be proved as Lemma 4 of [41]. For the completeness, we give its proof.

Lemma 12. *For $(x,y,w,\theta,u)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{U}$, we put*

$$\Sigma(x,y,w,\theta) = \begin{pmatrix} \sigma(x,y,u) & 0 \\ w\sigma(x,y,u) & \theta \end{pmatrix} \quad \text{and} \quad \beta(x,y,w,u) = \begin{pmatrix} b(x,y,w\sigma(x,y,u),u) \\ -f(x,y,w\sigma(x,y,u),u) \end{pmatrix}.$$

*Under assumption **(H)** we have*

$$\overline{co}\{((\Sigma\Sigma^*)(x,y,w,0),\beta(x,y,w,u))|(u,w)\in U\times\bar{B}_C(0)\}$$

$$\subset \{((\Sigma\Sigma^*)(x,y,w,\theta),\beta(x,y,w,u)|(u,w,\theta)\in\mathbb{U}\times\bar{B}_C(0)\times[0,K]\}$$

where, for any set $\mathcal{E}$, $co(\mathcal{E})$ denotes the convex hull of $\mathcal{E}$.

Proof. Let μ be a probability measure on the set $\mathbb{U} \times \bar{B}_C(0)$. Our goal is to find a triplet $(\bar{w}, \bar{\theta}, \bar{u}) \in \mathbb{R}^d \times [0, K] \times \mathbb{U}$ which satisfies :

$$\begin{aligned} &\textstyle\int_{\mathbb{U}\times \bar{B}_C(0)} ((\Sigma\Sigma^*)(x,y,w,0), \beta(x,y,w)\mu(du,dw) \\ &\qquad = \Big((\Sigma\Sigma^*)(x,y,\bar{w},\bar{\theta}), \beta(x,y,\bar{w},\bar{\theta},\bar{u})\Big). \end{aligned} \tag{2.1.3}$$

Let $\Phi(u,w) = ((\sigma\sigma^*)(x,y,u), w\sigma\sigma^*(x,y,u), b(x,y,u), f(x,y,w\sigma(x,y,u),u))$. According to assumption (**H**) and the continuity of Φ, there exists $(\bar{u}, \bar{w})$ in $\mathbb{U} \times \bar{B}_C(0)$ such that

$$\int_{\mathbb{U}\times \bar{B}_C(0)} \Phi(u,w)\mu(du,dw) = \Phi(\bar{u},\bar{w}). \tag{2.1.4}$$

A simple computation gives,

$$\Sigma\Sigma^*(x,y,w,\theta) = \begin{pmatrix} \sigma\sigma^*(x,y,u) & \sigma\sigma^*(x,y,u)w^* \\ w\sigma\sigma^*(x,y,u) & w\sigma\sigma^*(x,y,u)w^* + \theta^2 \end{pmatrix}$$

The expression of $(\Sigma\Sigma^*)(x,y,w,0)$ shows that, to obtain (2.1.3), it suffices to find $\bar{\theta} \in [0,K]$ such that

$$\bar{\theta}^2 = \int_{\mathbb{U}\times \bar{B}_C(0)} w\sigma\sigma^*(x,y,u)w^*\mu(du,dw) - \bar{w}\sigma\sigma^*(x,y,u)\bar{w}^* := \alpha. \tag{2.1.5}$$

Since $\sigma\sigma^*(x,y,\bar{u}) = \int_{\mathbb{U}\times \bar{B}_C(0)} \sigma\sigma^*(x,y,u)\mu(du,dw)$, then we can write α as follows

$$\alpha = \int_{\mathbb{U}\times \bar{B}_C(0)} w\sigma\sigma^*(x,y,u)w^*\mu(du,dw) - \int_{\mathbb{U}\times \bar{B}_C(0)} \bar{w}\sigma\sigma^*(x,y,u)\mu(du,dw)\bar{w}^* \tag{2.1.6}$$

$$= \int_{\mathbb{U}\times \bar{B}_C(0)} ((w-\bar{w})\sigma(x,y,u))((w-\bar{w})\sigma(x,y,u))^*\mu(du,dw) \tag{2.1.7}$$

It follows that $\alpha \geq 0$. Hence, it suffices now to choose $\bar{\theta} = \sqrt{\alpha}$.

Now, from (2.1.5) we have

$$\int_{\mathbb{U}\times \bar{B}_C(0)} |w\sigma(x,y,u)|^2\mu(du,dw) = |\bar{w}\sigma(x,y,u)|^2 + \bar{\theta}^2.$$

Since $|\sigma(x,y,u)|$ is bounded and the support of μ is included in $\mathbb{U}\times\bar{B}_C(0)$, it follows that $\bar{\theta}$ is bounded, that is : there exists $K>0$ such that $\bar{\theta}$ belongs to $[0,K]$. ■

2.2 The Hamilton-Jacobi-Bellman equation

Let $\mathbb{S}^d$ denote the space of the symmetric matrices in $\mathbb{R}^{d^2}$. For a function V, we denote by $\nabla_x V$ the gradient and $\nabla_{xx}V$ the Hessian matrix of V. Let H be the real function define on $\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d\times\mathbb{U}$ by :

$$H(x,y,p,A,u):=\frac{1}{2}\mathrm{tr}\left((\sigma\sigma^*)(x,y)A\right)+b(x,y,u)p+f(x,y,p\,\sigma(x,y),u) \tag{2.2.1}$$

According to Li and Wei [102], the value function $V(t,x)$, define by (2.1.2), solves the following Hamilton-Jacobi-Bellman equation in viscosity sense.

$$\begin{cases}\dfrac{\partial}{\partial t}V(t,x)+\inf_{u\in\mathbb{U}}H(x,V(t,x),\nabla_x V(t,x),\nabla_{xx}V(t,x),u)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ V(T,x)=\Phi(x),\ x\in\mathbb{R}^d,\end{cases} \tag{2.2.2}$$

2.3 The main results

Definition 13. *(**Relaxed control**) : Let $\mathbb{Q}(U)$ be the space of probability measures on U equipped with the topology of stable convergence. We denote $M(\Omega)$ the space of all $\mathcal{F}_t$-adapted processes $\nu_t(du)$ taking values in $\mathbb{Q}(U)$. A relaxed control is an $M(\Omega)-$valued process (ν_t),*

Theorem 14. *Assume that (B) and (H) are satisfied and the uniqueness holds for bounded viscosity solution of equation (2.1.2). Then, there exist a strict optimal control to the stochastic optimal control problem (2.0.1)–(2.0.3) in some reference stochastic system $\bar{\nu}=(\bar{\Omega},\bar{\mathcal{F}},\bar{P},(\bar{\mathcal{F}}_t),\bar{W})$*

2.4 Proof of the main results

The proof consists to construct an approximating sequence of controlled systems for which we prove the existence of a sequence of feedback controls. To this end, we have to approximate the coefficients of our original control problem by smooth ones. The existence of an optimal control is then obtained by passing to the limit. More precisely, we approximate the controlled FBSDE (2.0.1) by a sequence of FBSDEs, with smooth data $b_\delta, \sigma_\delta, f_\delta$ and Φ_δ and consider a sequence of value functions V^δ, which is associated to the FBSDE with these regularized coefficients. According to Krylov [94] (Theorems 6.4.3 and 6.4.4), V^δ is sufficiently smooth and satisfies an HJB equation. Since all admissible controls take their values in a compact set, we then deduce the existence of a feedback control u^δ. Next, we prove that the sequence V^δ converges uniformly to a function V which is the value function of our initial control problem.

2.4.1 Construction of an approximating control problem

The functions $b_\delta, \sigma_\delta, f_\delta$ and Φ_δ respectively denotes the mollifier of the functions b, σ, f and Φ [1].

Let $\delta \in (0, 1]$. Let H^δ be the approximating Hamiltonian define on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times U$

1. See Preliminaries for the definition of the mollifier

by :

$$H^{\delta}\left(x,y,p,A,v\right)=\frac{1}{2}\left(\operatorname{tr}\left(\left(\sigma_{\delta}\sigma_{\delta}^{*}\right)\left(x,y,v\right)+\delta^{2}I_{\mathbb{R}^{d}}\right)A\right)+b_{\delta}\left(x,y,p\sigma_{\delta}\left(x,y,v\right),v\right)p \qquad (2.4.1)$$

$$+f_{\delta}\left(x,y,p\sigma_{\delta}\left(x,y,v\right),v\right),$$

and consider the approximating HJB equation

$$\begin{cases}\frac{\partial}{\partial t}V^{\delta}\left(t,x\right)+\inf_{v\in U}H^{\delta}\left(x,(V^{\delta},\nabla_{x}V^{\delta},\nabla_{xx}V^{\delta})(t,x),v\right)=0,\ (t,x)\in[0,T]\times\mathbb{R}^{d},\\ V^{\delta}\left(T,x\right)=\Phi_{\delta}(x),\ x\in\mathbb{R}^{d},\end{cases} \qquad (2.4.2)$$

since H^{δ} is smooth and ($\left(\sigma_{\delta}\sigma_{\delta}^{*}\right)(x,y,v)+\delta^{2}I_{\mathbb{R}^{d}}$) is strictly elliptic, then according to [94] (Theorems 6.4.3 and 6.4.4) the PDE (2.4.2) admits a unique solution which belongs to $C_{b}^{1,2}([0,T]\times\mathbb{R}^{d})$.

The compactness of the control set U and the regularity of the solution allow us to prove the existence of a measurable function $v^{\delta}:[0,T]\times\mathbb{R}^{d}\to U$ which minimizes the Hamiltonian H^{δ} for each $(t,x)\in[0,T]\times\mathbb{R}^{d}$, that is :

$$H^{\delta}\left(x,(V^{\delta},\nabla_{x}V^{\delta},\nabla_{xx}V^{\delta})(t,x),v^{\delta}(t,x)\right):=\inf_{v\in U}H^{\delta}\left(x,(V^{\delta},\nabla_{x}V^{\delta},\nabla_{xx}V^{\delta})(t,x),v\right).$$

Let B be an $\mathbb{R}^{d}$–valued Brownian motion which is independent from W. For $(t,x)\in[0,T]\times\mathbb{R}^{d}$, let X^{δ} be a solution of the following SDE :

$$\begin{cases}dX_{s}^{\delta}=b_{\delta}(X_{s}^{\delta},V^{\delta}(s,X_{s}^{\delta}),\nabla_{x}V^{\delta}(s,X_{s}^{\delta})\sigma_{\delta}(X_{s}^{\delta},V^{\delta}(s,X_{s}^{\delta}),v^{\delta}(s,X_{s}^{\delta})),v^{\delta}(s,X_{s}^{\delta}))ds\\ +\sigma_{\delta}(X_{s}^{\delta},V^{\delta}(s,X_{s}^{\delta}),v^{\delta}(s,X_{s}^{\delta}))dW_{s}+\delta dB_{s},\quad s\in[t,T],\\ X_{t}^{\delta}=x.\end{cases} \qquad (2.4.3)$$

Since the matrix $\left(\sigma_{\delta}\sigma_{\delta}^{*}\right)\left(x,v^{\delta}(s,x)\right)+\delta^{2}I_{R^{d}}$ is uniformly elliptic and the coefficients $b_{\delta}(x,V^{\delta}(s,x),\nabla_{x}V^{\delta}(s,x)\sigma_{\delta}(x,V^{\delta}(s,x),v^{\delta}(s,x))$ and $\sigma_{\delta}(x,V^{\delta}(s,X_{s}^{\delta}),v^{\delta}(s,x))$ are measurable and bounded in (s,x), then according to Theorem 1 of Section 2.6 in [93] we get the existence of a weak solution. That is, there exists some reference stochastic system $\nu^\delta = (\Omega^\delta, \mathcal{F}^\delta, P^\delta, \mathcal{F}_t^\delta, W^\delta, B^\delta)$ and an $\mathcal{F}_t^\delta$-adapted continuous process X^δ which is a solution of (2.4.3).

For $s \in [t,T]$, we put :

$$Y_s^\delta := V^\delta(s, X_s^\delta),\; Z_s^\delta := \nabla_x V^\delta(s, X_s^\delta)\sigma_\delta(X_s^\delta, V^\delta(s, X_s^\delta), u_s^\delta),\; U_s^\delta := \delta\nabla_x V^\delta(s, X_s^\delta).$$

For an arbitrarily given admissible control $u \in \mathcal{U}_{\nu^\delta}(t)$, we consider the following coupled FBSDE equation, for $s \in [t,T]$,

$$\begin{cases} dX_s^{\delta,x,u} = b_\delta\left(X_s^{\delta,x,u}, Y^{\delta,x,u}, Z^{\delta,x,u}, u_s\right)dt + \sigma_\delta\left(X_s^{\delta,x,u}, Y^{\delta,x,u}, u_s\right)dW_s^\delta + \delta dB_s^\delta \\ dY_s^{\delta,x,u} = -f_\delta(X_s^{\delta,x,u}, Y_s^{\delta,x,u}, Z_s^{\delta,x,u}, u_s)ds + Z_s^{\delta,x,u}dW_s^\delta + U_s^{\delta,x,u}dB_s^\delta + dM_s^{\delta,x,u} \\ Y_T^{\delta,x,u} = \Phi_\delta(X_T^{\delta,x,u}),\; X_t^{\delta,x,u} = x, \\ (Y^{\delta,x,u}, Z^{\delta,x,u}, U^{\delta,x,u}) \in \mathcal{S}^2_{\nu^\delta}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d), \\ M^{\delta,x,u} \in \mathcal{M}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \text{ is orthogonal to } W^\delta \text{ and to } B^\delta. \end{cases} \tag{2.4.4}$$

According to [125], the previous FBSDE has a unique $\mathcal{F}_t^\delta$-adapted solution $(X^{\delta,x,u}, Y^{\delta,x,u}, Z^{\delta,x,u}, U^{\delta,x,u}, M^{\delta,x,u})$. The cost functional associated to the controlled system (2.4.4) is then defined by :

$$J^\delta(u) := Y_t^{\delta,x,u},\; u \in \mathcal{U}_{\nu^\delta}(t).$$

Proposition 2.4.1. *Let assumptions (B) be satisfied. Then,*

1. *for every $\delta \in (0,1]$, there exists an admissible control $u_s^\delta := v^\delta(s, X_s^\delta)$, $s \in [0,T]$, such that :*

$$J^\delta(u^\delta) = V^\delta(t,x) = essinf_{u\in\mathcal{U}_{\nu^\delta}(t)} J^\delta(u),$$

2. *(i) for all $t \in [0,T]$; $x, x' \in \mathbb{R}^d$ and $\delta, \delta' \in (0,1]$, there exits a constant C which*

depends from K, T and the bounds of the coefficients such that,

$$|V^{\delta'}(t,x') - V^{\delta}(t,x)| \leq C\ (|\delta - \delta'|^{\frac{1}{2}}\ + |x - x'|), \tag{2.4.5}$$

(ii) *for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$.*

$$|V^{\delta}(t,x) - V(t,x)| \leq C\sqrt{\delta}.$$

i.e V^{δ} converges uniformly to the unique viscosity solution of the HJB equation (2.1.2).

Proof. 1) We observe that from the uniqueness of the solution of (2.4.4) with the control process u^{δ}, it follows that $X^{\delta,x,u^{\delta}} = X^{\delta}$. Let,

$$Y_s^{\delta} = V^{\delta}(s, X_s^{\delta}),\ Z_s^{\delta} = \nabla_x V^{\delta}(s, X_s^{\delta})\sigma_{\delta}(X_s^{\delta}, V^{\delta}(s, X_s^{\delta}), u_s^{\delta}),\ U_s^{\delta} = \delta \nabla_x V^{\delta}(s, X_s^{\delta}),\ \ s \in [t,T].$$

Since $V^{\delta} \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$, Itô's formula applied to $V^{\delta}(s, X_s^{\delta})$ shows that $(Y^{\delta}, Z^{\delta}, U^{\delta})$ satisfies the backward component of (2.4.4) for $u = u^{\delta}$. Hence, from the uniqueness of the solution of (2.4.4), we get $(Y^{\delta,x,u^{\delta}}, Z^{\delta,x,u^{\delta}}, U^{\delta,x,u^{\delta}}) = (Y^{\delta}, Z^{\delta}, U^{\delta})$ and $M^{\delta,x,u^{\delta}} = 0$, in particular

$$Y_t^{\delta,x,u^{\delta}} = Y_t^{\delta} = V^{\delta}(t,x).$$

2) Let $\delta' > 0$ and $x' \in \mathbb{R}^d$. Let $X^{\delta',x',u^{\delta}} \in \mathcal{S}^2_{\nu^{\delta}}(t,T;\mathbb{R}^d)$ denote the unique solution of the following forward equation :

$$\begin{cases} dX_s^{\delta',x',u^{\delta}} = b_{\delta'}(X_s^{\delta',x',u^{\delta}}, V^{\delta'}(s, X_s^{\delta',x',u^{\delta}}), \nabla_x V^{\delta'}(s, X_s^{\delta',x',u^{\delta}}) \\ \sigma_{\delta'}(X_s^{\delta',x',u^{\delta}}, V^{\delta'}(s, X_s^{\delta',x',u^{\delta}}), u_s^{\delta}), u_s^{\delta})ds + \sigma_{\delta'}\left(X_s^{\delta',x',u^{\delta}}, V^{\delta'}(s, X_s^{\delta',x',u^{\delta}}), u_s^{\delta}\right) dW_s^{\delta} \\ +\delta' dB_s^{\delta},\ s \in [t,T], \\ X_t^{\delta',x',u^{\delta}} = x'. \end{cases}$$

We extend this solution to the whole interval $[0,T]$ by putting $X_s^{\delta',x',u^\delta} = x'$, for $s<t$. We set,

$$\widetilde{f}_s^{\delta',x',u^\delta} := -\frac{\partial}{\partial s} V^{\delta'}(s, X_s^{\delta',x',u^\delta}) \tag{2.4.6}$$
$$- \frac{1}{2}\mathrm{tr}\left((\sigma_{\delta'}\sigma_{\delta'}^*)(X_s^{\delta',x',u^\delta}, V^{\delta'}(s, X_s^{\delta',x',u^\delta}), u_s^\delta) + \delta'^2 I_{R^d} \right) \times \nabla_{xx} V^\delta(s, X_s^{\delta',x',u^\delta})$$
$$- b_{\delta'}(X_s^{\delta',x',u^\delta}, V^{\delta'}(s, X_s^{\delta',x',u^\delta}), \nabla_x V^{\delta'}(s, X_s^{\delta',x',u^\delta})\sigma_{\delta'}(X_s^{\delta',x',u^\delta}, V^{\delta'}(s, X_s^{\delta',x',u^\delta}), u_s^\delta), u_s^\delta)$$

Itô's formula applied to $V^{\delta'}(s, X_s^{\delta,x',u^\delta})$ shows that

$$Y_s^{\delta',x'} := V^{\delta'}(s, X_s^{\delta',x',u^\delta}),$$
$$Z_s^{\delta',x'} := \nabla_x V^{\delta'}(s, X_s^{\delta',x',u^\delta})\sigma_{\delta'}(X_s^{\delta',x',u^\delta}, V^{\delta'}(s, X_s^{\delta',x',u^\delta}), u_s^\delta),$$
$$U_s^{\delta',x'} := \delta' \nabla_x V^{\delta'}(s, X_s^{\delta',x',u^\delta}), \quad M_s^{\delta',x'} := 0,\ s \in [t,T],$$

is the unique solution of the BSDE :

$$\begin{cases} dY_s^{\delta',x'} = -\widetilde{f}_s^{\delta',x',u^\delta} ds + Z_s^{\delta',x'} dW_s^\delta + U_s^{\delta',x'} dB_s^\delta,\ s\in[t,T], \\ Y_T^{\delta',x'} = \Phi_{\delta'}(X_T^{\delta',x',u^{\delta'}}), \\ (Y^{\delta',x'}, Z^{\delta',x'}, U^{\delta',x'}) \in \mathcal{S}^2_{\nu^\delta}(t',T;\mathbb{R}) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d), \\ M^{\delta',x'} \in \mathcal{M}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \text{ is orthogonal to both } W^\delta \text{ and to } B^\delta. \end{cases} \tag{2.4.7}$$

We consider the BSDE :

$$\begin{cases} dY_s^{\delta',x',u^\delta} = -f_{\delta'}\left(X_s^{\delta',x',u^\delta}, Y_s^{\delta',x',u^\delta}, Z_s^{\delta',x',u^\delta}, u_s^\delta\right) ds \\ \qquad\qquad + Z_s^{\delta',x',u^\delta} dW_s^\delta + U_s^{\delta',x',u^\delta} dB_s^\delta + dM_s^{\delta',x',u^\delta},\ s\in[t',T], \\ Y_T^{\delta',x',u^\delta} = \Phi_{\delta'}(X_T^{\delta',x',u^\delta}), \\ (Y^{\delta',x',u^\delta}, Z^{\delta',x',u^\delta}, U^{\delta,x',u^\delta}, u^\delta) \in \mathcal{S}^2_{\nu^\delta}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \times \mathcal{H}^2_{\nu^\delta}(t,T;\mathbb{R}^d), \\ M^{\delta',x',u^\delta} \in \mathcal{M}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \text{ is orthogonal to } W^\delta \text{ and to } B^\delta. \end{cases}$$

From the HJB equation (2.4.2) with the classical solution $V^{\delta'}$ we observe that

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V^{\delta'}\left(s, X_s^{\delta',x',u^\delta}\right) + \inf_{v\in U} H^{\delta'}\left(X_s^{\delta',x',u^\delta}, (V^\delta, \nabla_x V^{\delta'}, \nabla_{xx} V^{\delta'})(s, X_s^{\delta',x',u^\delta}), v\right) \\ &\le \frac{\partial}{\partial t} V^{\delta'}\left(s, X_s^{\delta',x',u^\delta}\right) + H^{\delta'}\left(X_s^{\delta',x',u^\delta}, (V^\delta, \nabla_x V^{\delta'}, \nabla_{xx} V^{\delta'})(s, X_s^{\delta',x',u^\delta}), u_s^\delta\right) \\ &\le f_{\delta'}\left(X_s^{\delta',x',u^\delta}, Y_s^{\delta',x'}, Z_s^{\delta',x'}, u_s^\delta\right) - \widetilde{f}_s^{\delta',x',u^\delta},\ s \in [t,T]. \end{aligned}$$

Therefore, the comparison theorem shows that

$$\forall s \in [t,T], \qquad Y_s^{\delta',x'} \le Y_s^{\delta',x',u^\delta} \quad P^\delta\text{.a.s.}$$

A symmetric argument allows us deduce that :

$$|V^{\delta'}(t,x') - V^\delta(t,x)| = |Y_t^{\delta',x'} - Y_t^{\delta,x,u^\delta}| \le |Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x,u^\delta}|, \quad P^\delta\text{-a.s.}$$

Since V^δ and $V^{\delta'}$ are deterministic, we have

$$|V^{\delta'}(t,x') - V^\delta(t,x)| \le E(|Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x,u^\delta}|\,|\mathbb{F}_t^\delta).$$

Hence, it suffices to estimate $E(|Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x,u^\delta}|\,|\mathbb{F}_t^\delta)$. We assume that for $s < t$, $Y_s^{\delta',x',u^\delta} = Y_t^{\delta',x',u^\delta}$, $Z_s^{\delta',x',u^\delta} = 0$ and $M_s^{\delta',x',u^\delta} = 0$. Using Lemmas 16 and 17 (in Appendix), it follows that there exists a constant C which depends upon K, T and the bounds of the coefficients such that :

$$\begin{aligned} \mathbb{E}[|Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta] &\le 2\mathbb{E}[|Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x',u^\delta}|^2 + |Y_t^{\delta,x',u^\delta} - Y_t^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta] \\ &\le C(|x-x'|^2 + |\delta - \delta'|). \end{aligned}$$

In particular,

$$|V^\delta(t,x') - V^\delta(t,x)| \le C|x - x'|, \tag{2.4.8}$$

and

$$|V^{\delta'}(t,x) - V^{\delta}(t,x)| \leq C|\delta - \delta'|^{1/2}. \tag{2.4.9}$$

We prove assertion (ii).

According to assertion (i), (V^{δ}) is a Cauchy sequence with respect to the uniform convergence norm, in $(t,x) \in [0,T] \times \mathbb{R}^d$. It then converges uniformly to a function $\bar{V}$ as $\delta \to 0$. Since V^{δ} is uniformly bounded in (t,x,δ), then $\bar{V} \in C_b([0,T] \times \mathbb{R}^d)$.

Since H^{δ} converges uniformly on compact sets to H, then using the stability of viscosity solutions, it follows that $\bar{V}$ is a bounded viscosity solution to equation (2.1.2). the uniqueness of the viscosity solution, within the class of bounded continuous function, we get that $\bar{V} \equiv V$. This shows that the sequence $(V^{\delta'})$ converges to V, as $\delta' \to 0$. Using inequality (2.4.9), it follows that for each $\delta \in (0,1]$ and $(t,x) \in [0,T] \times \mathbb{R}^d$, $|V^{\delta}(t,x) - V(t,x)| \leq C\sqrt{\delta}$. ■

2.4.2 Auxiliary sequence and the passing to the limits

We will establish the convergence of the approximating control problem to the original one. To this end, let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers which are decreasing to 0. We put $w_s^n := \nabla_x V^{\delta_n}(t, X_s^{\delta_n})$, $Z_s^{\delta_n} := w_s^n \sigma(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n})$ and $U_s^{\delta_n} := \delta_n w_s^n$. Let $(X^{\delta_n}, Y^{\delta_n}, Z^{\delta_n}, U^{\delta_n}, u^{\delta_n})$ be a sequence of an approximating controlled systems. Since w^n is uniformly bounded (see Proposition 2.4.1), the idea consists to consider the couple (u^{δ_n}, w^n) as a relaxed control. This allows to overcome the difficulties related to the convergence of the component Z^{δ_n}. We then show that the system $(X^{\delta_n}, Y^{\delta_n}, Z^{\delta_n}, U_s^{\delta_n}, u^{\delta_n})$ has a subsequence which converges in law to some controlled system which solves our initial problem. Assumption **(H)** and the result of [63] allow us to prove the existence of a strict optimal control.

For $n \in \mathbb{N}$, $(X_s^{\delta_n}, Y_s^{\delta_n})_{t\leq s\leq T}$ is a solution to the following controlled system :

$$\begin{cases} dX_s^{\delta_n} = b_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, w_s^n \sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n})ds + \sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n})dW_s^{\delta_n} + \delta_n dB^{\delta_n}, \\ dY_s^{\delta_n} = -f_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, w_s^n \sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}), u_s^{\delta_n})ds \\ \qquad\qquad +w_s^n \sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n})dW_s^{\delta_n} + U_s^{\delta_n}\ dB_s^{\delta_n}. \\ X_t^{\delta_n} = x, \quad Y_t^{\delta_n} = V^{\delta_n}(t,x). \end{cases} \tag{2.4.10}$$

where $w_s^n := \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$.

To show that $(X_s^{\delta_n}, Y_s^{\delta_n})_{t\leq s\leq T}$ has a subsequence denoted also by $(X_s^{\delta_n}, Y_s^{\delta_n})_{t\leq s\leq T}$ which converges in law to a process $(\bar{X}_s, \bar{Y}_s)_{t\leq s\leq T}$ which solves our initial problem, we will construct a sequence of an auxiliary processes $(X_s^n, Y_s^n)_{t\leq s\leq T}$ which converges to $(\bar{X}_s, \bar{Y}_s)_{t\leq s\leq T}$ and such that the difference between $(X_s^n, Y_s^n)_{t\leq s\leq T}$ and $(X_s^{\delta_n}, Y_s^{\delta_n})_{t\leq s\leq T}$ goes to zero as n goes to infinity.

Let $(X_s^n, Y_s^n)_{t\leq s\leq T}$ be the unique solution of the following controlled forward system :

$$\begin{cases} dX_s^n = b(X_s^n, Y_s^n, w_s^n \sigma(X_s^n, Y_s^n, u_s^{\delta_n})ds + \sigma(X_s^n, Y_s^n, u_s^{\delta_n})dW_s^{\delta_n}, \\ dY_s^n = -f(X_s^n, Y_s^n, w_s^n \sigma(X_s^n, Y_s^n, u_s^{\delta_n}), u_s^{\delta_n})ds + w_s^n \sigma(X_s^n, Y_s^n, u_s^{\delta_n})dW_s^{\delta_n}. \\ X_t^n = x, \quad Y_t^n = V^{\delta_n}(t,x). \end{cases} \tag{2.4.11}$$

We define the processes χ^n, r^n and the Brownian motion $\mathcal{W}^n$ as follows

$$\chi_s^n = \begin{pmatrix} X_s^n \\ Y_s^n \end{pmatrix}, \; r_s^n = (w_s^n, 0, u_s^{\delta_n}) \text{ and } \mathcal{W}^n = \begin{pmatrix} W^{\delta_n} \\ B^{\delta_n} \end{pmatrix}.$$

We rewrite the system (2.4.11) as follows :

$$\begin{cases} d\chi_s^n = \beta(\chi_s^n, r_s^n)ds + \Sigma(\chi_s^n, r_s^n)d\mathcal{W}_s^n, \ s \in [t,T], \\ \chi_t^n = \begin{pmatrix} x \\ V^{\delta_n}(t,x) \end{pmatrix}, \end{cases} \tag{2.4.12}$$

where β and Σ are the functions defined in lemma 12.

According to Proposition 2.4.1, $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$ is uniformly bounded by C. Hence, we can interpret $(r_s^n, s \in [t,T])$ as a control with values in the compact set $A := \overline{B}(0_{\mathbb{R}^d}, C) \times [0,K] \times U$. Now, as usual, we embed the controls r^n in the set of relaxed controls, i.e. we consider r^n as random variable with values in the space ϑ of all Borel measures q on $[0,T] \times A$, whose projection $q(\cdot \times A)$ coincides with the Lebesgue measure. For this, we identify the control process r^n with the random measure

$$q^n(\omega, ds, da) = \delta_{r_s^n(\omega)}(da)ds, \ (s,a) \in [0,T] \times A, \ \omega \in \Omega.$$

From the boundedness of $\{(\Sigma(x,y,z,\theta,v), \beta(x,y,z,\theta,v)), (x,y,z,\theta,v) \in \mathbb{R}^d \times \mathbb{R} \times A\}$ and the compactness of ϑ with respect to the topology induced by the weak convergence of measures, we get the tightness of the laws of $(\chi^n, q^n), n \geq 1$, on $C([0,T]; \mathbb{R}^d \times \mathbb{R}) \times \vartheta$. Therefore, we can find a probability measure Q on $C([0,T]; \mathbb{R}^d \times \mathbb{R}) \times \vartheta$ and extract a subsequence -still denoted by (χ^n, q^n)- that converges in law to the canonical process (χ, q) on the space $C([0,T]; \mathbb{R}^d \times \mathbb{R}) \times \vartheta$ endowed with the measure Q. Thanks to assumption (H) and the result of [63], it follows that there exists a stochastic reference system $\bar{\nu} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{\mathbb{F}}, \bar{\mathcal{W}})$ enlarging $(C([0,T]; \mathbb{R}^d \times \mathbb{R}) \times \vartheta; Q)$ and an $\bar{\mathbb{F}}$-adapted process $\bar{r}$ with values in A, such that the process χ is a solution of

$$\begin{cases} d\chi_s = \beta(\chi_s, \bar{r}_s)ds + \Sigma(\chi_s, \bar{r}_s)d\bar{\mathcal{W}}_s, \ s \in [t,T], \\ \chi_t = \begin{pmatrix} x \\ V(t,x) \end{pmatrix}, \end{cases}$$

and has the same law under $\bar{P}$ as under Q. Replacing Σ and β by their definition and setting $\chi = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$, $\bar{\mathcal{W}} = \begin{pmatrix} \bar{W} \\ \bar{B} \end{pmatrix}$ and $\bar{r} = (\bar{w}, \bar{\theta}, \bar{u})$, this system is equivalent to

$$\begin{cases} d\bar{X}_s = b(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)ds + \sigma(\bar{X}_s, \bar{Y}_s, \bar{u}_s)d\bar{W}_s, \\ dY_s = -f(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)ds + \bar{Z}_s d\bar{W}_s + \bar{\theta}_s d\bar{B}_s, \quad s \in [t,T] \\ \bar{X}_t = x, \ \bar{Y}_t = V(t,x). \end{cases}$$

To continue our proof, we need the following :

Lemma 15. *Let $(X^{\delta_n}, Y^{\delta_n}, w^n\sigma^{\delta_n}(X^{\delta_n}, Y^{\delta_n}, u^{\delta_n}))$ (resp. X^n, Y^n) be the solution of the approximating FBSDE* (2.4.4) *for δ_n and u^{δ_n} (resp. the FBSDE* (2.4.11)*). Then there are two positive constants K_1 and K_2 such that for every $n \in \mathbb{N}$,*

$$\begin{aligned} &\mathbb{E}[\sup_{s\in[t,T]} |X_s^{\delta_n} - X_s^n|^2] \le K_1 \ \delta_n^2, \\ &\mathbb{E}[\sup_{s\in[t,T]} |Y_s^{\delta_n} - Y_s^n|^2] \le K_2 \ \delta_n^2. \end{aligned} \tag{2.4.13}$$

Before to give proof of this lemma, we first use it to finish the proof of the main theorem. Since $(X^n, Y^n)_{n\in\mathbb{N}}$ converges in law to $(\bar{X}, \bar{Y})_{n\in\mathbb{N}}$, then Lemma 15 implies that the same holds true for $(X^{\delta_n}, Y^{\delta_n})_{n\in\mathbb{N}}$ and the limits of these two sequences have the same law. Further, we deduce from (2.4.13) and Proposition 2.4.1, that $\bar{Y}_s = V(s, \bar{X}_s)$ for each $s \in [t,T]$, $\bar{P}$-a.s. In particular $Y_T = \Phi(X_T)$ $\bar{P}$-a.s. Thus, if we set $\bar{M}_s = \int_t^s \bar{\theta}_r d\bar{B}_r$, then $\langle \bar{M}, \bar{W} \rangle_s = \int_t^s \bar{\theta}_r d\langle \bar{B}, \bar{W} \rangle_r = 0$ and $(\bar{X}, \bar{Y}, \bar{Z}, \bar{M})$ satisfies (2.0.1) in the stochastic reference system $\bar{\nu} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{\mathbb{F}}, \bar{\mathcal{W}})$. According to Li and Wei [102] the unique bounded viscosity solution V of the HJB equation (2.1.2) satisfies $V(t,x) = \operatorname{essinf}_{u\in\mathcal{U}_{\bar{\nu}}(t)} J(t,x,u)$, $\bar{P}$-a.s. ■

Proof. of Lemma 15 We put

$$\overline{X}_s^n := X_s^{\delta_n} - X_s^n, \quad \textit{and} \quad \overline{Y}_s^n := Y_s^{\delta_n} - Y_s^n.$$

For $l = b,\ \sigma,\ f,\ A$, and $l_{\delta_n} = b_{\delta_n},\ \sigma_{\delta_n},\ f_{\delta_n},\ A_{\delta_n}$ Let :

$$\Delta l^n(s) := l_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^n \sigma_{\delta_n}\left(X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\right), u_s^{\delta_n}) - l(s, X_s^n, Y_s^n, w_s^n \sigma\left(X_s^n, Y_s^n, u_s^{\delta_n}\right), u_s^{\delta_n}),$$

Since w^n is uniformly bounded [2], it follows that there exists a constant $\bar{K}$ independent from δ_n such that :

$$\begin{aligned} |\Delta l^n(r)| &\leq |l_{\delta_n}(r, X_r^{\delta_n}, Y_r^{\delta_n}, w_r^n \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, u_r^{\delta_n}), u_r^{\delta_n}) - l(s, X_r^{\delta_n}, Y_r^{\delta_n}, w_r^n \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, u_r^{\delta_n}), u_r^{\delta_n})| \\ &\quad + |l(s, X_r^{\delta_n}, Y_r^{\delta_n}, w_r^n \sigma^{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, u_r^{\delta_n}), u_r^{\delta_n}) - l(r, X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n}), u_r^{\delta_n})| \\ &\leq \bar{K}\ (\delta_n + |\overline{X}_r^n| + |\overline{Y}_r^n|). \end{aligned}$$

Using Itô's formula and Young's inequality we can find a constant C_1 which depends only from K, T such that :

$$\begin{aligned} \mathbb{E}[|\overline{X}_s^n|^2 | \mathbb{F}_t^\delta] &= \mathbb{E}[\int_t^s (2\overline{X}_r^n \Delta b(r) + |\Delta\sigma(r)|^2) dr | \mathbb{F}_t^\delta] \\ &\leq C_1 \delta_n^2 + C_1 \mathbb{E}[\int_t^s (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2) dr | \mathbb{F}_t^\delta] \\ &\leq C_1 \delta_n^2 + C_1 \mathbb{E}[\int_t^T (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2) dr | \mathbb{F}_t^\delta]. \end{aligned} \qquad (2.4.14)$$

Using again Itô's formula and Young's inequality, it yields that there exists a constant C_2

2. See Proposition 2.4.1), then using proposition 1.1.1.

which does not depends from δ_n such that for any $t \le s \le T$,

$$\mathbb{E}[|\overline{Y}_s^n|^2|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |w_r^n|\Delta\sigma(r)|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |U_r^{\delta_n,t,x,u^{\delta_n}}|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\langle M^{\delta',t,x',u^\delta}\rangle_T|\mathbb{F}_t^\delta]$$
$$\le C_2(\delta_n^2 + \mathbb{E}[|\overline{X}_T^n|^2|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2 dr|\mathbb{F}_t^\delta]).$$

Putting $s = T$ in (2.4.14) and modifying C_2 if necessary it follows that

$$\mathbb{E}[|\overline{Y}_s^n|^2|\mathbb{F}_t^\delta] \quad \le \quad C_2\ (\delta_n^2 + \mathbb{E}[\int_t^T (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2) dr|\mathbb{F}_t^\delta]). \tag{2.4.15}$$

In the other hand, Itô's formula applied to $\langle G\overline{X}_s^n, \overline{Y}_s^n\rangle$ combined with assumption (B2) shows that :

$$\mathbb{E}[\langle G\overline{X}_s^n, \overline{Y}_s^n\rangle|\mathbb{F}_t^\delta] = \mathbb{E}[\langle G\overline{X}_T^n, \overline{Y}_T^n\rangle|\mathbb{F}_t^\delta] - \mathbb{E}[\int_s^T \langle\Delta A(r), (\overline{X}_r^n, \overline{Y}_r^n, w_r^n\Delta\sigma_n(r))\rangle dr|\mathbb{F}_t^\delta]$$
$$- \mathbb{E}[\delta_n \int_s^T GU_r^{\delta_n,x,u^{\delta_n}} dr|\mathbb{F}_t^\delta]$$
$$\ge \mathbb{E}[\langle G\overline{X}_T^n, \overline{Y}_T^n\rangle|\mathbb{F}_t^\delta] + \mathbb{E}[\beta_1 \int_s^T |G\overline{X}_r^n|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T \beta_2(|G^T\overline{Y}_r^n|^2$$
$$+ |G^T\ w_r^n\Delta\sigma_n(r)|^2 dr|\mathbb{F}_t^\delta] - \bar{K}^2\ \delta_n^2 - \delta_n\ \mathbb{E}[\int_s^T GU_r^{\delta_n,x,u^{\delta_n}} dr|\mathbb{F}_t^\delta]. \tag{2.4.16}$$

We shall estimate $\mathbb{E}[\langle G\overline{X}_T^n, \overline{Y}_T^n\rangle|\mathbb{F}_t^\delta]$. We use Young's inequality to get for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}[\langle G\overline{X}_T^n, \overline{Y}_T^n\rangle|\mathbb{F}_t^\delta] \quad &= \quad \mathbb{E}[\langle G\ (X_T^{\delta_n} - X_T^n), \Phi_{\delta_n}(X_T^{\delta_n}) - \Phi_{\delta_n}(X_T^n)\rangle|\mathbb{F}_t^\delta] \\
&- \quad \mathbb{E}[\langle G\ (X_T^{\delta_n} - X_T^n), \Phi(X_T^n) - \Phi_{\delta_n}(X_T^n)\rangle|\mathbb{F}_t^\delta] \\
&\ge \quad \mathbb{E}[\mu_1|G(X_T^{\delta_n} - X_T^n)|^2|\mathbb{F}_t^\delta] - |G|\ K\ \mathbb{E}[\ |X_T^{\delta_n} - X_T^n|\ \delta_n\ |\mathbb{F}_t^\delta] \\
&\ge \quad (\mu_1|G|^2 - |G|\ K\ \varepsilon)\mathbb{E}[\ |X_T^{\delta_n} - X_T^n|^2\ \delta_n\ |\mathbb{F}_t^\delta] - \frac{|G|\ K}{\varepsilon}\ \delta_n^2.
\end{aligned}$$

We choose $\varepsilon = \frac{\mu_1\ |G|}{K}$ in the previous inequality to obtain

$$\mathbb{E}[\langle G\overline{X}_T^n, \overline{Y}_T^n\rangle|\mathbb{F}_t^\delta] \ge -\frac{K^2}{\mu_1}\ \delta_n^2. \tag{2.4.17}$$

Since $\nabla_x V^{\delta_n}(t,x)$ is uniformly bounded in n,t,x (see Proposition 2.4.1) and $U_r^{\delta_n,t,x,u^{\delta_n}} = \delta^n \nabla_x V^{\delta_n}(r, X_r^{\delta_n,t,x,u^{\delta_n}})$, we deduce that there exists a constant $\bar{C}(K,T)$ such that,

$$\delta_n \ \mathbb{E}[\int_s^T |GU_r^{\delta_n,t,x,u^{\delta_n}}|dr|\mathbb{F}_t^\delta] \leq \bar{C}(K,T)\delta_n^2. \tag{2.4.18}$$

Putting $\widetilde{C} = (\bar{C}(K,T) + K^2 + \mathbf{C})$ and combining (2.4.16), (2.4.17) and (2.4.18), we get for $t \leq s \leq T$,

$$\begin{aligned}\mathbb{E}[\langle G\overline{X}_s^n, \overline{Y}_s^n\rangle|\mathbb{F}_t^\delta] + \widetilde{C}\delta_n^2 &\geq \mathbb{E}[\beta_1 \int_s^T |G\overline{X}_r^n|^2 dr|\mathbb{F}_t^\delta] \\ &\quad + \mathbb{E}[\int_s^T \beta_2(|G^T\overline{Y}_r^n|^2|\mathbb{F}_t^\delta] + \underbrace{\mathbb{E}[\int_s^T \beta_2 |G^T w_r^n \ \Delta\sigma_n(r)|^2 dr|\mathbb{F}_t^\delta]}_{\geq 0} \\ &\geq C_3 \ \mathbb{E}[\int_s^T (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2)dr|\mathbb{F}_t^\delta].\end{aligned}$$

Putting $s = t$, it follows that

$$\mathbb{E}[\int_t^T (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2)dr|\mathbb{F}_t^\delta] \leq C_4 \ (\mathbb{E}[\langle G\overline{X}_t^n, \overline{Y}_t^n\rangle|\mathbb{F}_t^\delta] + \delta_n^2).$$

where C_4 is some constant which depends only from $K, T, |G|, \beta_1$ and β_2.

Since $\overline{X}_t = 0$ (the two processes start from the same point x), we deduce that

$$\mathbb{E}[\int_t^T (|\overline{X}_r^n|^2 + |\overline{Y}_r^n|^2)dr|\mathbb{F}_t^\delta] \leq C_4\delta_n^2. \tag{2.4.19}$$

Using (2.4.14) and (2.4.19), one can show that there exists a constant K_1 which depends only upon $K, T, |G|, \beta_1$ and β_2 such that for any $t \leq s \leq T$, $\mathbb{E}[|\overline{X}_s^n|^2|\mathbb{F}_t^\delta] \leq K_1\delta_n^2$. Finally, using (2.4.15) and (2.4.19), it follows that there exists a constant K_2 which depends only from $K, T, |G|, \beta_1$ and β_2 such that $\mathbb{E}[|\overline{Y}_s^n|^2|\mathbb{F}_t^\delta] \leq K_2\delta_n^2$. ∎

Lemma 16. *(Gradient estimate) Let assumption (B) be satisfied. Then, for any $t \in [0,T]$, $\delta \in (0,1]$ and $x, x' \in \mathbb{R}^d$, there exits a constant C which depends from K, T and the bounds of the coefficients, such that*

$$\mathbb{E}[|Y_t^{\delta,x,u^\delta} - Y_t^{\delta,x',u^\delta}|^2|\mathbb{F}_t^\delta] \leq C|x-x'|^2. \tag{2.4.20}$$

Proof. Let $(X^{\delta,x,u^\delta}, Y^{\delta,x,u^\delta}, Z^{\delta,x,u^\delta})$ (resp. $(X^{\delta,x',u^\delta}, Y^{\delta,x',u^\delta}, Z^{\delta,x',u^\delta})$) be the solution of the FBSDE (2.4.4) with the initial value x (resp. x'). We put

$$\overline{X}_s := X_s^{\delta,x,u^\delta} - X_s^{\delta,x',u^\delta}, \qquad \overline{Y}_s := Y_s^{\delta,x,u^\delta} - Y_s^{\delta,x',u^\delta}, \qquad \overline{Z}_s := Z_s^{\delta,x,u^\delta} - Z_s^{\delta,x',u^\delta},$$

$$\overline{U}_s := U_s^{\delta,x,u^\delta} - U_s^{\delta,x',u^\delta}.$$

For $l = b,\ \sigma,\ f,\ A$, we put

$$\Delta l_\delta(s) := l_\delta(s, X_s^{\delta,x,u^\delta}, Y_s^{\delta,x,u^\delta}, Z_s^{\delta,t,x,u^\delta}) - l_\delta(s, X_s^{\delta,x',u^\delta}, Y_s^{\delta,x',u^\delta}, Z_s^{\delta,x',u^\delta}).$$

Using to Itô's formula, Young's inequality and Proposition 1.1.1, we get for any $s \in [t,T]$,

$$\mathbb{E}[|\overline{X}_s|^2|\mathbb{F}_t^\delta] \leq \mathbb{E}[\int_t^s |\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] \tag{2.4.21}$$

Again by using Itô's formula, Young's inequality, Proposition 1.1.1 and the Lipschitz assumption on Φ, we can find a constant C_2 which depends only upon K such that :

$$\begin{aligned}\mathbb{E}[|\overline{Y}_s|^2|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |\overline{U}_r|^2 dr|\mathbb{F}_t^\delta] + \ \mathbb{E}[\langle M^{\delta',t,x',u^\delta}\rangle_T|\mathbb{F}_t^\delta] \\ \leq C_2 \ \left(\mathbb{E}[|\overline{X}_T|^2|\mathbb{F}_t^\delta] + \ \mathbb{E}[\int_s^T \left(|\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2\right) dr|\mathbb{F}_t^\delta]\right).\end{aligned}$$

Putting $s = T$ in (2.4.21) then plugging it in the previous inequality, it follows (by modifying C_2 if necessary) that :

$$\begin{aligned}\mathbb{E}[|\overline{Y}_s|^2|\mathbb{F}_t^\delta] \;&+\; \mathbb{E}[\int_s^T |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |\overline{U}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\langle M^{\delta',x',u^\delta}\rangle_T|\mathbb{F}_t^\delta] \\ &\leq K^2|x-x'|^2 + C_2(\,\mathbb{E}[\int_t^T |\overline{Y}_r| + |\overline{X}_r|^2 + |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta]) \qquad (2.4.22)\end{aligned}$$

In the other hand, Itô's formula combined with assumption (B2), allows us to find a positive constant C_3 depending upon K, β_1 and β_2 such that :

$$\begin{aligned}\mathbb{E}[\langle G\overline{X}_s, \overline{Y}_s\rangle|\mathbb{F}_t^\delta] =&\; \mathbb{E}[\langle G\overline{X}_T, \overline{Y}_T\rangle|\mathbb{F}_t^\delta] - \mathbb{E}[\int_s^T \langle \Delta A(r), (\overline{X}_r, \overline{Y}_r, \overline{Z}_r) dr|\mathbb{F}_t^\delta] \\ \geq&\; \mathbb{E}[\mu_1|G\overline{X}_T|^2|\mathbb{F}_t^\delta] + \mathbb{E}[\beta_1 \int_s^T |G\overline{X}_r|^2 dr|\mathbb{F}_t^\delta] \\ &+ \mathbb{E}[\int_s^T \beta_2(|G^T\overline{Y}_r|^2 + |G^T\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] \\ \geq&\; C_3\,\mathbb{E}[\int_s^T |\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta],\end{aligned}$$

Therefore, for $s = t$ we have

$$\mathbb{E}[\int_t^T |\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] \leq \frac{1}{C_3}\,\mathbb{E}[\langle G\overline{X}_t, \overline{Y}_t\rangle|\mathbb{F}_t^\delta]. \qquad (2.4.23)$$

Using (2.4.22) and (2.4.23) we obtain, by putting $s = t$, that :

$$\mathbb{E}[|\overline{Y}_t|^2|\mathbb{F}_t^\delta] \leq K^2|x-x'|^2 + C_4\,\mathbb{E}[\langle G\overline{X}_t, \overline{Y}_t\rangle|\mathbb{F}_t^\delta]$$

By Young's inequality, we can find a constant $C_5(K, |G|, \beta_1, \beta_2)$ such that :

$$\mathbb{E}[|\overline{Y}_t|^2\;|\mathbb{F}_t^\delta] \;\leq\; K^2|x-x'|^2 + C_5\,\mathbb{E}[|\overline{X}_t|^2|\mathbb{F}_t^\delta] + \frac{1}{2}\,\mathbb{E}[|\overline{Y}_t|^2|\mathbb{F}_t^\delta]$$

Since $\overline{X}_t = x - x'$, it follows that : that $\mathbb{E}[|\overline{Y}_t|^2\;|\mathbb{F}_t^\delta] \leq 2(K^2 + C_5)|x-x'|^2$. ∎

Lemma 17. *(Stability). Let assumptions* (*B*) *be satisfied. Then for any* $t \in [0,T]$, $\delta, \delta' \in (0,1]$ *and* $x \in \mathbb{R}^d$, *there exits a constant* C *which depends from* K, T *and the bounds of the coefficients, such that*

$$\mathbb{E}[|Y_t^{\delta',x',u^\delta} - Y_t^{\delta,x',u^\delta}|^2|\mathbb{F}_t^\delta] \le C|\delta - \delta'|. \tag{2.4.24}$$

Proof. Let $(X^{\delta,x',u^\delta}, Y^{\delta,x',u^\delta}, Z^{\delta,x',u^\delta})$ (resp. $(X^{\delta',x',u^\delta}, Y^{\delta',x',u^\delta}, Z^{\delta',x',u^\delta})$) be the solution of the FBSDE (2.4.4)–(2.4.4) associated to δ (resp. δ'). We put,

$$\overline{X}_s := X_s^{\delta',x',u^\delta} - X_s^{\delta,x',u^\delta}, \qquad \overline{Y}_s := Y_s^{\delta',x',u^\delta} - Y_s^{\delta,x',u^\delta}, \qquad \overline{Z}_s := Z_s^{\delta',x',u^\delta} - Z_s^{\delta,x',u^\delta}$$

Let $l = b,\ \sigma,\ f,\ A$,

$$\Delta l(s) := l_{\delta'}(s, X_s^{\delta',x',u^\delta}, Y_s^{\delta',x',u^\delta}, Z_s^{\delta',x',u^\delta}, u_s^\delta) - l_\delta(s, X_s^{\delta,x',u^\delta}, Y_s^{\delta,x',u^\delta}, Z_s^{\delta,x',u^\delta}, u_s^\delta),$$

$$\overline{U}_s := U_s^{\delta',x',u^\delta} - U_s^{\delta,x,u^\delta},$$

from Proposition 1.1.1 we have

$$|\Delta l(s)|^2 \quad \le \quad 2K^2\ (|\delta - \delta'|^2 + |\overline{X}_s|^2 + |\overline{Y}_s|^2 + |\overline{Z}_s|^2).$$

Arguing as in the proof of Lemma 16, one can show that there exists two constants C_1 and C_2 which depend upon K, T but not from δ, δ' such that for any $t \le s \le T$,

$$\mathbb{E}[|\overline{X}_s|^2|\mathbb{F}_t^\delta] \ \le \ C_1|\delta - \delta'|^2 + C_1\mathbb{E}[\textstyle\int_t^s (|\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2)dr|\mathbb{F}_t^\delta]. \tag{2.4.25}$$

and

$$\begin{aligned} \mathbb{E}[|\overline{Y}_s|^2|\mathbb{F}_t^\delta] \ &+ \ \mathbb{E}[\textstyle\int_s^T |\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T |\overline{U}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\langle M^{\delta',x',u^\delta}\rangle_T|\mathbb{F}_t^\delta] \\ &\le \ C_2|\delta - \delta'|^2 + C_2\ \mathbb{E}[\int_t^T (|\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2)dr|\mathbb{F}_t^\delta]. \end{aligned} \tag{2.4.26}$$

In the other hand, we successively use Itô's formula and assumption (B2) to get

$$\begin{aligned}\mathbb{E}[\langle G\overline{X}_s, \overline{Y}_s\rangle|\mathbb{F}_t^\delta] &= \mathbb{E}[\langle G\overline{X}_T, \overline{Y}_T\rangle|\mathbb{F}_t^\delta] - \mathbb{E}[\int_s^T \langle \Delta A(r), (\overline{X}_r, \overline{Y}_r, \overline{Z}_r)\rangle dr|\mathbb{F}_t^\delta] \\ &\quad - \mathbb{E}[(\delta-\delta')\int_s^T G\overline{U}_r dr|\mathbb{F}_t^\delta] \\ &\geq \mathbb{E}[\langle G\overline{X}_T, \overline{Y}_T\rangle|\mathbb{F}_t^\delta] + \mathbb{E}[\beta_1\int_s^T |G\overline{X}_r|^2 dr|\mathbb{F}_t^\delta] + \mathbb{E}[\int_s^T \beta_2(|G^T\overline{Y}_r|^2 \\ &\quad + |G^T\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] - K^2|\delta-\delta'|^2 - (\delta-\delta')\mathbb{E}[\int_s^T G\overline{U}_r dr|\mathbb{F}_t^\delta]. \qquad (2.4.27)\end{aligned}$$

Since $|\Phi_\delta(x) - \Phi_{\delta'}(x)| \leq |\delta-\delta'|$, we use Young's inequality to obtain

$$\begin{aligned}\mathbb{E}[\langle G\overline{X}_T, \overline{Y}_T\rangle|\mathbb{F}_t^\delta] &= \mathbb{E}[\langle G\,(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}), \Phi_\delta(X_T^{\delta,x,u^\delta}) - \Phi_\delta(X_T^{\delta',x,u^\delta})\rangle|\mathbb{F}_t^\delta] \\ &- \mathbb{E}[\langle G\,(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}), \Phi_{\delta'}(X_T^{\delta',x,u^\delta}) - \Phi_\delta(X_T^{\delta',x,u^\delta})\rangle|\mathbb{F}_t^\delta] \\ &\geq -\mathbb{E}[|\langle G\,(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}), \Phi_{\delta'}(X_T^{\delta',x,u^\delta}) - \Phi_\delta(X_T^{\delta',x,u^\delta})\rangle|\,|\mathbb{F}_t^\delta] \\ &+ \mathbb{E}[\mu_1|G(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta})|^2|\mathbb{F}_t^\delta] \\ &\geq -|G|\,\mathbb{E}[\,|X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}|\,|\Phi_\delta(X_T^{\delta',x,u^\delta}) - \Phi_{\delta'}(X_T^{\delta',x,u^\delta})|\,|\mathbb{F}_t^\delta] \\ &+ \mathbb{E}[\mu_1|G(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta})|^2|\mathbb{F}_t^\delta] \\ &\geq -|G|\,K\,\mathbb{E}[\,|X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}|\,|\delta-\delta'|\,|\mathbb{F}_t^\delta] \\ &+ \mathbb{E}[\mu_1|G(X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta})|^2|\mathbb{F}_t^\delta] \\ &\geq (\mu_1|G|^2 - |G|\,K\,\varepsilon_1)\mathbb{E}[\,|X_T^{\delta,x,u^\delta} - X_T^{\delta',x,u^\delta}|^2\,|\mathbb{F}_t^\delta] - \frac{|G|\,K}{\varepsilon_1}|\delta-\delta'|^2.\end{aligned}$$

Putting $\varepsilon_1 = \frac{\mu_1\,|G|}{K}$, we get

$$\mathbb{E}[\langle G\overline{X}_T, \overline{Y}_T\rangle|\mathbb{F}_t^\delta] \;\geq\; -\frac{K^2}{\mu_1}|\delta-\delta'|^2. \qquad (2.4.28)$$

Since $\nabla_x V^\delta(t,x)$ is uniformly bounded in t,x (see Lemma 16) and $\overline{U}_r = \delta\nabla_x V^\delta(r, X_r^{\delta,x,u^\delta}) - \delta'\nabla_x V^{\delta'}(r, X_r^{\delta',x,u^\delta})$, it follows that there exists a positive constant $\bar{C} = \bar{C}(T,|G|)$ such that :

$$\begin{aligned}
\mathbb{E}[\int_s^T |G\overline{U}_r| dr|\mathbb{F}_t^\delta] &\leq |G|\ E[\int_s^T |\delta\nabla_x V^\delta(r, X_r^{\delta,x,u^\delta}) - \delta'\nabla_x V^{\delta'}(r, X_r^{\delta',t,x,u^\delta})|\ dr|\mathbb{F}_t^\delta] \\
&\leq |G|\ \mathbb{E}[\int_s^T |\delta\nabla_x V^\delta(r, X_r^{\delta,x,u^\delta})| + |\delta'\nabla_x V^{\delta'}(r, X_r^{\delta',t,x,u^\delta})|\ dr|\mathbb{F}_t^\delta] \\
&\leq |G|\ \mathbb{E}[\int_s^T |\nabla_x V^\delta(r, X_r^{\delta,x,u^\delta})| + |\nabla_x V^{\delta'}(r, X_r^{\delta',t,x,u^\delta})|\ dr|\mathbb{F}_t^\delta] \\
&\leq \bar{C}. \qquad (2.4.29)
\end{aligned}$$

Since $\delta, \delta' \in (0,1]$, then $|\delta - \delta'| \geq |\delta - \delta'|^2$. Therefore, by (2.4.28), (2.4.29) and (2.4.27) we show that,

$$\begin{aligned}
\mathbb{E}[\langle G\overline{X}_s, \overline{Y}_s\rangle|\mathbb{F}_t^\delta] + (\bar{C} + K^2 + \frac{K^2}{\mu_1})|\delta - \delta'| \geq\ & \mathbb{E}[\beta_1 \int_s^T |G\overline{X}_r|^2 dr|\mathbb{F}_t^\delta] \\
&+ \mathbb{E}[\int_s^T \beta_2(|G^T\overline{Y}_r|^2 + |G^T\overline{Z}_r|^2 dr|\mathbb{F}_t^\delta] \\
&\geq C_3\ \mathbb{E}[\int_s^T (|\overline{X}_r|^2 + |\overline{Y}_r|^2 + |\overline{Z}_r|^2) dr|\mathbb{F}_t^\delta].
\end{aligned} \qquad (2.4.30)$$

Using (2.4.26) and (2.4.30), it follows that there exists a constant $C_5(K,|G|,\beta_1,\beta_2)$,

$$|\overline{Y}_t|^2 \leq C_5(|\delta - \delta'| + \langle G\overline{X}_t, \overline{Y}_t\rangle). \qquad (2.4.31)$$

Since $\overline{X}_t = 0$ (the two processes start from the same point x), we deduce that : $\mathbb{E}[|\overline{Y}_t|^2|\mathbb{F}_t^\delta] \leq C|\delta - \delta'|, \quad P - a.s.$ ∎

Lemma 18. *Let assumption (B) be satisfied. Then, for any $t \in [0,T]$, $x, x' \in \mathbb{R}^d$ and*

$\delta, \delta' \in (0,1]$ *there exists a constant* C *which does not depends from* δ, δ' *such that :*

$$\mathbb{E}\Big(\sup_{t\le s\le T}[|X_s^{\delta,x,u^\delta}|^2+|X_s^{\delta',x,u^\delta}|^2+|Y_s^{\delta,x,u^\delta}|^2+|Y_s^{\delta',x,u^\delta}|^2]+\int_t^T|Z_r^{\delta,x,u^\delta}|^2dr \tag{2.4.32}$$
$$+\int_t^T|Z_r^{\delta',x,u^\delta}|^2dr+\int_t^T|U_r^{\delta,x,u^\delta}|^2dr+\int_t^T|U_r^{\delta',x,u^\delta}|^2dr+\langle M^{\delta,x,u^\delta}\rangle_T\ |\mathbb{F}_t^\delta\Big)\le C(1+|x|^2).$$

Proof. For $l=b,f,\sigma$ we denote by C_l the bound of l. Itô's formula gives for any $s\in[t,T]$,

$$\begin{aligned}|X_s^{\delta,x,u^\delta}|^2=|X_t^{\delta,x,u^\delta}|^2&+\int_t^s 2X_r^{\delta,x,u^\delta}b_\delta(X_r^{\delta,x,u^\delta},Y_r^{\delta,x,u^\delta},Z_r^{\delta,x,u^\delta},u_r^\delta)dr\\&+\int_t^s 2X_r^{\delta,x,u^\delta}\sigma_\delta(X_r^{\delta,x,u^\delta},Y_r^{\delta,x,u^\delta},u_r^\delta)dW_r^\delta\\&+\int_t^s 2X_r^{\delta,x,u^\delta}\delta dB_r^\delta\\&+\int_t^s\Big(|\sigma_\delta(X_r^{\delta,x,u^\delta},Y_r^{\delta,x,u^\delta},u_r^\delta)|^2+\delta^2\Big)dr.\end{aligned}$$

We successively use Burkholder-Davis-Gundy's and Young's inequalities to get, for any ε_1, ε_2 and $\varepsilon_3>0$,

$$\begin{aligned}\mathbb{E}[\sup_{t\le s\le T}|X_s^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta]&\le|x|^2+(2\varepsilon_1+\varepsilon_2+2\varepsilon_3)\ \mathbb{E}[\int_t^T|X_r^{\delta,x,u^\delta}|^2dr|\mathbb{F}_t^\delta]\\&\quad+\frac{2}{\varepsilon_1}C_b^2+(T+\frac{1}{\varepsilon_2})\ C_\sigma^2+\frac{2}{\varepsilon_3}\delta^2\\&\le|x|^2+(2\varepsilon_1+\varepsilon_2+2\varepsilon_3)\ T\ \mathbb{E}[\sup_{t\le s\le T}|X_s^{\delta,x,u^\delta}|^2dr|\mathbb{F}_t^\delta]\\&\quad+\frac{2}{\varepsilon_1}C_b^2+(T+\frac{1}{\varepsilon_2})\ C_\sigma^2+(\frac{2}{\varepsilon_3}+1)\delta^2\end{aligned}$$

Choosing $2\varepsilon_1+\varepsilon_2+2\varepsilon_3=\frac{1}{2\,T}$, it follows that (since $\delta\le1$) there exists a constant $\overline{C}_1>0$ independent from δ such that :

$$\mathbb{E}[\sup_{t\le s\le T}|X_s^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta]\le\ \overline{C}_1(1+|x|^2). \tag{2.4.33}$$

Arguing as in the proof of (2.4.33), one can show that there exists a constant $\overline{C}_2 > 0$ independent from δ, δ' such that :

$$\mathbb{E}[\sup_{t\leq s\leq T}|X_s^{\delta',x,u^\delta}|^2|\mathbb{F}_t^\delta] \leq \overline{C}_2(1+|x|^2). \tag{2.4.34}$$

In the other hand, Itô's formula gives

$$\begin{aligned}
&|Y_s^{\delta,x,u^\delta}|^2 + \int_s^T |Z_r^{\delta,x,u^\delta}|^2 dr + \int_s^T |U_r^{\delta,x,u^\delta}|^2 dr + \langle M^{\delta,x,u^\delta}\rangle_T \\
&= |Y_T^{\delta,x,u^\delta}|^2 + \int_t^s 2Y_r^{\delta,x,u^\delta} f_\delta(X_r^{\delta,x,u^\delta}, Y_r^{\delta,x,u^\delta}, Z_r^{\delta,x,u^\delta}, u_r^\delta)dr \\
&\quad - \int_s^T 2Y_r^{\delta,x,u^\delta} Z_r^{\delta,x,u^\delta} dW_r^\delta - \int_t^s 2Y_r^{\delta,x,u^\delta} U_r^{\delta,x,u^\delta} dB_r^\delta - \int_t^s 2Y_r^{\delta,x,u^\delta} dM_r^{\delta,x,u^\delta}.
\end{aligned}$$

Using Burkholder-Davis-Gundy's and Young's inequalities, we show that there exists a positive constant C_2^* such that :

$$\begin{aligned}
&\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta] + \frac{1}{2}\mathbb{E}[\int_t^T |Z_r^{\delta,x,u^\delta}|^2 dr|\mathbb{F}_t^\delta] + \frac{1}{2}\mathbb{E}[\int_t^T |U_r^{\delta,x,u^\delta}|^2 dr|\mathbb{F}_t^\delta] + \frac{1}{2}\ \mathbb{E}[\langle M^{\delta,x,u^\delta}\rangle_T|\mathbb{F}_t^\delta] \\
&\qquad \leq (1/2 + 2(C_2^*)^2)\mathbb{E}[\int_t^T |\sup_{t\leq r\leq T} Y_r^{\delta,x,u^\delta}|^2 dr|\mathbb{F}_t^\delta] + C_\Phi^2 + T\ C_f^2.
\end{aligned}$$

Therefore Gronwall's inequality yields

$$\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta] \leq (C_\Phi^2 + T\ C_f^2)\exp[(1/2 + 2(C_2^*)^2)T].$$

Thanks to the previous two estimates, we have

$$\begin{aligned}
\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{\delta,x,u^\delta}|^2|\mathbb{F}_t^\delta] \quad &+ \quad \frac{1}{2}\ \mathbb{E}[\int_t^T |Z_r^{\delta,x,u^\delta}|^2 dr|\mathbb{F}_t^\delta] + \frac{1}{2}\ \mathbb{E}[\int_t^T |U_r^{\delta,x,u^\delta}|^2 dr|\mathbb{F}_t^\delta] \\
&+ \quad \frac{1}{2}\ \mathbb{E}[\langle M^{\delta,x,u^\delta}\rangle_T|\mathbb{F}_t^\delta] \ \leq\ \overline{C}_4.
\end{aligned} \tag{2.4.35}$$

where $\overline{C}_4 := C_\Phi^2 + T\ C_f^2 + (C_\Phi^2 + T\ C_f^2)\exp[(1/2 + 2(C_2^*)^2)T]$.

The same arguments allow to show that there exists a constant $\overline{C}_5$ which does not depends from δ, δ' such that :

$$\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{\delta',x,u^\delta}|^2|\mathbb{F}_t^\delta]+\mathbb{E}[\int_t^T|Z_r^{\delta',x,u^\delta}|^2dr|\mathbb{F}_t^\delta]+\mathbb{E}[\int_t^T|U_r^{\delta',x,u^\delta}|^2dr|\mathbb{F}_t^\delta]+\mathbb{E}[\langle M^{\delta',x,u^\delta}\rangle_T|\mathbb{F}_t^\delta]$$
$$\leq \overline{C}_5.$$

Lemma 19. *Let assumption (B) be satisfied. Then, for any $t\in[0,T]$, there exists a constant C such that*

$$\sup_n \mathbb{E}\ (\ \sup_{t\leq s\leq T}[|X_s^{\delta_n}|^2+|X_s^n|^2+|Y_s^{\delta_n}|^2+|Y_s^n|^2]|\mathbb{F}_t^{\delta_n}\ )\leq C\ (1+|x|^2). \tag{2.4.36}$$

Proof. For $l=b,f,\sigma,\phi$ we denote by C_l the bounds of l. Itô's formula gives

$$|X_s^n|^2=|x|^2+\ 2\int_t^s X_r^n b(X_r^n,Y_r^n,w_r^n\sigma(X_r^n,Y_r^n,u_r^{\delta_n}),u_r^{\delta_n})dr+\int_t^s|\sigma(X_r^n,Y_r^n,u_r^{\delta_n})|^2dr$$
$$+\int_t^s(X_r^n\sigma(X_r^n,Y_r^n,u_r^{\delta_n})dW_r^{\delta_n}.$$

Using Burkholder-Davis-Gundy's and Young's inequalities, we get for any ε_1, $\varepsilon_2>0$,

$$\mathbb{E}[\sup_{t\leq s\leq T}|X_s^n|^2|\mathbb{F}_t^{\delta_n}\ ]\leq|x|^2+\ 2\ \varepsilon_1\mathbb{E}[\int_t^T|X_r^n|^2dr|\mathbb{F}_t^{\delta_n}\ ]+\frac{2}{\varepsilon_1}\ T\ C_b^2$$
$$+\ C^*\varepsilon_2\ \mathbb{E}[\int_t^s|X_r^n|^2dr|\mathbb{F}_t^{\delta_n}\ ]+(T+\frac{C^*}{\varepsilon_2})\ C_\sigma^2.$$

Choosing $\varepsilon_1=1/8$ and $\varepsilon_2=\frac{1}{4C^*}$, one can find $\tilde{C}_1$ which depends only from $C^*,\ C_b,\ T,C_\sigma$ such that :

$$\sup_n\mathbb{E}[\sup_{t\leq s\leq T}|X_s^n|^2|\mathbb{F}_t^{\delta_n}\ ]\leq\tilde{C}_1\ (1+|x|^2). \tag{2.4.37}$$

Again, by Itô's formula we have

$$
\begin{aligned}
|Y_s^n|^2 + \int_s^T |w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n})|^2 dr &= |Y_T^n|^2 + \int_s^T Y_r^n w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n}) dW_r^{\delta_n} \\
&+ 2\int_s^T Y_r^n f(X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n}), u_r^{\delta_n}) dr \\
&\le K^2 |X_T^n|^2 + \int_s^T Y_r^n w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n}) dW_r^{\delta_n} \\
&+ 2\int_s^T |Y_r^n| |f(X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n}), u_r^{\delta_n})| dr.
\end{aligned}
$$

We use Burkholder-Davis-Gundy's and Young's inequalities to get for any ε_3, $\varepsilon_4 > 0$,

$$
\begin{aligned}
&\mathbb{E}[\sup_{t\le s\le T} |Y_s^n|^2 | \mathbb{F}_t^{\delta_n}] + \mathbb{E}[\int_t^T |w_r^n \sigma(X_r^n, Y_r^n, u_r^{\delta_n})|^2 dr | \mathbb{F}_t^{\delta_n}] \\
&\le C_\Phi^2 + C^* \varepsilon_3 \, \mathbb{E}[\int_t^T |Y_r^n|^2 dr | \mathbb{F}_t^{\delta_n}] + \frac{T C_\sigma^2 C^* C_w^2}{\varepsilon_3} \\
&+ 2\, \varepsilon_4 \, \mathbb{E}[\int_s^T |Y_r^n| dr | \mathbb{F}_t^{\delta_n}] + \frac{C_f^2 \, T}{\varepsilon_4}.
\end{aligned}
$$

We successively choose $\varepsilon_3 = 1/(4C^*)$ and $\varepsilon_4 = 1/8$ then we use Gronwall's inequality to show that there exists a constant $\tilde{C}_2$ independent from δ_n such that :

$$
\sup_n \mathbb{E}[\sup_{t\le s\le T} |Y_s^n|^2 | \mathbb{F}_t^{\delta_n}] \le \tilde{C}_2 \tag{2.4.38}
$$

We conclude the proof by using the estimates (2.4.33) and (2.4.35). ■

Remark 20. (i) *As explained in the introduction, the uniform Lipschitz condition is not sufficient to guarantee the existence of solutions and hence the existence of optimal controls fails also. Nevertheless, there are results on the existence and uniqueness of solutions to coupled FBSDEs under the uniform Lipschitz condition and supplementary assumptions on the coefficients, see e.g. [59, 103, 136, 141].*

(ii) *When the coefficients are uniformly Lipschitz and σ is non degenerate, the existence and uniqueness of solutions were established in [59] for equation* (2.0.1). *In this case, the* *existence of an optimal control was recently established in [22] when the coefficients* σ *and* b *are independent from* z*, and* σ *is independent from the control* u*. The case where* b *depends from* z *and* σ *is independent from* z *and* u*, the existence of an optimal control can be performed as in [22]. In the case where* σ *depends upon* (x, y, u)*, the problem of existence of an optimal control seems difficult to obtain. Indeed, when the control enters the diffusion coefficient* σ*, we lead to an FBSDE with a measurable diffusion matrix and, in this case, the uniqueness of solution (even in the law sense) may fails. It is known from [92] that when the diffusion coefficient is merely measurable, then even the uniqueness in law fails in general for Itô's forward SDEs in dimension strictly greater than 2, see [92] for more details.*

(iii) *The existence of an optimal control under the conditions used in [141] can be obtained by using the method we developed in the present thesis.*

(iv) *However, the supplementary condition given in [103] consists to assuming the existence of a decoupling function. This condition is rather implicit and abstract, and hence can not be easily exploited in the problem of control.*

(iv) *The problem of existence of an optimal control for a fully coupled FBSDE when the coefficient* σ *depends from* z *and* u *remains open and is a challenge. In this case, the existence of solutions follows from [136] and the Bellman dynamic programming principle is given in [102]. In this case, Bellman dynamic programming principle leads to an HJB equation coupled with a constraint given by an algebraic equation.*

Chapitre 3

Existence of an optimal control for a system of fully coupled FBSDE in the non-degenerate case

In this chapter we studies as the first section of this chapter the existence of optimal control (2.0.1)–(2.0.3) but, now the diffusion is non-degenerate, the conditions here are deferent from the one of the degenerate case, because the existence and uniqueness results of the solution itself defers from the last section, also here we are not obligate to transform coefficient of the hessian uniformly elliptic by adding a strictly positive number $((\sigma_\delta\sigma_\delta^*)\left(x, v^\delta(s,x)\right) + \delta^2 I_{R^d})$ because the diffusion is already non-degenerate, this end will change the form of the FBSDE, (2.4.7)

The chapter is organized as follows : In next section, we introduce some notations, the controlled system, and the assumptions. In section 2, we present the cost functional and the value function. This value function verified the Hamilton-Jacobi-Bellman equation. In section 3, we give the main result and its proof. This section contains two subsections. The first one is devoted to study the approximating control problem together with its associated Hamilton-Jacobi-Bellman equation. In the second subsection, we prove our main result. In the last section we present the convergence of the approximating problem zn so the existence of an optimal control.

3.1 Lipshitz and non degenerate Hypothesis

There exists two constants K and $\lambda > 0$, such that the functions b, σ, f and Φ satisfy the following assumptions **(B)** :

— **(**B1)

1) For any $u \in \mathbb{U}$, (x, y, z) and $(x', y', z') \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$

$$|\sigma(x, y) - \sigma(x', y')|^2 \leq K^2(|x - x'|^2 + |y - y'|^2),$$

$$|\Phi(x) - \Phi(x')| \leq K|x - x'|,$$

$$|b(x, y, z, u) - b(x', y', z', u)| \leq K(|x - x'| + |y - y'| + |z - z'|),$$

$$|f(x, y, z, u) - f(x', y', z', u)| \leq K(|x - x'| + |y - y'| + |z - z'|).$$

2) The functions b, σ, f and Φ are bounded.

— **(**B2) For every $(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ the functions $b(x, y, z, .)$ and $f(x, y, z, .)$ are continuous in $u \in \mathbb{U}$.

— **(**B3) For every $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$,

$$\forall \zeta \in \mathbb{R}^d \quad \langle \zeta, \sigma(t, x, y)\zeta \rangle \geq \lambda|\zeta|^2,$$

When the control u is constant, one can show (by arguing as in [59]) that under assumptions (B1) and (B2), equation (2.0.1) has a unique solution $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, M^{t,x,u})$ in the space $\mathcal{S}^2_\nu(t, T; \mathbb{R}^d) \times \mathcal{S}^2_\nu(t, T; \mathbb{R}) \times \mathcal{H}^2_\nu(t, T; \mathbb{R}^d) \times \mathcal{M}^2_\nu(t, T; \mathbb{R}^d)$

Now we set the main result of this section

3.1.1 The main result

Theorem 21. *Assume that the assumptions **(B)** and **(H)** are satisfied, then there exists a strict control which solves the problem (2.0.1)–(2.0.3).*

For the prove we proceed as the last section, the difference here is that the diffusion is degenerate, and so we do not need to transform the HJB to strictly elliptic

3.2 Proof of the main results

As the last section we approximate our coefficients by a smooth one, first let write the HJB corresponding to the approximating system

3.2.1 The approximating Hamilton-Jacobi-Bellman equation

Let $\delta \in (0,1]$ be an arbitrarily fixed number. For $(x,y,p,A,v) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{U}$, we define the function H^δ by :

$$\begin{aligned} H^{\delta}\left(x,y,p,A,v\right) &= \frac{1}{2}\left(\mathrm{tr}\left(\left(\sigma_{\delta}\sigma_{\delta}^{*}\right)\left(x,y\right)\right)A\right) + b_{\delta}\left(x,y,p\sigma_{\delta}\left(x,y\right),v\right)p \\ &\quad + f_{\delta}\left(x,y,p\sigma_{\delta}\left(x,y\right),v\right), \end{aligned} \tag{3.2.1}$$

and consider the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \dfrac{\partial}{\partial t}V^{\delta}\left(t,x\right) + \inf\limits_{v\in\mathbb{U}} H^{\delta}\left(x,(V^{\delta},\nabla_x V^{\delta},\nabla_{xx}V^{\delta})(t,x),v\right) = 0,\ (t,x)\in[0,T]\times\mathbb{R}^d, \\ V^{\delta}\left(T,x\right) = \Phi_{\delta}(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{3.2.2}$$

Since H^δ is smooth and $(\sigma_\delta\sigma_\delta^*)(x,y)$ is uniformly elliptic, then according to the regularity results by Krylov [94] (Theorems 6.4.3 and 6.4.4 in [94]), the unique bounded continuous viscosity solution V^δ of the equation (3.2.2) is with regularity $C_b^{1,2}([0,T]\times\mathbb{R}^d)$. The regularity of V^δ and the compactness of the control state space $\mathbb{U}$ allow to find a measurable function

$v^\delta : [0,T] \times \mathbb{R}^d \longmapsto \mathbb{U}$ such that, for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$H^\delta\left(x, (V^\delta, \nabla_x V^\delta, \nabla_{xx} V^\delta)(t,x), v^\delta(t,x)\right) = \inf_{v\in\mathbb{U}} H^\delta\left(x, (V^\delta, \nabla_x V^\delta, \nabla_{xx} V^\delta)(t,x), v\right).$$

Lemma 22. *Assume that the hypothesis (B) is satisfied. Then :*

$$J^\delta(u^\delta) = V^\delta(t,x) = essinf_{u\in\mathcal{U}_{\nu^\delta}(t)} J^\delta(u),$$

Moreover $u^\delta_s := v^\delta(s, X^\delta_s)$, $s \in [0,T]$, *is an admissible control.*

Proof. We fix now an arbitrary initial datum $(t,x) \in [0,T] \times \mathbb{R}^d$ and define the process $(X^\delta_s, Y^\delta_s, Z^\delta_s)_{s\in[t,T]}$ by :

$$\begin{cases} dX^\delta_s \;=\; b_\delta(X^\delta_s, V^\delta(s,X^\delta_s), \nabla_x V^\delta(s,X^\delta_s)\sigma_\delta(X^\delta_s, V^\delta(s,X^\delta_s)),\; v^\delta(s,X^\delta_s))ds \\ \qquad\quad + \sigma_\delta(X^\delta_s, V^\delta(s,X^\delta_s))dW^\delta_s, \qquad s\in[t,\; T], \\ X^\delta_t = x. \end{cases} \tag{3.2.3}$$

Since $b_\delta(x, V^\delta(s,x), \nabla_x V^\delta(s,x)\sigma_\delta(x, V^\delta(s,x)), v^\delta(s,x))$ and $\sigma_\delta(x, V^\delta(s,x))$ are bounded measurable in (t,x) and $\sigma_\delta(x, V^\delta(s,x))$ is Lipschitz in x and uniformly elliptic, then according to [9] (Theorem 2.1 pp 56, see also [4]), equation (3.2.3) has a pathwise unique solution X^δ. We define Y^δ and Z^δ by :

$$Y^\delta_s = V^\delta(s, X^\delta_s) \qquad \text{and} \qquad Z^\delta_s = \nabla_x V^\delta(s,X^\delta_s)\sigma_\delta(X^\delta_s, V^\delta(s,X^\delta_s)), \quad s\in[t,T]. \tag{3.2.4}$$

Applying Itô's formula to $V^\delta(s, X^\delta_s)$, we obtain :

$$\begin{cases} dX^\delta_s = b_\delta(X^\delta_s, Y^\delta_s,\; Z^\delta_s,\; v^\delta(s,X^\delta_s))ds + \sigma_\delta(X^\delta_s, Y^\delta_s)dW^\delta_s, \\ dY^\delta_s = -f_\delta(X^\delta_s, Y^\delta_s, Z^\delta_s, v^\delta(s,X^\delta_s))ds + Z^\delta_s dW^\delta_s, \\ X^\delta_t = x,\; Y^\delta_T = \Phi_\delta(X^\delta_T),\; s\in[t,\; T] \end{cases} \tag{3.2.5}$$

Since f_δ is uniformly Lipschitz in (y,z), then according to [119] the backward component of equation (3.2.5) has a unique solution (Y^δ, Z^δ) in $\mathcal{S}^2_\nu(t,T;\mathbb{R}) \times \mathcal{H}^2_\nu(t,T;\mathbb{R}^d)$. Therefore $(X^\delta, Y^\delta, Z^\delta)$ is the unique solution of FBSDE (3.2.5) in $\mathcal{S}^2_\nu(t,T;\mathbb{R}^d) \times \mathcal{S}^2_\nu(t,T;\mathbb{R}) \times \mathcal{H}^2_\nu(t,T;\mathbb{R}^d)$.

Let $u \in \mathcal{U}_{\nu^\delta(t)}$ be an admissible control. Let $(X^{\delta,t,x,u}, Y^{\delta,t,x,u}, Z^{\delta,t,x,u})$ be the unique $\mathbb{F}$-adapted continuous solution of the following FBSDE :

$$\begin{cases} dX^{\delta,t,x,u}_s = b_\delta\left(X^{\delta,t,x,u}_s, Y^{\delta,t,x,u}_s, Z^{\delta,t,x,u}_s, u_s\right)ds + \sigma_\delta\left(X^{\delta,t,x,u}_s, Y^{\delta,t,x,u}_s\right)dW^\delta_s,\ s \in [t,T], \\ dY^{\delta,t,x,u}_s = -f_\delta(X^{\delta,t,x,u}_s, Y^{\delta,t,x,u}_s, Z^{\delta,t,x,u}_s, u_s)ds + Z^{\delta,t,x,u}_s dW^\delta_s + dM^{\delta,t,x,u}_s,\ \ s \in [t,T], \\ X^{\delta,t,x,u}_t = x,\ Y^{\delta,t,x,u}_T = \Phi_\delta(X^{\delta,t,x,u}_T), \\ M^\delta \in \mathcal{M}^2_{\nu^\delta}(t,T;\mathbb{R}^d) \text{ is orthogonal to } W^\delta. \end{cases} \tag{3.2.6}$$

We define the cost functional for the approximating control problem by :

$$J^\delta(u) := Y^{\delta,t,x,u}_t,\ u \in \mathcal{U}_{\nu^\delta}(t).$$

$(X^\delta, Y^\delta, Z^\delta)$ satisfies the FBSDE (3.2.6) for $u = u^\delta$, with $M^\delta = 0$. Hence, by the uniqueness of equation (3.2.6), we have $(X^\delta, Y^\delta, Z^\delta) = (X^{\delta,t,x,u^\delta}, Y^{\delta,t,x,u^\delta}, Z^{\delta,t,x,u^\delta})$. In particular $Y^{\delta,t,x,u^\delta}_t = Y^\delta_t = V^\delta(t,x)$. ∎

Proposition 3.2.1. *Under the assumption (B), there exists a universal constant C only depending on the Lipshitz constants of the functions σ, b, f and Φ such that for every $t, t' \in [0,T]$; $x, x' \in \mathbb{R}^d$ and $\delta, \delta' > 0$:*

$$|V^\delta(t,x) - V^{\delta'}(t',x')| \leq C(|\delta - \delta'| + |t-t'|^{\frac{1}{2}} + |x - x'|).$$

In particular :

$$|V^\delta(t,x) - V(t,x)| \leq C\delta, \ \ \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^d \textit{ and for each } \delta > 0.$$

and

$$\begin{aligned} |V^\delta(t,x)| + |\nabla_x V^\delta(t,x)| &\leq C, \\ |V^\delta(t,x) - V^\delta(t',x)| &\leq C|t-t'|^{1/2}. \end{aligned} \tag{3.2.7}$$

Proof. Let $(t,x),\ (t',x') \in [0,T]\times\mathbb{R}^d$ and $\delta, \delta' \in (0,1]$ be fixed. The value function $V^{\delta'}(t',x')$ satisfies the same Hamilton-Jacobi-Bellman equation (3.2.2) with the Hamiltonian $H^{\delta'}$ associated to the coefficients $f_{\delta'}$, $b_{\delta'}$, $\sigma_{\delta'}$ and $\Phi_{\delta'}$. Hence, the same arguments of regularity allow to find $v^{\delta'} : [0,T]\times\mathbb{R}^d \to \mathbb{U}$ such that, for any $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$H^{\delta'}\left(x, (V^{\delta'}, \nabla_x V^{\delta'}, \nabla_{xx} V^{\delta'})(t,x), v^{\delta'}(t,x)\right) = \inf_{v\in\mathbb{U}} H^{\delta'}\left(x, (V^{\delta'}, \nabla_x V^{\delta'}, \nabla_{xx} V^{\delta'})(t,x), v\right).$$

We use Lemma 22, to show that $u^{\delta'} = v^{\delta'}(t',x')$ is an admissible control.

Thanks to [9], let $X^{\delta',t',x',u^{\delta'}}$ be the unique solution of the following forward SDE :

$$\begin{cases} dX_s = b_{\delta'}(X_s, V^{\delta'}(s,X_s), \nabla_x V^{\delta'}(s,X_s)\sigma_{\delta'}(X_s^{\delta',t',x',u^{\delta'}}, V^{\delta'}(s,X_s))ds \\ \qquad\qquad + \sigma_{\delta'}(X_s^{\delta',t',x',u^{\delta'}}, V^{\delta'}(s,X_s))dW_s^\delta,\ s\in[t',T], \\ X_{t'} = x', . \end{cases}$$

We extend this solution in the interval $[0,T]$ by putting $X_r^{\delta',t',x',u^\delta} = x'$, for $r < t'$.

We put

$$Y_s^{\delta',t',x',u^{\delta'}} := V^{\delta'}(s, X_s^{\delta',t',x',u^{\delta'}})$$

$$Z_s^{\delta',t',x',u^{\delta'}} := \nabla_x V^{\delta'}(s, X_s^{\delta',t',x',u^{\delta'}})\sigma_{\delta'}(X_s^{\delta',t',x',u^{\delta'}}, V^{\delta'}(s, X_s^{\delta',t',x',u^{\delta'}})),$$

Itô's formula applied to $V^{\delta'}(s, X_s^{\delta,t',x',u^{\delta'}})$ shows that :

$$\begin{cases} dY_s^{\delta',t',x',u^{\delta'}} = -f_{\delta'}(X^{\delta',t',x',u^{\delta'}}, Y^{\delta',t',x',u^{\delta'}}, Z^{\delta',t',x',u^{\delta'}}, u^{\delta'})ds + Z_s^{\delta',t',x',u^{\delta'}} dW_s^\delta, \\ Y_T^{\delta',t',x',u^{\delta'}} = \Phi_{\delta'}(X_T^{\delta',t',x',u^{\delta'}}),\ s\in[t',T]. \end{cases} \tag{3.2.8}$$

Since $f_{\delta'}$ is Lipschitz, then the previous BSDE has a unique solution $(Y^{\delta',t',x',u^{\delta'}}, Z^{\delta',t',x',u^{\delta'}})$ in $\mathcal{S}^2_{\nu^\delta}(t',T;\mathbb{R}) \times \mathcal{H}^2_{\nu^\delta}(t',T;\mathbb{R}^d)$.

The following lemmas show that the variable Z^δ. This allows us to consider Z^{delta} as a control.

Lemma 23. *Assume that assumptions **(B)** is satisfied. Then there exists a non-negative constant $\tilde{C}$ only depending on the Lipshitz constants of the coefficients, verified the following estimation :*

$$|V^{\delta'}(t,x) - V^\delta(t,x)| \leq \tilde{C}|\delta' - \delta|. \tag{3.2.9}$$

The proof will be given later.

Since f_δ, b_δ, Φ_δ and σ_δ are bounded $\mathbb{C}^\infty$ functions with bounded derivatives of every order and satisfy Assumption **(B)** with the same constant K, we have the following lemma.

Lemma 24. *Let H^δ be defined by formula* (3.2.1)*. Then, the PDE*

$$\begin{cases} \dfrac{\partial}{\partial t}V^\delta(t,x) + H^\delta\left(x, (V^\delta, \nabla_x V^\delta, \nabla_{xx}V^\delta)(t,x), v^\delta(t,x)\right) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ V^\delta(T,x) = \Phi_\delta(x), \ x \in \mathbb{R}^d, \end{cases} \tag{3.2.10}$$

has a unique bounded solution V^δ in $C^{1,2}_b([0,T]\times\mathbb{R}^d)$.

Moreover, there exists a constant $\bar{C}$, only depending on λ and T, and two constants $\bar{\Gamma}$ and $\bar{\kappa}$, only depending on K, λ and T, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |V^\delta(t,x)| \leq \bar{C}, \tag{3.2.11}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla_x V^\delta(t,x)| \leq \bar{\Gamma}, \tag{3.2.12}$$

$$\forall (t,t') \in [0,T]^2, \quad \forall x \in \mathbb{R}^d, \qquad |V^\delta(t',x) - V^\delta(t,x)| \leq \bar{\kappa}|t'-t|^{1/2}. \tag{3.2.13}$$

The proof will be given later.

We shall show that V^δ converges uniformly to a bounded function $\bar{V}$, which is the unique viscosity solution of the initial HJB equation (2.1.2).

We have,

$$|V^{\delta'}(t',x') - V^\delta(t,x)| \le |V^{\delta'}(t',x') - V^\delta(t',x')| + |V^\delta(t',x') - V^\delta(t,x)|.$$

By (3.2.12) and (3.2.13), we have

$$|V^\delta(t',x') - V^\delta(t,x)| \le \kappa|t'-t|^{1/2} + \Gamma|x'-x|. \tag{3.2.14}$$

Therefore, using (3.2.9) and modifying the constants if necessary we obtain

$$|V^\delta(t,x) - V^{\delta'}(t',x')| \le C(|\delta-\delta'| + |t-t'|^{\frac{1}{2}} + |x-x'|).$$

Using standard arguments in the BSDE, one can show that V^δ is uniformly bounded in (t,x,δ). Hence, V^δ converges uniformly (as $\delta \to 0$) to a function in $\bar{V} \in C_b([0,T]\times\mathbb{R}^d)$. Using the stability of viscosity solutions and the fact that the Hamiltonian H^δ converges uniformly on compacts set to H, we get that $\bar{V}$ is a viscosity solution of equation (2.1.2). Thanks to the uniqueness of the solution of equation (2.1.2) with in the class of continuous function with at most polynomial growth, we get that $\bar{V} = V$. This proves the convergence of the approximating value function $V^{\delta'}$ to V, as $\delta' \to 0$. And hence

$$|V^\delta(t,x) - V(t,x)| \le C\delta, \quad \text{for all } \ \delta \in (0,1] \text{ and } (t,x) \in [0,T]\times\mathbb{R}^d. \qquad \blacksquare$$

3.2.2 Convergence of the Approximating Control Problems

We will prove the convergence of the approximating control problem to the original one. We adapt the idea of [41] to or situation. Put $w_s^n := \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$ and $Z_s^{\delta_n} := w_s^n \sigma\left(X_s^{\delta_n}, Y_s^{\delta_n}\right)$. Consider the sequence of approximating stochastic controlled systems $(X^{\delta_n}, Y^{\delta_n}, Z^{\delta_n}, u^{\delta_n})$. Since u^{δ_n} and u^{δ_n} are uniformly bounded, we see the couple (u^{δ_n}, w^n) as a relaxed control. We show that the system $(X^{\delta_n}, Y^{\delta_n}, Z^{\delta_n}, u^{\delta_n})$ has a subsequence which converges in law to some controlled system. And, since we have assumption **(H)**, we use the result of [63] to prove that the limiting process is a strict control.

Theorem 25. *Assume that the assumption **(H)** is satisfied. Let $(t,x) \in [0,T] \times \mathbb{R}^d$ and $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers which tends to 0. Then, there exists a reference stochastic system $\bar{\nu} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}, \bar{W})$, a process $(\bar{X}, \bar{Y}, \bar{Z}, \bar{M}) \in \mathcal{S}^2_{\bar{\nu}}(t,T;\mathbb{R}^d) \times \mathcal{S}^2_{\bar{\nu}}(t,T;\mathbb{R}) \times \mathcal{S}^2_{\bar{\nu}}(t,T;\mathbb{R}^d) \times \mathcal{M}^2_{\bar{\nu}}(t,T;\mathbb{R}^d)$, with $\bar{M}$ orthogonal to $\bar{W}$, and an admissible control $\bar{u} \in \mathcal{U}_{\bar{\nu}}(t)$, such that :*

1) There is a subsequence of $(X^{\delta_n}, Y^{\delta_n})_{n\in\mathbb{N}}$ which converges in distribution to $(\bar{X}, \bar{Y})$,

2) $(\bar{X}, \bar{Y}, \bar{Z}, \bar{M})$ is a solution of the following system

$$\begin{cases} d\bar{X}_s = b(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)ds + \sigma(\bar{X}_s, \bar{Y}_s)d\bar{W}_s, \\ d\bar{Y}_s = -f(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)ds + \bar{Z}_s d\bar{W}_s + d\bar{M}_s, \quad s \in [t,T] \\ \bar{X}_t = x, \ \bar{Y}_T = \Phi(X_T), \end{cases} \tag{3.2.15}$$

3) For every $(t,x) \in [0,T] \times \mathbb{R}^d$, it holds that

$$\bar{Y}_t = V(t,x) = essinf_{u \in \mathcal{U}_{\bar{\nu}}(t)} J(t,x,u) \ ,$$

i.e. the admissible control $\bar{u} \in \mathcal{U}_{\bar{\nu}}(t)$ is optimal for (3.2.15).

Proof. The idea consists to introduce an auxiliary sequence of processes (denoted by (X^n, Y^n))

which satisfied forward-system, for each n, and for which the existence of a relaxed holds according to [63]. We then show that (X^n, Y^n) has a subsequence which converges in law to a couple $(\bar{X}, \bar{Y})$. using the convexity assumption **(H)**, we prove that $(\bar{X}, \bar{Y})$ is associated to a strict control that is optimal for the original control problem. We finally show that the initial sequence $(X^{\delta_n}, Y^{\delta_n})_{n\in\mathbb{N}}$ and the auxiliary one have the same limits, by proving an estimation between the auxiliary and the appro solution.

For $n \in \mathbb{N}$, we define the sequence of auxiliary processes (X^n_s, Y^n_s) as the pathwise unique solution to the following controlled forward system :

$$\begin{cases} dX^n_s = b(X^n_s, Y^n_s, w^n_s\sigma(X^n_s, Y^n_s), u^{\delta_n}_s)ds + \sigma(X^n_s, Y^n_s)dW^{\delta_n}_s, \\ dY^n_s = -f(X^n_s, Y^n_s, w^n_s\sigma(X^n_s, Y^n_s), u^{\delta_n}_s)ds + w^n_s\sigma(X^n_s, Y^n_s)dW^{\delta_n}_s. \\ \ X^n_t = x, \quad Y^n_t = V^{\delta_n}(t, x), \quad s \in [t, T]. \end{cases} \tag{3.2.16}$$

where $u^{\delta_n}_s := v^{\delta_n}(s, X^{\delta_n}_s)$ and $w^n_s = \nabla_x V^{\delta_n}(s, X^{\delta_n}_s)$.

Note that for every n, the process $(X^{\delta_n}_s, Y^{\delta_n}_s)$ is a weak solution to the following controlled forward system :

$$\begin{cases} dX^{\delta_n}_s = b_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s, w^n_s\sigma_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s), u^{\delta_n}_s)ds + \sigma_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s)dW^{\delta_n}_s, \\ dY^{\delta_n}_s = -f_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s, w^n_s\sigma_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s), u^{\delta_n}_s)ds + w^n_s\sigma_{\delta_n}(X^{\delta_n}_s, Y^{\delta_n}_s)dW^{\delta_n}_s. \\ \ X^{\delta_n}_t = x, \quad Y^{\delta_n}_t = V^{\delta_n}(t, x), \ s \in [t, T]. \end{cases} \tag{3.2.17}$$

From (3.2.4) we have, for $t \leq s \leq T$,

$$Y^{\delta_n}_s = V^{\delta_n}(s, X^{\delta_n}) \qquad \text{and} \qquad u^{\delta_n}_s = v^{\delta_n}(t, X^{\delta_n}_s).$$

Since $(s, x) \mapsto V^{\delta_n}(s, x)$ is of class $\mathcal{C}^{1,2}$ and satisfies equation (2.1.2), then using Itô's formula we get for $t \le s \le T$

$$Y_s^{\delta_n} = \Phi_{\delta_n}(X_T^{\delta_n}) + \int_s^T f_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^n \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})dr$$
$$- \int_s^T w_r^n \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}) dW_r^{\delta_n}. \tag{3.2.18}$$

If we put

$$\chi_s^n := \begin{pmatrix} X_s^n \\ Y_s^n \end{pmatrix}, \quad r_s^n := (w_s^n, 0, u_s^{\delta_n}) \quad \text{and} \quad \mathcal{W}^n := \begin{pmatrix} W^{\delta_n} \\ B^{\delta_n} \end{pmatrix},$$

then the system (3.2.16) becomes :

$$\begin{cases} d\chi_s^n = \beta(\chi_s^n, r_s^n)ds + \Sigma(\chi_s^n, r_s^n)d\mathcal{W}_s^n, \quad s \in [t,T], \\ \chi_t^n = \begin{pmatrix} x \\ V^{\delta_n}(t,x) \end{pmatrix}. \end{cases} \tag{3.2.19}$$

Since $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$ is uniformly bounded (Proposition 3.2.1), we can interpret $(r_s^n, s \in [t,T])$ as a control with values in the compact set $A := \mathbb{U} \times \bar{B}_C(0) \times [0,K]$.

The next step is to take $n \to +\infty$, for this let's consider the random measure :

$$q^n(\omega, ds, da) = \delta_{r_s^n(\omega)}(da)ds, \ (s,a) \in [0,T] \times A, \omega \in \Omega.$$

we identify the control process r^n with the measure q^n, this end show us that the controls r^n is in the set of relaxed controls, looking consider r^n as random variable with values in the space V of all Borel measures q^n on $[0,T] \times \mathbb{U} \times \bar{B}_C(0) \times [0,K]$, whose projection $q^n(\cdot \times \mathbb{U} \times \bar{B}_C(0) \times [0,K])$ coincides with the Lebesgue measure, we need now to show a convergence result given in :

Lemma 26. *There exist a probability measure Q on $C([0,T];\mathbb{R}^d \times \mathbb{R}) \times V$ and a subsequence of $(\Upsilon^{\phi(n)}, q^{\phi(n)})$ of (Υ^n, q^n), such that $(\Upsilon^{\phi(n)}, q^{\phi(n)}) \xrightarrow[\mathcal{L}]{(C([0,T];\mathbb{R}^d \times \mathbb{R}) \times V, Q)} (\Upsilon, q)$ where, (Υ, q) is the canonical process.*

Proof. by the contraction above our conditions guarantee that $\{(\Sigma(x,y,z,\pi,v),\beta(x,y,z,\pi,v)),(x,y,z,\pi,v$

$\mathbb{R}^d\times\mathbb{R}\times A\}$ are bounded and by the compactness of V with respect to the topology induced by the weak convergence of measures, we get the tightness of the laws of $(\Upsilon^n,q^n), n\geq 1$, on $C([0,T];\mathbb{R}^d\times\mathbb{R})\times V$. and hence there exit a probability measure Q on $C([0,T];\mathbb{R}^d\times\mathbb{R})\times V$ and extract a subsequence that converges in law on the space $C([0,T];\mathbb{R}^d\times\mathbb{R})\times V$ donated with the measure Q to (Υ,q). ■

Since the coefficients of system (3.2.19) satisfy assumption **(H)**, then, according to [63], there exists a stochastic reference system $\bar{\nu}=(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}},\bar{\mathbb{F}},\bar{\mathcal{W}})$ enlarging $(C([0,T];\mathbb{R}^d\times\mathbb{R})\times V;Q)$ and an $\bar{\mathbb{F}}$-adapted process $(\chi,\bar{r})$ [$\bar{r}$ with values in A] which satisfies

$$\begin{cases} d\chi_s=\beta(\chi_s,\bar{r}_s)ds+\Sigma(\chi_s,\bar{r}_s)d\bar{\mathcal{W}}_s,\ s\in[t,T],\\ \chi_t=\begin{pmatrix} x\\ V^{\delta_n}(t,x)\end{pmatrix}. \end{cases} \tag{3.2.20}$$

Moreover, χ has the same law under $\bar{\mathbb{P}}$ as under Q.

Replacing Σ and β by their definition and setting $\chi:=\begin{pmatrix}\bar{X}\\ \bar{Y}\end{pmatrix}$, $\bar{\mathcal{W}}:=\begin{pmatrix}\bar{W}\\ \bar{B}\end{pmatrix}$ and $\bar{r}:=(\bar{w},\bar{\theta},\bar{u})$, the system (3.2.20) can be rewritten as follows :

$$\begin{cases} d\bar{X}_s=b(\bar{X}_s,\bar{Y}_s,\bar{Z}_s,\bar{u}_s)ds+\sigma(\bar{X}_s,\bar{Y}_s)d\bar{W}_s,\\ d\bar{Y}_s=-f(\bar{X}_s,\bar{Y}_s,\bar{Z}_s,\bar{u}_s)ds+\bar{Z}_sd\bar{W}_s+\bar{\theta}_sd\bar{B}_s,\quad s\in[t,T]\\ \bar{X}_t=x,\ \bar{Y}_t=V(t,x). \end{cases}$$

Assertion 1) is proved. To prove assertion 2), we need the Following lemma.

Lemma 27. *For some constant $L>0$ and for all $n\in\mathbb{N}$,*

$$\begin{aligned} \mathbb{E}[\sup\nolimits_{s\in[t,T]}|X_s^{\delta_n}-X_s^n|^2]\leq L\delta_n^2,\\ \mathbb{E}[\sup\nolimits_{s\in[t,T]}|Y_s^{\delta_n}-Y_s^n|^2]\leq L\delta_n^2. \end{aligned} \tag{3.2.21}$$

Lemma 27 shows that if the sequence $(X^n,Y^n)_{n\in\mathbb{N}}$ converges in law, the same holds true for $(X^{\delta_n},Y^{\delta_n})_{n\in\mathbb{N}}$, and the limits have same law. Further we deduce from (3.2.21) and

Proposition 3.2.1, that $\bar{Y}_s = V(s, \bar{X}_s)$ for each $s \in [t, T]$ $\bar{\mathbb{P}}$-a.s. In particular, $Y_T = \Phi(X_T)$ $\bar{\mathbb{P}}$-a.s. Thus, if we set $\bar{M}_s = \int_t^s \bar{\theta}_r d\bar{B}_r$, then $\langle \bar{M}, \bar{W}\rangle_s = \int_t^s \bar{\theta}_r d\langle \bar{B}, \bar{W}\rangle_r = 0$ and $(\bar{X}, \bar{Y}, \bar{Z}, \bar{M})$ satisfies (3.2.15). Assertion 2) is proved.

We shall prove assertion 3). We have already seen that $\bar{Y}_s = V(s, \bar{X}_s)$ for all $s \in [t, T]$ $\bar{\mathbb{P}}$-a.s. On the other hand, it is well known that, for the unique bounded viscosity solution V of the Hamilton-Jacobi-Bellman equation (2.1.2),

$$V(t,x) = \text{essinf}_{u \in \mathcal{U}_{\bar{\nu}\delta}(t)} J(t,x,u), \ \bar{\mathbb{P}}\text{-a.s.}$$

(see, e.g., Li and Wei [102]). This proves assertion 3. ■

3.3 Appendices

3.3.1 convexity hypothesis

Lemma 28. *1) Let the assumption* ***(H1)*** *as follow :*

$$\textbf{(H1)} \begin{cases} \text{For all } (x,y) \in \mathbb{R}^d \times \mathbb{R}, \text{ the set} \\ \{((\Sigma\Sigma^*)(x,y,w,\theta), \beta(x,y,w,u) | (u,w,\theta) \in \mathbb{U} \times \bar{B}_C(0) \times [0,K]\} \text{ , is convex.} \end{cases}$$

where, for all $(x,y,w,\theta,u) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{U}$, we have set

$$\Sigma(x,y,w,\theta) = \begin{pmatrix} \sigma(x,y) & 0 \\ w\sigma(x,y) & \theta \end{pmatrix} \quad \text{and} \quad \beta(x,y,w,u) = \begin{pmatrix} b(x,y,w\sigma(x,y),u) \\ -f(x,y,w\sigma(x,y),u) \end{pmatrix}.$$

If $\theta = 0$, then the assumption ***(H1)*** *is equivalent to* ***(H)****.*

2) Let us fix $(x,y) \in \mathbb{R}^d \times \mathbb{R}$. We will show that, under assumption ***(H1)****,*

$$\overline{co}\{((\Sigma\Sigma^*)(x,y,w,0), \beta(x,y,w,u)) | (u,w) \in U \times \bar{B}_C(0)\}$$

$$\subset \{((\Sigma\Sigma^*)(x,y,w,\theta),\beta(x,y,w,u)|(u,w,\theta)\in \mathbb{U}\times \bar{B}_C(0)\times[0,K]\}$$

where, for any set $\mathbb{E}$, $\overline{co}E$ stands for the convex hull of $\mathbb{E}$.

Proof. 1) The explicit calculus of $(\Sigma\Sigma^*)(x,y,w,\theta)$ gives

$$\Sigma\Sigma^*(x,y,w,\theta)=\begin{pmatrix}\sigma\sigma^*(x,y) & \sigma\sigma^*(x,y)w^*\\ w\sigma\sigma^*(x,y) & w\sigma\sigma^*(x,y)w^*+\theta^2\end{pmatrix}$$

If $\theta=0$, we can see that the assumption **(H1)** is equivalent to **(H)**.

2) We consider an arbitrarily chosen probability measure μ on the set $\mathbb{U}\times\bar{B}_C(0)$

Our goal is to find a triplet $(\bar{w},\bar{\theta},\bar{u})\in\mathbb{R}^d\times[0,K]\times\mathbb{U}$ satisfies :

$$\begin{aligned}\textstyle\int_{\mathbb{U}\times\bar{B}_C(0)}&((\Sigma\Sigma^*)(x,y,w,0),\beta(x,y,w)\mu(du,dw)\\ &=\Big((\Sigma\Sigma^*)(x,y,\bar{w},\bar{\theta}),\beta(x,y,\bar{w},\bar{\theta},\bar{u})\Big).\end{aligned}\tag{3.3.1}$$

Let $\Phi(u,w)=((\sigma\sigma^*)(x,y),w\sigma\sigma^*(x,y),b(x,y,w\sigma(x,y),u),f(x,y,w\sigma(x,y),u))$. The assumption **(H)** and by the continuity of Φ there exists $(\bar{u},\bar{w})$ in $\mathbb{U}\times\bar{B}_C(0)$ such that

$$\int_{\mathbb{U}\times\bar{B}_C(0)}\Phi(u,w)\mu(du,dw)=\Phi(\bar{u},\bar{w}).\tag{3.3.2}$$

The calculus of $(\Sigma\Sigma^*)(x,y,w,0)$ shows that, to obtain (3.3.1), it suffices to find $\bar{\theta}\in[0,K]$ such that

$$\alpha:=\int_{\mathbb{U}\times\bar{B}_C(0)}w\sigma\sigma^*(x,y)w^*\mu(du,dw)-\bar{w}\sigma\sigma^*(x,y)\bar{w}^*=\bar{\theta}^2.\tag{3.3.3}$$

Again by the calculus of $(\Sigma\Sigma^*)(x,y,w,0)$

we have $\sigma\sigma^*(x,y,\bar{u})=\int_{\mathbb{U}\times\bar{B}_C(0)}\sigma\sigma^*(x,y)\mu(du,dw)$, then we can rewrite α as follow

$$\alpha=\int_{\mathbb{U}\times\bar{B}_C(0)}w\sigma\sigma^*(x,y)w^*\mu(du,dw)-\bar{w}\int_{\mathbb{U}\times\bar{B}_C(0)}\sigma\sigma^*(x,y)\mu(du,dw)\bar{w}^*$$

Therefore,

$$\alpha = \int_{\mathbb{U}\times \bar{B}_C(0)} ((w-\bar{w})\sigma(x,y))((w-\bar{w})\sigma(x,y))^* \mu(du,dw) \geq 0.$$

Then α is non-negative, we choose $\bar{\theta} = \sqrt{\alpha}$ satisfying (3.3.3). Further let us rewrite (3.3.3) as

$$\int_{\mathbb{U}\times \bar{B}_C(0)} |w\sigma(x,y)|^2 \mu(du,dw) = |\bar{w}\sigma(x,y)|^2 + \bar{\theta}^2.$$

Since $|\sigma(x,y)|$ is bounded and the support of μ is included in $\mathbb{U}\times \bar{B}_C(0)$, it follow that $\bar{\theta}$ is bounded, then there exists $K>0$ such that $\bar{\theta}$ belongs to $[0,K]$ ■

3.3.2 Stability of the solutions Hamilton-Jacobi-Bellman equation

Lemma 29. *Assume that, the assumptions **(B)** are satisfied. Then there exists a non-negative constant $\tilde{C}$ only depending on the Lipshitz constants of the coefficients, verified the following estimation :*

$$|V^{\delta'}(t,x) - V^{\delta}(t,x)| \leq \tilde{C}|\delta' - \delta|. \tag{3.3.4}$$

Proof.

We start this proof by some notations :

$$X^{\delta,t,x,u^\delta}_{.} = X_{.}, \;\; Y^{\delta,t,x,u^\delta}_{.} = Y_{.}, \;\; Z^{\delta,t,x,u^\delta}_{.} = Z_{.}$$

And

$$X^{\delta',t,x,u^{\delta'}}_{.} = X'_{.}, \;\; Y^{\delta',t,x,u^{\delta'}}_{.} = Y'_{.}, \;\; Z^{\delta',t,x,u^{\delta'}}_{.} = Z'_{.}$$

Applying Itô's formula to the map $|Y^{'}_s - Y_s|^2$ then using standard arguments of FBSDEs, we obtain

$\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)+\mathbb{E}(\int_t^T|Z_r^{'}-Z_r|^2dr)$

$$\begin{aligned}
&\leq \mathbb{E}(|Y_T^{'}-Y_T|^2)\\
&+ 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta'}(X_r',Y_r',Z_r',u_r^{\delta'}),Y_r^{'}-Y_r > dr|)\\
&+ 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < Z_r-Z_r',Y_r^{'}-Y_r > dW_r^\delta|)
\end{aligned}$$

Now the estimate

$\mathbb{E}(\int_0^T|Z_r-Z_r'|^2|Y_r^{'}-Y_r|^2)^{\frac{1}{2}}$

$$\leq \mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2+\int_0^T|Z_r-Z_r'|^2dr)<\infty,$$

by Burkholder-Davis-Gundy we deduce

$$\forall s\in[0,T],\qquad \mathbb{E}(|\int_s^T < Z_r-Z_r',Y_r^{'}-Y_r > dW_r^\delta|)=0.$$

Then

$\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)+\mathbb{E}(\int_t^T|Z_r^{'}-Z_r|^2dr)$

$$\begin{aligned}
&\leq 2\mathbb{E}(\sup_{s\in[t,T]}\int_s^T|f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta'}(X_r',Y_r',Z_r',u_r^{\delta'})|\;|Y_r^{'}-Y_r|dr)\\
&+ \mathbb{E}(|Y_T^{'}-Y_T|^2)
\end{aligned}$$

The first part : by the Proposition 1.1.1 and the function σ is K-Lipshitz

$$\begin{aligned}
\mathbb{E}(|Y_T^{'}-Y_T|^2) &\leq \mathbb{E}(|\Phi_{\delta'}(X_T^{'})-\Phi_\delta(X_T)|^2)\\
&\leq 2\;\mathbb{E}(|\Phi_{\delta'}(X_T^{'})-\Phi_\delta(X_T')|^2)+2\;\mathbb{E}(|\Phi_\delta(X_T')-\Phi_\delta(X_T)|^2) \qquad (3.3.5)\\
&\leq 2\;K^2\;|\delta-\delta'|^2+\;2\;K^2\;\mathbb{E}(|X_T^{'}-X_T|^2).
\end{aligned}$$

The second part : by the Young inequality

$\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^{\delta'})||Y_r^{'}-Y_r|dr)$

$$
\begin{aligned}
\leq\ & \varepsilon_1\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^{\delta})|^2dr)\\
+\ & \frac{1}{\varepsilon_1}\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |Y_r^{'}-Y_r|^2dr)
\end{aligned}
$$

Using the fact the function f is K-Lipshitz, bounded by b_f and the proposition 1.1.1

$\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^{\delta'})||Y_r^{'}-Y_r|dr)$

$$
\begin{aligned}
\leq\ & 2\ \varepsilon_1\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_\delta(X_r,Y_r,Z_r,u_r^\delta)-f_{\delta}(X'_r,Y'_r,Z'_r,u_r^{\delta})|^2dr)\\
+\ & 2\ \varepsilon_1\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_\delta(X'_r,Y'_r,Z'_r,u_r^\delta)-f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^{\delta})|^2dr)\\
+\ & 2\ \varepsilon_1\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^\delta)-f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^{\delta'})|^2dr)\\
+\ & \frac{1}{\varepsilon_1}\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |Y_r^{'}-Y_r|^2dr)\\
\leq\ & 2\ K^2\ \varepsilon_1\left(\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |X_r^{'}-X_r|^2dr)+\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |Y_r^{'}-Y_r|^2dr)\right.\\
+\ & \left.\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |Z_r^{'}-Z_r|^2dr)\right)+2\ K^2\ \varepsilon_1(T-t)|\delta-\delta'|^2\\
+\ & \frac{1}{\varepsilon_1}\ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T |Y_r^{'}-Y_r|^2dr)
\end{aligned}
$$

Then

$$\begin{array}{lll}\mathbb{E} & (& \sup_{s\in[t,T]} \int_s^T |f_\delta(X_r,Y_r,Z_r,u_r^\delta) - f_{\delta'}(X'_r,Y'_r,Z'_r,u_r^\delta)||Y'_r - Y_r|dr) \\ & \leq & 2\ K^2\ \varepsilon_1(T-t)|\delta-\delta'|^2\ + (2\ K^2\ \varepsilon_1 + \frac{1}{\varepsilon_1})\ \mathbb{E}(\int_t^T |Y'_r - Y_r|^2 dr) \\ & + & 2\ b_f^2\ \varepsilon_1\ (T-t) + 2\ K^2\ \varepsilon_1\ \mathbb{E}(\int_t^T |X'_r - X_r|^2 dr)\ + 2\ K^2\ \varepsilon_1\ \mathbb{E}(\int_t^T |Z'_r - Z_r|^2 dr).\end{array} \tag{3.3.6}$$

Since

$$\mathbb{E}(\int_t^T |Y'_r - Y_r|^2 dr) \leq E(\int_t^T \sup_{s\in[t,r]} |Y'_s - Y_s|^2 ds),$$

$$\mathbb{E}(\int_t^T |X'_r - X_r|^2 dr) \leq (T-t)E(\sup_{s\in[t,T]} |X'_s - X_s|^2),$$

and

$$\mathbb{E}(|X'_T - X_T|^2) \leq \mathbb{E}(\sup_{s\in[t,T]} |X'_s - X_s|^2).$$

Then, by the inequalities (3.3.6) and (3.3.5) we have the following estimation :

$$\mathbb{E}(\sup_{s\in[t,T]} |Y'_s - Y_s|^2) + \mathbb{E}(\int_t^T |Z'_r - Z_r|^2 dr)$$

$$\begin{array}{ll} \leq & (4\ K^2\ \varepsilon_1(T-t) + 2\ K^2)|\delta-\delta'|^2\ + (2\ K^2\ \varepsilon_1 + \dfrac{1}{\varepsilon_1})\ \mathbb{E}(\displaystyle\int_t^T \sup_{s\in[t,r]} |Y'_s - Y_s|^2 ds) \\ + & (2\ K^2\ \varepsilon_1\ (T-t) + 2\ K^2)\mathbb{E}(\displaystyle\sup_{s\in[t,T]} |X'_s - X_s|^2)\ + 2\ K^2\ \varepsilon_1\ \mathbb{E}(\displaystyle\int_t^T |Z'_r - Z_r|^2 dr) \\ + & 4\ b_f^2\ \varepsilon_1\ (T-t).\end{array}$$

Then,

$\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)+\mathbb{E}(\int_t^T|Z_r^{'}-Z_r|^2dr)$

$$
\begin{aligned}
&\leq \mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)+C_z\mathbb{E}(\int_t^T|Z_r^{'}-Z_r|^2dr)\\
&\leq C_1\,|\delta-\delta'|^2\;+C_2\,\mathbb{E}(\int_t^T\sup_{s\in[t,r]}|Y_s^{'}-Y_s|^2ds)\\
&+ C_3\,\mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2)+\;C_f.
\end{aligned}
$$

With

$C_1=4\;K^2\;\varepsilon_1(T-t)+2\;K^2$

$C_2=2\;K^2\;\varepsilon_1+\frac{1}{\varepsilon_1}$

$C_3=2\;K^2\;\varepsilon_1\;(T-t)+2\;K^2$

$C_z=2\;K^2\;\varepsilon_1$

$C_f=4\;b_f^2\;\varepsilon_1\;(T-t).$

We choose ε_1 small as follow :

$$\varepsilon_1\leq\inf\{\frac{1}{8K^2}\;,\;\frac{2\;K^2\;\varepsilon_1(T-t)+\;K^2}{2\;b_f^2\;(T-t)}\;|\delta-\delta'|^2\}$$

This implied

$$
\begin{aligned}
C_z &\leq \frac{1}{4},\\
C_f &\leq C_1\,|\delta-\delta'|^2.
\end{aligned}
$$

Therefore, by modifying C_1 we obtain :

$$\begin{aligned}\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2) &\leq \mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)+\\ &\leq C_1\ |\delta-\delta'|^2\ +C_2\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]}|Y_s^{'}-Y_s|^2dr)\\ &+ C_3\ \mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2).\end{aligned}$$

We applied the Gronwall inequality we obtain

$$\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2)\leq e^{C_2(T-t)}\Bigg(C_1\ |\delta-\delta'|^2\ +C_3\ E(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2)\Bigg). \quad (3.3.7)$$

Again by Itô's formula applied to the function $|X_s^{'}-X_s|^2$ then

$$\begin{aligned}\mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2) &\leq \mathbb{E}(|X_t^{'}-X_t|^2)\\ &+ 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < b_\delta(X_r,Y_r,Z_r,u_r^\delta)-b_{\delta'}(X_r',Y_r',Z_r',u_r^{\delta'}),X_r^{'}-X_r > dr|)\\ &+ 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < \sigma_\delta(X_r,Y_r)-\sigma_{\delta'}(X_r',Y_r'),X_r^{'}-X_r > dW_r^\delta|)\\ &+ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T|\sigma_\delta(X_r,Y_r)-\sigma_{\delta'}(X_r',Y_r')|^2dr)\end{aligned}$$

Since X_t' and X_t have the same initial value x in the initial time t then $\mathbb{E}(|X_t^{'}-X_t|^2)=0$

therefore

$$\begin{aligned}\mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2) &\leq 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < b_\delta(X_r,Y_r,Z_r,u_r^\delta)-b_{\delta'}(X_r',Y_r',Z_r',u_r^{\delta'}),X_r^{'}-X_r > dr|)\\ &+ 2\mathbb{E}(\sup_{s\in[t,T]}|\int_s^T < \sigma_\delta(X_r,Y_r)-\sigma_{\delta'}(X_r',Y_r'),X_r^{'}-X_r > dW_r^\delta|)\\ &+ \mathbb{E}(\sup_{s\in[t,T]}\int_s^T|\sigma_\delta(X_r,Y_r)-\sigma_{\delta'}(X_r',Y_r')|^2dr).\end{aligned}$$

The first part : by our assumptions and the Young inequality we have

$\mathbb{E}(\sup_{s\in[t,T]} | \int_s^T < b_\delta(X_r, Y_r, Z_r, u_r^\delta) - b_{\delta'}(X'_r, Y'_r, Z'_r, u_r^{\delta'}), X'_r - X_r > dr|)$

$$\begin{aligned}
&\leq \varepsilon_2 \mathbb{E}(\sup_{s\in[t,T]} | \int_s^T |b_\delta(X_r, Y_r, Z_r, u_r^\delta) - b_{\delta'}(X'_r, Y'_r, Z'_r, u_r^{\delta'})|^2 dr) \\
&+ \frac{1}{\varepsilon_2} \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |X'_r - X_r|^2 dr) \\
&\leq 2\ \varepsilon_2 \mathbb{E}(\sup_{s\in[t,T]} | \int_s^T |b_\delta(X_r, Y_r, Z_r, u_r^\delta) - b_{\delta'}(X_r, Y_r, Z_r, u_r^\delta)|^2 dr) \\
&+ 2\ \varepsilon_2 \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |b_{\delta'}(X_r, Y_r, Z_r, u_r^\delta) - b_{\delta'}(X_r, Y_r, Z'_r, u_r^\delta)|^2 dr) \\
&+ 2\ \varepsilon_2 \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |b_{\delta'}(X_r, Y_r, Z'_r, u_r^\delta) - b_{\delta'}(X'_r, Y'_r, Z'_r, u_r^{\delta'})|^2 dr) \\
&+ \frac{1}{\varepsilon_2} \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |X'_r - X_r|^2 dr) \\
&\leq 2\ K^2 \varepsilon_2\ (T-t)|\delta - \delta'|^2 + (\frac{1}{\varepsilon_2} + 2\ K^2 \varepsilon_2) \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |X'_r - X_r|^2 dr) \\
&+ 2\ b_b^2\ \varepsilon_2\ (T-t) + 2\ K^2 \varepsilon_2 \mathbb{E}(\sup_{s\in[t,T]} \int_s^T |Y'_r - Y_r|^2 dr).
\end{aligned}$$

the second part : by the Burkholder-Davis-Gundy inequality and Young inequality with the constant C_2^*

$\mathbb{E}(\sup_{s\in[t,T]} | \int_s^T < \sigma_\delta(X_r,Y_r) - \sigma_{\delta'}(X'_r,Y'_r), X'_r - X_r > dW_r^\delta|)$

$$\begin{aligned}
&\leq C_2^*\mathbb{E}(\sup_{s\in[t,T]}\int_s^T |\sigma_\delta(X_r,Y_r) - \sigma_{\delta'}(X'_r,Y'_r)|^2|X'_r - X_r|^2dr)^{\frac{1}{2}} \\
&\leq \frac{C_2^*}{\varepsilon_3}\mathbb{E}(\int_t^T |X'_r - X_r|^2dr) + C_2^*\varepsilon_3\mathbb{E}(\int_t^T |\sigma_\delta(X_r,Y_r) - \sigma_{\delta'}(X'_r,Y'_r)|^2dr) \\
&\leq \frac{C_2^*}{\varepsilon_3}\mathbb{E}(\int_t^T |X'_r - X_r|^2dr) + 2C_2^*\varepsilon_3\mathbb{E}(\int_t^T |\sigma_\delta(X_r,Y_r) - \sigma_\delta(X'_r,Y'_r)|^2dr) \\
&+ 2C_2^*\varepsilon_3\mathbb{E}(\int_t^T |\sigma_\delta(X'_r,Y'_r) - \sigma_{\delta'}(X'_r,Y'_r)|^2dr) \\
&\leq (\frac{C_2^*}{\varepsilon_3} + 2\ K^2C_2^*\varepsilon_3)\mathbb{E}(\int_t^T |X'_r - X_r|^2dr) + 2\ K^2\ C_2^*\varepsilon_3\mathbb{E}(\int_t^T |Y_r - Y'_r|^2dr) \\
&+ 2C_2^*K^2\varepsilon_3(T-t)|\delta - \delta'|^2.
\end{aligned}$$

The Last part : by the σ assumptions we have

$\mathbb{E}(\sup_{s\in[t,T]} \int_s^T |\sigma_\delta(X_r,Y_r) - \sigma_{\delta'}(X'_r,Y'_r)|^2dr)$

$$\begin{aligned}
&\leq 2\ \mathbb{E}(\int_t^T |\sigma_\delta(X_r,Y_r) - \sigma_\delta(X'_r,Y'_r)|^2dr) \\
&+ 2\ \mathbb{E}(\int_t^T |\sigma_\delta(X'_r,Y'_r) - \sigma_{\delta'}(X'_r,Y'_r)|^2dr) \\
&\leq 2\ K^2\mathbb{E}(\int_t^T |X_r - X'_r|^2dr) + 2\ K^2\mathbb{E}(\int_t^T |Y_r - Y'_r|^2dr) \\
&+ 2\ K^2\ (T-t)|\delta - \delta'|^2.
\end{aligned}$$

Then

$$
\begin{aligned}
\mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2) &\leq (2\ K^2\varepsilon_2+2\ C_2^*\ \varepsilon_3+\ 2\ K^2)(T-t)|\delta-\delta^{'}|^2\\
&+ (\frac{1}{\varepsilon_2}+2\ K^2\varepsilon_2+2\ K^2C_2^*\varepsilon_3\frac{C_2^*}{\varepsilon_3}+2\ K^2)\mathbb{E}(\int_t^T|X_r^{'}-X_r|^2dr)\\
&+ (2\ K^2\varepsilon_2+2\ K^2C_2^*\varepsilon_3+2\ K^2)\mathbb{E}(\int_t^T|Y_r^{'}-Y_r|^2dr).
\end{aligned}
$$

We note

$C_4=(2\ K^2\varepsilon_2+2\ C_2^*\ \varepsilon_3+\ 2\ K^2)(T-t)$,

$C_5=\frac{1}{\varepsilon_2}+2\ K^2\varepsilon_2+2\ K^2C_2^*\varepsilon_3\frac{C_2^*}{\varepsilon_3}+2\ K^2$,

$C_6=2\ K^2\varepsilon_2+2\ K^2C_2^*\varepsilon_3+2\ K^2$.

$C_b=4\ b_b^2\ \varepsilon_2\ (T-t)$.

Then we can rewrite the last estimation as follow :

$$
\begin{aligned}
\mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2) &\leq \mathbb{E}(\sup_{s\in[t,T]}|X_s^{'}-X_s|^2)\\
&\leq C_4\ |\delta-\delta^{'}|^2+C_5\ \mathbb{E}(\int_t^T\sup_{s\in[t,r]}|X_s^{'}-X_s|^2ds)\\
&+ \quad C_b+C_6\ \mathbb{E}(\int_t^T\sup_{s\in[t,r]}|Y_s^{'}-Y_s|^2dr).
\end{aligned}
$$

We choose ε_2 small as follow :

$$
\varepsilon_2\leq\frac{2\ K^2\varepsilon_2+2\ C_2^*\ \varepsilon_3+\ 2\ K^2}{4\ b_b^2}\ |\delta-\delta^{'}|^2.
$$

We obtain $C_b \leq C_4\ |\delta - \delta^{'}|^2$. Therefore, by modifying C_4 :

$$\begin{aligned}
\mathbb{E}(\sup_{s\in[t,T]} |X_s^{'} - X_s|^2) &\leq \mathbb{E}(\sup_{s\in[t,T]} |X_s^{'} - X_s|^2) \\
&\leq C_4\ |\delta - \delta^{'}|^2 + C_5\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]} |X_s^{'} - X_s|^2 ds) \\
&+ \quad C_6\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]} |Y_s^{'} - Y_s|^2 dr).
\end{aligned}$$

We applied the Gronwall inequality we obtain

$$\mathbb{E}(\sup_{s\in[t,T]} |X_s^{'} - X_s|^2) \leq e^{C_5(T-t)}\Bigg(C_4\ |\delta - \delta^{'}|^2 +\ C_6\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]} |Y_s^{'} - Y_s|^2 ds)\Bigg). \tag{3.3.8}$$

We replace (3.3.8) in (3.3.7) we easily show the following estimation :

$$\begin{aligned}
\mathbb{E}(\sup_{s\in[t,T]} |Y_s^{'} - Y_s|^2) &\leq (C_1 e^{C_2(T-t)} + C_3 C_4 e^{(C_2+C_5)(T-t)})\ |\delta - \delta'|^2 \\
&+ \quad C_3 C_6 e^{(C_2+C_5)(T-t)}\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]} |Y_s^{'} - Y_s|^2 ds)
\end{aligned}$$

We note

$C_7 = C_1 e^{C_2(T-t)} + C_3 C_4\ e^{(C_2+C_5)(T-t)}$

$C_8 = C_3 C_6\ e^{(C_2+C_5)(T-t)}$

Then

$$\begin{aligned}
\mathbb{E}(\sup_{s\in[t,T]} |Y_s^{'} - Y_s|^2) &\leq C_7\ |\delta - \delta'|^2 \\
&+ \quad C_8\ \mathbb{E}(\int_t^T \sup_{s\in[t,r]} |Y_s^{'} - Y_s|^2 dr)\ .
\end{aligned}$$

Finally from the Gronwall inequality we have

$$\mathbb{E}(\sup_{s\in[t,T]}|Y_s^{'}-Y_s|^2) \;\leq\; C_7\; e^{C_8(T-t)}\;|\delta-\delta'|^2$$

■

3.3.3 proprieties of the solutions Hamilton-Jacobi-Bellman equation

Lemma 30. *Assume that f_δ, b_δ, Φ_δ and σ_δ are bounded $\mathbb{C}^\infty$ functions with bounded derivatives of every order and satisfy Assumption **(B)**.*

Then, setting the following system of PDEs :

$$\begin{cases} \dfrac{\partial}{\partial t}V^\delta(t,x)+\bar{H}^\delta\left(x,(V^\delta,\nabla_x V^\delta,\nabla_{xx}V^\delta)(t,x),v^\delta(t,x)\right)=0,\;\;(t,x)\in[0,T]\times\mathbb{R}^d,\\ V^\delta(T,x)=\Phi_\delta(x),\;\;x\in\mathbb{R}^d, \end{cases} \tag{3.3.9}$$

With the Hamiltonian

$$\bar{H}^\delta(x,y,p,A,v)=\frac{1}{2}\left(tr\left((\sigma_\delta\sigma_\delta^*)(x,y)\right)A\right)+b_\delta\left(x,y,p\sigma_\delta(x,y),v\right)p$$

$$+f_\delta\left(x,y,p\sigma_\delta(x,y),v\right),$$

for $(x,y,p,A,v)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d\times\mathbb{U}$.

admits a unique bounded solution $V^\delta\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. It satisfies

$$\nabla_x V^\delta \text{ and } \nabla_{xx}^2 V^\delta \text{ are bounded on } \mathbb{R}^d. \tag{3.3.10}$$

In addition, there exists a constant $\bar{C}$, only depending on λ and T, and two constants $\bar{\Gamma}$ and $\bar{\kappa}$, only depending on K, λ and T, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |V^\delta(t,x)| \le \bar{C}, \tag{3.3.11}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla_x V^\delta(t,x)| \le \bar{\Gamma}, \tag{3.3.12}$$

$$\forall (t,t') \in [0,T]^2, \quad \forall x \in \mathbb{R}^d, \qquad |V^\delta(t',x) - V^\delta(t,x)| \le \bar{\kappa}|t'-t|^{1/2}. \tag{3.3.13}$$

Proof. Since the Hamiltonian is smooth and $(\sigma_\delta\sigma_\delta^*)(x,y)$ is strictly elliptic, we can conclude that the unique bounded continuous viscosity solution V^δ of the above equation is smooth with regularity $C_b^{1,2}([0,T]\times\mathbb{R}^d)$. For this we can apply the regularity results by Krylov [94] (see the Theorems 6.4.3 and 6.4.4 in [94]). Then V^δ satisfies (3.3.10).

Let us show by means of probabilistic tools that (3.3.11) holds. To this end, let us define for every $(t,x) \in [0,T]\times\mathbb{R}^d$:

$$B(t,x) = b_\delta(t,x,V^\delta(t,x),\nabla_x V^\delta(t,x)\sigma(t,x,V^\delta(t,x))),$$

$$\Xi(t,x) = \sigma_\delta(t,x,V^\delta(t,x)).$$

For every $t \in [0,T]$, the SDE

$$X_s^{t,x,\delta} = x + \int_t^s B(r,X_r^{t,x,\delta})dr + \int_t^s \Xi(r,X_r^{t,x,\delta})dW_r^\delta,$$

admits a weak solution. We then define $\forall t \le s \le T$,

$$Y_s^{t,x,\delta} = V^\delta(s,X^{t,x,\delta}), \qquad Z^{t,x,\delta} = \nabla_x V^\delta(s,X_s^{t,x,\delta})\sigma_\delta(s,X_s^{t,x,\delta},Y_s^{t,x,\delta}).$$

Therefore, Itô's formula to the function $(s,x) \to V^\delta(s,x)$ with x is the processes $(X_s^{t,x,\delta})_{t\leq s\leq T}$ and system (3.3.9) show that $\forall t \leq s \leq T$

$$Y_s^{t,x,\delta} = \Phi_\delta(X_T^{t,x,\delta}) + \int_s^T f_\delta(r, X_r^{t,x,\delta}, Y_r^{t,x,\delta}, Z_r^{t,x,\delta}, v^\delta)dr - \int_s^T Z_r^{t,x,\delta} dW_r^\delta$$

Hence, the process $(X_s^{t,x,\delta}, Y_s^{t,x,\delta}, Z_s^{t,x,\delta})$ is a solution of the FBSDE associated to the coefficients b_δ, f_δ, Φ_δ, σ_δ and to the initial condition (t,x). Moreover, applying the Itô formula to the function $(s,x) \to |x|^2$ with x is process $(Y_s^{t,x,\delta})_{t\leq s\leq T}$, applied the Young inequality and by the boundedness of the data f_δ and Φ_δ , we deduce that for every $t \leq s \leq T$ and for every $x \in \mathbb{R}^d$:

$$\begin{aligned}
|Y_s^{t,x,\delta}|^2 &+ \int_s^T |Z_r^{t,x,\delta}|^2 dr \\
&\leq |Y_T^{t,x,\delta}|^2 + \int_s^T |f_\delta(X_r^{t,x,\delta}, Y_r^{t,x,\delta}, Z_r^{t,x,\delta}, v^\delta(t, X_r^{t,x,\delta}))|\ |Y_r^{t,x,\delta}| dr \\
&- 2\int_s^T \langle Y_r^{t,x,\delta}, Z_r^{t,x,\delta} dW_r^\delta \rangle, \\
&\leq |\Phi_\delta(X_T^{t,x,\delta})|^2 + \frac{1}{\varepsilon}\int_s^T |f_\delta(X_r^{t,x,\delta}, Y_r^{t,x,\delta}, Z_r^{t,x,\delta}, v^\delta(t, X_r^{t,x,\delta}))|^2 dr + \int_s^T \varepsilon\ |Y_r^{t,x,\delta}|^2 dr \\
&- 2\int_s^T \langle Y_r^{t,x,\delta}, Z_r^{t,x,\delta} dW_r^\delta \rangle, \\
&\leq C_\Phi^2 + \frac{C_f^2}{\varepsilon}\ (T-s) + \ \varepsilon \int_s^T \ |Y_r^{t,x,\delta}|^2 dr \\
&- 2\int_s^T \langle Y_r^{t,x,\delta}, Z_r^{t,x,\delta} dW_r^\delta \rangle.
\end{aligned}$$

Since

$$E(\int_0^T |Z_r^{t,x,\delta}|^2 |Y_r^{t,x,\delta}|^2)^{\frac{1}{2}} \leq E(\sup_{s\in[t,T]} |Y_s^{t,x,\delta}|^2 + \int_0^T |Z_r^{t,x,\delta}|^2 dr) < \infty,$$

by Burkholder-Davis-Gundy we deduce

$$\forall s \in [0,T], \quad E(|\int_s^T \langle Y_r^{t,x,\delta}, Z_r^{t,x,\delta} dW_r^{\delta}\rangle) = 0$$

We choose $\varepsilon = \frac{1}{4(T-t)}$, then

$$E(\sup_{s\in[t,T]} |Y_s^{t,x,\delta}|^2) \leq \frac{4\ C_\Phi^2}{3} + \frac{16\ C_f^2\ (T-t)^2}{3}$$

we deduce that there exists a constant $\bar{C}$, only depending on C_f, C_Φ and T, Such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \qquad |V^\delta(t,x)| \leq \bar{C}. \tag{3.3.14}$$

Then using Theorem 6.14 chapter VII of Ladyzenskaya et al. (1968), we can estimate the supremum norm of $|\nabla_x V^\delta(t,x)|^2$ on every compact of $[0,T]\times\mathbb{R}^d$. Inded for every $n \in \mathbb{N}^*$, we can apply this theorem to the cylinders $[0,T]\times\{x \in \mathbb{R}^d, |x| \leq n\}$ and $[0,T]\times\{x \in \mathbb{R}^d, |x| \leq n+1\}$. In particular, the quantity $\sup_{\{t\in[0,T],|x|\leq n\}} |\nabla_x V^\delta(t,x)|^2$ is estimated in terms of $\bar{C}$, k, λ, Λ and d, the distance between $\{x \in \mathbb{R}^d, |x| \leq n\}$ and $\{x \in \mathbb{R}^d, |x| \leq n+1\}$ being equal to 1. In particular, there exists a constant $\bar{\Gamma}$, only depending on K, λ and T such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \qquad |\nabla_x V^\delta(t,x)| \leq \bar{\Gamma}. \tag{3.3.15}$$

Lastly, let us prove (3.3.13). Let $0 \leq s \leq r \leq T$. Then using (3.3.14) and (3.3.15), we show that there exist $\bar{\kappa}$ only depending on K, λ and T, such that

$$E|Y_r^{\delta,s,x} - Y_s^{\delta,s,x}|^2 \leq \bar{\kappa}(r-s), \qquad E|X_r^{\delta,s,x} - X_s^{\delta,s,x}|^2 \leq \bar{\kappa}(r-s).$$

Hence by modifying $\bar{\kappa}$ if necessary, and using $Y_r^{\delta,s,x} = V^\delta(r, X_r^{\delta,s,x})$

$$\begin{aligned} E\ |V^\delta(s,x) - V^\delta(r,x)|^2 &\leq 2[E|V^\delta(s,x) - Y_r^{\delta,s,x}|^2 + E|Y_r^{\delta,s,x} - V^\delta(r,x)|^2], \\ &\leq 2\ \bar{\kappa}(r-s) + 2\ E|X_r^{\delta,s,x} - x|^2, \\ &\leq 2\ \bar{\kappa}(r-s) + 2\ E|X_r^{\delta,s,x} - X_s^{\delta,s,x}|^2, \\ &\leq 4\ \bar{\kappa}(r-s). \end{aligned}$$

This shows (3.3.13). ■

3.3.4 Convergence.

Lemma 31. *For all $n \in \mathbb{N}$, There exists a constant $L > 0$ such that*

$$\begin{aligned} &\mathbb{E}[\sup_{s\in[t,T]} |X_s^{\delta_n} - X_s^n|^2] \leq L\delta_n^2, \\ &\mathbb{E}[\sup_{s\in[t,T]} |Y_s^{\delta_n} - Y_s^n|^2] \leq L\delta_n^2. \end{aligned} \tag{3.3.16}$$

Proof. The sequence of processes (X_s^n, Y_s^n) satisfied the following controlled forward system :

$$\begin{cases} dX_s^n = b(X_s^n, Y_s^n, w_s^n\sigma(X_s^n, Y_s^n)ds + \sigma(X_s^n, Y_s^n)dW_s^{\delta_n}, \\ dY_s^n = -f(X_s^n, Y_s^n, w_s^n\sigma(X_s^n, Y_s^n), u_s^{\delta_n})ds + w_s^n\sigma(X_s^n, Y_s^n)dW_s^{\delta_n}. \\ \ X_t^n = x, \ \ Y_t^n = V^{\delta_n}(t,x), \ \ s \in [t,T]. \end{cases} \tag{3.3.17}$$

with $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$, and the subsequence $(X_s^{\delta_n}, Y_s^{\delta_n})$ satisfied the following controlled forward system :

$$\begin{cases} dX_s^{\delta_n} = b_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, w_s^n\sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n})ds + \sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n})dW_s^{\delta_n}, \\ dY_s^{\delta_n} = -f_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}, w_s^n\sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n}), u_s^{\delta_n})ds + w_s^n\sigma_{\delta_n}(X_s^{\delta_n}, Y_s^{\delta_n})dW_s^{\delta_n}. \\ \ X_t^{\delta_n} = x, \ \ Y_t^{\delta_n} = V^{\delta_n}(t,x), \ s \in [t,T]. \end{cases} \tag{3.3.18}$$

with $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$.

We applied the Itô's formula to the function $(t,x) \to |x|^2$, we obtain

$$
\begin{aligned}
|X_s^n - X_s^{\delta_n}|^2 &= 2\int_t^s \langle b\ (X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \\
&- b_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n} \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n}), X_r^n - X_r^{\delta_n}\rangle dr \\
&+ 2\int_t^s \langle \sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), X_r^n - X_r^{\delta_n}\rangle dW_r^{\delta_n} \\
&+ 2\int_t^s |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2\ dr.
\end{aligned}
$$

Then, by the Burkholder-Davis-Gundy inequality

$$
\begin{aligned}
\mathbb{E}(\sup_{t\le s\le T} |X_s^n - X_s^{\delta_n}|^2) &\le 2\mathbb{E}(\sup_{t\le s\le T} \int_t^s |b\ (X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \\
&- b_{\delta_n}(X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n))|\ |X_r^n - X_r^{\delta_n}|dr) \\
&+ 2\mathbb{E}(\sup_{t\le s\le T} \int_t^s |b_{\delta_n}(X_r^n, Y_r^n, w_r^n \sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \\
&- b_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n} \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|\ |X_r^n - X_r^{\delta_n}|dr) \\
&+ 2\ C_2^* \mathbb{E}(\sup_{t\le s\le T} \int_t^s |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 \\
&\quad |X_r^n - X_r^{\delta_n}|^2 dr)^{\frac{1}{2}} \\
&+ 2\mathbb{E}(\sup_{t\le s\le T} \int_t^s |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2\ dr).
\end{aligned}
$$

We applied the Young inequalities then there exists ε_1, ε_2 and ε_3,

$$\begin{aligned}
\mathbb{E}(\sup_{t\leq s\leq T}|X_s^n - X_s^{\delta_n}|^2) &\leq 2\varepsilon_1\ \mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |b\ (X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 dr)\\
&+ 2\varepsilon_2\ \mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- b_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2 dr)\\
&+ \frac{2}{\varepsilon_1}\ \mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |X_r^n - X_r^{\delta_n}|^2 dr) + \frac{2}{\varepsilon_2}\ \mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |X_r^n - X_r^{\delta_n}|^2 dr\\
&+ 2\ \varepsilon_3\ C_2^*\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 dr)\\
&+ \frac{2C_2^*}{\varepsilon_3}\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |X_r^n - X_r^{\delta_n}|^2 dr)\\
&+ 2\ \mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2\ dr).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}(\sup_{t\leq s\leq T}|X_s^n - X_s^{\delta_n}|^2) &\leq (\tfrac{2}{\varepsilon_1} + \tfrac{2}{\varepsilon_2} + \tfrac{2C_2^*}{\varepsilon_3})\mathbb{E}(\sup_{t\leq s\leq T}\textstyle\int_t^s |X_r^n - X_r^{\delta_n}|^2 dr)\\
&+ 2\varepsilon_1\ \mathbb{E}(\sup_{t\leq s\leq T}\textstyle\int_t^s |b\ (X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 dr)\\
&+ 2\varepsilon_2\ \mathbb{E}(\sup_{t\leq s\leq T}\textstyle\int_t^s |b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \qquad (3.3.19)\\
&- b_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2 dr)\\
&+ (2\ \varepsilon_3\ C_2^* + 2)\mathbb{E}(\sup_{t\leq s\leq T}\textstyle\int_t^s |\sigma\ (X_r^n, Y_r^n)\\
&- \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 dr)
\end{aligned}$$

Since the σ_{δ_n} is K-Lipshitz and by the Proposition 1.1.1.

$$|\sigma(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2$$

$$\begin{aligned}
&\leq 2\ |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^n, Y_r^n)|^2 + 2|\sigma_{\delta_n}(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^n, Y_r^n)|^2\\
&\leq 2C^2\delta_n^2 + 2K^2|X_r^n - X_r^{\delta_n}|^2 + 2K^2|Y_r^n - Y_r^{\delta_n}|^2,
\end{aligned}$$

the b_{δ_n} is K-Lipshitz and the w_r^n is bounded then

$$|b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) - b_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2$$

$$\begin{aligned}
&\leq K^2(|X_r^n - X_r^{\delta_n}|^2 + |Y_r^n - Y_r^{\delta_n}|^2 + |w_r^n|^2|\sigma(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2) \\
&\leq K^2\ |X_r^n - X_r^{\delta_n}|^2 + \ K^2\ |Y_r^n - Y_r^{\delta_n}|^2 + 2K^2C^4\delta_n^2 + \ 2K^4C^2\ |X_r^n - X_r^{\delta_n}|^2 \\
&+ \ 2K^4C^2|Y_r^n - Y_r^{\delta_n}|^2 \\
&\leq 2K^2C^4\delta_n^2 + (K^2 + 2K^4C^2)\ |X_r^n - X_r^{\delta_n}|^2 + \ (K^2 + 2K^4C^2)\ |Y_r^n - Y_r^{\delta_n}|^2,
\end{aligned}$$

and by the Proposition 1.1.1

$$|b\ (X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) - b_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 \leq K^2\delta_n^2.$$

By the three last estimation the inequality(3.3.19) be

$$\begin{aligned}
\mathbb{E}(\sup_{t\leq s\leq T}|X_s^n - X_s^{\delta_n}|^2) &\leq (2\ K^2\varepsilon_1\ + 4\varepsilon_2K^2C^4 + 2C^2(2\ \varepsilon_3\ C_2^* + 2))(T-t)\ \ \delta_n^2 \\
&+ \mathbb{E}(\textstyle\int_t^T|Y_r^n - Y_r^{\delta_n}|^2dr)\times \\
&\quad (2\varepsilon_2(K^2 + 2K^4C^2) + 2K^2(2\ \varepsilon_3\ C_2^* + 2)) \\
&+ \mathbb{E}(\textstyle\int_t^T|X_r^n - X_r^{\delta_n}|^2dr)\times \\
&\quad (\tfrac{2}{\varepsilon_1} + \tfrac{2}{\varepsilon_2} + \tfrac{2C_2^*}{\varepsilon_3} + 2\varepsilon_2(K^2 + 2K^4C^2) + 2K^2(2\ \varepsilon_3\ C_2^* + 2)).
\end{aligned} \tag{3.3.20}$$

Then

$$\begin{aligned}\mathbb{E}(\sup_{t\le s\le T}|X_s^n-X_s^{\delta_n}|^2) &\le (2\ K^2\varepsilon_1\ +4\varepsilon_2K^2C^4+2C^2(2\ \varepsilon_3\ C_2^*+2))(T-t)\ \ \delta_n^2\\ &+ \mathbb{E}(\int_t^T|Y_r^n-Y_r^{\delta_n}|^2dr)\times\\ &(2\varepsilon_2(K^2+2K^4C^2)+2K^2(2\ \varepsilon_3\ C_2^*+2))\\ &+ \mathbb{E}(\int_t^T\sup_{t\le s\le r}|X_s^n-X_s^{\delta_n}|^2ds)\times\\ &(\frac{2}{\varepsilon_1}+\frac{2}{\varepsilon_2}+\frac{2C_2^*}{\varepsilon_3}+2\varepsilon_2(K^2+2K^4C^2)+2K^2(2\ \varepsilon_3\ C_2^*+2)).\end{aligned}$$

We can rewrite the last estimation follow

$$\mathbb{E}(\sup_{t\le s\le T}|X_s^n-X_s^{\delta_n}|^2)\le C_1\ \ \delta_n^2+C_2\times\mathbb{E}(\int_t^T|Y_r^n-Y_r^{\delta_n}|^2dr)+C_3\times\mathbb{E}(\int_t^T\sup_{t\le s\le r}|X_s^n-X_s^{\delta_n}|^2ds),$$

with

$C_1=(2\ K^2\varepsilon_1\ +4\varepsilon_2K^2C^4+2C^2(2\ \varepsilon_3\ C_2^*+2))(T-t)$

$C_2=(2\varepsilon_2(K^2+2K^4C^2)+2K^2(2\ \varepsilon_3\ C_2^*+2))$

$C_3=(\frac{2}{\varepsilon_1}+\frac{2}{\varepsilon_2}+\frac{2C_2^*}{\varepsilon_3}+2\varepsilon_2(K^2+2K^4C^2)+2K^2(2\ \varepsilon_3\ C_2^*+2))$

Applied the Gronwall inequality we have

$$\mathbb{E}(\sup_{t\le s\le T}|X_s^n-X_s^{\delta_n}|^2)\le e^{C_3(T-t)}(\ C_1\ \ \delta_n^2+C_2\times\mathbb{E}(\int_t^T|Y_r^n-Y_r^{\delta_n}|^2dr)\ ).$$

Since

$\mathbb{E}(\int_t^T|Y_r^n-Y_r^{\delta_n}|^2dr)\le(T-t)\ \mathbb{E}(\sup_{t\le s\le T}|Y_s^n-Y_s^{\delta_n}|^2ds)$, Therefore,

$$\mathbb{E}(\sup_{t\leq s\leq T}|X_s^n - X_s^{\delta_n}|^2) \leq e^{C_3(T-t)}(\ C_1\ \ \delta_n^2 + C_2 \times (T-t)\ \mathbb{E}(\sup_{t\leq s\leq T}|Y_s^n - Y_s^{\delta_n}|^2 ds)\ ). \quad (3.3.21)$$

Again by the Ito's formula and Burkholder-Davis-Gundy we have,

$$\begin{aligned}
\mathbb{E}(\sup_{t\leq s\leq T}|Y_s^n - Y_s^{\delta_n}|^2) &\leq 2\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |f\ (X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \\
&- f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|\ |Y_r^n - Y_r^{\delta_n}|dr) \\
&+ 2\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \\
&- f_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|\ |Y_r^n - Y_r^{\delta_n}|dr) \\
&+ 2\ C_2^*\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |w_r^n|^2|\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 \\
&\quad |Y_r^n - Y_r^{\delta_n}|^2 dr)^{\frac{1}{2}} \\
&+ 2\mathbb{E}(\sup_{t\leq s\leq T}\int_t^s |w_r^n|^2|\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2\ dr).
\end{aligned}$$

By the fact w_r^n is bounded and applied the Young inequalities then there exists ε_4, ε_5 and

ε_6,

$$\begin{aligned}
\mathbb{E}(\sup_{t\le s\le T}|Y_s^n - Y_s^{\delta_n}|^2) &\le 2\varepsilon_4\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |f(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 dr)\\
&+ \frac{2}{\varepsilon_4}\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |Y_r^n - Y_r^{\delta_n}| dr)\\
&+ 2\varepsilon_5\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- f_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2 dr)\\
&+ \frac{2}{\varepsilon_5}\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |Y_r^n - Y_r^{\delta_n}|^2 dr)\\
&+ (2\ C^2C_2^*\varepsilon_6 + 2C^2)\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |\sigma\ (X_r^n, Y_r^n)\\
&- \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 dr)\\
&+ \frac{2\ C^2C_2^*}{\varepsilon_6}\ \mathbb{E}(\sup_{t\le s\le T}\int_t^s |Y_r^n - Y_r^{\delta_n}|^2 dr).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}(\sup_{t\le s\le T}|Y_s^n - Y_s^{\delta_n}|^2) &\le (\tfrac{2}{\varepsilon_4} + \tfrac{2}{\varepsilon_5} + \tfrac{2\ C^2C_2^*}{\varepsilon_6})\ \mathbb{E}(\sup_{t\le s\le T}\textstyle\int_t^s |Y_r^n - Y_r^{\delta_n}|^2 dr)\\
&+ 2\varepsilon_4\ \mathbb{E}(\sup_{t\le s\le T}\textstyle\int_t^s |f(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})\\
&- f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 dr)\\
&+ 2\varepsilon_5\ \mathbb{E}(\sup_{t\le s\le T}\textstyle\int_t^s |f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) \qquad (3.3.22)\\
&- f_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2 dr)\\
&+ (2\ C^2C_2^*\varepsilon_6 + 2C^2)\ \mathbb{E}(\sup_{t\le s\le T}\textstyle\int_t^s |\sigma\ (X_r^n, Y_r^n)\\
&- \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2 dr).
\end{aligned}$$

Since the σ_{δ_n} is K-Lipshitz and by the Proposition 1.1.1.

$$|\sigma(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2$$

$$\begin{aligned} &\leq \ 2\ |\sigma\ (X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^n, Y_r^n)|^2 + 2|\sigma_{\delta_n}(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^n, Y_r^n)|^2 \\ &\leq \ 2C^2\delta_n^2 + 2K^2|X_r^n - X_r^{\delta_n}|^2 + 2K^2|Y_r^n - Y_r^{\delta_n}|^2, \end{aligned}$$

the f_{δ_n} is K-Lipshitz and the w_r^n is bounded then

$$|f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) - f_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}, w_r^{\delta_n}\sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n}), u_r^{\delta_n})|^2$$

$$\begin{aligned} &\leq \ K^2(|X_r^n - X_r^{\delta_n}|^2 + |Y_r^n - Y_r^{\delta_n}|^2 + |w_r^n|^2|\sigma(X_r^n, Y_r^n) - \sigma_{\delta_n}(X_r^{\delta_n}, Y_r^{\delta_n})|^2) \\ &\leq \ K^2\ |X_r^n - X_r^{\delta_n}|^2 + \ K^2\ |Y_r^n - Y_r^{\delta_n}|^2 + 2K^2C^4\delta_n^2 + \ 2K^4C^2\ |X_r^n - X_r^{\delta_n}|^2 \\ &+ \ \ 2K^4C^2|Y_r^n - Y_r^{\delta_n}|^2 \\ &\leq \ 2K^2C^4\delta_n^2 + (K^2 + 2K^4C^2)\ |X_r^n - X_r^{\delta_n}|^2 + \ (K^2 + 2K^4C^2)\ |Y_r^n - Y_r^{\delta_n}|^2, \end{aligned}$$

and by the Proposition 1.1.1

$$|f\ (X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n}) - f_{\delta_n}(X_r^n, Y_r^n, w_r^n\sigma(X_r^n, Y_r^n), u_r^{\delta_n})|^2 \leq K^2\delta_n^2.$$

By the three last estimation the inequality(3.3.22) be

$$
\begin{aligned}
\mathbb{E}(\sup_{t\le s\le T}|Y_s^n - Y_s^{\delta_n}|^2) &\le (2K^2\varepsilon_4 + 4K^2C^4\varepsilon_5 + 4\ C^4C_2^*\varepsilon_6 + 4C^4)(T-t)\ \delta_n^2 \\
&+ \mathbb{E}(\int_t^T |Y_r^n - Y_r^{\delta_n}|^2 dr) \times \\
&(\frac{2}{\varepsilon_4} + \frac{2}{\varepsilon_5} + \frac{2\ C^2C_2^*}{\varepsilon_6} + 2(K^2 + 2K^4C^2)\varepsilon_5 + 2K^2(2C^2C_2^*\varepsilon_6 + 2C^2)) \\
&+ \mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr) \times \\
&(2K^2(2C^2C_2^*\varepsilon_6 + 2C^2) + 2\varepsilon_5(K^2 + 4K^2C^2)
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}(\sup_{t\le s\le T}|Y_s^n - Y_s^{\delta_n}|^2) &\le (2K^2\varepsilon_4 + 4K^2C^4\varepsilon_5 + 4\ C^4C_2^*\varepsilon_6 + 4C^4)(T-t)\ \delta_n^2 \\
&+ \mathbb{E}(\int_t^T \sup_{t\le s\le r}|Y_s^n - Y_s^{\delta_n}|^2 ds) \times \\
&(\frac{2}{\varepsilon_4} + \frac{2}{\varepsilon_5} + \frac{2\ C^2C_2^*}{\varepsilon_6} + 2(K^2 + 2K^4C^2)\varepsilon_5 + 2K^2(2C^2C_2^*\varepsilon_6 + 2C^2)) \\
&+ \mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr) \times \\
&(2K^2(2C^2C_2^*\varepsilon_6 + 2C^2) + 2\varepsilon_5(K^2 + 4K^2C^2).
\end{aligned}
$$

We can rewrite the last estimate as follow

$$
\mathbb{E}(\sup_{t\le s\le T}|Y_s^n - Y_s^{\delta_n}|^2) \le C_4 \times \delta_n^2 + C_5 \times \mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr) + C_6 \times \mathbb{E}(\int_t^T \sup_{t\le s\le r}|Y_s^n - Y_s^{\delta_n}|^2 ds),
$$

with

$C_4 = (2K^2\varepsilon_4 + 4K^2C^4\varepsilon_5 + 4\ C^4C_2^*\varepsilon_6 + 4C^4)(T-t),$

$C_5 = (2K^2(2C^2C_2^*\varepsilon_6 + 2C^2) + 2\varepsilon_5(K^2 + 4K^2C^2),$

$C_6 = (\frac{2}{\varepsilon_4} + \frac{2}{\varepsilon_5} + \frac{2\ C^2 C_2^*}{\varepsilon_6} + 2(K^2 + 2K^4C^2)\varepsilon_5 + 2K^2(2C^2C_2^*\varepsilon_6 + 2C^2))$.

Applied the Gronwall inequality we have

$$\mathbb{E}(\sup_{t\leq s\leq T} |Y_s^n - Y_s^{\delta_n}|^2) \leq\ e^{C_6(T-t)}(\ C_4\ \times \delta_n^2 +\ C_5\ \times \mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr)\ ). \qquad (3.3.23)$$

Since

$\mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr) \leq\ \ \mathbb{E}(\int_t^T \sup_{t\leq s\leq r} |X_s^n - X_s^{\delta_n}|^2 ds)$, therefore,

$$\mathbb{E}(\sup_{t\leq s\leq T} |Y_s^n - Y_s^{\delta_n}|^2) \leq\ e^{C_6(T-t)}(\ C_4\ \times \delta_n^2 +\ C_5\ \times \mathbb{E}(\int_t^T \sup_{t\leq s\leq r} |X_s^n - X_s^{\delta_n}|^2 ds)\ ). \qquad (3.3.24)$$

We replace in (3.3.21), we obtain

$$\begin{aligned}\mathbb{E}(\sup_{t\leq s\leq T} |X_s^n - X_s^{\delta_n}|^2) \ &\leq\ e^{C_3(T-t)}(\ C_1\ \ \delta_n^2 + C_2 \times (T-t)\ \ e^{C_6(T-t)} \\ &\times(\ C_4 \times \delta_n^2 +\ C_5\ \times \mathbb{E}(\int_t^T \sup_{t\leq s\leq r} |X_s^n - X_s^{\delta_n}|^2 ds)\ )\ )\end{aligned}$$

Then

$$\mathbb{E}(\sup_{t\leq s\leq T} |X_s^n - X_s^{\delta_n}|^2) \leq C_7\ \ \delta_n^2 +\ C_8\ \times \mathbb{E}(\int_t^T \sup_{t\leq s\leq r} |X_s^n - X_s^{\delta_n}|^2 ds),$$

with

$C_7 = C_1 e^{C_3(T-t)} + C_2C_4(T-t) \times e^{(C_3+C_6)(T-t)}$.

$C_8 = C_2C_5(T-t) \times e^{(C_3+C_6)(T-t)}$

Finally, again by the Gronwall inequality we deduce

$$\mathbb{E}(\sup_{t\leq s\leq T} |X_s^n - X_s^{\delta_n}|^2) \leq C_7\ e^{C_8(T-t)}\ \delta_n^2. \qquad (3.3.25)$$

Now, recall to the Y estimate (3.3.23), by the fact

$$\mathbb{E}(\int_t^T |X_r^n - X_r^{\delta_n}|^2 dr) \leq\ \ (T-t)\ \mathbb{E}(\sup_{t\leq s\leq T} |X_s^n - X_s^{\delta_n}|^2 ds),$$

and the estimation (3.3.25) we have :

$$\mathbb{E}(\sup_{t\leq s\leq T}|Y_s^n - Y_s^{\delta_n}|^2) \leq (\ C_4 e^{C_6(T-t)} +\ C_5\ C_7\ (T-t)\ e^{(C_6+C_8)(T-t)}\ )\ \delta_n^2. \tag{3.3.26}$$

■

Chapitre 4

One dimensional BSDEs with logarithmic growth : Application to PDEs

In this chapter we study the existence and uniqueness of solutions to one dimensional BSDEs with generator allowing a logarithmic growth $(|y||\ln|y|| + |z|\sqrt{|\ln|z||})$ in the state variables y and z. This is done with an L^p- integrable terminal value, for some $p > 2$. As byproduct, we obtain the existence of viscosity solutions to PDEs with logarithmic nonlinearities.

4.1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a probability space on which is defined a standard d-dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the P-null sets of $\mathcal{F}$. Let $f(t, \omega, y, z)$ be a real valued $\mathcal{F}_t$–progressively measurable process defined on $[0,\ T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Let ξ be an $\mathcal{F}_T$–measurable $\mathbb{R}$–valued random variable. The backward stochastic differential equations (BSDEs) under consideration is :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad t \in [0, T] \tag{4.1.1}$$

The previous equation will be denoted by $eq(\xi, f)$. The data ξ and f are respectively called the terminal condition and the coefficient or the generator of $eq(\xi, f)$.

Due to the applications of BSDEs, many efforts have been made to relax the assumptions on the driver f and/or on the terminal value. Few results are known for multidimensional BSDEs with local assumptions on the generator, see for instance [7, 8, 15, 16, 15, 21, 39, 46, 88]. Closer to our concern here, the one dimensional BSDEs have been more intensively studied and the quasi-totality of works are based on a comparison theorem. The later allows to prove the existence of solution when the generator is merely continuous, see for instance [99, 100, 97, 87]. Roughly speaking, when the generator is at most of linear growth in the variables y and z, the existence of solutions holds under a square integrable (or even integrable) terminal datum, see for instance [99]. When the generator is of quadratic growth in the variable z (QBSDE), the boundedness or at least the exponential integrability of the terminal value in order to ensure the existence of solutions, see for instance [5, 13, 37, 66, 97, 100]. Note however that, recently, a large class of QBSDEs which have solutions under merely square integrable terminal datum were given in [11, 12, 6].

In this section, we consider a one-dimensional BSDE with a continuous generator f which is of logarithmic growth like $(|y||\ln|y|| + |z|\sqrt{|\ln|z||})$. Neither the uniform continuity nor the local monotony (hence nor the locally Lipschitz) condition will be required to the generator. These kind of generators are between the linear growth and the quadratic one. In this case, the square integrability of the terminal datum is not sufficient to ensure the existence of solutions while the exponential integrability seems strong enough. In our situation, one should require some p–integrability of the terminal datum ξ with $p > 2$. It should be noted that we do not need the comparison theorem in our proofs. We derive the existence of solution by an approach used in [12] and lately more developed in [5, 6]. This method allows us to deduce the solvability of BSDE without barriers from the solvability of BSDEs with barriers. Stochastic optimal control and BSDE with logarithmic growth were studied in [13] We merely need the following two assumptions to get the existence of solutions,

(H1) There exists a positive constant λ large enough such that $\mathbb{E}[|\xi|^{e^{\lambda T}+1}] < +\infty$,

(H2) (i) f is continuous in (y,z) for almost all (t,w),

(ii) There exist a positive process η_t satisfying

$$\mathbb{E}\left[\int_0^T \eta_s^{e^{\lambda T}+1} ds\right] < +\infty, \tag{4.1.2}$$

and two positive constants c_0 and K such that for every t, ω, y, z :

$$| f(t,\omega,y,z) | \leq \eta_t + K|y||\ln|y|| + c_0|z|\sqrt{|\ln(|z|)|} := g(t,\omega,y,z).$$

To establish the uniqueness, we use a localization procedure introduced in [7, 8] and more developed in [16, 15]. However, in contrast to [16, 15], we do not impose the well known condition $yf(s,y,z) \leq \eta_t + M|y|^2 + K|y||z|$ on the generator. Therefore, our generator is of super-linear growth in its two sides. In return, we assume that the terminal data ξ is $\mathbb{L}^p-$ integrable, for some $p > 2$. The method we use allows to establish the uniqueness as well as the stability of solutions by the same calculus. To this end, we moreover need the following assumption

(H3) There exist $v \in \mathbb{L}^{q'}(\Omega \times [0,T]; \mathbb{R}_+))$ (for some $q' > 0$) and a real valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that :

i) $\forall N > 1, \quad 1 < A_N \leq N^r$,

ii) $\lim_{N\to\infty} A_N = \infty$,

iii) For every $N \in \mathbb{N}$, and every $y,\ y'\ z,\ z'$ such that $|y|,\ |y'|,\ |z|,\ |z'| \leq N$, we have

$$\Big(y-y'\Big)\Big(f(t,\omega,y,z)-f(t,\omega,y',z')\Big)\mathbb{1}_{\{v_t(\omega)\leq N\}} \leq M_2|y-y'|^2\ln A_N + M_2|y-y'||z-z'|\sqrt{\ln A_N} + M_2\frac{\ln A_N}{A_N}.$$

The main objective of the first part of this section is to prove the existence of solutions under assumptions **(H1)**, **(H2)**. In a first step, we establish the existence and uniqueness of solutions to equation (4.1.1) under the three assumptions **(H1)**, **(H2)** and **(H3)** then we deduce the existence of solutions by assuming merely the two conditions **(H1)**, **(H2)**. Let us give more details : We use a suitable localization procedure to establish the existence and uniqueness of solutions. Since the functions g and $-g$ satisfies **(H1)**, **(H2)** and **(H3)**, we then deduce from the first step that $eq\,(\xi^+, g)$ and $eq\,(-\xi^-, -g)$ have unique solutions which we will respectively denote by (Y^g, Z^g) and (Y^{-g}, Z^{-g}). Since $Y^{-g} \leq Y^g$, we use them as reflecting barriers. Using the result of [67], we show that the two barriers reflected BSDE with parameters (ξ, f, Y^{-g}, Y^g) has a solution. We then deduce that $eq\,(\xi, f)$ has a solution (Y, Z) such that $Y^{-g} \leq Y \leq Y^g$ by showing that the increasing processes, which forces the solution to stay between the barriers, are null.

In the second part, we establish as application the existence of a continuous viscosity solution to a semilinear PDE with logarithmic growth nonlinear term. For instance, the simple Markovian version of $eq\,(\xi, f)$ is related to the semilinear PDE,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \log|u| = 0 \;\; \text{on} \;\; (0,\ \infty) \times \mathbb{R}^d, \\ \\ u(0^+) = \varphi > 0. \end{cases} \tag{4.1.3}$$

This kind of PDEs appears in physics (see e.g. [44, 56, 57, 33, 126, 133]) as well as in the theory of continuous branching processes where it is related to the Neveu branching mechanism, see e.g. [31, 68, 110]. The logarithmic nonlinearity $u \log|u|$ is interesting in its own, since it is neither locally Lipschitz nor uniformly continuous.

The proofs of the following main results are given in last section of this chapter.

4.2 The main results

4.2.1 BSDEs with logarithmic growth

Theorem 32. *Assume that* (**H1**)–(**H3**) *are satisfied. Then, equation (4.1.1) has a unique solution in* $\mathcal{S}^{e^{\lambda T}+1} \times \mathcal{M}^2$.

Theorem 33. *Assume that* (**H1**) *and* (**H2**) *are satisfied. Then, equation (4.1.1) has a least one solution* $(Y,\, Z)$ *which belongs to* $\mathcal{S}^{e^{\lambda T}+1} \times \mathcal{M}^2$.

In the following, we give a stability result for the solution of $eq(f,\xi)$. Roughly speaking, if f_n converges to f in the metric defined by the family of semi-norms (ρ_N) and ξ_n converges to ξ in $L^p(\Omega)$ for $p>2$ then (Y^n, Z^n) converges to (Y, Z) in some $L^q(\Omega)$ for $1<q<2$. Let (f_n) be a sequence of functions which are measurable for each n. Let (ξ_n) be a sequence of random variables which are $\mathcal{F}_T$-measurable for each n and such that $\sup_n \mathbb{E}\Big(|\xi_n|^{e^{\lambda T}+1}\Big) < +\infty$. We will assume that for each n, the BSDE $eq(f_n, \xi_n)$ has a (not necessarily unique) solution.

Each solution of the BSDE $eq(f_n, \xi_n)$ will be denoted by (Y^n, Z^n). We consider the following assumptions,

a) For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$,

b) $\mathbb{E}(|\xi_n - \xi|^{e^{\lambda T}+1}) \longrightarrow 0$ as $n \to \infty$,

c) There exist a positive constant c_0 and a positive process η_t such that

$$\mathbb{E}\left[\int_0^T \eta_s^{e^{\lambda T}+1} ds\right] < +\infty,$$

and

$$\sup_n |f_n(t,\omega,y,z)| \leq \eta_t + K|y||\ln|y|| + c_0|z|\sqrt{|\ln(|z|)|}.$$

Theorem 34. *Let (f,ξ) be as in Theorem 32. Assume that* ***a)****,* ***b)****, and* ***c)*** *are satisfied. Then, for every $q < 2$ we have*

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leq t\leq T}|Y_t^n - Y_t|^q + \mathbb{E}\int_0^T |Z_s^n - Z_s|^q ds\right) = 0.$$

4.3 Application to PDEs.

The Markovian version of BSDE (4.1.1) is defined by the following system of SDE-BSDE, for $0 \leq s \leq T$,

$$\begin{cases} X_s = x + \int_t^s b(X_r)dr + \int_t^s \sigma(X_r)dW_r, \\ Y_s = H(X_T) + \int_s^T f(X_r, Y_r, Z_r)dr - \int_s^T Z_r dW_r. \end{cases} \tag{4.3.1}$$

where $b : \mathbb{R}^k \longmapsto \mathbb{R}^k$, $\sigma : \mathbb{R}^k \longmapsto \mathbb{R}^{kd}$, $H : \mathbb{R}^k \longmapsto \mathbb{R}$, and $f : [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \longmapsto \mathbb{R}$ are measurable functions.

Let the PDE associated to the Markovian BSDE (4.3.1) is given by,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x) + f(x, u(t,x), \sigma(x)\nabla u(t,x)) \quad 0 \le t \le T >, \\ \\ u(T,x) = H(x), \end{cases} \tag{4.3.2}$$

where,

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}, \qquad \text{and} \qquad a(x) := \frac{1}{2}(\sigma\sigma^*)(x).$$

Consider the following assumptions :

(H4) σ, b are uniformly Lipschitz functions,

(H5) σ, b are continuous functions and a is uniformly elliptic,

(H6) σ, b are of linear growth,

(H7) H is continuous and satisfies $\mathbb{E}\Big(\Big[H(X_T)\Big]^{e^{\lambda T}+1}\Big) < \infty$.

Theorem 35. *Assume that* (**H1**)-(**H4**) *and* (**H7**) *are satisfied. Then, equation (4.3.2) has a viscosity solution v such that $v(t,x) = Y_t^{(t,x)}$.*

Remark 36. (i) *The conclusion of Theorem 35 remains valid if we replace the Lipschitz condition* (**H4**) *by the assumptions given in [9] or that of [18].*

(ii) *What happens about the conclusion of Theorem 35 when assumption* (**H4**) *is replaced by : the martingale problem is well–posed for $a := \frac{1}{2}\sigma\sigma^*$ and b ?*

(iii) *If a is uniformly elliptic, b and/or σ are discontinuous and the martingale problem is well–posed for a and b as in [92] for instance, is it possible to get the existence of an L^p-viscosity solution to equation* (4.3.1) *by arguing as in [14] ?*

4.4 Proofs

4.4.1 A priori estimations.

To prove Theorem 32 and Theorem 34, we need the following lemmas and the first one is quite technical. We can assume that $(|Y_t|)$ is large enough.

Lemma 37. *Let $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ be such that y is large enough. Then, for every $C_1 > 0$ there exists $C_2 > 0$ such that,*

$$C_1 |y| |z| \sqrt{|\ln(|z|)|} \leq \frac{|z|^2}{2} + C_2 |y|^2 \ln(|y|). \tag{4.4.1}$$

Proof. If $|z| \leq |y|$, (4.4.1) is obvious. Assume now that $|z| > |y|$. The number $a := \frac{|z|}{|y|}$ is then strictly greater than 1. Since $|y|$ is assumed to be large enough and $|z| > |y|$, then $|z|$ is also large enough, and it yields

$$C_1 |y| |z| \sqrt{\ln(|z|)} \leq C_1 a |y|^2 \left[\sqrt{\ln(a)} + \sqrt{\ln(|y|)} \right],$$

and

$$\frac{|z|^2}{2} + C_2 \ln(|y|) |y|^2 = |y|^2 \left[\frac{a^2}{2} + C_2 \ln(|y|) \right].$$

Obviously

$$C_1 a \sqrt{\ln(|y|)} \leq \frac{1}{2} \left[\frac{a^2}{2} + 2C_1^2 \ln(|y|) \right].$$

So that we have simply to prove that,

$$\frac{a^2}{4} + C_1 a \sqrt{\ln(a)} + C_1^2 \ln(|y|) \leq \frac{a^2}{2} + C_2 \ln(|y|).$$

Let r be a constant such that $r = \max\left\{z \in \mathbb{R},\ C_1\sqrt{\ln(z)} = \frac{z}{4}\right\}$.

If $a \geq r$, then $\quad C_1 a\sqrt{\ln(a)} \leq \dfrac{a^2}{4}$.

If $a < r$, then we have $\quad C_1 a\sqrt{\ln(a)} \leq C_1 r\sqrt{\ln(r)} \leq C_1' \leq C_1' \ln(|y|)$.

This proves inequality (4.4.1). ∎

Lemma 38. *Let* (Y, Z) *be a solution of the BSDE (4.1.1). Let* $\lambda \geq 2K+1$. *Assume moreover that* (ξ, f) *satisfies conditions* (**H1**) *and* (**H2**). *Then there exists a constant* C_T, *such that :*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^{e^{\lambda t}+1}\right) \leq C_T \mathbb{E}\left(|\xi|^{e^{\lambda T}+1} + \int_0^T \eta_s^{e^{\lambda s}+1} ds\right).$$

Proof. Let λ be a positive number large enough. Let $u(t,x) := |x|^{e^{\lambda t}+1}$. We define $sgn(x) := -\mathbf{1}_{\{x\leq 0\}} + \mathbf{1}_{\{x>0\}}$. We have,

$$u_t = Ce^{\lambda t}\ln(|x|)\,|x|^{e^{\lambda t}+1},\ u_x = (e^{\lambda t}+1)\,|x|^{e^{\lambda t}}\,sgn(x) \ \text{ and } u_{xx} = (e^{\lambda t}+1)e^{\lambda t}\,|x|^{e^{\lambda t}-1}.$$

For $k \geq 0$, let τ_k be the stopping time defined as follows :

$$\tau_k := \inf\left\{t \geq 0,\ \left[\int_0^t (e^{\lambda s}+1)^2\,|Y_s|^{2e^{\lambda s}}\,|Z_s|^2\,ds\right] \vee |Y_t| \geq k\right\}.$$

By Itô's formula, we have :

$$
\begin{aligned}
|Y_{t\wedge\tau_k}|^{e^{\lambda(t\wedge\tau_k)}+1} &= |Y_{T\wedge\tau_k}|^{e^{\lambda(T\wedge\tau_k)}+1} - \int_{t\wedge\tau_k}^{T\wedge\tau_k} \lambda e^{\lambda s}\ln(|Y_s|)\,|Y_s|^{e^{\lambda s}+1}\,ds \\
&\quad - \frac{1}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} |Z_s|^2(e^{\lambda s}+1)e^{\lambda s}\,|Y_s|^{e^{\lambda s}-1}\,ds \\
&\quad + \int_{t\wedge\tau_k}^{T\wedge\tau_k}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,sgn(Y_s)f(s,Y_s,Z_s)ds \\
&\quad - \int_{t\wedge\tau_k}^{T\wedge\tau_k}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,sgn(Y_s)Z_s dW_s, \\
&\leq |Y_{T\wedge\tau_k}|^{e^{\lambda(T\wedge\tau_k)}+1} - \int_{t\wedge\tau_k}^{T\wedge\tau_k} \lambda e^{\lambda s}\ln(|Y_s|)\,|Y_s|^{e^{\lambda s}+1}\,ds \\
&\quad - \frac{1}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} |Z_s|^2(e^{\lambda s}+1)e^{\lambda s}\,|Y_s|^{e^{\lambda s}-1}\,ds \\
&\quad + \int_{t\wedge\tau_k}^{T\wedge\tau_k}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\left(\eta_s + K\,|Y_s|\ln(\,|Y_s|\,) + c_0|Z_s|\sqrt{|\ln(|Z_s|)|}\right)ds \\
&\quad - \int_{t\wedge\tau_k}^{T\wedge\tau_k}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,sgn(Y_s)Z_s dW_s.
\end{aligned}
$$

By Young's inequality it holds :

$$
(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\eta_s \leq |Y_s|^{e^{\lambda s}+1} + (e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}.
$$

For $|Y_s|$ large enough and thanks to the last inequality we have :

$$\begin{aligned}
|Y_{t\wedge\tau_k}|^{e^{\lambda(t\wedge\tau_k)+1}} &\leq |Y_{T\wedge\tau_k}|^{e^{\lambda(T\wedge\tau_k)}+1} - \int_{t\wedge\tau_k}^{T\wedge\tau_k} \lambda e^{\lambda s}(\ln|Y_s|)\,|Y_s|^{(e^{\lambda s}+1)}\,ds \\
&- \frac{1}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} |Z_s|^2(e^{\lambda s}+1)e^{\lambda s}\,|Y_s|^{e^{\lambda s}-1}\,ds \\
&+ \int_{t\wedge\tau_k}^{T\wedge\tau_k} |Y_s|^{e^{\lambda s}+1}ds \\
&+ \int_{t\wedge\tau_k}^{T\wedge\tau_k} (e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}\,ds \\
&+ \int_{t\wedge\tau_k}^{T\wedge\tau_k} K(e^{\lambda s}+1)|Y_s|^{e^{\lambda s}+1}\ln(|Y_s|)ds. \\
&+ \int_{t\wedge\tau_k}^{T\wedge\tau_k} c_0(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,|Z_s|\sqrt{|\ln(|Z_s|)|}ds \\
&- \int_{t\wedge\tau_k}^{T\wedge\tau_k} (e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,sgn(Y_s)Z_s dW_s,
\end{aligned}$$

Note that for $\lambda > 2K+1$, we have $(\lambda e^{\lambda s} - K(e^{\lambda s}+1)-1) > 0$ and hence using Lemma 37, we deduce, for λ large enough, that :

$$\begin{aligned}
c_0(e^{\lambda s}+1)\,|Y_s|\,|Z_s|\sqrt{|\ln(|Z_s|)|} &\leq (e^{\lambda s}+1)e^{\lambda s}\frac{|Z_s|^2}{2} \\
&+ (\lambda e^{\lambda s} - K(e^{\lambda s}+1)-1)\ln(|Y_s|)\,|Y_s|^2\,.
\end{aligned} \tag{4.4.2}$$

Hence,

$$\begin{aligned}
|Y_{t\wedge\tau_k}|^{e^{\lambda(t\wedge\tau_k)}+1} &\leq |Y_{T\wedge\tau_k}|^{e^{\lambda(T\wedge\tau_k)}+1} + \int_{t\wedge\tau_k}^{T\wedge\tau_k} (e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}ds \\
&- \int_{t\wedge\tau_k}^{T\wedge\tau_k} (e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,sgn(Y_s)Z_s dW_s.
\end{aligned}$$

Taking expectation, we have

$$\mathbb{E}(|Y_{t\wedge\tau_k}|^{e^{\lambda(t\wedge\tau_k)}+1}) \leq \mathbb{E}(|Y_{T\wedge\tau_k}|^{e^{\lambda(T\wedge\tau_k)}+1}) + (e^{\lambda T}+1)^{e^{\lambda T}+1}\mathbb{E}\int_0^T \eta_s^{e^{\lambda s}+1}ds.$$

Passing to the limits in k and using Fatou's Lemma we get

$$\mathbb{E}(|Y_t|^{e^{\lambda t}+1}) \leq \mathbb{E}(|\xi|^{e^{\lambda T}+1}) + (e^{\lambda T}+1)^{e^{\lambda T}+1}\mathbb{E}\int_0^T \eta_s^{e^{\lambda s}+1} ds.$$

The proof is completed by using the Burkholder-Davis-Gundy inequality. ■

Lemma 39. *Let* (Y, Z) *be a solution of BSDE* (1.1)*. Assume that* (**H1**) *and* (**H2**) *are satisfied. Then, there exists a positive constant* $C(T, c_0, K)$ *such that :*

$$\mathbb{E}\int_0^T |Z_s|^2 ds \leq C(T, c_0, K)\mathbb{E}\left[|\xi|^2 + \sup_{s\in[0,T]} |Y_s|^{e^{\lambda T}+1} + \int_0^T |\eta_s|^2 ds\right].$$

Proof. Itô's formula shows that :

$$\begin{aligned}|Y_t|^2 + \int_t^T |Z_s|^2 ds &= |\xi|^2 + 2\int_t^T Y_s f(s, Y_s, Z_s) ds - 2\int_t^T Y_s Z_s dW_s \\ &\leq |\xi|^2 + 2\int_t^T |Y_s| \left(\eta_s + K|Y_s||\ln(|Y_s|)| + c_0|Z_s|\sqrt{|\ln(|Z_s|)|}\right) ds \\ &\quad - 2\int_t^T Y_s Z_s dW_s.\end{aligned}$$

Since for $|Y_s|$ large enough, we have for any $\varepsilon > 0$, $|Y_s|^2 |\ln(|Y_s|)| \leq |Y_s|^{2+\varepsilon}$, we use Lemma 37 to show that there exists a positive constat K_1 depending upon c_0 and K such that :

$$\frac{1}{2}\int_t^T |Z_s|^2 ds \ \leq \ |\xi|^2 + T\sup_{s\in[0,T]} |Y_s|^2 + \int_t^T |\eta_s|^2 ds + 2TK_1 \sup_{s\in[0,T]} |Y_s|^{2+\varepsilon} - 2\int_t^T Y_s Z_s dW_s.$$

Since $|Y_s|^{2+\varepsilon} \geq |Y_s|^2$ for $|Y_s|$ large enough, then there exists a positive constant $K_2 = K_2(T, c_2, K)$ such that :

$$\int_t^T |Z_s|^2 ds \ \leq \ K_2\left(|\xi|^2 + \sup_{s\in[0,T]} |Y_s|^{2+\varepsilon} + \int_t^T |\eta_s|^2 ds + \left|\int_t^T Y_s Z_s dW_s\right|\right).$$

If we put $\varepsilon = e^{\lambda T} - 1$, we get

$$\mathbb{E}\int_0^T |Z_s|^2\, dt \le K_2 \mathbb{E}\Big[|\xi|^2 + \sup_{s\in[0,T]} |Y_s|^{e^{\lambda T}+1} + \int_0^T |\eta_s|^2\, ds + \sup_{t\in[0,T]}\Big|\int_t^T Y_s Z_s dW_s\Big|\Big].$$

Thanks to the Burkhölder-Davis-Gundy inequality we have for any $\beta > 0$

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T Y_s Z_s dW_s\right|\right] &\le \bar{C}\mathbb{E}\left[\left(\int_0^T |Y_s|^2 |Z_s|^2\, dt\right)^{\frac{1}{2}}\right] \\
&\le \bar{C}\mathbb{E}\left[\sup_{s\in[0,T]} |Y_s| \left(\int_0^T |Z_s|^2\, ds\right)^{\frac{1}{2}}\right] \\
&\le \frac{\bar{C}}{2\beta}\mathbb{E}\left(\sup_{s\in[0,T]} |Y_s|^2\right) + \frac{\bar{C}\beta}{2}\mathbb{E}\left(\int_0^T |Z_s|^2\, ds\right).
\end{aligned}$$

Choosing β small enough, we get the desired result. ∎

Lemma 40. *Let* (**H1**)*,* (**H2**)*–(ii) be satisfied. Then,*

$$\mathbb{E}\int_0^T |f(s, Y_s, Z_s)|^{\bar{\alpha}} ds \ \le K\Big[1 + \mathbb{E}\int_0^T \left(\eta_s^2 + |Y_s|^2\right) ds + \mathbb{E}\int_0^T |Z_s|^2 ds\Big].$$

where $\bar{\alpha} = \min(2, \frac{2}{\alpha})$ *and* K *is a positive constant which depends from* c_0 *and* T*.*

Proof. Observe that assumption (**H2**) implies that there exist positives constants c_1, c_2 and α with $1 < \alpha < 2$ and a process $\bar{\eta} := \eta + c_1$ such that :

$$|f(t, \omega, y, z)| \le \bar{\eta}_t + c_1|y|^{\alpha} + c_2|z|^{\alpha}. \tag{4.4.3}$$

For simplicity, we assume that $\bar{\eta} := \eta$. We successively use inequality (4.4.3) and Assumption

(H.3) to show that

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(s,Y_s,Z_s)|^{\bar{\alpha}}ds\\
&\leq \mathbb{E}\int_0^T (\eta_s + c_1|Y_s|^{\alpha} + c_2|Z_s|^{\alpha})^{\overline{\alpha}}ds,\\
&\leq 3(1 + c_1^{\bar{\alpha}}+c_2^{\overline{\alpha}})\mathbb{E}\int_0^T \Big((\eta_s)^{\bar{\alpha}} + |Y_s|^{\alpha\overline{\alpha}} + (|Z_s|^{\alpha\overline{\alpha}}\Big)\, ds,\\
&\leq 3(1 + c_1^{\bar{\alpha}}+c_2^{\overline{\alpha}})\mathbb{E}\int_0^T \Big(\Big(1+\eta_s)^{\bar{\alpha}} + (1+|Y_s|)^{\alpha\overline{\alpha}} + (1+|Z_s|)^{\alpha\overline{\alpha}}\Big)\, ds,\\
&\leq 6(1 + c_1^{\bar{\alpha}}+c_2^{\overline{\alpha}})\Big(3T + \mathbb{E}\int_0^T (\eta_s^2+|Y_s|^2 + |Z_s|^2)ds\Big) < \infty.
\end{aligned}$$

Lemma 40 is proved. ■

Lemma 41. *There exists a sequence of functions* (f_n) *such that,*

(a) *For each* n*,* f_n *is bounded and globally Lipschitz in* (y,z) *a.e.* t *and* P*-a.s.*ω*.*

(b) $\sup_n |f_n(t,\omega,y,z)| \leq \eta_t + K|Y_s||\ln(|Y_s|)| + c_0|z|\sqrt{|\ln(|Z_s|)|}$, P*-a.s., a.e.* $t \in [0,T]$*.*

(c) *For every* N*,* $\rho_N(f_n - f) \longrightarrow 0$ *as* $n \longrightarrow \infty$*.*

Proof. Let $\alpha_n : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \alpha_n(u)du = 1$. Let ψ_n from $\mathbb{R}^2$ to $\mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq |\psi_n| \leq 1$, $\psi_n(u) = 1$ for $|u| \leq n$ and $\psi_n(u) = 0$ for $|u| \geq n+1$. We put, $\varepsilon_{q,n}(t,y,z) = \int f(t,(y,z)-u)\alpha_q(u)du\psi_n(y,z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq n + n^{\alpha}$. It is not difficult to see that the sequence $f_n := \varepsilon_{q(n),n}$ satisfies all the assertions (a)-(c). ■

Arguing as in the proofs of Lemma 38, Lemma 39, Lemma 40, Lemma 41 and standard arguments of BSDEs, one can prove the following estimates.

Lemma 42. *Let f and ξ be as in Theorem 32. Let (f_n) be the sequence of functions associated to f by Lemma 41. Denote by (Y^{f_n}, Z^{f_n}) the solution of equation (E^{f_n}). Then, there exit constants $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$ such that*

a) $\sup_n \mathbb{E}\int_0^T |Z_s^{f_n}|^2 ds \leq \bar{K}_1$

b) $\sup_n \mathbb{E}\left[\sup_{0\leq t\leq T}\left(|Y_t^{f_n}|^{e^{\lambda T}+1}\right)\right] \leq \bar{K}_2$

c) $\sup_n \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n})|^{\bar{\alpha}} ds \leq \bar{K}_3$

where $\bar{\alpha} = \min(2, \frac{2}{\alpha})$

The following lemma (which established in [16]) is a direct consequence of Hölder's and Schwartz's inequalities.

Lemma 43. *For every $\beta \in]1,2]$, $A > 0$, $(y)_{i=1..d} \subset \mathbb{R}$, $(z)_{i=1..d,j=1..r} \subset \mathbb{R}$ we have,*

$$A|y||z| - \frac{1}{2}|z|^2 + \frac{2-\beta}{2}|y|^{-2}|yz|^2 \leq \frac{1}{\beta-1}A^2|y|^2 - \frac{\beta-1}{4}|z|^2.$$

4.4.2 Estimate between two solutions

Proposition 4.4.1. *For every $R \in \mathbb{N}$, $\beta \in]1, \min\left(3-\frac{2}{\bar{\alpha}}, 2\right)[$, $\delta < (\beta-1)\min\left(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leq T$:*

$$\limsup_{n,m\to+\infty} E \sup_{(T'-\delta)^+\leq t\leq T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + E\int_{(T'-\delta)^+}^{T'} \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{\beta-1} e^{C_N\delta} \limsup_{n,m\to+\infty} E|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta.$$

where $\nu_R = \sup\{(A_N)^{-1}, N \geq R\}$, $C_N = \frac{2M_2^2\beta}{(\beta-1)}\ln A_N$ and ℓ is a universal positive constant.

For $N \in \mathbb{N}^\star$, we put

$$\Delta_t := \left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + (A_N)^{-1} \text{ and } \Phi(s) := |Y_s^{f_n}| + |Y_s^{f_m}| + |Z_s^{f_n}| + |Z_s^{f_m}| \qquad (4.4.4)$$

Lemma 44. *Let assumptions of Proposition 4.4.1 be satisfied and let* $\kappa := 3 - \frac{2}{\bar{\alpha}} - \beta$ *. Then, for any* $C > 0$ *we have,*

$$\begin{aligned}
&e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \\
&\leq e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right)dW_s\rangle \\
&- \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
&+ \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left((Y_s^{f_n} - Y_s^{f_m})(Z_s^{f_n} - Z_s^{f_m})\right)^2 ds \\
&+ J_1 + J_2 + J_3,
\end{aligned}$$

where

$$J_1 := \beta e^{CT'}\frac{1}{N^\kappa}\int_t^{T'} \Delta_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)|f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m}|ds,$$

$$\begin{aligned}
J_2 := \beta e^{CT'}[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\Bigg[&\int_t^{T'} \sup_{|y|,|z|\leq N}|f_n(s,y,z) - f(s,y,z)|ds \\
&+ \int_t^{T'} \sup_{|y|,|z|\leq N}|f_m(s,y,z) - f(s,y,z)|ds\Bigg].
\end{aligned}$$

$$J_3 := \beta M_2\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left[\Delta_s \ln A_N + |Y_s^{f_n} - Y_s^{f_m}||Z_s^{f_n} - Z_s^{f_m}|\sqrt{\ln A_N}\right]ds.$$

Proof. To simplify the computations, we assume (without loss of generality) that assumption **(H3)**-(iii) holds without the multiplicative term $\mathbf{1}_{\{v_t(\omega)\leq N\}}$. Let $C > 0$. Itô's formula shows that,

$$\begin{aligned}
&e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \\
&= e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\right)ds \\
&- \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right)dW_s\rangle - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
&- \beta(\frac{\beta}{2} - 1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left((Y_s^{f_n} - Y_s^{f_m})(Z_s^{f_n} - Z_s^{f_m})\right)^2 ds.
\end{aligned}$$

Using the fact that $\Phi(s) = |Y_s^{f_n}| + |Y_s^{f_m}| + |Z_s^{f_n}| + |Z_s^{f_m}|$, we get

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \\
& = e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right)dW_s\rangle \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
& + \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left((Y_s^{f_n} - Y_s^{f_m})(Z_s^{f_n} - Z_s^{f_m})\right)^2 ds \\
& + \acute{J}_1 + \acute{J}_2 + \acute{J}_3 + \acute{J}_4,
\end{aligned}
$$

where

$$\acute{J}_1 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\right)1\!\!1_{\{\Phi(s)>N\}}ds.$$

$$\acute{J}_2 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})\right)1\!\!1_{\{\Phi(s)\leq N\}}ds.$$

$$\acute{J}_3 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_m}, Z_s^{f_m})\right)1\!\!1_{\{\Phi(s)\leq N\}}ds.$$

$$\acute{J}_4 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f(s, Y_s^{f_m}, Z_s^{f_m}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\right)1\!\!1_{\{\Phi(s)\leq N\}}ds.$$

We now proceed to estimate $\acute{J}_1,\ \acute{J}_2,\ \acute{J}_3,\ \acute{J}_4$. We use the fact that $|Y_s^{f_n} - Y_s^{f_m}| \leq \Delta_s^{\frac{1}{2}}$ to obtain

$$
\begin{aligned}
\acute{J}_1 & \leq \beta e^{CT'}\int_t^{T'}\Delta_s^{\frac{\beta-1}{2}}|f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m}|1\!\!1_{\{\Phi(s)>N\}}ds, \\
& \leq J_1,
\end{aligned}
$$

and

$$\acute{J}_2 + \acute{J}_4 \leq J_2.$$

Using assumption **(H3)**, we get

$$\acute{J}_3 \leq \beta M_2 \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} \Big[|Y_s^{f_n} - Y_s^{f_m}|^2 \ln A_N$$

$$+ |Y_s^{f_n} - Y_s^{f_m}||Z_s^{f_n} - Z_s^{f_m}| \sqrt{\ln A_N} + \frac{\ln A_N}{A_N} \Big] \mathbb{1}_{\{\Phi(s)<N\}} ds$$

$$\leq J_3.$$

Lemma 44 is proved. ■

Lemma 45. *Let assumptions of Proposition 4.4.1 be satisfied and let* $\gamma := \frac{2M_2^2\delta\beta}{\beta-1}$. *Then, there exists a universal constant* ℓ *such that,*

$$\mathbb{E} \sup_{(T'-\delta)^+ \leq t \leq T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + \mathbb{E} \int_{(T'-\delta)^+}^{T'} \frac{\left| Z_s^{f_n} - Z_s^{f_m} \right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R \right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \frac{\ell}{\beta - 1} e^{C_N \delta} \mathbb{E} |Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta + \frac{\ell}{\beta - 1} \frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}$$
$$+ \frac{4\ell}{\beta - 1} \beta K_3^{\frac{1}{\bar{\alpha}}} (4TK_2 + T\nu_R)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} \frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}$$
$$+ \frac{\ell}{\beta - 1} e^{C_N \delta} \beta [2N^2 + \nu_1]^{\frac{\beta-1}{2}} \Big[\rho_N(f_n - f) + \rho_N(f_m - f) \Big].$$

Proof. We choose $C := C_N := \dfrac{2M_2^2\beta}{\beta - 1} \ln A_N$ in Lemma 44. Using Lemma 43, Burkholder's inequality and Hölder's inequality (since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\bar{\alpha}} = 1$), we show that there exists a universal constant $\ell > 0$ such that for any $\delta > 0$,

$$\mathbb{E} \sup_{(T'-\delta)^+ \le t \le T'} \left[e^{C_N t} \Delta_t^{\frac{\beta}{2}} \right] + \mathbb{E} \int_{(T'-\delta)^+}^{T'} e^{C_N s} \Delta_s^{\frac{\beta}{2}-1} \left| Z_s^{f_n} - Z_s^{f_m} \right|^2 ds$$
$$\le \frac{\ell}{\beta-1} e^{C_N T'} \Bigg\{ \mathbb{E}\left[\Delta_{T'}^{\frac{\beta}{2}}\right] + \frac{\beta}{N^\kappa} \left[\mathbb{E}\int_0^T \Delta_s ds\right]^{\frac{\beta-1}{2}} \left[\mathbb{E}\int_0^T \Phi(s)^2 ds\right]^{\frac{\kappa}{2}}$$
$$\times \left[\mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m}|^{\bar{\alpha}} ds\right]^{\frac{1}{\bar{\alpha}}}$$
$$+ \beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}} \mathbb{E}\Bigg[\int_0^T \sup_{|y|,|z| \le N} |f_n(s,y,z) - f(s,y,z)| ds$$
$$+ \int_0^T \sup_{|y|,|z| \le N} |f_m(s,y,z) - f(s,y,z)| ds \Bigg] \Bigg\}.$$

We use Lemma 41 and Lemma 42 to obtain for any $N > R$,

$$\mathbb{E} \sup_{(T'-\delta)^+ \le t \le T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + \mathbb{E} \int_{(T'-\delta)^+}^{T'} \frac{\left| Z_s^{f_n} - Z_s^{f_m} \right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R \right)^{\frac{2-\beta}{2}}} ds$$
$$\le \frac{\ell}{\beta-1} e^{C_N \delta} \Bigg\{ \mathbb{E}|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta + (A_N)^{\frac{-\beta}{2}}$$
$$+ \frac{\beta}{N^\kappa} (4TK_2 + T\nu_R)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} (4K_3^{\frac{1}{\bar{\alpha}}})$$
$$+ \beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}} \Big[\rho_N(f_n - f) + \rho_N(f_m - f) \Big] \Bigg\}$$
$$\le \frac{\ell}{\beta-1} e^{C_N \delta} \mathbb{E}|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta + \frac{\ell}{\beta-1} \frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}$$
$$+ \frac{4\ell}{\beta-1} \beta K_3^{\frac{1}{\bar{\alpha}}} (4TK_2 + T\nu_R)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} \frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}$$
$$+ \frac{\ell}{\beta-1} e^{C_N \delta} \beta [2N^2 + \nu_1]^{\frac{\beta-1}{2}} \Big[\rho_N(f_n - f) + \rho_N(f_m - f) \Big].$$

Lemma 45 is proved. ■

Proof of Proposition 4.4.1 Taking $\delta < (\beta - 1) \min\left(\frac{1}{4M_2^2}, \frac{\kappa}{2rM_2^2\beta} \right)$ we derive

$$\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}} \longrightarrow_{N \to \infty} 0,$$

and

$$\frac{A_N^{\gamma}}{(A_N)^{\frac{\kappa}{r}}} \longrightarrow_{N\to\infty} 0.$$

To finish the proof of Proposition 4.4.1 we pass to the limits first on n and next on N using assertion (c) of lemma 41. ■

Remark 46. *To deal with the case which take account of the process v_t appearing in assumption* (**H3**)*, it suffices to take* $\Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s$ *in the proof of proposition 4.4.1.*

4.4.3 Existence and uniqueness

Proof. of Theorem 32 Taking successively $T' = T$, $T' = (T-\delta)^+$, $T' = (T-2\delta)^+$... in Proposition 4.4.1, we show that for any $\beta \in \left]1, \quad \min(3-\frac{2}{\bar{\alpha}}, 2)\right[$

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}|Y_t^{f_n} - Y_t^{f_m}|^{\beta} + \mathbb{E}\int_0^T \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds\right) = 0.$$

Using Schwartz's inequality we have,

$$\mathbb{E}\int_0^T |Z_s^{f_n} - Z_s^{f_m}|ds \le \left(\mathbb{E}\int_0^T \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}ds\right)^{\frac{1}{2}}.$$

Lemma 42 shows that

$$\left(\mathbb{E}\int_0^T \left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}ds\right)^{\frac{1}{2}} < \infty$$

It follows that :

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}|Y_t^{f_n} - Y_t^{f_m}|^{\beta} + \mathbb{E}\int_0^T |Z_s^{f_n} - Z_s^{f_m}|ds\right) = 0$$

Hence, there exists (Y, Z) satisfying

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t|^\beta + \mathbb{E}\int_0^T |Z_s|ds < \infty$$

and

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leq t\leq T}|Y_t^{f_n} - Y_t|^\beta + \mathbb{E}\int_0^T |Z_s^{f_n} - Z_s|ds\right) = 0$$

In particular, there exists a subsequence, which we still denote $(Y^{f_n},\, Z^{f_n})$, such that

$$\lim_{n\to+\infty}\left(|Y_t^{f_n} - Y_t| + |Z_t^{f_n} - Z_t|\right) = 0 \quad a.e.\ (t,\omega).$$

We shall prove that $\int_0^T [f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^f, Z_s^f)]ds$ tends in probability to 0 as n tends to ∞. Triangular inequality gives

$$\begin{aligned}\mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^f, Z_s^f)|ds &\leq \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})|ds \\ &\quad + \mathbb{E}\int_0^T |f(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^f, Z_s^f)|ds\end{aligned}$$

Since $\mathbb{1}_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\geq N\}} \leq |\frac{(|Y_s^{f_n}| + |Z_s^{f_n}|)^{(2-\frac{2}{\bar{\alpha}})}}{N^{(2-\frac{2}{\bar{\alpha}})}}\mathbb{1}_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\geq N\}}$, it follows that :

$$\begin{aligned}&\mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})|ds \\ &\leq \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})|\mathbb{1}_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\leq N\}}ds \\ &+ \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})|\frac{(|Y_s^{f_n}| + |Z_s^{f_n}|)^{(2-\frac{2}{\bar{\alpha}})}}{N^{(2-\frac{2}{\bar{\alpha}})}}\mathbb{1}_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\geq N\}}ds \\ &\leq \rho_N(f_n - f) + \frac{2\bar{K}_3^{\frac{1}{\bar{\alpha}}}\left[T\bar{K}_2 + \bar{K}_1\right]^{1-\frac{1}{\bar{\alpha}}}}{N^{(2-\frac{2}{\bar{\alpha}})}}.\end{aligned}$$

Passing to the limit first on n and next on N we get,

$$\lim_n E\int_0^T |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})|ds = 0.$$

We use Lemma 42 and the Lebesgue dominated convergence theorem to show that,

$$\lim_n E \int_0^T |f(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s, Z_s)| ds = 0.$$

The existence is proved. ■

Uniqueness. Let (Y, Z) and (Y', Z') be two solutions of equation $eq(f, \xi)$. Arguing as previously, one can show that :

For every $R > 2$, $\beta \in]1, \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$, $\delta < (\beta - 1) \min\left(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$

there exists $N_0 > R$ such that for every $N > N_0$ and every $T' \leq T$

$$\mathbb{E} \sup_{(T'-\delta)^+ \leq t \leq T'} |Y_t - Y_t^{'}|^\beta + \mathbb{E} \int_{(T'-\delta)^+}^{T'} \frac{\left|Z_s - Z_s^{'}\right|^2}{(|Y_s - Y_s^{'}|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta} \mathbb{E}|Y_{T'} - Y_{T'}^{'}|^\beta.$$

We successively take $T' = T$, $T' = (T' - \delta)^+$, ... to complete the proof of uniqueness. ■

Proof. of Theorem 33 Clearly both the functions $g(t, y, z) := \eta_t + K|y|\left|\ln(|y|)\right| + c_0|z|\sqrt{|\ln(|z|)|}$ and $-g$ satisfy assumptions **(H2)** and **(H3)**. Hence, according to Theorem 32, $eq(\xi, g)$ (resp.$eq(-\xi, -g)$) has unique solution which belong to $\mathcal{S}^{e^{\lambda T}+1} \times \mathcal{M}^2$. We now consider the following reflected BSDE,

$$\begin{cases} i)\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, \\ ii)\ \forall t \leq T,\ Y_t^{-g} \leq Y_t \leq Y_t^g, \\ iii) \int_0^T \left(Y_t - Y_t^{-g}\right) dK_s^+ = \int_0^T \left(Y_t^g - Y_t\right) dK_s^- = 0 \text{ a.s.}, \\ iv)\ K^+,\ K^- \text{ are continuous nondecreasing, and } K_0^+ = K_0^- = 0, \\ v)\ dK^+ \perp dK^-. \end{cases}$$

For every $(t,w) \in [0,\,T] \times \Omega$, every $y \in \left[Y_t^{-g}(\omega),\ Y_t^g(\omega)\right]$ and every $z \in \mathbb{R}^d$ we have

$$\begin{aligned} f\left(t,y,z\right) &\leq \eta_t + K\left|y\right|\left|\ln\left(\left|y\right|\right)\right| + c_0\left|z\right|\sqrt{\left|\ln\left(\left|z\right|\right)\right|} \\ &\leq \eta_t + K\left(1+\left|y\right|^2\right) + c_0(1+\left|z\right|^2) \\ &\leq \left[\eta_t + K + c_0 + K\left(|Y_t^{-g}|^2 + |Y_t^g|^2\right)\right] + c_0\left|z\right|^2. \end{aligned}$$

Therefore, according to Theorem 3.2 of [67], the previous reflected BSDE, has a solution (Y, Z, K^+, K^-) such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$. In order to show that (Y, Z) is a solution to our non–reflected BSDE $eq(\xi, f)$, it is enough to prove that $dK^+ = dK^- = 0$. Since (Y^g, Z^g) is a solution to $eq(\xi, g)$, then Tanaka's formula shows that :

$$\begin{aligned} (Y_t^g - Y_t)^+ &= (Y_0^g - Y_0)^+ + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}[f(s,Y_s,Z_s) - g(Y_s^g, Z_s^g)]ds \\ &\quad + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(dK_s^+ - dK_s^-) + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(Z_s^g - Z_s)dW_s \\ &\quad + L_t^0(Y^g - Y) \end{aligned}$$

where $L_t^0(Y^g - Y)$ denotes the local time at time t and level 0 of the semimartingale $(Y^g - Y)$.

Identifying the terms of $(Y_t^g - Y_t)^+$ with those of $(Y_t^g - Y_t)$, we show that $(Z_s - Z_s^g)\mathbf{1}_{\{Y_s^g = Y_s\}} = 0$ for *a.e.* (s,ω). Since $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^+ = 0$ and $f(s,y,z) \leq g(y,z)$, we deduce that :

$$\begin{aligned} 0 &\leq L_t^0(Y^g - Y) + \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(Y_s^g, Z_s^g) - f(s,Y_s,Z_s)]ds \\ &= -\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- \leq 0 \end{aligned}$$

It follows that $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- = 0$, which implies that $dK^- = 0$. Arguing symmetrically, one can show that $dK^+ = 0$. Since both Y^g and Y^{-g} belong to $\mathcal{S}^{e^{\lambda T}+1}$, so does for Y. Arguing as the proof of Lemma (3.3), we can check that $Z \in \mathcal{M}^2$.

Proof. of Theorem 34. Arguing as in the proof of Theorem 32, we show that for every $R > 2$, $\beta \in]1, \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$, $\delta < (\beta - 1)\min\left(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for every $N > N_0$ and every $T' \leq T$

$$\limsup_{n\to+\infty} \mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t^n - Y_t|^\beta + \mathbb{E}\int_{(T'-\delta)^+}^{T'} \frac{|Z_s^n - Z_s|^2}{\left(|Y_s^n - Y_s|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta} \limsup_{n\to+\infty} \mathbb{E}|Y_{T'}^n - Y_{T'}|^\beta.$$

Taking successively $T' = T$, $T' = (T' - \delta)^+$, ... , we get the convergence in the whole interval $[0, T]$. In particular, we have for every $q < 2$, $\lim_{n\to+\infty}(|Y^n - Y|^q) = 0$ and $\lim_{n\to+\infty}(|Z^n - Z|^q) = 0$ *in measure* $P \times dt$. Since (Y^n) and (Z^n) are square integrable, the proof is finished by using an uniform integrability argument. Theorem 34 is proved. ■

4.4.4 Proof of Theorem 35

4.4.5 Continuity of the map $(t, x) \longmapsto Y_t^{(t,x)}$

Proposition 4.4.2. *Assume* (**H1**)–(**H4**) *and* (**H6**) *hold. Then, the map* $(t, x) \longmapsto Y_t^{(t,x)}$ *is continuous.*

Proof. Let $(t_n, x_n) \to (t, x)$ such that $t_n \leq t$ for each n. The proof goes symmetrically when $t_n \geq t$. Since b and σ are Lipshitz and of linear growth, there exists a positive constant

$C' = C'(x,T,k)$ such that for n large enough and for every $k \in \mathbb{N}$

$$\mathbb{E}\left(\sup_{0\leq\ s\leq T} |X_s^{t_n,x_n}|^k + |X_s^{t,x}|^k\right) \leq C'. \tag{4.4.5}$$

and

$$\mathbb{E}(\sup_{0\leq\ s\leq T} |X_s^{t_n,x_n} - X_s^{t,x}|^2) \leq C'\left(|t_n - t| + |x_n - x|^2\right). \tag{4.4.6}$$

In the other hand, since $|Y_{t_n}^{t_n,x_n} - Y_t^{t,x}|$ is deterministic, we have

$$\begin{aligned}
|Y_t^{t,x} - Y_{t_n}^{t_n,x_n}| &\leq \mathbb{E}\left(|Y_t^{t,x} - Y_{t_n}^{t_n,x_n}|\right) \\
&\leq \mathbb{E}\Big(|Y_t^{t,x} - Y_{t_n}^{t,x}|\Big) + \mathbb{E}\Big(|Y_{t_n}^{t,x} - Y_{t_n}^{t_n,x_n}|\Big), \\
&:= I_1^n + I_2^n.
\end{aligned}$$

where

$$I_1^n := \mathbb{E}\Big(|Y_t^{t,x} - Y_{t_n}^{t,x}|\Big) \qquad \text{and} \qquad I_2^n := \mathbb{E}\Big(|Y_{t_n}^{t,x} - Y_{t_n}^{t_n,x_n}|\Big)$$

Using Lemma 38, Lemma 39 and Lemma 40, we get $lim_{n\to\infty} I_1^n = 0$. We shall show that $lim_{n\to\infty} I_2^n = 0$. Since $I_2^n \leq \mathbb{E}\Big(\sup_{0\leq s\leq T} |Y_s^{t_n,x_n} - Y_s^{t,x}|\Big)$, we proceed as in the proofs of Lemmas 44 and 45 to get,

$$\begin{aligned}
&\mathbb{E}\sup_{(T'-\delta)^+\leq s\leq T'} |Y_s^{t_n,x_n} - Y_s^{t,x}|^\beta + \mathbb{E}\int_{(T'-\delta)^+}^{T'} \frac{|Z_s^{t_n,x_n} - Z_s^{t,x}|^2}{\left(|Y_s^{t_n,x_n} - Y_s^{t,x}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds \\
&\leq \frac{\ell}{\beta-1} e^{C_N\delta}\mathbb{E}|H\left(X_{T'}^{t_n,x_n}\right) - H\left(X_{T'}^{t,x}\right)|^\beta + \frac{\ell}{\beta-1}\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}} \\
&+ \frac{2\ell}{\beta-1}\beta K_3^{\frac{1}{\bar\alpha}}\left(4T'K_2 + T'\ell\right)^{\frac{\beta-1}{2}}\left(8T'K_2 + 8K_1\right)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}} \\
&+ \frac{2\ell}{\beta-1} e^{C_N\delta}\beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\left[\mathbb{E}\int_t^{T'} |f(s, X_s^{t_n,x_n}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})| ds\right].
\end{aligned}$$

Since f and H are continuous in the x–variable, we successively pass to the limit on n and N to get

$$\limsup_{n,N\to+\infty}\mathbb{E}\sup_{(T'-\delta')^+\le t\le T'}|Y_t^{t_n,x_n}-Y_t^{t,x}|^\beta+\mathbb{E}\int_{(T'-\delta)^+}^{T'}\frac{|Z_s^{t_n,x_n}-Z_s^{t,x}|^2}{\left(|Y_s^{t_n,x_n}-Y_s^{t,x}|^2+\nu_R\right)^{\frac{2-\beta}{2}}}ds=0.$$

Taking successively $T'=T, T'=(T'-\delta)^+, \ldots$, we show that for every $\beta\in\left]1,\quad\min\left(3-\frac{2}{\bar{\alpha}},2\right)\right[$,

$$\lim_{n\to+\infty}\mathbb{E}\sup_{0\le s\le T}|Y_s^{t_n,x_n}-Y_s^{t,x}|^\beta=0.$$

Since $\beta>1$, we then conclude the proof by using Holder's inequality. ■

4.4.6 $u(t,x):=Y_t^{(t,x)}$ is a viscosity solution to PDE (4.3.2)

We will follow the method of [97]. We then need the following touching lemma which allows to avoid the comparison theorem. The proof of the touching lemma can be found for instance in [97].

Lemma 47. *Let $(\xi_t)_{0\le t\le T}$ be a continuous adapted process such that,*

$$d\xi_t=\beta(t)dt+\alpha(t)dW_t,$$

where β and α are continuous adapted processes such that β and $|\alpha|^2$ are integrable. If $\xi_t\ge 0$ a.s. for all t, then for all t,

$$\mathbf{1}_{\{\xi_t=0\}}\alpha(t)=0\quad a.s.,$$

$$\mathbf{1}_{\{\xi_t=0\}}\beta(t)\ge 0\quad a.s..$$

We now prove Theorem 35. From Proposition 4.4.2, the map $v(t,x):=Y_t^{t,x}$ is continuous in (t,x). It remains to prove that $v(t,x)$ is a viscosity solution to PDE (4.3.2). To simplify the notations, we denote $(X_s, Y_s, Z_s) := (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$. Since $v(t,x) = Y_t^{t,x}$, then the Markov property of X and the uniqueness of Y show that for every $s \in [0,T]$,

$$v(s, X_s) = Y_s.$$

We show that v is a viscosity subsolution to PDE (4.3.2). Let $\phi \in \mathcal{C}^{1,2}$ and (t,x) be a local maximum of $(v - \phi)$ which we suppose global and equal to 0, that is :

$$\phi(t,x) = v(t,x) \quad \text{ and } \quad \phi(\bar{t}, \bar{x}) \geq v(\bar{t}, \bar{x}) \text{ for each } (\bar{t}, \bar{x}).$$

It follows that

$$\phi(s, X_s) \geq Y_s.$$

By Itô's formula we have

$$d\phi(s, X_s) = \left(\frac{\partial \phi}{\partial s} + L\phi \right)(s, X_s)ds + \sigma \nabla_x \phi(s, X_s) dW_s,$$

and Y satisfies the equation

$$-dY_s = f(s, Y_s, Z_s)ds - Z_s dW_s.$$

Since $\phi(s, X_s) \geq Y_s$, then the touching property shows that for each s,

$$\mathbf{1}_{\{\phi(s,X_s)=Y_s\}} \left[\left(\frac{\partial \phi}{\partial t} + L\phi \right)(s, X_s) + f(Y_s) \right] \geq 0,$$

and

$$\mathbf{1}_{\{\phi(s,X_s)=Y_s\}} |\sigma \nabla_x \phi(s, X_s) - Z_s| = 0 \quad a.s..$$

Since for $s = t$ we have $\phi(t, X_t) = Y_t$, then the second equation gives $Z_t = \sigma \nabla_x \phi(t, X_t) := \sigma \nabla_x \phi(t,x)$, and the first inequality gives the desired result. ■

Remark 48. ***Application to Quadratic BSDEs***. *Let* $f(t, y, z)$ *be continuous in* (y, z) *and satisfies the quadratic growth :* $|f(t, y, z)| \leq a + b|y| + \frac{1}{2}|z|^2 := h(t, y, z)$. *Arguing as in the proof of Theorem 33, the solvability of* $eq(\xi, f)$ *is reduced to that of* $eq(\xi, h)$. *Using an exponential transformation, it is clear that* $eq(\xi, h)$ *is equivalent to* $eq(e^{\xi},\ a|y| + b\,|y|| \ln|y||)$. *Thanks to Theorem 32 , this last logarithmic BSDE admits a solution whenever* e^{ξ} *has finite p-moment for some* $p > 0$. *This shows that we can deduce the solvability of Quadratic BSDEs from that of logarithmic ones.*

Chapitre 5

Singularly perturbed forward backward stochastic differential equations : application to the optimal control of bilinear systems

5.1 Introduction

In this chapter we present some applications of FBSDEs and its bridge with the field of stochastic optimal control, this end is one of the important field in Mathematics which has been subject of large literature. We mention among of them [127], [71], and [20] it found increasing applications in the domain of molecular dynamics [129], [79], [143] and financial mathematics [36], [117], [58]. High-dimensionality is the common question of huge ensemble of these applications, either the system is itself high dimensional like the case of molecular dynamics, or it is produced form the space discretization of a time-space PDE in the absence of the analytic solution, these systems have characteristic time scales, which lead us to the possibility of eliminate the fast dynamic in such a way controlling the reduced system is equivalent to control the hole system.

The question of the existence of an optimal control in some appropriate sense has been subject of many authors, but the in most cases there is no analytical technique to find the explicit optimal control, in this chapter we focus on the numerical studies, which lead us to find the optimal control and the value function as well, for the question of existence of optimal control the reader is referred to [138, 70, 96, 71, 116, 130, 17] and [19].

Model order redaction MOR, is an important tools to beat the curse of dimensionality, this end came form the fact that we do a space discretization of a time-space PDE, or it can be form the system it self, like the case of molecular dynamics, MOR has been studied by several authors see [77, 144, 129, 143]

This chapter is organized as follow, first we set up the problem, the next section is concerned to the model redaction of our problem, some numerical studies are given in the next section and the last one is concerned to a study the building model which is a LQ stochastic optimal control.

5.1.1 Set-up and problem statement

We consider the linear-quadratic (LQ) stochastic control problem of the following form : minimize the expected cost

$$J(u;t,x) = \mathbb{E}\Big[\int_t^\tau \Big(q_0(X_s^u) + |u_s|^2\Big)\,ds + q_1(X_\tau^u)\,\Big|\,X_t^u = x\Big] \tag{5.1.1}$$

over all admissible controls $u \in \mathcal{U}$ and subject to

$$dX_s^u = (a(X_s^u) + b(X_s^u)u_s)\,ds + \sigma(X_s^u)dW_s\,, \quad 0 \leqslant t \leqslant s \leqslant \tau\,. \tag{5.1.2}$$

Here $\tau < \infty$ is a bounded stopping time (specified below), and the set of admissible controls $\mathcal{U}$ is chosen such that (5.1.2) has a unique strong solution. The denomination *linear-quadratic* for (5.1.1)–(5.1.2) is due to the specific dependence of the system on the control variable u. The state vector $x \in \mathbb{R}^n$ is assumed to be high-dimensional, which is why we seek a low-dimensional approximation of (5.1.1)–(5.1.2).

Specifically, we consider the case that q_0 and q_1 are quadratic in x, a is linear and σ is constant, and the control term is an affine function of x, i.e.,

$$b(x)u = (Nx + B)\, u$$

In this case the system is called *bilinear* (including linear systems as a special case), and the aim is to replace (5.1.2) by a lower dimensional bilinear system

$$d\bar{X}^v_s = \bar{A}\bar{X}^v_s\, ds + \left(\bar{N}\bar{X}^v_s + \bar{B}\right) v_s\, ds + \bar{C} dw_s\,, \quad 0 \leqslant t \leqslant s \leqslant \tau\,,$$

with states $\bar{x} \in \mathbb{R}^{n_s}$, $n_s \ll n$ and an associated reduced cost functional

$$\bar{J}(v; \bar{x}, t) = \mathbb{E}\left[\int_t^\tau \left(\bar{q}_0(\bar{X}^v_s) + |v_s|^2\right) ds + \bar{q}_1(\bar{X}^v_\tau) \,\middle|\, \bar{X}^v_t = \bar{x}\right]\,,$$

that is solved in lieu of (5.1.1)–(5.1.2). Letting v^* denote the minimizer of $\bar{J}$, we require that v^* is a good approximation of the minimizer u^* of the original problem where "good approximation" is understood in the sense that

$$J(v^*; \cdot, t = 0) \approx J(u^*; \cdot, t = 0)\,.$$

In the last equation, closeness must be suitably interpreted, e.g. uniformly on all compact subsets of $\mathbb{R}^n \times [0, T)$ for some $T < \infty$. One situation in which the above approximation property holds is when $u^* \approx v^*$ uniformly in t and the cost is continuous in the control, but it turns out that this requirement will be too strong in general and overly restrictive. We will discuss alternative criteria in the course of this thesis.

5.2 Singularly perturbed bilinear control systems

We now specify the system dynamics (5.1.2) and the corresponding cost functional (5.1.1). Let $(x_1, x_2) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ with $n_s + n_f = n$ denote a decomposition of the state vector $x \in \mathbb{R}^n$ into relevant (slow) and irrelevant (fast) components. Further let $W = (W_t)_{t\geq 0}$ denote $\mathbb{R}^m$-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ that is endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W. For any initial condition $x \in \mathbb{R}^n$ and any $\mathcal{A}$-valued admissible control $u \in \mathcal{U}$, with $\mathcal{A} \subset \mathbb{R}$, we consider the following system of Itô stochastic differential equations

$$dX_s^\epsilon = AX_s^\epsilon\, ds + (NX_s^\epsilon + B)u_s\, ds + CdW_s\,,\; X_t^\epsilon = x\,, \tag{5.2.1}$$

that depends parametrically on a parameter $\varepsilon > 0$ via the coefficients

$$A = A^\varepsilon \in \mathbb{R}^{n\times n}\,,\; N = N^\varepsilon \in \mathbb{R}^{n\times n}\,,\; B = B^\varepsilon \in \mathbb{R}^n\,,\; \text{and } C = C^\varepsilon \in \mathbb{R}^{n\times m}\,,$$

where for brevity we also drop the dependence of the process on the control u, i.e. $X_s^\varepsilon = X_s^{u,\varepsilon}$. The stiffness matrix A in (5.2.1) is assumed to be of the form

$$A = \begin{pmatrix} A_{11} & \epsilon^{-1/2}A_{12} \\ \epsilon^{-1/2}A_{21} & \epsilon^{-1}A_{22} \end{pmatrix} \in \mathbb{R}^{(n_s+n_f)\times(n_s+n_f)}\,, \tag{5.2.2}$$

with $n = n_s + n_f$. Control and noise coefficients are given by

$$N = \begin{pmatrix} N_{11} & N_{12} \\ \\ \epsilon^{-1/2} N_{21} & \epsilon^{-1/2} N_{22} \end{pmatrix} \in \mathbb{R}^{(n_s+n_f)\times(n_s+n_f)} \tag{5.2.3}$$

and

$$B = \begin{pmatrix} B_1 \\ \epsilon^{-1/2} B_2 \end{pmatrix} \in \mathbb{R}^{(n_s+n_f)\times 1}, \quad C = \begin{pmatrix} C_1 \\ \epsilon^{-1/2} C_2 \end{pmatrix} \in \mathbb{R}^{(n_s+n_f)\times m}\,, \tag{5.2.4}$$

where $Nx + B \in range(C)$ for all $x \in \mathbb{R}^n$; often we will consider either the case $m = 1$ with $C_i = \sqrt{\rho} B_i$, $\rho > 0$, or $m = n$, with C being a multiple of the identity when $\varepsilon = 1$. All block matrices A_{ij}, N_{ij}, B_i and C_j are assumed to be order 1 and independent of ε.

The above ε-scaling of coefficients is natural for a system with n_s slow and n_f fast degrees of freedom and arises, for example, as a result of a balancing transformation applied to a large-scale system of equations ; see e.g. [78]. A is the linear system

$$dX_s^\epsilon = (AX_s^\epsilon + Bu_s)\, ds + CdW_s\,. \tag{5.2.5}$$

Our goal is to control the stochastic dynamics (5.2.1)—or (5.2.5) as a special variant—so that a given cost criterion is optimized. Specifically, given two symmetric positive semidefinite matrices $Q_0, Q_1 \in \mathbb{R}^{n_s \times n_s}$, we consider the quadratic cost functional

$$J(u; t, x) = \mathbb{E}\left[\frac{1}{2}\int_t^\tau ((X_{1,s}^\epsilon)^\top Q_0 X_{1,s}^\epsilon + |u_s|^2) ds + \frac{1}{2}(X_{1,\tau}^\epsilon)^\top Q_1 X_{1,\tau}^\epsilon\right], \tag{5.2.6}$$

that we seek to minimize subject to the dynamics (5.2.1). Here the expectation is understood over all realizations of $(X_s^\varepsilon)_{s\in[t,\tau]}$ starting at $X_t^\varepsilon = x$, and as a consequence J is a function of the initial data (t, x). The stopping time is defined as the minimum of some time $T < \infty$

and the first exit time of a domain $D = D_s \times \mathbb{R}^{n_f} \subset \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ where D_s is an open and bounded set with smooth boundary. Specifically, we set $\tau = \min\{\tau_D, T\}$, with

$$\tau_D = \inf\{s \geq t : X_s^\varepsilon \notin D\}\,.$$

In other words, τ is the stopping time that is defined by the event that either $s = T$ or X_s^ε leaves the set $D = D_s \times \mathbb{R}^{n_f}$, whichever comes first. Note that the cost function does not explicitly depend on the fast variables x_2. We define the corresponding value function by

$$V^\varepsilon(t,x) = \inf_{u\in\mathcal{U}} J(u;t,x)\,. \tag{5.2.7}$$

Remark 49. *1. As a consequence of the boundedness of $D_s \subset \mathbb{R}^{n_s}$, we may assume that all coefficients in our control problem are bounded or Lipschitz continuous, which makes some of the proofs in this work more transparent.*

2. All of the following considerations trivially carry over to the case $N = 0$ and a multi-dimensional control variable, i.e., $u \in \mathbb{R}^k$ and $B \in \mathbb{R}^{n\times k}$.

5.2.1 From LQ control to uncoupled forward-backward stochastic differential equations

We suppose that the matrix pair (A, C) satisfies the Kalman rank condition

$$rank(C|AC|A^2C|\ldots|A^{n-1}C) = n\,. \tag{5.2.8}$$

A necessary—and in this case sufficient—condition for optimality of our optimal control problem is that the value function (5.2.7) solves a semilinear parabolic partial differential equation of Hamilton-Jacobi-Bellman type (a.k.a. *dynamic programming equation*) [69]

$$-\frac{\partial V^\varepsilon}{\partial t} = L^\varepsilon V^\varepsilon + f(x, V^\varepsilon, C^\top \nabla V^\varepsilon)\,, \quad V^\varepsilon|_{E^+} = q_1\,, \tag{5.2.9}$$

where

$$q_1(x) = \frac{1}{2} x_1^\top Q_1 x_1$$

and E^+ is the terminal set of the augmented process (s, X_s^ε), precisely $E^+ = ([0,T) \times \partial D) \cup (\{T\} \times D)$. Here L^ε is the infinitesimal generator of the control-free process,

$$L^\varepsilon = \frac{1}{2} C C^\top : \nabla^2 + (Ax) \cdot \nabla\,, \tag{5.2.10}$$

and the nonlinearity f is independent of ε and given by

$$f(x,y,z) = \frac{1}{2} x_1^\top Q_0 x_1 - \frac{1}{2} \left| \left(x^\top N^\top + B^\top\right) \left(C^\top\right)^\sharp z \right|^2 . \tag{5.2.11}$$

Note that f is furthermore independent of y and that the Moore-Penrose pseudoinverse

$$\left(C^\top\right)^\sharp = C(C^\top C)^{-1}$$

is unambiguously defined since $z = C^\top \nabla V^\varepsilon$ and $(Nx + B) \in \mathit{range}(C)$, which by noting that $\left(C^\top\right)^\sharp C^\top$ is the orthogonal projection onto $\mathit{range}(C)$ implies that

$$\left|(x^\top N^\top + B^\top) \nabla V^\varepsilon\right|^2 = \left|(x^\top N^\top + B^\top) \left(C^\top\right)^\sharp z\right|^2 .$$

The specific semilinear form of the equation is a consequence of the control problem being linear-quadratic. As a consequence, the dynamic programming equation (5.2.9) admits a representation in form of an uncoupled forward backward stochastic differential equation

(FBSDE). To appreciate this point, consider the control-free process $X_s^\varepsilon = X_s^{\varepsilon,u=0}$ with infinitesimal generator L^ε and define an adapted process $Y_s^\varepsilon = Y_s^{\varepsilon,x,t}$ by

$$Y_s^\varepsilon = V^\varepsilon(s, X_s^\varepsilon)\,.$$

(We abuse notation and denote both the controlled and the uncontrolled process by X_s^ε.) Then, by definition, $Y_t^\varepsilon = V^\varepsilon(x,t)$. Moreover, by Itô's formula and the dynamic programming equation (5.2.9), the pair $(X_s^\varepsilon, Y_s^\varepsilon)_{s\in[t,\tau]}$ can be shown to solve the system of equations

$$\begin{aligned} dX_s^\varepsilon &= AX_s^\varepsilon\, ds + C\, dW_s\,, \quad X_t^\varepsilon = x \\ dY_s^\varepsilon &= -f(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)ds + Z_s^\varepsilon\, dW_s\,, \quad Y_\tau^\varepsilon = q_1(X_\tau^\varepsilon)\,, \end{aligned} \tag{5.2.12}$$

with $Z_s^\varepsilon = C^\top \nabla V^\varepsilon(s, X_s^\varepsilon)$ being the control variable. Here, the second equation is only meaningful if interpreted as a backward equation, since only in this case Z_s^ε is uniquely defined.

The BSDE (5.2.12) is a quadratic backward stochastic differential equation, by [10] it has at least one solution, and by [97] this solution is bounded by using the fact that the terminal condition is bounded [1]

Remark 50. *Equation (5.2.12) is called an* uncoupled FBSDE *because the forward equation for $\tilde{X}_s^\varepsilon$ is independent of Y_s^ε or Z_s^ε. The fact that the FBSDE is uncoupled furnishes a well-known duality relation between the value function of an LQ optimal control problem and the cumulate generating function of the cost [43, 55]; specifically, in the case that $N = 0$, $B = C$ and the pair (A, B) being completely controllable, it holds that*

$$V^\varepsilon(x,t) = -\log \mathbb{E}\left[\exp\left(-\int_t^\tau q_0(X_s^\epsilon)ds - q_1(X_\tau^\epsilon)\right)\right], \tag{5.2.13}$$

1. The boundedness of the terminal condition came from the fact that x leave in the bounded domain D

with

$$q_0(x) = \frac{1}{2} x_1^\top Q_0 x_1 \,.$$

Here the expectation on the right hand side is taken over all realisations of the control-free process $X_s^\varepsilon = X_s^{\varepsilon,u=0}$*, starting at* $X_t^\varepsilon = x$*. By the Feynman-Kac theorem, the function* $\psi^\varepsilon = \exp(-V^\varepsilon)$ *solves the linear parabolic boundary value problem*

$$\left(\frac{\partial}{\partial t} + L^\varepsilon\right)\psi^\varepsilon = q_0(x)\psi^\varepsilon \,, \quad \psi^\varepsilon|_{E^+} = \exp\left(-q_1\right) \,. \tag{5.2.14}$$

5.3 Model reduction

The idea now is to exploit the fact that (5.2.12) is uncoupled, which allows us to derive an FBSDE for the slow variables $\bar{X}_s^\varepsilon = X_{1,s}^\varepsilon$ only, by standard singular perturbation methods. The reduced FBSDE as $\varepsilon \to 0$ will then be of the form

$$\begin{aligned} d\bar{X}_s &= \bar{A}\bar{X}_s \, ds + \bar{C} \, dW_s \,, \quad \bar{X}_t = x_1 \\ d\bar{Y}_s &= -\bar{f}(\bar{X}_s, \bar{Y}_s, \bar{Z}_s) ds + \bar{Z}_s \, dW_s \,, \quad \bar{Y}_\tau = \bar{q}_1(\bar{X}_\tau) \,, \end{aligned} \tag{5.3.1}$$

where the limiting form of the backward SDE follows from the corresponding properties of the forward SDE. Specifically, assuming that the solution of the associated SDE

$$d\xi_u = A_{22}\xi_u du + C_2 dW_u \,, \tag{5.3.2}$$

that is governing the fast dynamics as $\varepsilon \to 0$, is ergodic with unique Gaussian invariant measure $\pi = \mathcal{N}(0, \Sigma)$, where $\Sigma = \Sigma^\top > 0$ is the unique solution to the Lyapunov equation

$$A_{22}\Sigma + \Sigma A_{22}^\top = -C_2 C_2^\top \,, \tag{5.3.3}$$

we obtain that, asymptotically as $\varepsilon \to 0$,

$$X_{2,s}^\varepsilon \sim \xi_{u/\varepsilon} \,, \quad s > 0 \,. \tag{5.3.4}$$

As a consequence, the limiting SDE governing the evolution of the slow process $X^{\varepsilon}_{1,s}$— in other words : the forward part of (5.3.1)—has the coefficients

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}\,, \quad \bar{C} = C_1 - A_{12}A_{22}^{-1}C_2\,, \tag{5.3.5}$$

as following from standard homogenization arguments [121]. By a similar reasoning we find that the driver of the limiting backward SDE reads

$$\bar{f}(x_1, y, z_1) = \int_{\mathbb{R}^{n_f}} f((x_1, x_2), y, (z_1, 0))\,\pi(dx_2)\,, \tag{5.3.6}$$

specifically,

$$\bar{f}(x_1, y, z_1) = \frac{1}{2}x_1^\top \bar{Q}_0 x_1 - \frac{1}{2}\Big|\left(x_1^\top \bar{N}^\top + \bar{B}^\top\right) z_1\Big|^2 + K_0\,, \tag{5.3.7}$$

with

$$\bar{Q}_0 = Q_0\,, \quad \bar{N} = C_1^\sharp N_{11}\,, \quad \bar{B} = C_1^\sharp\left(B_1 + N_{12}\Sigma^{1/2}\right)\,. \tag{5.3.8}$$

The limiting backward SDE is equipped with a terminal condition $\bar{q}_1$ that is equals q_1, namely,

$$\bar{q}_1(x_1) = \frac{1}{2}x_1^\top Q_1 x_1\,. \tag{5.3.9}$$

Interpretation as an optimal control problem

It is possible to interpret the reduced FBSDE again as the probabilistic version of a dynamic programming equation. To this end, note that (5.2.8) implies that the matrix pair $(\bar{A}, \bar{C})$ satisfies the Kalman rank condition

$$rank(\bar{C}|A\bar{C}|A^2\bar{C}|\ldots|A^{n_s-1}\bar{C}) = n_s\,.$$

As a consequence, the semilinear partial differential equation

$$-\frac{\partial V}{\partial t} = \bar{L}V + \bar{f}(x_1, V, \bar{C}^\top \nabla V)\,, \quad V|_{E_s^+} = \bar{q}_1\,, \tag{5.3.10}$$

with $E_s^+ = ([0,T) \times \partial D_s) \cup (\{T\} \times D_s)$ and

$$\bar{L} = \frac{1}{2}\bar{C}\bar{C}^\top : \nabla^2 + (\bar{A}x_1)\cdot\nabla \tag{5.3.11}$$

has a classical solution $V \in C^{1,2}([0,T)\times D) \cap C^{0,1}(E_s^+)$. Letting $\bar{Y}_s := V(s, \bar{X}_s)$, $0 \leqslant t \leqslant s \leqslant \tau$, with initial data $\bar{X}_t = x_1$ and $\bar{Z}_s = \bar{C}^\top \nabla V(s, \bar{X}_s)$, the limiting FBSDE (5.3.1) can be readily seen to be equivalent to (5.3.10). The latter is the dynamic programming equation of the following LQ optimal control problem : minimize the cost functional

$$\bar{J}(v; t, x_1) = \mathbb{E}\left[\frac{1}{2}\int_t^\tau (\bar{X}_s^\top \bar{Q}_0 \bar{X}_s + |v_s|^2)ds + \frac{1}{2}\bar{X}_\tau^\top \bar{Q}_1 \bar{X}_\tau\right], \tag{5.3.12}$$

subject to

$$d\bar{X}_s = \bar{A}\bar{X}_s ds + \left(\bar{M}\bar{X}_s + \bar{D}\right) v_s\, ds + \bar{C} dw_s\,, \quad \bar{X}_t = x_1\,, \tag{5.3.13}$$

where $(w_s)_{s\geq 0}$ denotes standard Brownian motion in $\mathbb{R}^{n_s}$ and we have introduced the new control coefficients $\bar{M} = \bar{C}\bar{N}$ and $\bar{D} = \bar{C}\bar{B}$.

5.3.1 Convergence of the control value

Before we state our main result and discuss its implications for the model reduction of linear and bilinear systems, we recall that basic assumptions that we impose on the system dynamics. Specifically, we say that the dynamics (5.2.1) and the corresponding cost functional (5.2.6) satisfy **Condition U** if the following holds :

1. (A, C) is controllable, and the range of $b(x) = Nx + B$ is a subspace of $range(C)$.

2. We suppose that one of the following conditions hold true

   — B=C, or

   — C^{-1} exist and have the form

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & \epsilon^{-1/2} C_2 \end{pmatrix} \tag{5.3.14}$$

3. The matrix A_{22} is Hurwitz (i.e., its spectrum lies entirely in the open left complex half-plane) and the matrix pair (A_{22}, C_2) is controllable.

4. The driver of the FBSDE (5.2.12) is continuous and quadratically growing in Z.

5. The terminal condition in (5.2.12) is bounded ; for simplicity we set $Q_1 = 0$ in (5.2.6).

Remark 51. *for assumption (2) we can explicitly compute the limiting coefficients, we can relax this condition by finding a larger class of couple (B,C), which hold the convergence of the original system to the limiting one and where we can explicitly compute the limiting PDE which is subject of future work.*

Assumption 3 implies that the fast subsystem (5.3.2) has a unique Gaussian invariant measure $\pi = \mathcal{N}(0, \Sigma)$ with full topological support, i.e., we have $\Sigma = \Sigma^\top > 0$. According to [28, Prop. 3.1], existence and uniqueness of (5.2.12) is guaranteed by Assumptions 4 and 5 and the controllability of (A, C) and the range condition, which imply that the transition probability densities of the (controlled or uncontrolled) forward process X_s^ε are smooth and strictly positive. As a consequence of the complete controllability of the original system, the reduced system (5.3.13) is completely controllable too, which guarantees existence and uniqueness of a classical solution of the limiting dynamic programming equation (5.3.10); see, e.g., [120].

Uniform convergence of the value function $V^\varepsilon \to V$ is now entailed by the strong convergence of the solution to the corresponding FBSDE as is expressed by the following Theorem.

Theorem 52. *Let the assumptions of Condition U hold. Further let V^ε be the classical solution of the dynamic programming equation (5.2.9) and V be the solution of (5.3.10). Then*

$$V^\varepsilon \to V\,,$$

uniformly on all compact subsets of $[0,T] \times D$.

The proof of the Theorem is given in the last section. For the reader's convenience, we present a formal derivation of the limit equation in the next subsection.

5.3.2 Formal derivation of the limiting FBSDE

Our derivation of the limit FBSDE follows standard homogenization arguments (see [74, 91, 121]), taking advantage of the fact that the FBSDE is uncoupled. To this end we consider the following linear evolution equation

$$\left(\frac{\partial}{\partial t} - L^\varepsilon\right)\phi^\varepsilon = 0\,, \quad \phi^\varepsilon(x_1, x_2, 0) = g(x_1) \tag{5.3.15}$$

for a function $\phi^\varepsilon\colon \bar{D}_s \times \mathbb{R}^{n_f} \times [0,T]$ where

$$L^\varepsilon = \frac{1}{\varepsilon}L_0 + \frac{1}{\sqrt{\varepsilon}}L_1 + L_2\,, \tag{5.3.16}$$

with

$$L_0 = \frac{1}{2} C_2 C_2^\top : \nabla^2_{x_2} + (A_{22} x_2) \cdot \nabla_{x_2} \tag{5.3.17a}$$

$$L_1 = \frac{1}{2} C_1 C_2^\top : \nabla^2_{x_2 x_1} + \frac{1}{2} C_2 C_1^\top : \nabla^2_{x_1 x_2} + (A_{12} x_2) \cdot \nabla_{x_1} + (A_{21} x_1) \cdot \nabla_{x_2} \tag{5.3.17b}$$

$$L_2 = \frac{1}{2} C_1 C_1^\top : \nabla^2_{x_1} + (A_{11} x_1) \cdot \nabla_{x_1} \tag{5.3.17c}$$

is the generator associated with the control-free forward process X_s^ε in (5.2.12). We follow the standard procedure of [121] and consider the perturbative expansion

$$\phi^\varepsilon = \phi_0 + \sqrt{\varepsilon}\phi_1 + \varepsilon\phi_2 + \ldots$$

that we insert into the Kolmogorov equation (5.3.15). Equating different powers of ε we find a hierarchy of equations, the first three of which read

$$L_0\phi_0 = 0\,, \quad L_0\phi_1 = -L_1\phi_0\,, \quad L_0\phi_2 = \frac{\partial \phi_0}{\partial t} - L_1\phi_1 - L_2\phi_0\,. \tag{5.3.18}$$

Assumption 3 on page 134 implies that L_0 has a one-dimensional nullspace that is spanned by functions that are constant in x_2, and thus the first of the three equations implies that ϕ_0 is independent of x_1. Hence the second equation—the cell problem—reads

$$L_0\phi_1 = -(A_{12} x_2) \cdot \nabla\phi_0(x_1, t)\,. \tag{5.3.19}$$

The last equation has a solution by the Fredholm alternative, since the right hand side averages to zero under the invariant measure π of the fast dynamics that is generated by the operator L_0, in other words, the right hand side of the linear equation is orthogonal to the nullspace of L_0^* spanned by the density of π.[2] The form of the equation suggests the general ansatz

$$\phi_1 = \psi(x_2) \cdot \nabla \phi_0(x_1, t) + R(x_1, t)$$

where the function R plays no role in what follows, so we set it equal to zero. Since $L_0 \psi = -(A_{12} x_2)^\top$, the function ψ must be of the form $\psi = Q x_2$ with a matrix $Q \in \mathbb{R}^{n_s \times n_f}$. Hence

$$Q = -A_{12} A_{22}^{-1} .$$

Now, solvability of the last of the three equations requires again that the right hand side averages to zero under π, i.e.

$$\int_{\mathbb{R}^{n_f}} \left(\frac{\partial \phi}{\partial t} + L_1 \left[\left(A_{12} A_{22}^{-1} x_2 \right) \cdot \nabla \phi \right] - L_2 \phi \right) \pi(dx_2) , \tag{5.3.20}$$

which formally yields the limiting equation for $\phi = \phi_0(x_1, t)$. Since π is a Gaussian measure with mean 0 and covariance Σ given by (5.3.3), the integral (5.3.20) can be explicitly computed :

$$\left(\frac{\partial}{\partial t} - \bar{L} \right) \phi , \quad \phi(x_1, 0) = g(x_1) , \tag{5.3.21}$$

where $\bar{L}$ is given by (5.3.11) and the initial condition $\phi(\cdot, 0) = g$ is a consequence of the fact that the initial condition in (5.3.15) is independent of ε. By the controllability of the pair $(\bar{A}, \bar{C})$, the limiting equation (5.3.21) has a unique classical solution and uniform convergence $\phi^\varepsilon \to \phi$ is guaranteed by standard results, e.g., [121, Thm. 20.1].

2. Here L_0^* is the formal L^2 adjoint of the operator L_0, defined on a suitable dense subspace of L^2.

Since the backward part of (5.2.12) is independent of ϵ, the final form of the homogenized FBSDE (5.3.1) is found by averaging over x_2, with the unique solution of the corresponding backward SDE satisfying $Z_{2,s} = 0$ as the averaged backward process is independent of x_2.

5.3.3 Zero viscosity limit

We consider the linear case $N = 0$ and consider the situation

$$C = \delta B \,, \quad \delta = \delta(\varepsilon) > 0 \,.$$

Now we suppose that our equation (5.2.5) is given with small noise as :

$$dX_t^\epsilon = A^\epsilon X_t^\epsilon + \delta(\epsilon) B^\epsilon dW_t, X_0^\epsilon = x. \tag{5.3.22}$$

where A is given by (5.2.2).

In this section we focus on the convergence of our system (5.3.22) when the both parameters : the homogenization parameter ϵ and the noise one δ goes to zero, we suppose that $\delta(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$, and then the question is to study $\lim_{\epsilon \longrightarrow 0} X^{1,\epsilon}$, we prove a large deviation principle. The convergence will be in the weak sense as given in [61], we suppose that ϵ go to zero faster then δ, means that $\lim_{\epsilon \downarrow 0} \frac{\epsilon}{\delta} = 0$, this is a special case when we do homogenization first and then we send the noise to zero.

In order to prove a large deviation principle upper bound we prove an analogue result which is the Laplace principle given in this :

Definition 53. *Let $\{X^\epsilon, \epsilon > 0\}$ be a family of random variables taking values in the space $\mathcal{S}$ and let I be a rate function on $\mathcal{S}$. We say that $\{X^\epsilon, \epsilon > 0\}$ satisfies the Laplace principle* *with rate function I if for every bounded and continuous function $h : \mathcal{S} \to \mathbb{R}$*

$$\lim_{\epsilon \downarrow 0} -\epsilon \ln \mathbb{E}\left[\exp\left\{-\frac{h(X^\epsilon)}{\epsilon}\right\}\right] = \inf_{x \in \mathcal{S}} \left[I(x) + h(x)\right].$$

For a Polish space $\mathcal{S}$, we have by Varadhans Lemma [132] and its converse Brycs lemma [60] equivalent between LDP the Laplace principle, then proving that $X^{1,\epsilon}$ hold a Laplace principle with the rate function I is equivalent to prove a Large Deviation Principle LDP with the same rate function.

With the same Girsanov representation, Theorem 8.6.6 [113] our uncontrolled stochastic process can represent by a controlled one $\bar{X}^{1,\epsilon}$ solution of :

$$d\bar{X}_t^\epsilon = A^\epsilon \bar{X}_t^\epsilon + B^\epsilon u_t^\epsilon + \delta(\epsilon) B^\epsilon dW_t, \bar{X}_0^\epsilon = x, \tag{5.3.23}$$

where the control process $u^\epsilon \in \mathcal{U}_{ad}$, is supposed such that

$$\sup_{\epsilon > 0} \mathbb{E} \int_0^1 \|u_t^\epsilon\|^2 \, dt < \infty$$

In order to study the limit in the weak sense as defined in [61] we defined for a Polish space $\mathcal{S}$, let $\mathcal{P}(\mathcal{S})$ be the space of probability measures on $\mathcal{S}$. Let $\Delta = \Delta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.

Let A, B, Γ be Borel sets of $D, \mathbb{R}^{n_f}, [0,1]$ respectively. Let $u^\epsilon \in U_{ad}$ and let $\bar{X}_s^\epsilon$ be the solution of the controlled dynamic. We associate with $\bar{X}^\epsilon$ and u^ϵ a family of occupation measures $\mathrm{P}^{\epsilon,\Delta}$ defined by

$$\mathrm{P}^{\epsilon,\Delta}(A \times B \times \Gamma) = \int_\Gamma \left[\frac{1}{\Delta} \int_t^{t+\Delta} 1_A(u_s^\epsilon) 1_B\left(\frac{\bar{X}_s^\epsilon}{\epsilon} \mod 1\right) ds\right] dt, \tag{5.3.24}$$

and an extension for $s > 1$ by putting $u^\epsilon_s = 0$. Now we set a result on the convergence of the pair $\{(\bar{X}^\epsilon, \mathrm{P}^{\epsilon,\Delta}), \epsilon > 0\}$ in the weak sense as defined in [61]

Theorem 54. *Given $x_0 \in \mathbb{R}^d$, consider any family $\{u^\epsilon, \epsilon > 0\}$ of controls in $\mathcal{U}_{ad}$ satisfying*

$$\sup_{\epsilon>0} \mathbb{E} \int_0^1 \|u^\epsilon_t\|^2 \, dt < \infty,$$

and that :

$$\int X^\epsilon_s \pi_{x_1}(x_2) dx_2 = 0,$$

then the family $\{(\bar{X}^\epsilon, \mathrm{P}^{\epsilon,\Delta}), \epsilon > 0\}$ is tight, given any subsequence of $\{(\bar{X}^\epsilon, \mathrm{P}^{\epsilon,\Delta}), \epsilon > 0\}$, there exists a subsequence that converges in distribution with limit $(\bar{X}, \mathrm{P})$.

5.4 Numerical studies

In this section we presents numerical results for linear and bilinear control systems and discuss the numerical discretization of uncoupled FBSDE associated with LQ stochastic control problems, discretization of stochastic dynamics were subject of many authors we mansion [35, 109, 27, 95]. We begin with the latter.

5.4.1 Numerical FBSDE discretization

The fact that the (5.2.12) or (5.3.1) are decoupled entails that they can be discredited by an explicit time-stepping algorithm. Here we utilize a variant of the least-squares Monte Carlo algorithm proposed in [26]. The convergence of numerical schemes for FBSDE with quadratic nonlinearities in the driver has been analysed in [131].

The least-squares Monte Carlo scheme is based on the Euler discretization of (5.2.12) :

$$\hat{X}_{n+1} = \hat{X}_n + \Delta t A \hat{X}_n + \sqrt{\Delta t} C \xi_{n+1}$$
$$\hat{Y}_{n+1} = \hat{Y}_n - \Delta t f(\hat{X}_n, \hat{Y}_n, \hat{Z}_n) + \sqrt{\Delta t} \hat{Z}_n \cdot \xi_{n+1} \tag{5.4.1}$$

where $(\hat{X}_n, \hat{Y}_n)$ denotes the numerical discretization of the joint process $(X_s^\epsilon, Y_s^\varepsilon)$, where we set $X_s^\epsilon = X_{\tau_D}^\epsilon$ for $s \in (\tau_D, T]$ when $\tau_D < T$, and $(\xi_k)_{k\geq 1}$ is an i.i.d. sequence of normalized Gaussian random variables. Now let

$$\mathcal{F}_n = \sigma\Big(\Big\{\hat{W}_k : 0 \leqslant k \leqslant n\Big\}\Big)$$

be the σ-algebra generated by the discrete Brownian motion $\hat{W}_n := \sqrt{\Delta t} \sum_{i \leqslant n} \xi_i$. By definition the joint process $(X_s^\epsilon, Y_s^\epsilon)$ is adapted to the filtration generated by $(W_u)_{0 \leqslant u \leqslant s}$, therefore

$$\hat{Y}_n = \mathbb{E}\Big[\hat{Y}_n | \mathcal{F}\Big] = \mathbb{E}\Big[\hat{Y}_{n+1} + \Delta t f(\hat{X}_n, \hat{Y}_n, \hat{Z}_n) | \mathcal{F}_n\Big]\,, \tag{5.4.2}$$

where we have used that $\hat{Z}_n$ is independent of ξ_{n+1}. In order to compute $\hat{Y}_n$ from $\hat{Y}_{n+1}$ we use the identification of Z_s^ϵ with $C^\top \nabla V^\epsilon(s, X_s^\epsilon)$ and replace (5.4.2) by the backward iteration

$$\hat{Y}_n = \mathbb{E}\Big[\hat{Y}_{n+1} + \Delta t f(\hat{X}_n, \hat{Y}_{n+1}, C^\top \hat{Y}_{n+1}) | \mathcal{F}_n\Big]\,, \tag{5.4.3}$$

which makes the overall scheme explicit in $\hat{X}_n$ and $\hat{Y}_n$.

Least-squares solution of the backward SDE

In order to evaluate the conditional expectation $\hat{Y}_n = \mathbb{E}[\cdot | \mathcal{F}_n]$ we recall that a conditional expectation can be characterised as the solution to the following quadratic minimization problem :

$$\mathbb{E}\Big[S | \mathcal{F}_n\Big] = \underset{Y \in L^2,\, \mathcal{F}_n\text{-measurable}}{\operatorname{argmin}} \mathbb{E}[|Y - S|^2]\,.$$

Given N independent realizations $\hat{X}_n^{(i)}$, $i = 1, \ldots, N$ of the forward process $\hat{X}_n$, this suggests the approximation scheme

$$\hat{Y}_n \approx \underset{Y=Y(\hat{X}_n)}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \left| Y - \hat{Y}_{n+1}^{(i)} - \Delta t f\left(\hat{X}_n^{(i)}, \hat{Y}_{n+1}^{(i)}, C^\top \hat{Y}_{n+1}^{(i)}\right) \right|^2, \tag{5.4.4}$$

where $\hat{Y}^{(i)}$ is defined by $\hat{Y}^{(i)} = Y\left(\hat{X}^{(i)}\right)$ with terminal values

$$\hat{Y}_M^{(i)} = q_1\left(X_M^{(i)}\right) \quad \tau = M\Delta t\,.$$

(Note that $M = M_D$ is random.) For simplicity, we assume in what follows that the terminal value is zero, i.e., we set $q_1 = 0$. (Recall that the existence and uniqueness result from [97] requires q_1 to be bounded.) To represent $\hat{Y}_n$ as a function $Y(\hat{X}_n)$ we use the ansatz

$$Y(\hat{X}_n) = \sum_{k=1}^{K} \alpha_k(n) \varphi_k(\hat{X}_n)\,, \tag{5.4.5}$$

with coefficients $\alpha_1(\cdot), \ldots, \alpha_K(\cdot) \in \mathbb{R}$ and suitable basis functions $\varphi_1, \ldots, \varphi_K \colon \mathbb{R}^n \to \mathbb{R}$ (e.g. Gaussians). Note that the coefficients α_k are the unknowns in the least-squares problem (5.4.4) and thus are independent of the realization. Now the least-squares problem that has to be solved in the n-th step of the backward iteration is of the form

$$\hat{\alpha}(n) = \underset{\alpha \in \mathbb{R}^K}{\operatorname{argmin}} \left\| A_n \alpha - b_n \right\|^2 , \tag{5.4.6}$$

with coefficients

$$A_n = \left(\varphi_k\left(\hat{X}_n^{(i)}\right)\right)_{i=1,\ldots,N; k=1,\ldots,K} \tag{5.4.7}$$

and data

$$b_n = \left(\hat{Y}_{n+1}^{(i)} - \Delta t f\left(\hat{X}_n^{(i)}, \hat{Y}_{n+1}^{(i)}, C^\top \hat{Y}_{n+1}^{(i)}\right)\right)_{i=1,\ldots,N} . \tag{5.4.8}$$

Assuming that the coefficient matrix $A_n \in \mathbb{R}^{N\times K}$, $K \leqslant N$ defined by (5.4.7) has maximum rank K, then the solution to the least-squares problem (5.4.6) is given by

$$\hat{\alpha}(n) = \left(A_n^\top A_n\right)^{-1} A_n^\top b_n \,. \tag{5.4.9}$$

The thus defined scheme is strongly convergent of order 1/2 as $\Delta t \to 0$ and $N, K \to \infty$ as has been analysed by [26]. Controlling the approximation quality for finite values $\Delta t, N, K$, however, requires a careful adjustment of the simulation parameters and appropriate basis functions, especially with regard to the condition number of the matrix A_n

5.4.2 Numerical solution of a FBSDEs

In this section we present some numerical analysis methods to solve decoupled and fully coupled FBSDEs, high friction example is studied.

5.4.3 Scheme for fully FBSDEs

Consider the fully coupled FBSDE (1.1.4), because of the strong coupling between the forward and the backward equations the scheme introduced by [142] cannot be implemented directly by compute first the solution of the forward and then inject it in the backward, an iterative discretization of a FBSDE is given by (see [27])

$$\begin{cases} u_i^{n,0} = 0, \\ X_0^{n,m} := x, \\ X_{i+1}^{n,m} := X_i^{n,m} + b(t_i, X_i^{n,m}, u_i^{n,m-1}(X_i^{n,m}))h + b(t_i, X_i^{n,m}, u_i^{n,m-1}(X_i^{n,m})) \vartriangle W_{i+1} \\ Y_n^{n,m} := g(X_n^{n,m}), \\ \bar{Z}_i^{n,m} := \frac{1}{h} E_{t_i}(Y_{i+1}^{n,m} \vartriangle W_{i+1}), \\ Y_i^{n,m} := E_{t_i}(Y_{i+1}^{n,m} + f(t_i, X_i^{n,m}, Y_{i+1}^{n,m}, \bar{Z}_i^{n,m})h), \\ u_i^{n,m}(X_i^{n,m}) := Y_i^{n,m}. \end{cases} \tag{5.4.10}$$

The gain of this scheme is that $Y_i^{n,m}$ depend only on $X_i^{n,m}$ and note to the other solution of the forward in the previous iterations. One of the important question in this stage is how to implement the above algorithm in a Matlab code, and the most important question is to compute the conditional expectation which appear in computing the solution of the backward SDE Y and Z, to this end we used the simulation based least squares regression estimator (see [75]), here we want to mention on our special chose of the basis as the Gaussian density defined as follows : We simulate in addition to the required number of iterations for the forward process an extra m iterations and the basis is defined as the Gaussian density with mean the extra iterations, by this way the dimension do not effect so much the speed of the algorithm

Remark 55. 1. *Note like the case of the grid basis which grow up exponentially with the dimension of the system, and this can not be computed by even a supercomputers, in our technic that results in (Figure :5.1) the dimension do not effect a lot, and this give results when working in high dimension problems.*

2. *We got the result given in (Figure :5.1) with 1000 realizations, but we plotted only 10 of them. The point behind taking a good number of realisations is to avoid the stiffness*

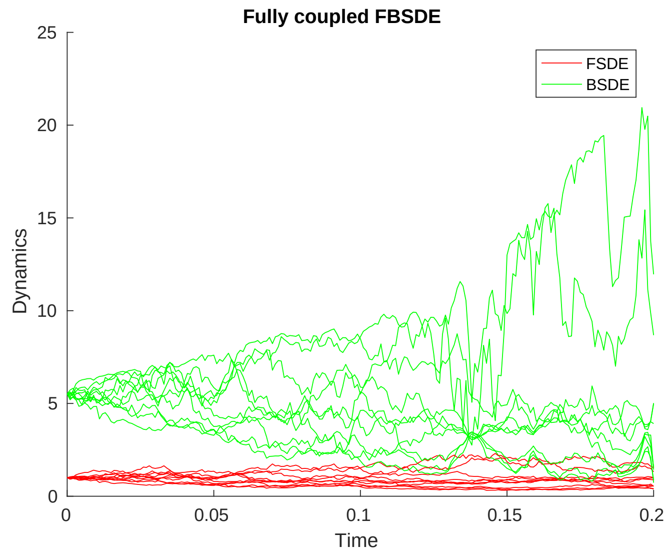

FIGURE 5.1 – Plot of 10 realizations of a solution of a Forward and Backward equation

of a matrix in the algorithm.

3. *(Figure :5.1) is for two dimensions, we plot only the first component of the FBSDE, the program take around 140s, if the dimension of the system increase to for example 10, the time will be around 290s (the dimension increase on 5 times the first but the time is only the double.*

5.5 Building Model :Los Angeles University Hospital

5.5.1 Homogenization of Quadratic Stochastic optimal Control via multiscaling a FBSDE

Now we present a special case of (5.2.1), is the Linear Quadratic Stochastic Optimal Control (LQSOC, in short) given by a fast and slow variables, our goal is to reduce the dimension of the system, the idea is to do homogenization of the system by using the links between forward backward stochastic differential equations (FBSDE) and SOC via HJB-equation, and hence we study a perturbed FBSDE which give us the limiting equation of the perturbed solution of the HJB-Equation which represent our SOC. An application in building model is given with numerical results, this last confirm our theoretical analysis of the SOC.

Introduction and Notations

Let $\epsilon > 0$, and consider the following probability space $(\Omega, \mathcal{F}, P, W)$, W is an $\mathbb{R}^m$-value Brownian mention, endowed with a filtration $\mathbb{F}$ satisfying the usual assumptions (i.e. $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$).

For any initial condition $x \in \mathbb{R}^n$ and any $\mathbb{R}^m$-value admissible control $u^\epsilon \in \mathcal{U}_{ad}$, we consider the following controlled stochastic linear and bilinear differential equation respectively :

$$dX_t^\epsilon = (A^\epsilon X_t^\epsilon + B^\epsilon u_t^\epsilon)dt + B^\epsilon dW_t, X_0^\epsilon = x, \tag{5.5.1}$$

The matrices A^ϵ and B^ϵ are given below

Our goal is to control the stochastic dynamic (5.5.1) by optimize a given functional but only when our dynamic live in a bounded set, which is the case in a large number of applications, for this, let consider a bounded set $D \in \mathbb{R}^n$ and define the stopping time :

$$\tau = \inf\{t \geq 0, X_t \notin D\}, \tag{5.5.2}$$

the initial condition $x \in D$, and the objective is to minimize over the controls u^ϵ the quadratic functional :

$$J(u(.),x)=\frac{1}{2}E[\int_0^{\tau}(X_t^{\epsilon T}Q_0X_t^{\epsilon}+|u_t^{\epsilon}|^2)dt+X_{\tau}^{\epsilon T}Q_1X_{\tau}^{\epsilon}]. \tag{5.5.3}$$

$V^{\epsilon}(x)=inf_{u(.)}J(u(.),x)$ where :

$$A^{\epsilon}=\underbrace{\begin{pmatrix}\underbrace{A_{11}}_{n_s} & \underbrace{\epsilon^{-\frac{1}{2}}A_{12}}_{n_f}\\ \\ \epsilon^{-\frac{1}{2}}A_{21} & \epsilon^{-1}A_{22}\end{pmatrix}}_{n}$$

$$B^{\epsilon}=\begin{pmatrix}B_1\\ \epsilon^{-1/2}B_2\end{pmatrix} \tag{5.5.4}$$

The matrices A_{11} and A_{22} are square matrices, in the applications later, we will consider slow and fast dynamics, this where came from the notations n_s, n_f which refers to the dimension of the slow and the fast variables respectively.

The notion of Ergodicity is an important propriety in the homogenization technic, which is ensured by Kalmann condition (5.2.8).

Now we set The Girsanov theorem, which is an important tools in our approach :

Theorem 56. *[113] Let $Y(t)$ be an itô process with value in $\mathbb{R}^n$ of the form :*

$$dY_t=a(t,w)dt+dB_t, t\leq T$$

with initial value $Y_0=0$, where T is a given constant and B is Brownian motion with value in $\mathbb{R}^n$, *set :*

$$M_t = exp(-\int_0^t a(s,w)dB_s - \frac{1}{2}\int_0^t a(s,w)^2 ds), \tag{5.5.5}$$

Assume that $a(s,.)$ satisfies Novikovs condition :

$$\mathbb{E}_P[exp(\frac{1}{2}\int_0^T a(s,w)^2 ds)],$$

Then $Y(t)$ is an n-dimensional Brownian motion w.r.t. the probability law Q, for $t = T$., where Q is the probability measure defined by :

$$dQ(w) = M_T(w)dP(w)$$

The transformation $P \longrightarrow Q$ called the Girsanov transformation of measures.

Limiting equation for the linear optimal control

Let now study the SOC (5.5.1), (5.5.3), and consider $dB_t = u_t dt + dW_t$, by Girsanov Theorem 56 B is a standard Wiener process under probability measure Q where :

$$\frac{dQ}{dP} = exp(-\int_0^\tau u_t dW_t - \frac{1}{2}\int_0^\tau |u_t|^2 dt), \tag{5.5.6}$$

and the equation (5.5.1) write as :

$$dX_t^\epsilon = A^\epsilon X_t^\epsilon + B^\epsilon dB_t, X_0^\epsilon = x, \tag{5.5.7}$$

let denote $\overline{E}$ the expectation under the probability Q , by the duality relation ([43], [55])

$$V^\epsilon(x) = -log\overline{E}[exp(-\int_0^\tau X_t^{\epsilon T} Q_0 X_t^\epsilon dt + X_\tau^{\epsilon T} Q_1 X_\tau^\epsilon)] \tag{5.5.8}$$

In order to take the limit as ϵ goes to zero, we suppose that the fast variable is Ergodic, and note by $\pi_{x_1}(x_2)$ for a fixed $x_1 \in \mathbb{R}^{n_s}$ the density probability w.r.t the Lebesgue measure of the invariant measure corresponding to the fast variable, using the reduced dynamics technique see ([74], [91], [121]) taking ϵ to zero, the system (5.5.8), (5.5.7) converge to :

$$dX_t = \overline{A}X_t + \overline{B}dB_t, X_0 = x_1, \tag{5.5.9}$$

$$-\,log\overline{E}[exp(-\int_0^\tau {X_t}^T\overline{Q}_0 X_t dt + {X_t}^T\overline{Q}_1 X_t)], \tag{5.5.10}$$

where :

$$\overline{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \overline{B} = B_1 - B_2A_{22}^{-1}A_{21}, \tag{5.5.11}$$

and for $i = 1, 2$

$$\overline{Q}_i = Q_i^{11} - Q_i^{12}A_{22}^{-1}A_{21} - A_{21}^T A_{22}^{-T} Q_i^{21} + A_{21}^T A_{22}^{-T} Q_i^{22} A_{22}^{-1} A_{21},$$

For the convergence see the last section. Hence by the inverse Girsenov Theorem [3] apply on (5.5.7),(5.5.8) , the system (5.5.10), (5.5.9) can be writen as :

$$dX_t^1 = (\overline{A}X_t^1 + \overline{B}u_t)dt + \overline{B}dW_t^1, X_0^1 = x_1, \tag{5.5.12}$$

and the quadratic functional :

$$J(u(.), x^1) = \frac{1}{2}E[\int_0^\tau ({X_t^1}^T\overline{Q}_0 X_t^1 + |u_t|^2)dt + {X_\tau^1}^T\overline{Q}_1 X_\tau^1]. \tag{5.5.13}$$

3. here we mean inverse by using the opposite way that used to transform of system (5.5.1), (5.5.3

System (5.5.12),(5.5.13) called the reduced dynamics of our original dynamics (5.5.1), (5.5.3), (reduced in the sense that controlling the hole system is equivalent to control the reduced dimensional one.

In order to solve the system (5.5.12),(5.5.13), we introduce the Ricatti equation :

$$dP(t) = P(t)\overline{B}\overline{B}^T P(t) - \overline{A}P(t) - P(t)\overline{A}^T - \overline{Q}_0, P(\tau) = \overline{Q}_1 \tag{5.5.14}$$

the equation (5.5.14) has a unique solution [4], then the optimal control of the system (5.5.12),(5.5.13) can given as an feedback control by :

$$u_t^* = -\overline{B}^T P(t)X^* \tag{5.5.15}$$

and therefore the optimal solution X^* is the solution of the following SDE :

$$dX_t^* = (\overline{A}X_t^* - \overline{B}\overline{B}^T P(t)X^*)dt + \overline{B}dW_t, X_0^* = x_1. \tag{5.5.16}$$

To solve equation (5.5.14) and (5.5.16) we use numerical analysis, (for (5.5.16) we use Euler Maruyama method), which is given in section 5.4

5.5.2 Numerical studies of the SOC

In this section we presents numerical results on the linear and bilinear SOC, first we show by numerical results that controlling the reduced linear SOC is equivalent to control the hole system, the applications are on the build model.

4. for more details on the existence and uniqueness of the Ricatti equation we refer to [1]

A two dimension example

Consider the following SOC in 2 dimensions :

$$dX_t^\epsilon = (A^\epsilon X_t^\epsilon + B^\epsilon u_t^\epsilon)dt + \sqrt{\sigma} B^\epsilon dW_t, X_0^\epsilon = X_0, \tag{5.5.17}$$

here $\sigma = 0.002, \epsilon = 0.01$

$$J(u(.), x) = \frac{1}{2} E[\int_0^\tau (X_t^{\epsilon T} Q_0 X_t^\epsilon + |u_t^\epsilon|^2)dt + X_T^{\epsilon\, T} Q_1 X_T^\epsilon]. \tag{5.5.18}$$

$V^\epsilon(x) = \inf_{u(.)} J(u(.), x)$ where :
$A^\epsilon = \begin{pmatrix} -2 & -\epsilon^{-1} \\ \epsilon^{-1} & -2\epsilon^{-2} \end{pmatrix}$,$B^\epsilon = \begin{pmatrix} 0.1 \\ \frac{2}{\epsilon} \end{pmatrix}$ and $Q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and
$X_0 = \begin{pmatrix} 1 \\ -A_{21}/A_{22} \end{pmatrix}$,

we will show the convergence of our high dimensional stochastic optimal control to a low dimension one in such a way controlling the limiting one is equivalent to control the original (full) dynamic, this is well illustrated in (Figure :5.2), where the blue line which represent the solution of the full dynamics is totally hidden by the read line representing the solution of the limiting equation.

The Building Model

Now let's use our limiting problem (5.5.12), (5.5.13) to show by a numerical study of the problem that controlling the reduced 4 dimensions problem is equivalent to control the hole

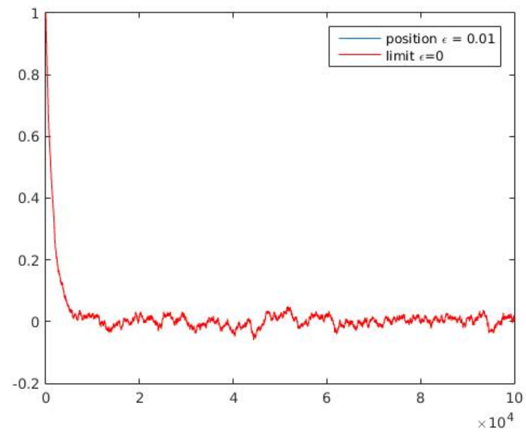

FIGURE 5.2 – Two dimension example of reducing dimension

48 dimensions, this by considering the Building Model example.

For the Building Model we study the build of Los Angeles hospital University, the 48 dimensions came from that we have 8 floors, each with 3 degrees of freedom, rotation and displacement in the plan, hence we have the 24 dimensions equation :

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = vu, \tag{5.5.19}$$

where q is the position and u is the control, M is the positive definite mass matrix, C and K denote the symmetric positive definite friction and stiffness matrices. Therefore by consider the traditional space by putting $x = (q, \dot{q})$ we get the following ODE :

$$dx(t)/dt = Ax(t) + Bu(t) \tag{5.5.20}$$

with 48 dimensions,

the out-put Y define by

$$Y = Cx, \tag{5.5.21}$$

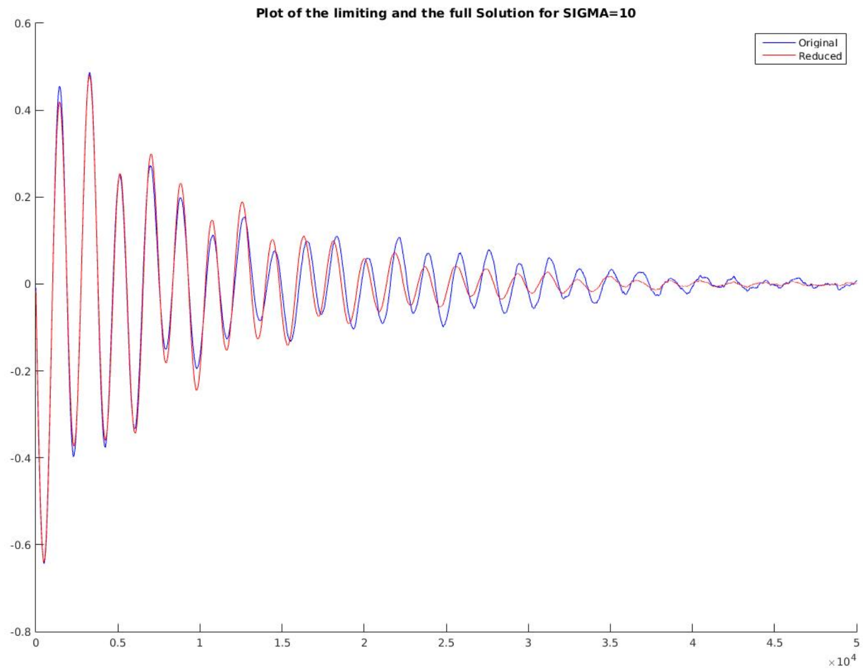

FIGURE 5.3 – Comparison between the limit and the full system

where the matrices A, B and C are in $\mathbb{R}^{48\times48}, \mathbb{R}^{48\times1}$ and $\mathbb{R}^{1\times48}$ resp., the data are given in [144], (for more details on the relation between (5.5.19) and (5.5.20), see [77])

The idea is to show that controlling a reduced 4 dimensions problem of our 48 dimensions is enough to control the hole problem and then the natural question that came is, which part of the system play the role of the reduced dimension, for this we use the balancing result of [77], and then we study the homogenization of a stochastic system by adding a noise to our ODE (5.5.20) and then, the result given in the previous section.

Figure 5.3 show that the trajectory of the out put of the reduced problem are very close to that of the hole problem.

Now we focus on the analytical results to prove the convergence of the solution of the

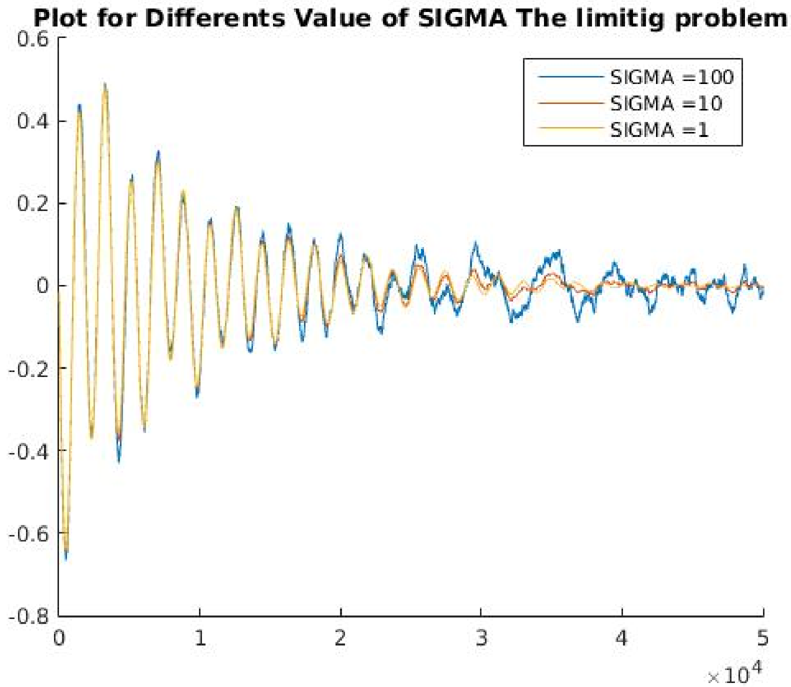

FIGURE 5.4 – Limiting out-put with different value of sigma

PDE corresponding to the hole problem (full dimension) to a reduced one, after writing our problem as a multiscale problem. The idea of the prove is the use the homogenization of FBSDE, for this end, we present first the bridge between stochastic optimal control and FBSDE, this connection is useful looking to the flexibility of applying homogenization of a FBSDEs, after that we present two approaches for the convergence of the hole system to the reduced one.

5.6 Proofs and technical lemmas

5.6.1 Convergence of the value function

The idea of the proof of Theorem 52 closely follows the work [38], with the main differences being (a) that we consider slow-fast systems exhibiting three time scales, in particular the slow equation contains singular $\mathcal{O}(\epsilon^{-1/2})$ terms, and (b) that the coefficients of the fast dynamics are not periodic, with the fast process being asymptotically Gaussian as $\epsilon \to 0$; in particular the n_f-dimensional fast process lives on the unbounded domain $\mathbb{R}^{n_f}$.

Theorem 52 rests on the following Lemma that is similar to a result in [29].

Lemma 57. *Suppose that the assumptions of Condition U on page 134 hold and define $h\colon [0,T] \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \to \mathbb{R}$ to be a function of the class $C_b^{1,2,2}$. Further assume that h is centered with respect to the invariant measure π of the fast process. Then for every $t \in [0,T]$ and initial conditions $(X^\epsilon_{1,u}, X^\epsilon_{2,u}) = (x_1, x_2) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$, $0 \leqslant u < t$, we have*

$$\lim_{\epsilon\to 0} \mathbb{E}\left[\left(\int_u^v h(s, X^\epsilon_{1,s}, X^\epsilon_{2,s}) ds\right)^2\right] = 0\,, \quad 0 \leqslant u < v \leqslant t\,. \tag{5.6.1}$$

Proof. We remind the reader of the definition (5.3.17) of the differential operators L_0, L_1 and L_2, and consider the Poisson equation

$$L_0 \psi = -h \tag{5.6.2}$$

on the domain $\mathbb{R}^{n_f}$. (The variables $x_1 \in \mathbb{R}^{n_s}$ and $t \in [0,T]$ are considered as parameters.) Since h is centered with respect to π, equation (5.6.2) has a solution by the Fredholm alternative. By Assumption 3 L_0 is a hypoelliptic operator in x_2 and thus by [123, Thm. 2], the Poisson equation (5.6.2) has a unique solution that is smooth and bounded. Applying

Itô's formula to ψ and introducing the shorthand $\delta\psi(u,v) = \psi(v, X^{\epsilon}_{1,v}, X^{\epsilon}_{2,v}) - \psi(u, x_1, x_2)$ yields

$$\begin{aligned}\delta\psi(u,v) = &\int_u^v (\partial_t\psi + L_2\psi)(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds + \frac{1}{\sqrt{\epsilon}}\int_u^v L_1\psi(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds \\ &+ \frac{1}{\epsilon}\int_u^v L_0\psi(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds + M_1(u,v) + \frac{1}{\sqrt{\epsilon}}M_2(u,v)\,,\end{aligned} \tag{5.6.3}$$

where M_1 and M_2 are square integrable martingales with respect to the natural filtration generated by the Brownian motion W_s. By the properties of the solution to (5.6.2) the first three integrals on the right hand side are uniformly bounded in u and v, and thus

$$\begin{aligned}\int_u^v h(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds = &- \varepsilon\delta\psi(u,v) + \epsilon\int_u^v (\partial_t\psi + L_2\psi)(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds \\ &+ \sqrt{\epsilon}\int_u^v L_1\psi(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds + \epsilon M_1(u,v) + \sqrt{\epsilon}M_2(u,v)\,.\end{aligned}$$

By the Itô isometry and the boundedness of the derivatives $\nabla_{x_1}\psi$ and $\nabla_{x_2}\psi$, the martingale term can be bounded by

$$\mathbb{E}\left[(M_i(u,v))^2\right] \leqslant C_i(v-u)\,, \quad 0 < C_i < \infty\,.$$

Hence

$$\mathbb{E}\left[\left(\int_u^v h(s, X^{\epsilon}_{1,s}, X^{\epsilon}_{2,s})ds\right)^2\right] \leqslant C\epsilon\,,$$

with a generic constant $0 < C < \infty$ that is independent of u, v and ϵ. ∎

Lemma 58 (Upper bound)**.** *Suppose that the Conditions U from page 134 hold true. Then*

$$|V^{\epsilon}(t,x) - V(t,x_1)| \leq C\sqrt{\epsilon}\,,$$

with $x = (x_1, x_2) \in D = D_s \times \mathbb{R}^{n_f}$, where V^{ϵ} is the solution of the original dynamic programming equation (5.2.9) and V is the solution of the limiting dynamic programming equation

(5.3.10). The constant and C depends on x and t, but is finite on every compact subset of $D \times [0,T]$.

Proof. The idea of the proof is to apply Itô's formula to $|y^\epsilon_s|^2$, where $y^\epsilon_s = Y^\epsilon_s - V(s, X^\epsilon_{1,s})$ satisfies the backward SDE

$$dy^\epsilon_s = -G^\epsilon(s, X^\epsilon_{1,s}, X^\epsilon_{2,s}, y^\epsilon_s, z^\epsilon_s)ds + z^\epsilon_s \cdot dW_s \tag{5.6.4}$$

where

$$z^\epsilon_s = Z^\epsilon_s - \left(\bar{C}^\top \nabla V(s, X^\epsilon_{1,s}),\, 0\right)^\top \qquad (\nabla V = \nabla_{x_1} V)$$

and

$$G^\epsilon(t, x_1, x_2, y, z) = G_1(t, x_1, x_2, y, z) + G^\epsilon_2(t, x_1, x_2, y, z)\,,$$

with

$$G_1 = f(t, x, y + V(t, x_1), z + (\bar{C}^\top \nabla V(t, x_1), 0)) - \bar{f}(t, x_1, V(t, x_1), \bar{C}^\top \nabla V(t, x_1))$$

$$G^\epsilon_2 = \left((A_{11} - \overline{A})x_1 + \frac{1}{\epsilon} A_{12} x_2\right) \cdot \nabla V(t, x_1) + \frac{1}{2}(C_1 C_1^\top - \bar{C}\bar{C}^\top)\nabla^2 V(t, x_1)\,.$$

We set $X^\epsilon_s = X^\epsilon_{\tau_D}$ for $s \in (\tau_D, T]$ when $\tau_D < T$. Then, by construction, $G_1(t, x, 0, 0)$, $x = (x_1, x_2) \in D_s \times \mathbb{R}^{n_f}$ is centered with respect to π and bounded (since the running cost is independent of x_2), therefore Lemma 57 implies that

$$\sup_{t \in [0,T]} \mathbb{E}\left[\left(\int_t^T G_1(s, X^\epsilon_{1,s}, X^\epsilon_{2,s}, 0, 0)ds\right)^2\right] \leqslant C_1 \epsilon\,, \tag{5.6.5}$$

The second contribution to the driver can be recast as $G^\epsilon_2 = (L - \bar{L})V$, with L_2 and $\bar{L}$ as given by (5.2.10) and (5.3.11) and thus, as $\epsilon \to 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(\int_t^T G_2^\epsilon(s, X_{1,s}^\epsilon, X_{2,s}^\epsilon, 0, 0)ds\right)^2\right] \leqslant C_2\epsilon \tag{5.6.6}$$

by the functional central limit theorem for diffusions with Lipschitz coefficients [74] ; cf. also Sec. 5.3.2. As a consequence of (5.6.5) and (5.6.6), we have $G^\epsilon \to 0$ in L^2, which, since $\mathbb{E}[|y_T^\epsilon|^2] \leqslant C_3\epsilon$, implies strong convergence of the solution of the corresponding backward SDE in L^2.

Specifically, since ∇V is bounded $\bar{D}_s$, Itô's formula applied to $|y_s^\epsilon|^2$, yields after an application of Gronwall's Lemma :

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |y_s^\epsilon|^2 + \int_t^T |z_s^\epsilon|^2\, ds\right] \leqslant \ell_D \mathbb{E}\left[\left(\int_t^T G^\epsilon(s, X_{1,s}^\epsilon, X_{2,s}^\epsilon, 0, 0)ds\right)^2\right] + \ell_D\mathbb{E}[|y_T^\epsilon|^2]$$

where the Lipschitz constant ℓ_D is independent of ϵ and finite for every compact subset $\bar{D}_s \subset \mathbb{R}^{n_s}$ by the boundedness of ∇V (since V is a classical solution and D_s in bounded). Hence $\mathbb{E}[|y_s^\epsilon|^2] \leq C_3\epsilon$ uniformly for $s \in [t, T]$, and by setting $s = t$, we obtain

$$|Y_t^\epsilon| = |V^\epsilon(t, x) - V(t, x_1)| \leq C\sqrt{\epsilon}$$

for a constant $C \in (0, \infty)$.

■

This proves Theorem 52.

Remark 59. *The condition of the periodicity of the coefficient w.r.t. the fast variable in [73] is to ensure the ergodicity of the fast variable that can holds in our case without periodicity by using the kalman rank condition (5.2.8)*

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عربي

هذه الأطروحة تهتم بدراسة وجود التحكم الأمثل لنظام معادلات تفاضلية عشوائية مباشرة-خلفية, بالإضافة الي ان معامل الانتشار يمكن أن يتغير (أي ليس بالضرورة موحد بيضاوي الشكل) وتطبيقاتها كدا و دراسة وجود حل وحيد للعادلة خلفية ذات بعد واحد .

في بادئ الأمر، تركز أطروحة علي النظرية العامة للمعادلات التفاضلية العشوائية مباشرة-خلفية ، ثم ، تتطرق الاطروحة الي مسألة وجود التحكم الأمثل.

وأخيرا، نسلط الضوء علي الدراسات العددية للمعادلات التفاضلية العشوائية المباشرة-الخلفية , كذا و دراسة التحكم الأمثل في الحالات الخطية و الغير الخطية، وقد تمت دراسة التطبيقات على نماذج من المباني، واخدت حالة مستشفى جامعة لوس انجلوس كمثال حي

Français

Cette thèse établit l'existence d'un contrôle optimal pour un système modélisé par une équation différentielle stochastique progressive-rétrograde (EDSPR) couplée, dans les cas, dégénéré et non- dégénéré. On montre l’existence et l’unicité pour des équations rétrogrades avec une condition logarithmique.

Dans une première partie, on présent la théorie générale des équations différentielle stochastiques progressive-rétrogrades, et on étudie la question d'existence d'un contrôle optimal.

Enfin, des études numériques portent sur les équations différentielles stochastiques progressive-rétrograde (EDSPR) couplées avec une homogénéisation des problèmes de control optimal stochastique dans le cas linéaire et non-linéaire. Des applications sur le modelé des bâtiments ont été étudiées, particulièrement le cas de l’hôpital universitaire de Los-Anglos a présenté.

English

The purpose of the present dissertation is to study existence of an optimal control whose dynamical system is driven by a coupled forward-backward stochastic differential equation. The thesis studied the case of possibly degenerate diffusion coefficient. An existence and uniqueness results on a one dimensional BSDE with logarithmic condition is also studied.

An application in high dimensional stochastic differential equations is given in the last chapter of the thesis with numerical results; a real case of the Los Angeles University hospital is studied. A numerical analysis of fully coupled FBSDEs is also presented.

Deutsch

Das Ziel dieser Arbeit ist die Untersuchung der Existenz einer optimalen Steuerung eines durch gekoppelte stochastische Vorwärts-/Rückwärtsdifferentialgleichungen gegebenen Systems. Dabei wird insbesondere der Fall degenerierter Diffusionskoeffizienten untersucht.

Eine Anwendung hochdimensionaler stochastischer Differentialgleichungen und numerische Ergebnisse werden anhand des Beispiels eines Universitätsklinikums von Los Angeles im letzten Kapitel der Arbeit vorgestellt.

Zusätzlich wird eine numerische Analyse der vollständig gekoppelten Vorwärts-Rückwärtsdifferentialgleichungen präsentiert.